Chapter 3
What Is a Number?

In a scientific technique there is almost always an arbitrary element, and a philosophical discussion which puts too much stress on the 'technical' aspects of the problem in question, exposes itself all too easily to the suspicion of resting for a part on purely arbitrary stipulations.

Evert W. Beth, *Aspects of Modern Logic* [5].

Abstract In Chap. 1, we used addition and multiplication of the natural numbers to introduce first-order logic. Now, equipped with formal logic, we will go back and we will *reconstruct* the natural numbers and other number systems that are built on them. This looks circular, and to some extent it is. The set of natural numbers with a set of two relations—addition and multiplication—is a fundamental mathematical structure. In the previous discussion, we took the structure of natural numbers for granted, and we saw how some of its features can be described using first-order logic. Now we will examine the notion of natural number more carefully. It will not be as easy as one could expect.

Keywords Natural numbers · Arithmetic operations · Decimal system · Zero

3.1 How Natural Are the Natural Numbers?

As often happens, the most common ideas turn out to be hardest to pinpoint exactly. In "New Athens," the first Polish encyclopedia, published in 1745, Benedykt Chmielowski wrote "What a horse is, everyone can see, and goats—smelly kind of animal."[1] Nowadays, Wikipedia informs us that "The horse (Equus ferus caballus) is one of two extant subspecies of Equus ferus. It is an odd-toed ungulate mammal belonging to the taxonomic family Equidae." Horses are complex creatures, but they can be given a more or less precise definition. But what is a number? According

[1] My translation.

R. Kossak, *Mathematical Logic*, Springer Graduate Texts in Philosophy 3,
https://doi.org/10.1007/978-3-319-97298-5_3

to Google, it is “an arithmetical value, expressed by a word, symbol, or figure, representing a particular quantity and used in counting and making calculations and for showing order in a series or for identification.” This looks fine until we ask: What is arithmetical value, or What is quantity? We read further: “A number is a mathematical object used to count, label, and measure.” So here we have even more of a challenge, we need to explain what is a “mathematical object.” Intuitive understanding is assumed. The encyclopedists seem to be saying: what is a number, everyone can see. This is perfectly fine for practical applications, but our discussion here is at a different level. Our goal is to understand mathematical structures. Since some of the most important structures are made of numbers, it would be good to know what those numbers are, so now I will outline a modern approach to numbers and number systems that was developed in nineteenth century by Georg Cantor, following important earlier work by Richard Dedekind.

Numeracy is a basic skill. We all know numbers, but keep in mind that most of us are familiar not with numbers as such, but rather with their representations. Think of a number, say 123. What is 123? It is a sequence of digits. To know what this sequence *represents*, we need to understand the decimal system. The symbols 1, 2, and 3 are digits. Digits represent the first ten counting numbers (starting with zero). The number corresponding to 123 is $1 \cdot 100 + 2 \cdot 10 + 3 \cdot 1$. In this representation, the number has been split into groups: three ones (units), two tens, and one hundred. The example illustrates how the decimal system works, but it does not explain what the number 123 is, and what natural numbers are in general.

There is nothing special about counting in groups of ten. If instead of groups of tens, we counted in groups of fives, the number 123 would be written as 443, since it is the sum of four twenty-fives, four fives, and three units. If we used the quinary system instead of the decimal, the powers of 5, $5^2 = 25$, $5^3 = 625$, would be known as 100, 1000, and they would have some special names.

The decimal system is very efficient, but the fact that we depend so much on it when we learn basic arithmetic strongly affects the common perception of numbers. To try to get to the essence of natural numbers, we will start with something more basic—counting.

