Chapter 9
Minimal and Order-Minimal Structures

Abstract Before we get into details regarding number structures, we will examine definability in cases that are easier to analyze. We define two important classes of structures: *minimal*, in Definition 9.1, and *order-minimal*, in Definition 9.4. The important concepts of *type* and *symmetry* were already introduced in Chap. 2; here we define them in general model-theoretic terms and use them to analyze the orderings of the sets of natural numbers, integers, and rationals.

Keywords Minimal structures · Order minimal structures · Types of elements in ordered sets · Symmetries of ordered sets · Order types of sets of numbers

9.1 Types, Symmetries, and Minimal Structures

For any structure $\mathfrak{A}$, any element of the domain is parametrically definable; hence all finite subsets of the domain are definable as well. For example, if a, b, and c are elements of the domain, then the formula $(x = a) \vee (x = b) \vee (x = c)$ defines $\{a, b, c\}$. If a set X is defined by a formula $\varphi(x)$, then its complement, i.e. the set of all elements of the domain that are not in X, is defined by $\neg\varphi(x)$. A set whose complement is finite is called *cofinite*. It follows that all finite and all cofinite subsets of the domain of any structure are parametrically definable. This brings us to the following definition.

Definition 9.1 A structure $\mathfrak{A}$ is *minimal* if every subset of its domain that is parametrically definable is either finite or cofinite.

Since all structures with finite domains are minimal, we will be only interested in structures with infinite domains. In a minimal structure, all parametrically definable subsets can be defined by formulas not involving the relation symbols of the language, other than $=$. Notice that the definition involves only subsets of the domain of the structure, and not subsets of its Cartesian powers. The definition cannot be made stronger by demanding that parametrically definable subsets of

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all Cartesian powers are also either finite or cofinite, because for structures with infinite domains it is never the case. For example, if A is infinite, then the subset of A^2 defined by the formula $x = y$ is neither finite nor cofinite.

In Chap. 2 we defined a notion of symmetry of a graph. Now we will generalize this definition to arbitrary structures. Informally, a symmetry is a rearrangement of the elements of a structure that does not change the way in which the elements are related. Recall that a permutation of a set A is a function $f : A \longrightarrow A$ that is one-to-one and onto.

Definition 9.2 Let $\mathfrak{A}$ be a structure with domain A. A permutation f of A is a *symmetry* of $\mathfrak{A}$ if for every relation symbol R of arity n in the language of $\mathfrak{A}$, for all $a_1, a_2, \ldots, a_n$ in A the sentence

$$R(a_1, a_2, \ldots, a_n) \Longleftrightarrow R(f(a_1), f(a_2), \ldots, f(a_n)) \qquad (*)$$

is true in $\overline{\mathfrak{A}}$.[1]

Symmetries not only preserve relations, they also preserve all definable properties. This is the content of the next theorem. It is a very useful tool for detecting undefinability of elements and sets.

Theorem 9.1 *Let $\mathfrak{A}$ be a structure with domain A and let f be a symmetry of $\mathfrak{A}$. Then, for any first-order formula $\varphi(x_1, x_2, \ldots, x_n)$ and all $a_1, a_2, \ldots, a_n$ in A the sentence*

$$\varphi(a_1, a_2, \ldots, a_n) \Longleftrightarrow \varphi(f(a_1), f(a_2), \ldots, f(a_n)) \qquad (**)$$

is true in $\overline{\mathfrak{A}}$.

We will use Theorem 9.1 to show that if X is a subset of A that is defined by a formula without parameters $\varphi(x)$, and f is a symmetry of $\mathfrak{A}$, then the image of X under f, i.e the set $\{f(a) : a \in X\}$ is X. Let us see why. This will be a routine argument often called "chasing the definitions."

