Chapter 14
First-Order Properties

Abstract A first-order property of a structure is a property that can be expressed in first-order logic. Some important properties are first-order but many are not. We will see why finiteness, minimality, order-minimality, and being well ordered are not first-order, and how some such properties can be expressed in higher-order logics.

Keywords Finiteness · Pseudofinite structures · Well-ordered sets · Strong minimality · Second-order logic

14.1 Beyond First-Order

This book is about mathematical structures and how logic is used to study them. The term that has been used most often is "first-order." By now we know what a structure is, but what is not clear at all is what is a property of a structure. There are many distinguishing features that can be chosen to describe a structure, but so far we have only paid attention to properties that are expressible by first-order formulas. The straightforward grammar of the language finds its counterpart in the geometry of definable relations, and gives rise to natural approaches to various classification problems. Moreover, this geometry is not a superficial construct imported from the world of logic into the world of mathematics. In the cases of the fields of real and complex numbers, it coincides with the geometries that are studied by algebraists and analysts. But first-order logic has its weaknesses.

Every structure with an infinite domain has a proper elementary extension. It follows that first-order properties cannot fully characterize infinite structures. For every structure with an infinite domain, there is a structure that has exactly the same first-order properties, but is not isomorphic to it. For a formal system in which structures can be fully characterized by their properties, one has to look beyond what is first-order expressible. It is a big subject and we will only touch it lightly, but before that let us see some more examples of properties that are first-order, and some that are not.

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14.1.1 Finiteness

Finiteness is a significant property; it is also a prime example of a property that is not first-order. There are sentences that force structure to be infinite. For example, one can say in one first-order statement that a binary relation is a linear ordering in which every element has an immediate successor, or that a linear ordering is dense. All structures equipped with such relations are infinite. However, first-order theories axiomatizing classical mathematical structure such as graphs, groups, rings, and fields, all have models with finite domains. But if a theory has only finite models, then there is an upper bound on their size. They cannot be arbitrarily large. If a theory has models with arbitrarily large finite domains, then it also must have infinite models, in fact it has models of arbitrarily large infinite cardinalities. Let s see why.

Let T be a theory[1] that has models with arbitrarily large finite domains. Then, T also has a model with an infinite domain. To prove it, onsider the theory T' consisting of T and the set of sentences φ_n, for $n = 1, 2, 3, \ldots$, where φ_n says that there are n different elements. Each finite fragment of T' contains only finitely many sentences φ_n; and since T has models with arbitrarily large finite domains, each finite fragment of T' has a model. By the compactness theorem, T' has a model. This model is a model of T, and its domain is infinite, because all sentences φ_n are true in it.

The fact that finiteness cannot be captured in a first-order way has interesting consequences. Let T be a theory that has models with arbitrarily large finite domains, and let T_{fin} be the set of sentences that are true in all finite models of T. Since all those finite models are models of T, T_{fin} contains T, and, as we saw above, T_{fin} must have infinite models. An infinite model of T_{fin} is called *pseudofinite* . Pseudofinite models are infinite, but retain some traces of finiteness. For example, let T be the theory of linear orderings. Every finite linear ordering has a smallest and a largest element. This can be expressed in a first-order way, hence all models of T_{fin} also must have a smallest and a largest element.

14.1.2 Minimality and Order-Minimality

Minimality is not a first-order property. In Chap. 12, we saw that $(\mathbb{N}, <)$ is minimal, and if $(\mathbb{N}^*, <)$ is one of its proper elementary extension, than $\mathbb{N}^*$ contains an isomorphic copy of $(\mathbb{Z}, <)$. If c is an element in that copy, then the set defined by the formula $x < c$ is neither finite nor cofinite in $\mathbb{N}^*$. If there were a sentence φ in the language with one relation symbol $<$, such that a structure is minimal if an only if φ is true in it, then φ would have to be true in $(\mathbb{N}, <)$ and in $(\mathbb{N}^*, <)$, and that can't be.

[1] A theory is a set of first-order sentences.

A structure is *strongly minimal* if it is minimal and every structure that has the same first-order theory is also minimal. The structure $(\mathbb{N}, <)$, and its dual $(\mathbb{N}, >)$, are rare examples of structures that are minimal but not strongly minimal. No other examples are known at the time I am writing these notes. It is easy to see that trivial structures are strongly minimal. It can be shown that the field of complex numbers is also strongly minimal.

There is no need for the notion of strongly order-minimal structure. Julia Knight, Anand Pillay, and Charles Steinhorn proved in 1986 that if $(M, <, \dots)$ is an order-minimal structure then so is every structure that has the same first-order theory [19]. In particular, if $(M, <, \dots)$ is order-minimal, then so is its every elementary extension. In this sense, order-minimality is a first-order property—the theory of the structure decides whether it is order-minimal or not, although there is no first-order theory in the language with just $<$ that decides that. If there were such a theory, it would be true in $(\mathbb{Q}, <)$, which is order-minimal, but then it would be also true in $(\mathbb{Q}, +, \cdot, <)$ (because it only speaks about $<$) which is as far from being order-minimal as possible.

