Chapter 13
Tame vs. Wild

Abstract In this chapter we will compare two classical structures: the field of complex numbers $(\mathbb{C}, +, \cdot)$ and the standard model of arithmetic $(\mathbb{N}, +, \cdot)$. The former is vast and mysterious, the latter deceptively simple. As it turns out, as far as the model-theoretic properties of both structures are concerned, the roles are reversed, the former is very tame while the latter quite wild, and those terms have well-understood meanings. In recent years, tameness has become a popular word in model theory. Tameness is not defined formally, but a structure is considered tame if the geometry of its definable sets is well-described and understood. Tameness has different levels. The most tame structures are the minimal ones. All parametrically definable unary relations in a minimal structure are either finite or cofinite. The examples of minimal structures that we have seen so far are the structures with no relations on them—the trivial structures—and $(\mathbb{N}, <)$. It is somewhat surprising that the ultimate number structure—the complex numbers, is also minimal. It is a fascinating example.

Keywords Complex numbers · Cantor's pairing function · Arithmetization of language · Tarski's undefinability of truth theorem · Gödel's second incompleteness theorem

13.1 Complex Numbers

In order to extend the field of the rational numbers to include numbers such as $\sqrt{2}$, π, and e, we employed the concept of Dedekind cut, and accepted all set-theoretic complexities it involved. Now we will enlarge the domain of numbers one more time, but this move will be easier. The set $\mathbb{C}$ of complex numbers is simply $\mathbb{R}^2$, the set of all ordered pairs of real numbers.[1] To make complex numbers numbers, we need to define how they are added and multiplied.

[1]Complex numbers are often defined as expressions of the form $a + bi$, where a and b are real number, and i is such that $i^2 = -1$. You will see below that our definition is equivalent.

R. Kossak, *Mathematical Logic*, Springer Graduate Texts in Philosophy 3,
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Definition 13.1 For the complex numbers (a, b) and (c, d)

$$(a, b) + (c, d) = (a + c, b + d)$$

and

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

Notice that addition and multiplication of complex numbers are definable in $(\mathbb{R}, +, \cdot)$. It follows that the logical complexity of the structure $(\mathbb{C}, +, \cdot)$ is no greater than that of $(\mathbb{R}, +, \cdot)$. The geometry of the complex numbers is a part of the geometry of the real numbers.

Addition of complex numbers is straightforward, but why is multiplication so strange? The answer is: Because it works. With this definition $(\mathbb{C}, +, \cdot)$ becomes a field. And what a field it is!

First, let us see why $\mathbb{C}$ can be considered an extension of $\mathbb{R}$. Indeed, each real number r can be identified with the pair $(r, 0)$, and it is easy to check that addition and multiplication of such pairs agrees with addition and multiplication of the corresponding real numbers. Thus $\mathbb{C}$ contains a copy of $\mathbb{R}$, and in this sense we consider it an extension. Also, for this reason, for a complex number (a, b), a is called its *real part*, and b its *imaginary part*.

Let i be the complex number $(0, 1)$. It is called the *imaginary unit*. Let us compute $i^2 = (0, 1) \cdot (0, 1)$. According to Definition 13.1,

$$(0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0).$$

Since we identified $(-1, 0)$ with -1, we can say that $i^2 = -1$. It follows that i is a solution of the equation $x^2 + 1 = 0$. This equation has another solution in $\mathbb{C}$. It is $-i = (0, -1)$, as can be easily checked. Thus the sentence $\exists x\ [x^2 + 1 = 0]$ is true in $(\mathbb{C}, +, \cdot)$, and false $(\mathbb{R}, +, \cdot)$. This shows that $(\mathbb{C}, +, \cdot)$ is not an elementary extension of $(\mathbb{R}, +, \cdot)$.

Not only $x^2 + 1 = 0$ has solutions in $\mathbb{C}$. All polynomial equations have. The field $(\mathbb{C}, +, \cdot)$ is *algebraically closed*. Any polynomial equation with coefficients in $\mathbb{C}$ has a solution in $\mathbb{C}$. This nontrivial fact, called the Fundamental Theorem of Algebra, was first proved rigorously by Jean-Robert Argand in 1806.

More polynomial equations are solvable in $\mathbb{R}$ than in $\mathbb{Q}$, and this is why the field $(\mathbb{R}, +, \cdot)$ is tamer than $(\mathbb{Q}, +, \cdot)$. In $\mathbb{C}$ all polynomial equations have solutions, and the structure becomes as tame as possible. The field of complex numbers is minimal. This is a consequence of a theorem of Chevalley and Tarski, who (independently) proved that a projection of a set that is definable in $(\mathbb{C}, +, \cdot)$ by a polynomial equation is itself definable by a polynomial equation. This allows to show that the field of complex numbers admits elimination of quantifiers, and from this it follows that every parametrically definable subset of $\mathbb{C}$ is either finite or cofinite.