If a is an element of X, then $\varphi(a)$ holds in $\overline{\mathfrak{A}}$, hence, because f is a symmetry, $\varphi(f(a))$ must hold as well, which means that $f(a)$ is in X. This shows that the image of X under the symmetry is contained in X. To see that it is all of X, suppose that b is in X. Because f is onto, there is an a in A such that $f(a) = b$. Because b is in X, $\varphi(b)$ holds, which means that $\varphi(f(a))$ holds, and this implies that $\varphi(a)$ holds. Now, since $\varphi(a)$ holds, a is in X and this concludes the argument.

A similar argument shows that if X is a subset of A defined by a formula with parameters $\varphi(x, b_1, b_2, \ldots, b_n)$, and f is a symmetry that fixes all parameters, i.e. $f(b_1) = b_1$, $f(b_2) = b_2, \ldots, f(b_n) = b_n$, then the image of X must be X. Hence, if for a given X and any finite set of parameters we can find a symmetry

[1] Symmetries of structures are known in mathematics as *automorphisms*.

that fixes the parameters and moves some element of X outside X, that will show that X is not parametrically definable in the structure. We will use this to establish non-minimality of certain basic structures.

In Chap. 2 we defined types of vertices in graphs, now we consider types in arbitrary structures.

Definition 9.3 If $a_1, a_2, \ldots, a_n$ is a finite sequence of elements of the domain of a structure $\mathfrak{A}$, then the *type* of $a_1, a_2, \ldots, a_n$ in $\mathfrak{A}$ is the set of all first-order formulas $\varphi(x_1, x_2, \ldots, x_n)$ such that $\varphi(a_1, a_2, \ldots, a_n)$ holds in $\overline{\mathfrak{A}}$.

The type of a finite sequence is the set of all its first-order properties. Types are easy to define, but it does not mean that they are easy to describe. The classification of all possible types realized in a structure is an important task. Sometimes all possible types can be easily classified according to some well-defined criteria, and we will see how it is done in a few cases. Sometimes such an analysis involves much hard work. Sometimes it cannot be achieved at all, and this is not because we do not know how to do it, but because we can prove that a classification of the kind we consider intelligible does not exist.

Theorem 9.1 can also be formulated as a statement about types: if $f : A \longrightarrow A$ is a symmetry of $\mathfrak{A}$, then for all $a_1, a_2, \ldots, a_n$ in A the type of $a_1, a_2, \ldots, a_n$ in $\mathfrak{A}$ is the same as the type of $f(a_1), f(a_2), \ldots, f(a_n)$. Symmetries preserve types.

9.2 Trivial Structures

Recall that a trivial structure consists of a domain with no relations. As one should expect, trivial structures are minimal. We will prove it. The proof is an argument about trivial structures, and it is rather trivial itself, but to a novice the details may not be that obvious. The argument uses Theorem 9.1 and some formal trickery. It is worth reading, as it shows the power of the methods that are the subject of this book. Paradoxically, when we get to discuss less trivial structures in the following section, the arguments may be easier to follow, so you may want to read about those other examples first.

Let A be the domain of a trivial structure $\mathfrak{A}$, and let $a_1, a_2, \ldots, a_n$ be a sequence of parameters from A. Let $\varphi(x, a_1, a_2, \ldots, a_n)$ be formula with one free variable x and the parameters as displayed, and let X be the subset of A defined by it. If X is a subset of $\{a_1, a_2, \ldots, a_n\}$, it is finite. Otherwise, it contains an element b outside the set of parameters. We will show that then X must contain all such elements; hence it is cofinite. So let us assume that b is not one of the parameters.

Since b is in the set defined by $\varphi(x, a_1, a_2, \ldots, a_n)$, $\varphi(b, a_1, a_2, \ldots, a_n)$ holds in $\overline{\mathfrak{A}}$. Let c be another element of A that is not in $\{a_1, a_2, \ldots, a_n\}$. Let $f : A \longrightarrow A$ be such that $f(b) = c$, $f(c) = b$, and for all other a in A, $f(a) = a$. Clearly, f is a permutation of A, and since $\mathfrak{A}$ has no relations, f vacuously satisfies the condition in Definition 9.2. Moreover, f is not only a symmetry of $\mathfrak{A}$. Since it fixes all parameters $a_1, a_2, \ldots, a_n$, it is also a symmetry of $(A, a_1, a_2, \ldots, a_n)$. Then, it follows from Theorem 9.1 that $\varphi(b, a_1, a_2, \ldots, a_n)$ holds in $\overline{\mathfrak{A}}$, proving that c is in the set defined by $\varphi(x, a_1, a_2, \ldots, a_n)$, and proving our claim.

