Chapter 4
Seeing the Number Structures

I thought, Prime Number. A positive integer not divisible. But what was the rest of it? What else about primes? What else about integers?

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Abstract In the previous chapter, we introduced and named an actually infinite set. The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. What is the *structure* of this set? We will give a simple answer to this question, and then we will proceed with a reconstruction of the arithmetic structures of the integers and the rational numbers in terms of first-order logic. The reconstruction is technical and rather tedious, but it serves as a good example of how some mathematical structures can bee seen with the eyes of logic inside other structures. This chapter can be skipped on the first reading, but it should not be forgotten.

Keywords Set notation · Linearly ordered sets · Integers · Fractions · Rational numbers · Equivalence relations · Densely ordered sets

4.1 What Is the Structure of the Natural Numbers?

From our point of view, the question in the title of this section is ill-posed. This whole book is about mathematical structures understood in a specific way. To study a mathematical structure one needs to define it first. A structure is a set with a set of relations on it. We have the set $\mathbb{N}$, but we have not yet defined any relations. The set $\mathbb{N}$ itself comes with an empty set of relations, and as such is not different (as a structure) from any other infinite set of the same size with no relations on it.[1] It is up to us what relations we want to consider. There is a vast supply of relations, and they can be put together to form many structures.

[1] We have to wait until Chap. 6 for a discussion of sizes of infinite sets.

R. Kossak, *Mathematical Logic*, Springer Graduate Texts in Philosophy 3,
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Typically, in mathematics, a relation is defined by a rule that allows one to verify whether given objects are related or not. Often, but not always, it is given a special name. Here is an example that I just made up: say that natural numbers m and n are *equidigital* if their decimal representations have the same number of different digits. According to this rule, 123123 and 987 are equidigital, and 2334 and 1234 are not. Equidigitality is a perfectly well-defined relation. As an exercise you may want to define a few similar relations on your own. Some such relations may be of special mathematical interest, most are meaningless. In this chapter we will examine some basic relations that are essential in the study of number structures. We will talk about the ordering, addition, and multiplication of numbers, but before we get into details, I need to introduce more notation.

4.1.1 Sets and Set Notation

Since most mathematical objects can be considered as sets, we need to say more about what sets are, and we will need some notation to do that. Initially, one can think of sets as just collections of objects, but not much is gained by replacing the word "set" by the word "collection." The question, "What is a set?" has many answers. Georg Cantor who created set theory in an effort to generalize concepts of counting and number beyond the finite, wrote, "A set is a Many that allows itself to be thought of as a One." What it could mean requires a thorough discussion, but for now we only need to introduce some notational conventions.

Consider the set of letters in the word "set." Let us call it S. We can also say, let $S = \{s, e, t\}$. We could also define the same set by saying that $S = \{e, s, t\}$ because in sets the order of elements is irrelevant. Another way to define the same set S is

$$S = \{x : x \text{ is a letter in the word "set"}\}.$$

Here we think of x as a variable in some formula expressing a property of unspecified objects of some kind. Often this formula will be written in a formal language, but not always, as is the case of this definition of S.

In general, $\{x : \varphi(x)\}$ is the set of elements x which have a certain property $\varphi(x)$. If a set is small, we can just list all its elements. For larger sets, the last form of the definition above is most convenient.

The elements of a set are also called its members. Thus, in the example above, e is a member of the set S we defined above, but a is not. The commonly used symbol for the membership relation is $\in$. In symbols: $e \in S$, and $a \notin S$.

Since we pay much attention to the separation of syntax and semantics, a caveat is needed. The set theoretic notation just introduced is not a part of the formalism of first-order logic. We use this notation and symbols to abbreviate mathematical statements. Later, we will discuss formal theories of sets, and then the membership relation symbol $\in$ will be also used for a relation symbol of first-order logic.

4.1.2 Language of Formal Arithmetic

In the chapter on first-order logic we introduced three relation symbols L, A, and M. The symbol L was interpreted in the domain of natural numbers as the relation "less than," and A and M as addition and multiplication both considered as relations. Using the set notation and logical symbols, L was interpreted as the set

$$\{(k, l) : k \in \mathbb{N} \wedge l \in \mathbb{N} \wedge k < l\},$$

A as the set

$$\{(k, l, m) : k \in \mathbb{N} \wedge l \in \mathbb{N} \wedge m \in \mathbb{N} \wedge k + l = m\},$$

and M as the set

$$\{(k, l, m) : k \in \mathbb{N} \wedge l \in \mathbb{N} \wedge m \in \mathbb{N} \wedge k \cdot l = m\}.$$

We will call those sets Less, Add, and Mult, respectively. For example (0, 0, 0) and $(1, 2, 3)$ are in Add, but $(1, 1, 1)$ is not.

Let me stress one more time that while L, A, and M are formal relation symbols of first-order logic, the names Less, Add, and Mult are informal mathematical abbreviations.

The three relations can be used in different configurations to form different structures. We will discuss four of them: the natural numbers with the ordering relation ($\mathbb{N}$, Less), the additive structure ($\mathbb{N}$, Add), the multiplicative structure ($\mathbb{N}$, Mult), and the most interesting ($\mathbb{N}$, Add, Mult) which we will call the *standard model of arithmetic*.

4.1.3 Linearly Ordered Sets

One does not need advanced mathematics to analyze ($\mathbb{N}$, Less). The picture below shows what it looks like. It is a sequence with a first element on the left, and it extends without a bound to the right. It is an infinitely long queue, and it is not important that it is illustrated as beginning on the left and going to the right, although this is how it is usually thought of, because we like to see the numbers on the number line as increasing from left to right. In ($\mathbb{N}$, Less), it is not important that the elements of the set $\mathbb{N}$ are numbers. It only matters that the set $\mathbb{N}$ is infinite and that the relation Less orders its elements in a particular way.