13.1.1 Real Numbers and Order-Minimality

More complex, but still tame, are the order-minimal structures. Those are the structures whose domains are linearly ordered by a definable binary relation, and all of whose parametrically definable subsets of the domain are finite unions of intervals. $(\mathbb{Q}, <)$, $(\mathbb{Q}, +, <)$, and $(\mathbb{R}, +, \cdot, <)$ are order-minimal, as are various other structures obtained by expanding $(\mathbb{R}, +, \cdot, <)$ by adding functions and relations to it. In 1991, Alex Wilkie solved an old problem posed by Tarski, by showing that $(\mathbb{R}, +, \cdot, <, \exp)$, where exp is the binary relation $y = 2^x$, is also order-minimal. Wilkie's result initiated a whole area of study of order-minimal expansions of the field of real numbers. There is a beautiful theorem due to Ya'acov Peterzil and Sergei Starchenko, which states that the structures listed above are in a sense the only order-minimal structures.[2]

There are other levels of tameness, and there is a well-developed theory of tame structures with applications in classical algebra and analysis. We cannot go into more details. Now it is time to see what is on the other side, where wild structures live.

13.2 On the Wild Side

The additive structure of the natural numbers $(\mathbb{N}, +)$ and the multiplicative structure $(\mathbb{N}, \cdot)$ are considered simple. Neither is minimal. The formula $\exists y(y+y = x)$ defines the set of even numbers in $(\mathbb{N}, +)$, and $\exists y(y \cdot y = x)$, the set of square numbers in $(\mathbb{N}, \cdot)$, and those sets are neither finite nor cofinite. Nevertheless, parametrically definable sets in $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$ are well-understood, hence tame. All hell breaks loose when the two operations are combined in the *standard model of arithmetic* $(\mathbb{N}, +, \cdot)$. We will denote $(\mathbb{N}, +, \cdot)$ by $\mathfrak{N}$.

Much of the elegant theory of minimal and order-minimal structures is based on considerations involving a notion of dimension for definable sets. No such notion is available for $\mathfrak{N}$.

In set theory, one can prove that for every infinite set X, there is a one-to-one correspondence between the set of pairs X^2 and X, i.e. there is a one-to-one and onto function $f : X^2 \longrightarrow X$. Such a function always exists, but it may not be easy to find an explicit definition for it. Georg Cantor noticed that for the set $\mathbb{N}$ there is such a function with a particularly simple definition. He defined $C : \mathbb{N}^2 \longrightarrow \mathbb{N}$ as follows

$$C(x, y) = \frac{1}{2}(x + y + 1) \cdot (x + y) + y.$$

[2]For the precise statement of the theorem and an interesting discussion see [28].

Relation $C(x, y) = z$ is definable in $\mathfrak{N}$.[3] It is a small technical fact, but it has important consequences. Speaking metaphorically, first-order logic of $\mathfrak{N}$ sees that $\mathbb{N}^2$ and $\mathbb{N}$ have the same size. This implies that there can be no good notion of dimension for sets which are definable over $\mathfrak{N}$. Every definable subset of $\mathbb{N}^n$, for any n, can be coded in a definable way by a subset of $\mathbb{N}$. The geometry of $\mathfrak{N}$ is flat. Nothing like that can happen in $(\mathbb{R}, +, \cdot)$ nor in $(\mathbb{C}, +, \cdot)$, no one-to-one correspondence between the domain and its Cartesian square is definable in these structures.

Nothing too wild is happening yet, but the fact that higher dimensions are compressed to one in $\mathfrak{N}$ is just a beginning. In 1931, Kurt Gödel proved that the theory of $\mathfrak{N}$ is not axiomatizable, which means that for any effectively presented set axioms for $\mathfrak{N}$, there is a first-order sentence that is true in $\mathfrak{N}$, but does not logically follow from the axioms. In the proof, Gödel exposed the great expressive power of the first-order language of arithmetic. It followed from his analysis that any set of natural numbers that can be generated by an effective process has a first-order definition in $\mathfrak{N}$. All computable sets of natural numbers are definable, and many noncomputable sets are definable as well.