9.3 The Ordering of the Natural Numbers

Next, we will consider $(\mathbb{N}, <)$, consisting of the natural numbers with their ordering. $(\mathbb{N}, <)$ is minimal, but we will not prove it yet. This will be done in Chap. 12, where a powerful technique is described that will allow us to give a slick proof. For now, let us examine the types of elements in $(\mathbb{N}, <)$.

We have already seen that every natural number is definable in $(\mathbb{N}, <)$. Let us repeat the argument in a more general setting. If a and b are elements of an ordered set $(A, <)$, $a < b$, and there are no elements between a and b, then we call b the *successor* of a, and we call a the *predecessor* of b. If b is a successor of a, then b is the only element x of A such that the following holds in $(A, <)$:

$$a < x \wedge \forall y[a < y \Longrightarrow [(x < y) \vee (x = y)].$$

This shows that if b is a successor of a, then b is definable in $(A, <, a)$. It follows that if a is definable in $(A, <)$, then so is b. A similar argument applies to predecessors, hence, if b is a successor of a, then a is definable in $(A, <)$ if and only if b is definable.[2]

Since 0 is the least element of $\mathbb{N}$ and this property is expressible by a first-order formula, 0 is definable in $(\mathbb{N}, <)$. Since 1 is a successor of 0, 1 is definable, then 2 is definable as a successor of 1, and so on. Every natural number is definable. For every natural number there is a first-order property that identifies that number uniquely in $(\mathbb{N}, <)$. It implies that if a and b are different natural numbers then the types of a and b in $(\mathbb{N}, <)$ are different. Every number has its own unique type. Also, since every symmetry of a structure maps elements to elements of the same type, it means that $(\mathbb{N}, <)$ has no symmetries, other than the identity function. For each structure the identity function $f(x) = x$ on the domain is a symmetry. We call it trivial. A structure whose only symmetry is the trivial symmetry is called *rigid*. We have shown that $(\mathbb{N}, <)$ is rigid.

[2] A reminder: when we just say *definable*, we mean definable without parameters in the original language of the structure. If parameters are involved we say *parametrically definable.*

9.4 The Ordering of the Integers

Let us now compare $(\mathbb{N}, <)$ with the ordered set of the integers $(\mathbb{Z}, <)$. While every natural number is definable in $(\mathbb{N}, <)$, no integer is definable in $(\mathbb{Z}, <)$. Let us see why. Suppose a number a has a certain first-order property $\varphi(x)$. Let b be any other number. Let the function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by $f(x) = x + b - a$. If a is less than b, f shifts all numbers up by $b - a$, otherwise, it shifts them down by $a - b$. Since f preserves the ordering, it is a symmetry of $(\mathbb{Z}, <)$. Also $f(a) = a+b-a = b$. Hence, by Theorem 9.1, b also has the property $\varphi(x)$, and this proves that the set defined by $\varphi(x)$ in $(\mathbb{Z}, <)$ contains all integers.

The same argument shows that in $(\mathbb{Z}, <)$ there is only one type of single elements. We have shown, that if a and b are integers, then there is a symmetry f of $(\mathbb{Z}, <)$ such that $f(a) = b$; hence a and b have the same type in $(\mathbb{Z}, <)$. You can see that types of elements very much depend on the ambient structure. Each natural number has its own distinct type in $(\mathbb{N}, <)$, but they all share the same type in $(\mathbb{Z}, <)$.