The picture below tells us the whole story.

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We will use $(\mathbb{N}, \mathsf{Less})$ to illustrate an approach that we are going to apply later to more complex structures.

For now, we will consider first-order logic in the language with only one binary relation symbol L. We will start with a formal definition involving three properties formalized in this language. Those statements are about structures of the forms (M, R), where M is a set, and R is a binary relation on M. In other words, we interpret the symbol L, as the relation R on M. The meaning of the three conditions is explained right after the definition.

Definition 4.1 Let R be a binary relation on a set M. We say that R *linearly orders* M, or that R is a *linear ordering*, if the following three statements are true in (M, R), when L is interpreted as R:

(O1) $\forall x \neg L(x, x)$.
(O2) $\forall x \forall y [(x = y) \vee L(x, y) \vee L(y, x)]$.
(O3) $\forall x \forall y \forall z [(L(x, y) \wedge L(y, z)) \Longrightarrow L(x, z)]$.

Think of the elements of the set M forming a line, like the natural numbers in the illustration of $(\mathbb{N}, \mathsf{Less})$ above. The idea here is that the line is determined by the relation R in the sense that a stands before b if and only if the pair (a, b) is in the set R. With this in mind, you can see that *O1* says that no element stands before itself; *O2* says for any two elements a and b, either a stands before b, or b before a; and *O3* says that if a stands before b and b before c, then a is before c as well.

Clearly, $(\mathbb{N}, \mathsf{Less})$ has all three properties, hence Less linearly orders $\mathbb{N}$. We will see other examples of linearly ordered sets later, but for just one example now, let us note that if we define the relation More to be the set of pairs (a, b) such that (b, a) is in Less, then More also linearly orders $\mathbb{N}$. The structure $(\mathbb{N}, \mathsf{More})$ has a largest element, and no least element.

4.1.4 The Ordering of the Natural Numbers

The relation Less linearly orders $\mathbb{N}$, but $(\mathbb{N}, \mathsf{Less})$ also has a number other of properties that can be expressed by the first-order statements that distinguish it from other linearly ordered sets. Here are some of those properties with the translations to the informal language provided below.

(1) $\exists m \forall n\ [(n = m) \vee L(m, n)]$.
(2) $\forall m \exists n\ L(m, n)$.
(3) $\forall m \exists n \{L(m, n) \wedge \forall k [L(k, n) \Longrightarrow ((k = m) \vee L(k, m))]\}$.

Here are the translations:

(1) There is a least natural number.
(2) There no greatest natural number.
(3) Every natural number has an immediate successor (see below for an explanation).

A linear ordering is *discrete* if for every element a, except for the last element if there is one, there is a b that is larger than a, and there are no elements between a and b. We call such a b an immediate successor of a.

The combined content of the statements *O1*, *O2*, *O3*, and (1), (2), (3) above, can be summarized with just one sentence: Less is a discrete linear ordering with a least element and no last element. That one sentence tells us almost everything there is to know about ($\mathbb{N}$, Less). This is made precise by the following theorem. For a proof of a similar result see Proposition 2.4.10 in [22].

Theorem 4.1 *If a binary relation* R *is a discrete linear ordering of a set M with a least element, and every element has an immediate successor, and every element, except the least one, has an immediate predecessor, then a first-order sentence φ is true in* (M, R) *if and only if it is true in* ($\mathbb{N}$, Less).

Theorem 4.1 says that (M, R) and ($\mathbb{N}$, Less) have exactly the same first-order properties. However, it does not preclude that there may be differences between (M, R) and ($\mathbb{N}$, Less), and if there are such differences, that means that they cannot be expressed in the first-order way. Let us see an example.

Here is a picture of a set M linearly ordered by a relation R, the ordering is from left to right:

The idea here is that (M, R) has a part (the one on the left) that looks like ($\mathbb{N}$, Less) and each element in it is smaller than all elements from the rest of the structure that is a discretely ordered set that stretches to infinity in both directions. It is easy to see that the statements (1), (2), and (3) are true about (M, R); hence, by Theorem 4.1 (M, R) is first-order indistinguishable from ($\mathbb{N}$, Less). There is a clear difference though. The domain of new structure has a subset without a least element (the infinite part on the right). There are no such subsets of $\mathbb{N}$. It follows that this difference is not first-order expressible.

4.2 The Arithmetic Structure of the Natural Numbers

We defined the addition relation Add on the set of natural numbers as the set of ordered triples (k, l, m) such that $k + l = m$. Think of the set Add as an oracle that knows correct results of all additions. To find out whether 253 plus 875 is 1274, there is no need to perform any operations. Just ask the oracle if the triple (253, 875, 1274) is in Add (the answer is "no"). But this is all an oracle that knows a relation can do. If we want to know what is $253 + 875$, we ca still use the oracle, but we cannot ask it for the answer directly. We need to ask all questions of the form (253, 875, $\star$) for different values of $\star$ until we get a "yes." To clarify, an oracle is a data base and it knows what it has, but it cannot automatically respond to queries of the kind that require more elaborate searches. It does not understand when we ask: Tell me what number is the third entry in the triple you have that starts with 253 and 875.

Recall that A is a ternary relation symbol, that we have already used to examine some of first-order statements about Add. Here are three more examples; their informal translations follow.

(1) $\exists x \forall y\ A(x, y, y)$.
(2) $\forall x \forall y \forall z\ [A(x, y, z) \implies A(y, x, z)]$.
(3) $\forall x \forall y \forall w \forall z\ [(A(x, y, w) \wedge A(x, y, z)) \implies w = z]$.