An important ingredient of Gödel's proof is the "arithmetization of language." Each formal symbol of the first-order language of arithmetic is assigned a natural number that serves as its code. Then, using Cantor's coding of finite sequences each formula is coded by a single number known as its *Gödel number*. If T is an effectively presented set of axioms for $\mathfrak{N}$, then the set of Gödel numbers of the sentences in T is definable in $\mathfrak{N}$, and so is the set of their logical consequences $Cons_T$. Moreover, if T is strong enough, for example if it is the set of Peano's axioms, then $Cons_T$ is not computable. This gives us examples of definable noncomputable sets. But there is more. For every number n, the set of Gödel numbers of sentences with no more than n quantifiers that are true in $\mathfrak{N}$ is definable, but, as Alfred Tarski showed, the set of Gödel numbers of all sentences that are true in $\mathfrak{N}$ is not—this is the famous Tarski's undefinability of truth theorem. A closely related argument shows that for every $n > 0$ there are sets of natural numbers that have definitions involving n quantifiers, but cannot be defined by a formula with fewer than n quantifiers. This is already a pretty wild picture, but it perhaps takes a mathematically trained eye to see that. Here comes something even more radical.

It has been already mentioned that the question whether the formal theory ZFC is consistent is delicate. Most mathematicians believe that it is, but we also know that, due to Gödel's Second Incompleteness theorem, we cannot prove that it is consistent within the theory itself. Consequently, when we talk about consistency of ZFC we have to treat it as a plausible, but unproven conjecture. If ZFC is consistent, then by Gödel's completeness theorem, there is a set V and a binary relation E on it, which is a model of the ZFC axioms. The completeness theorem has its arithmetized version which says that if an effectively presented set of axioms is consistent, then it has a model whose domain is $\mathbb{N}$ and whose relations are definable in $\mathfrak{N}$. Since

[3] The defining formula is $z + z = (x + y + 1) \cdot (x + y) + y$.

the axioms of ZFC are effectively presented, if ZFC is consistent, then there is a binary relation E on $\mathbb{N}$, definable in $\mathfrak{N}$, such that $(\mathbb{N}, E)$ is a model of ZFC. This is unexpected. Often in this book I stressed the enormous power of set theory. In ZFC we can reconstruct most of mathematics. All classical mathematical objects can be modeled as sets constructed using the axioms. Now it turns out that all those objects can be found as natural numbers (!) in $(\mathbb{N}, E)$. Now, this is wild.

We all learned in school that multiplication is repeated addition: $5 \cdot 2 = 2+2+2+2+2$. The addition of natural numbers completely determines their multiplication. An interesting corollary of the results that we discussed above is that multiplication is not definable in $(\mathbb{N}, +)$. If it were, this would allow us to reconstruct the whole wilderness of $\mathfrak{N}$ inside $(\mathbb{N}, +)$ and, because the latter structure is somewhat tame, this cannot be done.

Since $\mathfrak{N}$ is wild, any structure in which one can define an isomorphic copy of $\mathfrak{N}$ is wild as well. For example, $\mathfrak{N}$ can be defined in $(\mathbb{Z}, +, \cdot)$. The definition is not complicated, but it uses a nontrivial number-theoretic fact. A theorem of Lagrange states that every natural number can be written as a sum of four squares.[4] Hence, an integer n is nonnegative if and only if there are numbers n_1, n_2, n_3, and n_4 such that

$$n = n_1^2 + n_2^2 + n_3^2 + n_4^2.$$

It follows that the formula

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 [x = x_1^2 + x_2^2 + x_3^2 + x_4^2]$$

defines the set of nonnegative integers $\mathbb{N}$ in $(\mathbb{Z}, +, \cdot)$.

The natural numbers can also be defined in $(\mathbb{Q}, +, \cdot)$. It is a theorem of Julia Robinson, and we discussed it in Chap. 10.

Exercises

Exercise 13.1 *Use Definition 13.1 to verify that the distributive property $a \cdot (b + c) = a \cdot b + a \cdot c$ holds for all complex numbers a, b, and c.*

Exercise 13.2 *Write first-order formulas defining addition and multiplication of complex numbers in* $(\mathbb{R}, +, \cdot)$. *Hint: Remember that complex numbers are defined as pairs of real numbers.*

Exercise 13.3 *Let $C : \mathbb{N}^2 \longrightarrow \mathbb{N}$ be Cantor's pairing function. Compute $C(0, 0)$, $C(1, 0)$, $C(0, 1)$ and $C(1, 1)$.*

[4]For example $5 = 2^2 + 1^2 + 0^2 + 0^2$, $12 = 3^3 + 1^1 + 1^2 + 1^2$, and $98 = 9^2 + 3^2 + 2^2 + 2^2$.

Exercise 13.4 *Cantor's pairing function C can be used to define a one-to-one correspondence between $\mathbb{N}^n$ and $\mathbb{N}$, for any $n > 2$. For example, we can define $D : \mathbb{N}^3 \longrightarrow \mathbb{N}$ by $D(x, y, z) = C(x, C(y, z))$. Show that the relation $D(x, y, z) = w$ is definable in* $(\mathbb{N}, +, \cdot)$.