There is only one type of single elements in $(\mathbb{Z}, <)$, but there are many types of ordered pairs. First let us observe that in any ordered structure, if $a < b$, then the types of (a, b) and (b, a) are different. The formula $x_1 < x_2$ is in the type of (a, b), and its negation is in the type of (b, a). Because of this, to avoid notational complications, we will only consider types of ordered pairs and sequences that are ordered from the lowest to the highest with respect to the ordering of the structure. In other words, we will talk about types of sequences $(a_1, a_2, \ldots, a_n)$ such that $a_1 < a_2 < \cdots < a_n$. As we will see, in some ordered sets, every two such ordered sequences of the same length have the same type. Not in $(\mathbb{Z}, <)$ though. If a and b are integers, and $a < b$, then let the *distance* between a and b be $b - a$. Let n be a natural number. The property "the distance between a and b is n" is definable in $(\mathbb{Z}, <)$. There is a specific definition for each n, and those definitions grow longer as n gets larger, but they all follow the same pattern. Here is a defining formula for $n = 3$:

$$\exists v \exists w[(x < v)\wedge(v < w)\wedge(w < y)\wedge\forall z[(x < z)\wedge(z < y) \Longrightarrow ((z{=}v)\vee(z{=}w))]].$$

If a and b are integers, and $a < b$, than the type of (a, b) is completely determined by the distance between a and b. That is because if (a, b) and (c, d) are two ordered pairs such that $a < b$, $c < d$, and the distances between a and b, and c and d are equal, then the symmetry that maps a onto c, must also map b onto d, which shows that the types (a, b) and (c, d) are the same. A similar argument applies to arbitrary finite ordered sequences. The type of an ordered increasing sequence $a_1, a_2, \ldots, a_n$ is completely determined by the distances between consecutive elements in the sequence.

$(\mathbb{Z}, <)$ is not minimal. For example, the set of negative numbers is defined by $x < 0$, and it is neither finite nor cofinite. There are other examples of definable sets, but not that many; $(\mathbb{Z}, <)$ is almost minimal. This last statement has a precise meaning, and to make it precise we need a couple of definitions.

Let $(A, <)$ be a linearly ordered set. For all a and b in A, the formula $(a < x) \wedge (x < b)$ defines the interval of all numbers between a and b. We call it the *open interval* between a and b. If a is not less than b, then the set defined by $(a < x) \wedge (x < b)$ is empty. We consider the empty set an open interval. The set A and sets of the form $\{x : x < a\}$ and $\{x : b < x\}$ are also open intervals of $(A, <)$. All open intervals of $(A, <)$ are parametrically definable and, so are the intervals of one of the forms

$$\{x : [(a < x) \wedge (x < b)] \vee (x = a)\},$$

$$\{x : [(a < x) \wedge (x < b)] \vee (x = b)\},$$

$$\{x : [(a < x) \wedge (x < b)] \vee (x = a) \vee (x = b)\}.$$

We call all such sets *intervals* of $(A, <)$.

For a, b, c and d in A, the formula $[(a < x) \wedge (x < b)] \vee [(c < x) \wedge (x < d)]$ defines the union of two open intervals. In a similar fashion one can write definitions for every finite union of intervals, including those that are unbounded at one end. For every finite set of intervals in a linearly ordered set, its is parametrically definable.

Definition 9.4 An *ordered structure* is a structure that includes a relation linearly ordering its domain. In particular, linearly ordered sets are ordered structures. An ordered structure is *order-minimal* if every subset of its domain that is parametrically definable is the union of finitely many intervals.[3]

One can show that $(\mathbb{Z}, <)$ is order-minimal, and the proof is similar to the one given in Chap. 12 that shows that $(\mathbb{N}, <)$ is minimal.

There are general theories of minimal and order-minimal structures, with important applications in classical algebra and analysis. Parametrically definable subsets of the domain of minimal or order-minimal structure are as simple as definable sets can be, but this is not implying anything directly about multi-dimensional sets that are definable in higher Cartesian powers of the domain. Those sets are more complex, but the theory provides precise insights into what this complexity is.