The first sentence declares the existence of a natural number with a special property. It is a number that is "neutral" with respect to addition. There is only one such number—zero. The second expresses that addition is commutative, meaning that the order in which the numbers are added does not matter. The third sentence does not seem very interesting, but it is of some importance. We chose to formalize statements about addition in a relational language, i.e. instead of introducing a function symbol for the operation of adding numbers, we represent addition as a relation. Sentence (3) expresses that the addition is in fact a function; for any two numbers, their sum is unique.

The multiplicative structure $(\mathbb{N}, \mathsf{Mult})$ is interesting and there will be more about it later, but now we will turn to an even more interesting structure in which addition and multiplication are combined. It is $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$ which we called the *standard model of arithmetic*. We have seen before how certain number-theoretic facts can be expressed as first-order properties of $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$. The examples included Euclid's theorem on the infinitude of prime numbers. In two exercises in Chap. 1, the reader was asked to formalize the Twin Primes Conjecture, and the Goldbach Conjecture. The fact that such statements can be written just in terms of addition and multiplication already shows how rich the standard model is and how much of this richness can be expressed in first-order logic. In fact, the standard model is one of the most complex structures studied in mathematics. There is much that we know about it, and even more that we don't. In contrast, the structures $(\mathbb{N}, \mathsf{Less})$, $(\mathbb{N}, \mathsf{Add})$, and $(\mathbb{N}, \mathsf{Mult})$ are relatively simple and well-understood. All of this will be discussed later in the book, but our immediate goal is first to introduce other number systems that are extensions of $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$.

4.3 The Arithmetic Structure of the Integers

When we introduced the natural numbers, we appealed to the simple idea of counting. To introduce negative numbers, we will follow a formal route. We will start from the already introduced structure $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$, and construct over it a system of integers (positive and negative whole numbers) it using first-order logic. The purpose is to show how new structures can be built form the ones we already have. Be prepared that the presentation will be more formal and not particularly intuitive. There are good reasons why the presentation cannot be too straightforward. Although technically there is nothing difficult about the basic rules of arithmetic of integers numbers, the historical development of the foundations of arithmetic was quite convoluted, and it took a while until the mathematical community came to a consensus. Morris Kline writes about it in [18], and he quotes Augustus de Morgan:

> The imaginary expression $\sqrt{-a}$ and the negative expression $-b$ have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. As far as real meaning is concerned, both are equally imaginary, since $0 - a$ is as inconceivable as $\sqrt{-a}$, [18, p. 593].

Recall that the set on which the relations of a structure are defined is called its *domain*. Thus, the domain of ($\mathbb{N}$, Add, Mult) is $\mathbb{N}$. We will build a new structure with a domain $\mathbb{Z}$ and three relations on it: $\mathsf{Less}_{\mathbb{Z}}$, $\mathsf{Add}_{\mathbb{Z}}$ and $\mathsf{Mult}_{\mathbb{Z}}$.[2] Ultimately, $\mathbb{Z}$ will become a set of numbers, but initially its elements will not look like ordinary numbers at all.

Let $\mathbb{Z}$ be the set of all ordered pairs $(0, m)$ and $(1, m)$, where m is any natural number, except that we exclude $(0, 0)$. The idea here is that all pairs $(1, m)$ represent the nonnegative integers, and all pairs $(0, m)$ their negative opposites. Think of $(1, m)$ as the number m, and $(0, m)$, as its opposite $-m$. We have excluded $(0, 0)$ to avoid duplication. According to the definition, zero is represented by $(1, 0)$.

What is the purpose of defining integer numbers this way? The point is to seriously address the question: Where do mathematical objects come from? It is a difficult philosophical problem that does not have an ultimate answer. We have followed the route of grounding the concept of natural number in our intuitive grasp of counting. Then, we "made" the standard model ($\mathbb{N}$, Add, Mult), and now we are extending it to a larger number system of integers. We will continue extending it, culminating in the complex numbers, and we want to see what is this process based on, and how the newly constructed structures are justified. For a while no now tools will be needed. We will construct the integers and the rational numbers (fractions) by defining them in the standard model using first-order logic.

The set $\mathbb{Z}$ is first-order definable in ($\mathbb{N}$, Add, Mult). Here is a formal definition:

$$\mathbb{Z} = \{(i, j) : i \in \mathbb{N} \wedge j \in \mathbb{N} \wedge (i = 0 \vee i = 1) \wedge \neg(i = 0 \wedge j = 0)\}.$$

Since the domain $\mathbb{Z}$ is made of pairs of numbers, binary relations on it will be represented as a sets of pairs of pairs (not a typo) of numbers.

Definition 4.2 The relation $\mathsf{Less}_{\mathbb{Z}}$ consists of all pairs $((i, m), (j, n))$ such that one of the following holds:

(1) $i = 0 \wedge j = 1$;
(2) $i = 1 \wedge j = 1 \wedge L(m, n)$;
(3) $i = 0 \wedge j = 0 \wedge L(n, m)$.

Condition (1) says that all negative numbers $(0, m)$ are smaller than all nonnegative numbers $(1, n)$. Condition (2) says that the relation $\mathsf{Less}_{\mathbb{Z}}$ agrees with the relation

[2] $\mathbb{Z}$ for the German word Zahlen.

Less on the set of nonnegative numbers. Condition (3) says that the ordering of the negative numbers reverses the ordering of the positive ones. For example, $(0, 5)$ is less than $(0, 2)$, and this is what we want, because -5 is less than -2. Notice that all three conditions can be put together with the connective $\vee$ combining them into a single first-order statement.