14.2 Well-Ordered Sets and Second-Order Logic

Many properties of orderings are first-order. To have a first or last element, to be discrete, to be dense—these are all properties that can be expressed by a single first-order sentence about the ordering, but there is an important property of orderings that is not first-order. A linearly ordered set $(A, <)$ is *well-ordered* if every nonempty subset of A has least element. To see why this is not a first-order property, let us look again at $(\mathbb{N}, <)$ and its proper elementary extension $(\mathbb{N}^*, <)$. The natural numbers are well-ordered. Indeed, this is one of their most fundamental properties: every nonempty set of natural numbers has a least element. It is one of those clear and basic principles that do not require a proof. The integers $\mathbb{Z}$ are not well-ordered—the whole set $\mathbb{Z}$ does not have a least element, and hence every ordered set that contains a copy of $(\mathbb{Z}, <)$ is not well-ordered as well. Since $(\mathbb{N}, <)$ is well-ordered and its elementary extension $(\mathbb{N}^*, <)$ is not, it follows that the property of being well-ordered is not expressible by a first-order theory.

Let us now see how the property of being well-ordered can be expressed formally. We will do it in an extension of first-order logic, known as the *monadic second-order logic*. In the vocabulary of the monadic second-order logic, together with all the symbols of first-order logic, there are variables of the second-order sort X_1, X_2, $\dots$; and there is an additional relation symbol $\in$, giving rise to atomic formulas of the form $x_i \in X_j$, for all i and j. In a structure, the first-order variables are interpreted in the usual way by individual elements of the domain, and the second order variables are interpreted by subsets of the domain. If x_i is interpreted by an element a and the variable X_j is interpreted by set of elements A, then, under this evaluation, the atomic formula $x_i \in X_j$ is true if an only if a is an element of the set A.

Here is a sentence that in a direct way expresses that $<$ is a well-ordering:

$$\forall X[\exists x\ (x \in X) \Longrightarrow (\exists x(x \in X \wedge \forall y(y \in X \Longrightarrow (y = x \vee x < y))))].$$

In the monadic second-order logic we can only quantify over subsets of the domain, as in the sentence above. In full second-order logic we can also quantify over arbitrary subsets of all Cartesian powers of the domain.

Second-order logic has a tremendous expressive power. Many infinite structures, such as $(\mathbb{N}, +, \times)$, can be completely characterized by their second-order properties. In fact, much of modern mathematics can be formalized using an axiom systems just about the second-order properties of $(\mathbb{N}, +, \times)$. Why then have we not used this logic to begin with? There is absolutely nothing wrong with the second-order logic but one has to use it with caution and sharper technical skills are necessary to do it well. When we approach a structure such as $(\mathbb{N}, +, \times)$ in a first-order fashion, we stand on a reasonably firm ground. The quantifiers $\exists$ and $\forall$ range over individual elements of the domain—the natural numbers. We have a good sense of what those elements are, and what it means that there is an element with some property, or that all elements have it. The moment we use $\exists X$ or $\forall X$, we need to pause to think a bit. What is the range of those quantifiers? They range over arbitrary sets of natural numbers, i.e. over the power set of $\mathbb{N}$. This is a much more elusive domain. To begin with, it is much larger than $\mathbb{N}$. The set of natural numbers is countable, its power set is not. All kind of questions arise. Some are of quite subtle character, for example, if a set A of natural numbers is defined by a second-order formula involving a quantifier $\forall X$, should A be considered in the range of this quantifier or not? The problem here is that if A allowed, then one can suspect some circularity—it seems that A is used in its own definition. Is this something that we want? There are other complications, due to the fact that many useful tools of first-order logic, such as the compactness theorem, are no longer available in second-order logic.

There is also the third-order logic, in which one can quantify sets of subsets of the domain, the fourth-order logic with quantifiers ranging over sets of sets of subsets of the domain. There is a whole hierarchy of logics that are used to express properties of sets in the hierarchy of sets that can be built over the domain of a structure. There are many other formal systems that extend first-order logic. There are systems in which the rules allow us to build infinite conjunctions and disjunctions of formulas, and there are systems in which one can also quantify infinite strings of variables. There are powerful logics obtained by adding new quantifiers. There is much more, but all those other logics always stand in comparison to first-order logic. They are either extensions or fragments of it and none of them has a model theory that is as well-developed as the model theory of first-order logic.

Exercises

Exercise 14.1 *Find a second order sentence that is true in a structure if and only if the structure is finite. Hint: Use the fact that a set* X *is infinite if and only if there is a one-to-one function* $f : X \longrightarrow X$*, such that for some* $b \in X$*,* $f(a) \neq b$ *for all* $a \in X$.

Exercise 14.2 * *Use the previous exercise to express minimality of structures as an axiom schema of second order logic. Hint: For each formula* $\varphi(x)$ *of the language of the structure, include axioms of the form "either the set of* x *such that* $\varphi(x)$ *is finite, or the set of* x *such that* $\neg\varphi(x)$ *is finite."*

Exercise 14.3 * *Modify your answer to the previous exercise to express order-minimality of structures as an axiom schema of second order logic.*

Exercise 14.4 * *Suppose that all second-order sentences that are true in* $(\mathbb{N}, +, \cdot)$ *are also true in* $(M, \oplus, \odot)$*. Prove that* $(M, \oplus, \odot)$ *is isomorphic to* $(\mathbb{N}, +, \cdot)$*. Hint: Let* 0_M *and* 1_M *be the first two elements in the ordering of* M*. Define* $f : \mathbb{N} \longrightarrow M$ *by induction:* $f(0) = 0_M$*, and* $f(n+1) = f(n) \oplus 1_M$*. Let* $N = \{f(n) : n \in \mathbb{N}\}$*. Show that* $(N, \oplus, \odot)$ *is isomorphic to* $(\mathbb{N}, +, \cdot)$*, and that* $N = M$*.*