9.5 The Additive Structure of the Integers

Let us take a quick look and the additive structure of the integers. To fully describe the types and the subsets of $\mathbb{Z}$ that are parametrically definable in $(\mathbb{Z}, +)$ would take us too far into mathematics proper. Instead, we will make two important observations. The first is that the ordering is not definable in $(\mathbb{Z}, +)$, and the second is that $(\mathbb{Z}, +)$ is not minimal.

[3] In model theory, order-minimal structures are called *o-minimal*.

To prove that the ordering $<$ is not definable in $(\mathbb{Z}, +)$, we will use Theorem 9.1. Let f be the function that maps each element to its opposite: $f(x) = -x$. For all integers a, b, and c, if $a + b = c$, then by multiplying both sides by (-1) we get that $(-a) = (-b) + (-c)$. Hence, f is a symmetry of $(\mathbb{Z}, +)$; it preserves the addition relation, but does not preserve the ordering. It reverses it. If a formula $\varphi(x, y)$ in the first-order language of $(\mathbb{Z}, +)$ defined the ordering, then $\varphi(1, 2)$ would hold in $(\mathbb{Z}, +)$, and because f is a symmetry, $\varphi(-1, -2)$ would hold as well, which is a contradiction, showing that there can be no such definition.

The formula $\exists y[y + y = x]$ defines the set of even numbers and that set is neither finite nor cofinite in $\mathbb{Z}$. Hence, $(\mathbb{Z}, +)$ is not minimal. $(\mathbb{Z}, +)$ is not an ordered structure so we cannot ask whether it is order-minimal, but $(\mathbb{Z}, +, <)$ is ordered, and the same argument shows that this structure is not order-minimal.

One could ask whether addition is definable $(\mathbb{Z}, <)$. It is a legitimate question, especially since addition is in a strict sense determined by the ordering. For every integer a, $a + 1$ is the successor of a, and, as we have seen, the successor of a is definable in $(\mathbb{Z}, <)$ by a formula with parameter a. Similarly, for positive b, $a + b$ can be defined as the b-th successor of a. Let us see how it is done for $b = 3$. Let $s(x, y)$ be the formula of $(\mathbb{Z}, <)$ defining the relation "y is a successor of x." Then the following formula defines the relation $x + 3 = y$:

$$\exists v \exists w[s(x, v) \wedge s(v, w) \wedge s(w, y)].$$

For every natural number n we can define the relation $x + n = y$, but there is no single formula of $(\mathbb{Z}, <)$ that would define the relation $0 < y \wedge (x + y = z)$, even if parameters are allowed. This statement follows from order-minimality of $(\mathbb{Z}, <)$, but we can directly prove a weaker statement, namely that the relation $x + y = z$ is not definable without parameters in $(\mathbb{Z}, <)$. To this end, suppose $\varphi(x, y, z)$ defines addition in $(\mathbb{Z}, <)$. Consider $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(x) = x + 1$. Since f is a symmetry of $(\mathbb{Z}, <)$, $\varphi(0, 0, 0)$ holds in $(\mathbb{Z}, <)$, and $f(0) = 1$; hence $\varphi(1, 1, 1)$ must also hold, telling us that 1+1=1, which is a contradiction.

The above argument cannot be used to prove that addition is not parametrically definable in $(\mathbb{Z}, <)$, because that would require symmetries that fix finite sequences of parameters, but there are no such symmetries. As soon as one integer is fixed, the whole structure becomes rigid. For any a in $\mathbb{Z}$, every integer is definable in $(\mathbb{Z}, <, a)$, either as an iterated successor or predecessor of a.

9.6 The Ordering of the Rational Numbers

In $(\mathbb{N}, <)$ every element has its own unique type. This implies that $(\mathbb{N}, <)$ is rigid. It has no nontrivial symmetries. The ordering of $\mathbb{Z}$ has symmetries, but not many. Every symmetry of $(\mathbb{Z}, <)$ is a shift moving all elements up, or all elements down. In other words, every symmetry of $(\mathbb{Z}, <)$ is of the form $f(x) = x + n$, for some integer n. The order structure of $\mathbb{Z}$ becomes rigid once we fix one of the elements, since then no shift can be applied.