Why, you can ask, are we making things so complicated? Why don't we just say that the integers are the natural numbers and their opposites? In other words, why don't we say that $\mathbb{Z}$ is the *union* of two sets, one being $\mathbb{N}$ itself, and the other defined as $\{-m : m \in \mathbb{N}\}$. We could do that, but then our task would be to explain the meaning of $-m$. What is $-m$? What is this operation of putting the negative sign in front of a natural number? Putting a negative *sign* in front of an expression is a syntactic operation, i.e. it is an operation on symbols. We are not defining sets of symbols, we are defining honest to goodness sets of mathematical objects. It is all related to the separation of syntax and semantics. We need to keep a clear dividing line between syntactic symbols and mathematical objects they represent. But is it really an honest separation? Instead of $-m$ we wrote $(0, m)$, and we said that $(0, m)$ is an *ordered pair*. Why is $(0, m)$ better than $-m$? That is explained by a convenient feature the syntax of first-order logic: it allows us to express facts about ordered pairs, triples, and finite tuples in general in a natural way. If x and y are free (unquantified) variables in a formula $\varphi(x, y)$, then, in any structure in which the language is interpreted, $\varphi(x, y)$ defines the set of ordered pairs having the property expressed by the formula. Hence, to define the structure $(\mathbb{Z}, \mathsf{Less}_{\mathbb{Z}})$ we did not need to expand our formal language; what we already have suffices.

Now, in a similar fashion we will define addition $\mathsf{Add}_{\mathbb{Z}}$, and multiplication $\mathsf{Mult}_{\mathbb{Z}}$ as relations on $\mathbb{Z}$. (If you are not intrigued, you can skip the details.)

The definition of $\mathsf{Add}_{\mathbb{Z}}$ given below refers only to addition and the order relation on the set $\mathbb{N}$; in other words, $\mathsf{Add}_{\mathbb{Z}}$ is first-order definable in the structure $(\mathbb{N}, \mathsf{Less}, \mathsf{Add})$. Since, the ordering relation Less on $\mathbb{N}$ is first-order definable in the structure $(\mathbb{N}, \mathsf{Add})$, it follows that $\mathsf{Add}_{\mathbb{Z}}$ is also first-order definable in $(\mathbb{N}, \mathsf{Add})$, i.e. the reference to Less in the definition below can be eliminated. In fact, this way we could eliminate all references to Less, and hence all occurrences of the relation symbol L as well, but if we replaced all occurrences of statements of the form $L(x, y)$ by their definition: $\exists z[z\neg(z = 0) \wedge A(x, z, y)]$, that would make the already complicated formulas almost impossible to read.

We will now formally define addition of integers. Addition of signed numbers is usually explained by examples, including all different cases. This is exactly what Definition 4.3 below does. For given three integer numbers in our chosen format, (h, k), (i, l), and (j, m), the definition list all cases in which (j, m) is the sum of (h, k) and (i, l). For example, if $k = 7$ and $l = 3$, the cases we need to include are: $7 + 4 = 10$, $(-7) + (-3) = (-10)$, $(-3) + 7 = 4$, $3 + (-7) = (-4)$. Since we usually tacitly assume that addition of integers is commutative, i.e. the result does not depend on the order in which numbers are added, once we are told that $(-3) + 7 = 4$, we also know that $7 + (-3) = 4$. In a formal definition all tacit assumptions have to be made explicit. Nothing can be left our. The result is a definition that looks very technical. If you are not intrigued by all those details, skip it, and instead move on to Definition 4.4.

Definition 4.3 The relation $\mathsf{Add}_{\mathbb{Z}}$ is the collection of all ordered triples $((h,k),(i,l),(j,m))$ such that one of the following holds,

(1) $h=1 \wedge i=1 \wedge j=1 \wedge A(k,l,m)$;
(2) $h=0 \wedge i=0 \wedge j=0 \wedge A(k,l,m)$;
(3) $h=1 \wedge i=0 \wedge k=l \wedge j=1 \wedge m=0$;
(3′) $h=0 \wedge i=1 \wedge k=l \wedge j=1 \wedge m=0$;
(4) $h=1 \wedge i=0 \wedge L(k,l) \wedge j=0 \wedge A(m,k,l)$;
(4′) $h=0 \wedge i=1 \wedge L(l,k) \wedge j=0 \wedge A(m,l.k)$;
(5) $h=1 \wedge i=0 \wedge L(l,k) \wedge j=1 \wedge A(m,l,k)$;
(5′) $h=0 \wedge i=1 \wedge L(k,l) \wedge j=1 \wedge A(m,k,l)$.

For completeness of the presentation, let us also define the multiplication relation $\mathsf{Mult}_{\mathbb{Z}}$. The definition is mercifully simpler than that of $\mathsf{Add}_{\mathbb{Z}}$.

Definition 4.4 The relation $\mathsf{Mult}_{\mathbb{Z}}$ is the collection of all ordered triples $((h,k),(i,l),(j,m))$ such that one of the following holds,

(1) $h=1 \wedge i=1 \wedge j=1 \wedge M(k,l,m)$;
(2) $h=0 \wedge i=0 \wedge j=1 \wedge M(k,l,m)$;
(3) $h=1 \wedge i=0 \wedge j=0 \wedge M(k,l,m)$;
(4) $h=0 \wedge i=1 \wedge j=0 \wedge M(k,l,m)$.

Notice that (2) expresses the rule "$negative \cdot negative = positive$."

For us, the most important feature of Definitions 4.3 and 4.4 is that the new structure, $(\mathbb{Z}, \mathsf{Add}_Z, \mathsf{Mult}_{\mathbb{Z}})$, has been defined completely in terms of the structure $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$, or using our metaphor, $(\mathbb{Z}, \mathsf{Add}_Z, \mathsf{Mult}_{\mathbb{Z}})$ can be logically seen in the structure $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$. We will see later that these two structures are even more intimately connected. Namely, not only is $(\mathbb{Z}, \mathsf{Add}_Z, \mathsf{Mult}_{\mathbb{Z}})$ first-order definable in $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$, but also $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$ is first-order definable in $(\mathbb{Z}, \mathsf{Add}_Z, \mathsf{Mult}_{\mathbb{Z}})$. We need to develop more tools to show how it is done.