What is counting? We look at a collection of objects and we count: one, two, three, four, … . In the search for basic insights, this is not getting us far, since counting this way presupposes numbers already. So let us forget about numbers for a moment. Notice that we can still count. How? One way would be to point at objects one by one, and say something like: one, one, one… . This is a rudimentary method, but it can be of some use. Looking at a herd of sheep and a collection of pebbles, we can count both of them this way simultaneously, and we can learn something. If we stop one count before the other, we will know whether there are more sheep than pebbles, or more pebbles than sheep. If we stop at the same time, we know that both sets have the same size. This conclusion will be our point of departure. We will not go into whether such counting can always be performed, or what kinds of collections it can be applied to, but let us agree that we do have common intuition about the process itself. We will try to build on that.

To arrive at the notion of number, we will begin with comparison of sizes. To perform such comparisons, one does not even have to count. There is another way. Think of two collections of pebbles. To compare their sizes, one can make two rows, one next to the other, with the same distance between consecutive pebbles. Once we see the rows, we know how the sizes compare. To make such comparisons, we need to be able to organize collections of elements in rows, or, as we will be saying, we need to be able to *order* them. In each such row, there will be a first element, a second element, and so on. We are not *using* numbers here yet. We just see the distinct element at the beginning of the row, then the next right behind it, and the next, and so on. We are not talking here about practical aspects of ordering, counting, and comparing. Our goal is to use a general intuition to motivate a concept.

Since we appeal only to basic intuitions, the discussion so far seems rather trivial, but with a bit more thought, a more sophisticated concept emerges. We can observe that among all rows are those that are the shortest, and any two such shortest rows are of the same size. Let us call this size *two.* If we adjoin one object to one of those smallest rows, we obtain a collection that is *immediately* larger. We call the size of this collection *three*. Next, we generalize. We see that for each size there is a size immediately larger, obtained by adding another element to a collection whose size is already established. Once we learn how to recognize some initial sizes, it is natural to given them names: one, two, three, ...

You may be concerned that we did not start naming sizes starting withe the size of a single object, and naming it *one*. We could have, but we are trying hard to stay close to absolutely clear and basic intuitions, the concept of *one* requires a more careful justification, and we will not do it here. Once we see how to extract the notion of number from comparisons of sizes of ordered collections, it is reasonable to accept also this extreme case of the shortest one one-element row, but it is an afterthought. This is a subtle point, and I am certainly not the first to bring it up. It was Edmund Husserl, who in his insightful analysis of psychological roots of arithmetic concepts in *Philosophy of Arithmetic: Psychological and Logical Investigations* [16], first published in 1891, argued that 1 is a *negative* number since it signifies absence of multiplicity. And what about zero? If we root our understanding of numbers in counting, then the concept of zero does not arise in a natural way, but we can think of it as the size of a row that is still waiting for its first element; it has not started forming yet.

The discussion above serves as an illustration of how a more abstract notion, that of number, is grounded in a more rudimentary one, that of counting. We will not go into what makes a concept rudimentary or what phenomenology would call "immediately given." For us it is enough that the concept does not, in any obvious way involve anything already formally mathematical. There are many ways in which such grounding of higher level concepts can be described, and there will always be a debate on whether the lofty goal of complete clarification has indeed been achieved. There is always room for doubt. In order to make progress, the grounding process has to stop somewhere. For us, it stops at the concept of counting. We will also assume for now that we know intuitively what a collection of objects is, and what it means to order it.

In our approach to numbers, one could suspect some trickery. The point was to explain the nature of natural numbers, and to do so we talked about sizes of collections. We identified a few initial small sizes and gave them names that clearly sounded like numbers. Here again appears to be some circularity. There is a subtle difference though. Numbers can serve as measures of sizes, but the idea of size is more basic. It is more basic in the sense that we do not need to assign any abstract quantities to collections to be able to *compare* their sizes. We can see whether or not two collections are of the same size, without knowing the number of their elements.