The dense linear ordering $(\mathbb{Q}, <)$ has lots and lots of symmetries. There is no symmetry of $(\mathbb{Z}, <)$ that would fix 0 and move 1 to 2, but $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ defined by $f(x) = 2x$ is a symmetry of $(\mathbb{Q}, <)$ and does it. $(\mathbb{Q}, <)$ is much more elastic than $(\mathbb{Z}, <)$. It is not hard to show that if $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ are increasing sequences of rational numbers, then there is a symmetry f such that $f(p_1) = q_1$, $f(p_2) = q_2, \ldots, f(p_n) = q_n$. It follows that not only all single rational numbers share the same type, but for each n, all increasing sequence of n rational numbers share the same type. Each pair (p_1, q_1) has the same type as any other pair (p_2, q_2) as long as they are ordered the same way. The notion of distance that was used to show that there are many types of ordered pairs in $(\mathbb{Z}, <)$ is meaningless in $(\mathbb{Q}, <)$.[4]

Using symmetries, it is not difficult to show that the structure $(\mathbb{Q}, <)$ is order-minimal. A more advanced reader can try to prove it as an exercise. Hints are provided in the exercise section. It is a bit more difficult to show that the ordered additive structure $(\mathbb{Q}, +, <)$ is also order-minimal. We will not go over details of the proof, but let us just see why the formula $\exists y[y + y = x]$ that defines even numbers in $(\mathbb{Z}, +, <)$ does not do that in $(\mathbb{Q}, +, <)$. The simple reason is that while in $\mathbb{Z}$ only even numbers are divisible by 2, in $\mathbb{Q}$ all numbers are. This is what the rational numbers were made for. A fraction of a fraction is again a fraction. By extending the number system from $\mathbb{Z}$ to $\mathbb{Q}$, we made the domain larger, but less complex because some specific properties, such as evenness, lost their meaning. Every rational number is even.

Something interesting is happening here. While the set of rational numbers $\mathbb{Q}$ is richer than the set of integers $\mathbb{Z}$, and the discrete ordering of $\mathbb{Z}$ seems less complex the dense ordering of $\mathbb{Q}$, nevertheless, the logical structure of $(\mathbb{Q}, +, <)$ is simpler than that of $(\mathbb{Z}, +, <)$. This pattern continues. We will see that $(\mathbb{R}, +, \cdot)$ is simpler than $(\mathbb{Q}, +, \cdot)$, and at the end that the field of complex numbers $(\mathbb{C}, +, \cdot)$, despite its name, is the simplest of them all—it is in fact minimal. To talk about all of that we need to say more about the geometry of definable sets in all those structures. We will do this in the next chapter.

Exercises

Exercise 9.1 *Show that the only subsets of the domain of a trivial structure that are definable without parameters are the empty set and the whole domain. Hint: Suppose that a and b are elements of the domain, and $\varphi(a)$ holds in the structure. Define a symmetry f such that $f(a) = b$.*

Exercise 9.2 *Show that every integer is definable in* $(\mathbb{Z}, <, 0)$. *Hint: See Exercise* 8.4.

[4]This should not be confused with the natural notion of distance for rational numbers. It is definable in $(\mathbb{Q}, +, <)$, but not in $(\mathbb{Q}, <)$.

Exercise 9.3 * *Prove that* $(\mathbb{Q}, <)$ *is order-minimal. Hint: Suppose that* $p_1, p_2 \ldots, p_n$ *is an increasing sequence of rational numbers, and the set* X *defined by a formula* $\varphi(x, p_1, p_2, \ldots, p_n)$ *in* $(\mathbb{Q}, <)$ *is nonempty. By defining an appropriate symmetry of* $(\mathbb{Q}, <, p_1, p_2, \ldots, p_n)$ *show that if* X *has a nonempty intersection with one of the intervals* $\{x : x < p_1\}$*,* $\{x : p_n < x\}$*, and* $\{x : p_i < x < p_{i+1}\}$*, for* $i = 1, \ldots, n-1$*, then* X *contains that whole interval.*