4.3.1 Natural Numbers Are Integers, Naturally

According to our definition, integers are ordered pairs of pairs of natural numbers, so, as defined, the domains $\mathbb{N}$ and $\mathbb{Z}$ are disjoint. Usually, the natural numbers are identified with the non-negative integers. The structure with domain $\{(1,m) : m \in \mathbb{N}\}$ and the relations Add_Z and, $\mathsf{Mult}_{\mathbb{Z}}$ restricted to this domain, is a copy of the structure $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$; the only difference is that every natural number n is now represented by the pair $(1,n)$. Later, we will be saying, more precisely, that we identify these two structures *up to isomorphism*. In this sense, the set of natural numbers $\mathbb{N}$ becomes a *subset* of the set of integers $\mathbb{Z}$.

What does it mean to identify to structures? In a precise way, the previous paragraph can be rephrased as follows.[3] The function $f : \mathbb{N} \longrightarrow \mathbb{Z}$, defined by $f(n) = (1, n)$ is a one-to-one correspondence between sets $\mathbb{N}$ and $\mathbb{Z}$, and for all $k, l, m \in \mathbb{N}$, $(k, l, m) \in \mathsf{Add}$ if and only if $(f(k), f(l), f(m)) \in \mathsf{Add}_{\mathbb{Z}}$, and $(k, l, m) \in \mathsf{Mult}$ if and only if $(f(k), f(l), f(m)) \in \mathsf{Mult}_{\mathbb{Z}}$. This shows that the image of $\mathbb{N}$ under f is *isomorphic* to $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$.

The ordering relation Less is defined in $(\mathbb{N}, \mathsf{Add})$ by the formula $\exists z[z\neg(z = 0) \wedge A(x, z, y)]$. This formula does not define the $\mathsf{Less}_{\mathbb{Z}}$ in $(\mathbb{Z}, \mathsf{Add}_{\mathbb{Z}})$. In $(\mathbb{Z}, \mathsf{Add}_{\mathbb{Z}})$ it defines the set of all integers. In fact, it can be shown that Less_Z can not be defined in $(\mathbb{Z}, \mathsf{Add}_{\mathbb{Z}})$ by any first-order formula at all. This interesting result is not difficult to prove, and we will do it later after we discuss a general strategy for proving such results.

We are building numbers systems, from the natural numbers up. So far, we made one step, from the natural numbers to the integers. There will be more steps, each justified by a particular need in the historical development of mathematics. How the number systems came to be what they are now is a fascinating story, and we will not discuss it fully. For a comprehensive account, including technical details that are omitted here, see [32].

4.4 Fractions!

While natural numbers have their origin in counting and multiplying, the rational numbers, originate from cutting, slicing, and breaking. If you break a chocolate bar into eight equal pieces, and eat three of them, you will have eaten is $\frac{3}{8}$ of the whole bar. If you eat one more, this will add up to $\frac{3}{8} + \frac{1}{8} = \frac{4}{8}$ of the bar. The result is the same as if you broke the bar into two equal pieces, and ate one of them. The fraction $\frac{4}{8}$ represents the same quantity as $\frac{1}{2}$.

All basic properties of addition and multiplication of fractions can be explained on simple examples, as we did above. For practical applications, such explanations suffice to justify the rules for adding and multiplying fractions. It all seems simple and natural, but, as every teacher of mathematics knows, the arithmetic of fractions often instills fear and loathing in students, and this phenomenon is not that easy to explain. We will not get into this discussion here, but what follows is relevant. As you will see, a full description of the arithmetic structure of fractions, requires significantly more effort than what suffices for addition and multiplication of integers.

In the next section, we will perform an exercise in the formalism of first-order logic. We will define a structure of fractions inside $(\mathbb{Z}, \mathsf{Add}_Z, \mathsf{Mult}_{\mathbb{Z}})$, and therefore also in $(\mathbb{N}, \mathsf{Add}, \mathsf{Mult})$. This definition will be based on a useful feature of formal

[3] Skip this paragraph if the terms and notation are unfamiliar. This whole topic will be thoroughly discussed in Part II.

logic—its ability to deal with abstraction in a precise way. The problem we need to address is that fractions such as $\frac{6}{4}$ and $\frac{3}{2}$ are not equal as formal expressions. They are not the same, but they do represent the same quantity, so they are in this sense *equivalent*. There is great redundancy in fractions. The same quantity can be represented by infinitely many different but equivalent fractions. Because of that, we will make a distinction: by a fraction we will mean a formal expression of the form $\frac{m}{n}$, and by a *rational number* for now we will mean the quantity that a fraction represents. So we have to address another question: What is the quantity represented by a fraction? We will give a precise answer, but the reader should be warned again that this material is technically more advanced. If you have never liked fractions, you should skip the next section, and return to it later.

4.5 The Arithmetic Structure of the Rationals

We will define the structure $(\mathbb{Q}, \mathsf{Add}_{\mathbb{Q}}, \mathsf{Mult}_{\mathbb{Q}})$, whose domain is the set of rational numbers and $\mathsf{Add}_{\mathbb{Q}}$ and $\mathsf{Mult}_{\mathbb{Q}}$ are the addition and multiplication as relations on $\mathbb{Q}$. All definitions will take place in $(\mathbb{Z}, \mathsf{Add}_{\mathbb{Z}}, \mathsf{Mult}_{\mathbb{Z}})$.