In the process of grounding we can go a bit further. To arrive at the concept of number, we do not even need the notion of size. Instead, we can just rely on an intuitive understanding of *larger* and *smaller*. Then, the notion of size emerges as follows. Consider a collection of elements. We can compare this collection to other collections, as described above. Some collections will be smaller, some larger. For some collections, the simultaneous counting process will terminate at the same time. If this happens for two collections, then they are in a certain precise sense equivalent. To identify this sort of equivalence we will use the term *equinumerous*. For any collection, there is a multitude of collections equinumerous with it. The collection of fingers of my right hand is equinumerous with the collection of fingers of my left, and they both are equinumerous with the collection of the USB slots on my laptop, and equinumerous with the collection of all oceans. All those equinumerous collections have their individual features, but there is one thing that they all have in common. That is this *one thing* that we call the size. This common feature *is* the size of the collection and of all other collections that are equinumerous with it.

Now we can introduce the following, more formal definition: a *counting number* is the size of a finite collection. In the definition, the new concept of a counting number is defined in terms of concepts already clarified (or assumed to be clear), with an important exception, we have not yet said a word about what it means for a collection to be finite. One is tempted to say that a collection is finite if it has a finite number of elements, but that is an obvious circularity. In the discussion so far, we tacitly assumed that all collections were finite in the common understanding of this word. In fact, the qualification "finite" is needed only if one anticipates collections that may not be finite. Allowing infinite collections in the discussion is a big step towards abstraction. Up to this point, we could have assumed that all collections were finite, but soon we will be very much concerned with infinite collections. It is the admittance of infinite mathematical structures that gave a great boost to the developments that are the subject of this book. It turns out that the discussion of size, as we described it above, can be generalized to infinite collections in a precise way. It makes sense to talk about the size of any collection, and there is an interesting arithmetic of infinite numbers associated with those sizes. We will say more about it in Chap. 6.

3.1.1 Arithmetic Operations and the Decimal System

The decimal system assigns symbolic representations to all counting numbers. It also provides practical recipes for addition and multiplication. For example

$$\begin{array}{r} 123 \\ \underline{+88} \\ 211 \end{array}$$

Given symbolic representations of two or more numbers, we can, as in the example above, mechanically perform algorithmic operations on their digits and obtain a symbolic representation of their sums. The familiar algorithms for addition and multiplication do not operate on numbers, they operate on strings of digits, but addition and multiplication as such are defined independently of any system of representations. How? Let us go back to the definition of counting numbers. What is three plus two? Think of an ordered set of size three extended by a set of size two in such a way that all elements of the second set come after the elements of the first. The size of the ordered set thus obtained is the result:

$$\bullet\ \bullet\ \bullet + \bullet\ \bullet = \bullet\ \bullet\ \bullet\ \bullet\ \bullet$$

Multiplication is repeated addition. Three times two is $2 + 2 + 2$. To define addition of natural numbers, we employed counting. For multiplication, we need another concept—repetition. Is repetition an everyday notion that needs no formal clarification or are we importing a concept of mathematical nature? This calls for a deeper philosophical analysis, but, instead, we will use a different approach. Instead of repetition, we will invoke another basic idea, that of splitting or breaking into pieces. If three pebbles are each broken into two pieces, the result is six smaller pebbles. The pebbles have multiplied. See the illustration below.

$$\bullet\ \bullet\ \bullet \times \bullet\ \bullet = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array} = \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet$$

This is a set theoretic way of introducing multiplication. It allows to generalize the notion to the case of infinite numbers in a natural way. If m and n are natural numbers, then the product $m \cdot n$ can be defined to be the size of the set obtained from a collection of m elements, by replacing each element with a collection of size n. Notice that the concept of multiplication is defined without any reference to number representations.

3.1.2 How Many Numbers Are There?

In practice, there are natural limitations on the sizes of sets that can actually be counted. We live in a bounded region of the physical space, and our lives are not endless, but even regardless the human aspect, in modern physics the knowable universe is described as huge, but still finite collection of elementary particles, so, in this sense, it is finite. But could there be a largest number? This does not seem to make much sense. If we can count up to a number n, surely we can count up to the next number $n + 1$. The fact that we accept that there is no largest number has profound consequences. It forces us to consider potentially infinite processes, and hints at the necessity to consider mathematical objects that are actually infinite.