Let $\mathcal{F}$ be the set of all ordered pairs of integers (m, n), where n is not 0. The intention here is that (m, n) represents the fraction $\frac{m}{n}$. The domain of the rational numbers will be defined shortly, for now we will describe the additive and multiplicative structure on the set $\mathcal{F}$.

Recall that adding fractions requires a procedure. The sum of $\frac{k}{l}$ and $\frac{m}{l}$ is $\frac{k+m}{l}$, but if the denominators are different, we first need to find equivalent fractions with a common denominator, and then add their numerators. This whole process is summarized in the following addition formula:

$$\frac{k}{l} + \frac{m}{n} = \frac{k \cdot n + l \cdot m}{l \cdot n}.$$

For example $\frac{1}{2} + \frac{1}{3} = \frac{1 \cdot 3 + 1 \cdot 2}{2 \cdot 3} = \frac{5}{6}$.

Translated into the language of pairs and relations the rule for multiplication gives the following definition of the addition relation for $\mathcal{F}$: $((k, l), (m, n), (r, s)) \in \mathsf{Add}_{\mathcal{F}}$ whenever

(1) $k \cdot n + l \cdot m = r$ and
(2) $l \cdot n = s$.

In the two conditions above, $+$ and $\cdot$ are the usual addition and multiplication. They are functions, and since we do not allow function symbols in formal statements, we have to pay a price now, when we translate the above definition into the language with relation symbols A and M. The translated formula is not pretty. The addition relation $\mathsf{Add}_{\mathcal{F}}$ can be defined as the set of triples of pairs of integers $((k, l), (m, n), (r, s))$ such that for some a and b in $\mathbb{Z}$, $k \cdot n = a$, $l \cdot m = b$, $a + b = r$, and $l \cdot n = s$. Written in the language of relations, the first-order definition of $\mathsf{Add}_{\mathcal{F}}$

over $(\mathbb{Z}, \mathsf{Add}_{\mathbb{Z}}, \mathsf{Mult}_{\mathbb{Z}})$ is given by the formula

$$\exists a \exists b\ [M(k, n, a) \wedge M(l, m, b) \wedge A(a, b, r) \wedge M(l, n, s)].$$

We have defined the addition relation on the set $\mathcal{F}$ in terms of addition and multiplication on $\mathbb{Z}$.

The complications above are caused by the fact that the algorithm for addition of fractions requires finding their common denominator. The rule for multiplication of fractions is more user friendly. To define the multiplication relation $\mathsf{Mult}_{\mathcal{F}}$ we can use the familiar rule:

$$\frac{k}{l} \cdot \frac{m}{n} = \frac{k \cdot m}{l \cdot n}.$$

This shows that multiplication in $\mathcal{F}$ is fully determined by multiplication in $\mathbb{Z}$, and it is a routine exercise to write a first-order formula that defines $\mathsf{Mult}_{\mathcal{F}}$ in $(\mathbb{Z}, \mathsf{Mult}_{\mathbb{Z}})$.

Finally, let us define the ordering of fractions. Do you remember how to find out which of two given fractions is larger? For example, which one is larger, $\frac{7}{13}$ or $\frac{5}{11}$? The problem here is that the first fraction represents thirteenths of a unit, and the second, elevenths. They are like apples and oranges, you cannot compare them directly. To compare the fractions, we need convert them to equivalent fractions with a least common denominator, which in this case is $13 \cdot 11 = 143$. Lets do it: $\frac{7}{13} = \frac{7 \cdot 11}{13 \cdot 11} = \frac{77}{143}$; and $\frac{5}{11} = \frac{5 \cdot 13}{11 \cdot 13} = \frac{65}{143}$, and we can see that $\frac{7}{13}$ is larger.

The method described above gives us the following criterion for comparing fractions. If k, l, m, and n are positive integers, then $\frac{k}{l} < \frac{m}{n}$ if and only if $k \cdot n < l \cdot m$. This shows that the ordering of positive fractions can be defined in $(\mathbb{Z}, \mathsf{Less}_{\mathbb{Z}}, \mathsf{Mult}_{\mathbb{Z}})$. Then the definition can be extended to all fractions by considering cases, keeping in mind that if both m and n are negative, then $\frac{m}{n}$ is positive, and if only one of them is negative, then $\frac{m}{n}$ is negative as well.

4.5.1 Equivalence Relations and the Rationals

Abstraction is a process in which new concepts are arrived at by stripping the objects of discourse of some of their features. Mathematical logic has a formalized approach to this kind of abstraction and this approach is essential to many developments in modern mathematics.

In abstracting, we identify object that are considered equivalent according to some criteria. The following definition makes this precise.

Definition 4.5 A set E of ordered pairs of elements of a set A is an *equivalence relation* on A if it satisfies the following three conditions:

(1) E is *reflexive* i.e. for all $a \in A$, $(a, a) \in E$, i.e. every element is related to itself;

(2) E is *symmetric* i.e. for all $a, b \in A$, if $(a, b) \in E$, then $(b, a) \in E$, i.e. if a is related to b, then b is related to a;
(3) E is *transitive* i.e. for all $a, b, c \in A$, if $(a, b) \in E$ and $(b, c) \in E$, then $(a, c) \in E$, i.e. if a is related to b, and b is related to c, then a is related to c.

If E is an equivalence relation and (a, b) is in E, then we say that a and b are *E-equivalent*.

To discuss properties of equivalence relations, we need to introduce some notation. The next two paragraphs should be read as an exercise. Try to see how the facts stated below follow directly from Definition 4.5.

Let E be an equivalence relation on a set A. Then for each $a \in A$, the set $\{b : b \in A \wedge (a, b) \in E\}$ is called the *equivalence class* of a, and is denoted by $[a]_E$. For each equivalence class any of its elements is a *representative* of the class; in particular, a is always a representative of $[a]_E$. If a and b are E-equivalent, then $[a]_E = [b]_E$. Moreover, for all a and b in A, either the equivalence classes $[a]_E$, $[b]_E$ coincide, or they are disjoint. It follows that the set of all equivalence classes forms a partition of the set A into disjoint nonempty subsets.

For another exercise, you can go over Definition 4.5 and convince yourself that the relation of being related (in the family sense) is an equivalence relation on the domain of all people. Notice that one has to make a somewhat controversial assumption that everyone is related to herself or himself. What are the equivalence classes of this relation? What is your equivalence class? How many representatives does it have? This last question has a theoretical answer, but the exact answer is impossible to pin down in practice.

Now let us define the following relation Eq on the set of fractions $\mathcal{F}$

$$((k, l), (m, n)) \in \mathrm{Eq} \text{ if and only if } k \cdot n = l \cdot m.$$

One can show, and the reader is encouraged to do it as an exercise, that Eq is an equivalence relation. Observe that transitivity property (3) holds because pairs of the form $(m, 0)$ are not fractions. See what goes wrong if they were.

Since Eq is an equivalence relation, we can talk about its equivalence classes. For example $[(1, 2)]_{\mathrm{Eq}} = \{(1, 2), (-1, -2), (2, 4), (-2, -4), \dots\}$.

So far we talked about fractions, now it is time to define the rational numbers.

Definition 4.6 *Rational numbers* are equivalence classes of the relation Eq on the set of fractions $\mathcal{F}$. In the set notation, the set of rational numbers $\mathbb{Q}$ is

$$\{[(m, n)]_{\mathrm{Eq}} : (m, n) \in \mathcal{F}\}.$$

In mathematics, when we say $\frac{1}{2}$, we often mean the one-half as a rational number, as in our example $[(1, 2)]_{\mathrm{Eq}}$ above. In other words, we think of it not just as a single fraction, but also as the whole set of fractions that are equivalent to it. This practice is formalized with the help of the equivalence relation Eq.

4.5.2 Defining Addition and Multiplication of the Rational Numbers Formally

Addition and multiplication of the integers determines addition and multiplication on the set of fractions $\mathcal{F}$. Because of the more complex nature of the rational numbers, defining their arithmetic operations is harder. This section shows how it is done. This is the most technical section of this chapter and it can be skipped on the first reading.

We define addition and multiplication of rational numbers as ternary relations as follows:

$$\mathsf{Add}_{\mathbb{Q}} = \{([(h,k)]_{\mathrm{Eq}}, [(i,l)]_{Eq}, [(j,m)]_{\mathrm{Eq}}) : ((h,k),(i,l),(j,m)) \in \mathsf{Add}_{\mathcal{F}}\},$$

$$\mathsf{Mult}_{\mathbb{Q}} = \{([(h,k)]_{\mathrm{Eq}}, [(i,l)]_{\mathrm{Eq}}, [(j,m)]_{\mathrm{Eq}}) : ((h,k),(i,l),(j,m)) \in \mathsf{Mult}_{\mathcal{F}}\}.$$

Notation becomes really heavy here, and that is because we are talking about a domain whose elements are equivalence classes of elements from another domain. Those two relations on $\mathbb{Q}$ are defined in terms of the already defined addition and multiplication on the set of fractions $\mathcal{F}$. It is not immediately clear that the definitions are correct. We defined sets of ordered triples of equivalence classes of fractions, but in the conditions defining the relations instead of properties of classes, we use properties of their representatives. A proof is needed that those conditions do not depend on the choice of representatives. This is one of the exercises at the end of this chapter.

Similarly, the ordering of $\mathbb{Q}$ is determined by the ordering of $\mathcal{F}$:

$$\mathsf{Less}_{\mathbb{Q}} = \{([(k,l)]_{\mathrm{Eq}}, [(m,n)]_{\mathrm{Eq}}) : ((k,l),(m,n)) \in \mathsf{Less}_{\mathcal{F}}\}.$$

As above, one can check that the definition does not depend on the choice of representatives of equivalence classes.

In elementary mathematics such a level of pedantry as we exercised above is not necessary, and it could be harmful. It is worth noting though that some specialists in mathematics education maintain that one of the reasons that the arithmetic of rational numbers is a difficult topic is that the definition of $\mathbb{Q}$ involves equivalence classes, rather than just individual fractions.

The move from a set with an equivalence relation on it to the set of equivalence classes, called "taking the quotient," is an important mathematical operation. It is a mathematical formalization of the process of passing from the particular to the general.

As formally defined, the sets $\mathbb{N}$ and $\mathbb{Z}$ are disjoint, but, as we saw, there is a way in which one can identify natural numbers in $\mathbb{N}$ with the positive numbers in $\mathbb{Z}$; hence we can consider $\mathbb{Z}$ as an extension of $\mathbb{N}$. Similarly, by identifying each integer m with the rational number $[(m,1)]_{\mathrm{Eq}}$, we consider $\mathbb{Q}$ to be an extension of $\mathbb{Z}$.

Notice that in the definition of $\mathbb{Q}$ at the end of the previous section, we moved away from the structure of the integers. We changed the domain quite significantly. It is defined as a set of equivalence classes of pairs of integers, and those classes are infinite sets. Every fraction has infinitely many fractions that are equivalent to it, and hence, each equivalence class $[(m, n)]_{\mathrm{Eq}}$ is infinite. First-order definitions can be used to define sets of elements, ordered pairs, ordered triples, and, in general, ordered sequences of any fixed finite length, but not sets of infinite sets. Fractions, the elements of $\mathcal{F}$ are just certain pairs of integers, but the domain $\mathbb{Q}$ is the set of infinite sets. Still, there is a way to ground the new structure in the old one in a first-order fashion, by selecting a representative for each equivalence class in $\mathbb{Q}$ in a certain way.