Thinking in the other direction, one can imagine a number system without infinity. For example, in a society in which sizes 1, 2, 3, and 4 are recognized, and each collection of more than four elements is considered large, counting and the number system would be simple: one, two, three, four, many. There still would be some general mathematical rules, such as $m + n = m + n$, or many + many = many, but there would be no need to invoke any concept of infinity.

3.1.3 Zero

In the discussion above the size of a smallest size was called one. In a move towards abstraction, we can introduce the concept of *empty* collection. An empty collection is a collection that has no elements, for example the collection of chairs in an empty room. An empty collection is not nothing. It is an object. The size of an empty collection is called zero. Thus we extended the domain of counting numbers, by adding the number zero. It follows from the definitions of addition and multiplication we gave above, that for each number n, $n + 0 = n$ and $0 \cdot n = 0$. While it is clear that if we consider multiplication as repeated addition, then $n \cdot 0$ is the size of the n empty sets put together, hence it is zero. To justify that $0 \cdot n = 0$ using the interpretation that involves splitting, one can say that since there are no elements to split in an empty set, nothing has been added to the set; hence its size remains 0.

Whether or not to include zero as a counting number is a philosophical question, but for practical applications and further developments in algebra, the concept of zero is crucial. For example, in the decimal system 0 is a placeholder marking the absence of certain powers of 10 in the decimal representation of a number. For example, 2017 denotes the size of a set that can be split into $2 \cdot 1,000 + 0 \cdot 100 + 1 \cdot 10 + 7 \cdot 1$. There are no hundreds. The collection of hundreds in the decimal representation of 2017 is empty.

3.1.4 The Set of Natural Numbers

In mathematics, the counting numbers are called *natural*. Also, instead of the term "collection" we will more often use the word *set*. The set of natural numbers is usually denoted by $\mathbb{N}$ and we will follow this convention. In other words, $\mathbb{N} = \{0, 1, 2, \dots\}$.[2]

In the last sentence above, we did something profound. We declared that the symbol $\mathbb{N}$ denotes a mathematical object and, as indicated by the dots, that object is not finite. The set $\mathbb{N}$ is *actually infinite*. We could have said, the natural numbers are: $0, 1, 2, \dots$, having in mind a never-ending process in which for each natural number we can find or construct a number that is one larger. That would involve the idea of *potential infinity* —a process that has no end. Instead we wrote $\mathbb{N} = \{0, 1, 2, \dots\}$. Just by adding two curly brackets, we brought to life a new kind of object—an infinite set. Formal set theory which we will discuss soon, provides a framework for such acts of creation.

Exercises

The arithmetic of natural numbers is governed by a set of simple rules. Addition is *commutative* and *associative*, which means that for all natural numbers a, b, and c, $a + b = b + a$, and $a + (b + c) = (a + b) + c$. Based on the direct intuition of addition, these two rules can be easily justified. Multiplication is also commutative and associative, but how do we know that? For example $3 \cdot 2 = 2 + 2 + 2 = 6$, and $2 \cdot 3 = 3 + 3 = 6$, but how do we know that it is always the case? In the following exercises, you are asked to provide convincing justifications for the following statements. Hint: Think about rectangles and boxes.

Exercise 3.1 *Explain why, for all natural numbers a and b,* $a \cdot b = b \cdot a$.

Exercise 3.2 *Explain why, for all natural numbers a, b,and c,* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Exercise 3.3 *Explain why, for all natural numbers a, b,and c,* $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. *Hint: Draw a rectangle with on side of length a and the other of length* $b + c$.

[2] In the set-theoretic tradition, I included 0 in the set of all natural numbers. Another popular convention, adopted in many textbooks, is to start natural numbers with 1, and then to call the set $\mathbb{N}$ defined above either the set of whole numbers, or the set of nonnegative integers.