A fraction $\frac{m}{n}$ is *reduced* if m and n do not have common factors. For example, $\frac{2}{5}$ is reduced, but $\frac{6}{15}$ is not, because 3 is a common factor of 6 and 15. Each equivalence class $[(m, n)]_{\mathrm{Eq}}$ has only one reduced fraction in it, and it can be chosen to be its representative. Moreover, the set of reduced fractions is first-order definable in $(\mathbb{Z}, \mathsf{Add}_Z, \mathsf{Mult}_{\mathbb{Z}})$ (this is left as an exercise). Addition and multiplication of reduced fractions matches addition and multiplication of the equivalence classes they represent, so we obtain a copy of the structure of the rational numbers that is logically visible in $(\mathbb{Z}, \mathsf{Add}_{\mathbb{Z}}, \mathsf{Mult}_{\mathbb{Z}})$.

4.5.3 Dense Ordering of the Rationals

The ordering of the natural numbers is easy do describe. The natural numbers begin with zero, and progress up in increments of one. The ordering of the integers is similar, they also can be thought of as beginning with zero, and then going in two directions up and down (or left and right) in increments of one. The key property here is that every natural number and every integer m has a unique successor $m + 1$. This property makes $(\mathbb{N}, \mathsf{Less})$ and $(\mathbb{Z}, \mathsf{Less}_{\mathbb{Z}})$ discrete orderings.

The ordering of $\mathbb{Q}$ is different. Between any two rational numbers there is another one. For example, if p and q are rational and $p < q$, then $\frac{p+q}{2}$ is a rational number between p and q.

In general, if a relation R linearly orders a set M, the ordering R is called *dense* if between any two elements of M there is another element. Formally, R is dense if for all a and b in M, if $(a, b) \in \mathsf{R}$, then there is a c such that $(a, c) \in \mathsf{R}$ and $(c, b) \in \mathsf{R}$.

Density of the ordering of $\mathbb{Q}$ has important consequences, and they will be explored in the next chapter. For now, let us just notice that density implies that not only is the whole set $\mathbb{Q}$ actually infinite, but also that already between any two rational numbers there are infinitely many rational numbers. Because of that, illustrating the ordering of rational numbers as points on a number line is not easy. On the one hand, there should not be any visible gaps in the line, as between any two points there have to be (infinitely many!) other points. On the other hand, as you will see in the next chapter, the line made of points corresponding to rational numbers cannot be solid. For reasons having to do with geometry it must have gaps. This is a serious problem.

Exercises

Exercise 4.1 *Show that if an element in a linearly ordered set has an immediate successor, than it has only one immediate successor.*

Exercise 4.2 *Let* (M, R) *be a linearly ordered set. Assume that the statement* $\forall m \exists n\ L(m, n)$ *holds in* (M, R). *Prove that the set* M *is infinite.*

Exercise 4.3 *The argument below is a sequence of true statements about* 0, 1, *and* -1. *Provide a justification for each step.*

$$\begin{aligned} 1 + (-1) &= 0, \\ (-1) \cdot (1 + (-1)) &= 0, \\ (-1) \cdot 1 + (-1) \cdot (-1) &= 0, \\ (-1) + (-1) \cdot (-1) &= 0, \\ 1 + (-1) + (-1) \cdot (-1) &= 0 + 1, \\ (-1) \cdot (-1) &= 1. \end{aligned}$$

Exercise 4.4 *According to the definition of* $\mathbb{Z}$ *in this chapter, the integers* 2, 5, -2, *and* -5 *are represented by* $(1, 2)$, $(1, 5)$, $(0, 2)$, *and* $(0, 5)$, *respectively. Use this representation and Definition 4.3 to compute* $(-2) + (-5)$, $(2) + (-5)$, *and* $(-2) + (5)$.

Exercise 4.5 ** Verify that Definition 4.3 is correct.*

Exercise 4.6 *Define* $\mathsf{Mult}_{\mathbb{Q}}$ *in terms of* $\mathsf{Mult}_{\mathbb{Z}}$.

Exercise 4.7 *Prove that if* E *is an equivalence relation on a set* A, *then for all* a *and* b *in* A, *either the equivalence classes* $[a]_{\mathrm{E}}$ *and* $[b]_{\mathrm{E}}$ *coincide, or they are disjoint.*

Exercise 4.8 *Show that the relation* E *on the set of fractions* $\mathcal{F}$ *defined by:* $(\frac{k}{l}, \frac{m}{n}) \in E$ *if and only if* $k \cdot n = l \cdot m$, *is an equivalence relation.*

Exercise 4.9 *Define the ordering* $\mathsf{Less}_{\mathbb{Q}}$ *in terms of* $\mathsf{Less}_{\mathbb{Z}}$ *and* $\mathsf{Mult}_{\mathbb{Z}}$.

Exercise 4.10 ** Find a first-order formula defining the set of reduced fractions in* $(\mathbb{Z}, \mathsf{Add}_Z, \mathsf{Mult}_{\mathbb{Z}})$.

Exercise 4.11 *This exercise requires some elementary algebra. Use formal definitions of* $\mathsf{Less}_{\mathbb{Q}}$, $\mathsf{Add}_{\mathbb{Q}}$ *and* $\mathsf{Mult}_{\mathbb{Q}}$ *to verify that for all distinct rational numbers* p *and* q, $\frac{1}{2} \cdot (p + q)$ *is between* p *and* q.