Chapter 11
Where Do Structures Come From?

Abstract The compactness theorem, Theorem 11.2, is one of the most frequently used basic tools of model theory. It implies that for every structure with an infinite domain there is another structure that is very similar but not isomorphic to the given one. We will see a toy example that shows how such structure could be used to study number-theoretic problems. A more advanced application is given in Appendix A.5.

Keywords Completeness theorem · Compactness theorem · Elementary extensions · Twin primes conjecture · Nonstandard models

11.1 The Compactness Theorem

A mathematical structure is a set with a set of relations on it. The universe of sets is rich and diverse, hence so is the world of all structures. The axioms of set theory allow us to prove that there exist set-theoretic representations of all objects that modern mathematics needs. This is good, but one may become concerned that the axioms are too powerful. Do we really need all those structures that set theory provides? The answer is positive, but not straightforward. We have seen how set theory is used to construct representations of structures that we intend to study. In this chapter we will see how it can also be used to generate the whole range of other structures often with unexpected properties.

For a given structure, one can ask whether some familiar specific property is expressible in first-order logic, but now we will ask a more ambitious question: Given a structure $\mathfrak{A}$, can we describe *all* properties of $\mathfrak{A}$ that are first-order expressible? Put in this way, the problem is somewhat vague. To make it precise let us define the *first-order theory of* $\mathfrak{A}$ to be the set of all first-order sentences that are true in $\mathfrak{A}$. There are infinitely many sentences that are true in $\mathfrak{A}$, and infinitely many that are false. How can we possibly know the truth or falsity for all of them?

In some cases, the task is not that hopeless. It can happen that there is a finite set of sentences in the theory of a structure from which all other sentences that

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are true in the structure logically follow.[1] If this happens, we say that the theory of $\mathfrak{A}$ is *finitely axiomatizable*. There are structures whose theories are not finitely axiomatizable, but for which there is an infinite, effectively listed set of sentences from which all other first-order truths about the structure follow. In such cases, we say that the theory of the structure is *axiomatizable*.[2] If the theory of a structure is axiomatizable, it may not be a trivial matter to derive this or that particular true statement from the axioms, but at least we do know where all those true statements come from.

Examples of structures with axiomatizable theories are $(\mathbb{Q}, <)$, and $(\mathbb{Z}, +)$; the former is axiomatized by a single sentence expressing that the ordering of $\mathbb{Q}$ is dense and has no least and no last element; the latter by a set of sentences expressing elementary properties of addition. An example of a structure whose theory is not axiomatizable is $(\mathbb{Z}, +, \cdot)$.

Axiomatizability of theories of structures is a big topic and a fuller discussion would take us too far. We will move in another direction.

By a *first-order theory*, we simply mean a set of sentences in a first-order language. Let T be a theory. A structure in which all sentences in T are true is called a *model* of T.

For any structure $\mathfrak{A}$, the theory of $\mathfrak{A}$, denoted $\mathrm{Th}(\mathfrak{A})$, is the set of all sentences in the language of $\mathfrak{A}$ that are true in $\mathfrak{A}$. For each $\mathfrak{A}$, $\mathrm{Th}(\mathfrak{A})$ is a *complete theory*, i.e. for each sentence φ of the language of $\mathfrak{A}$, either φ is in $\mathrm{Th}(\mathfrak{A})$, or $\neg\varphi$ is in $\mathrm{Th}(\mathfrak{A})$. Clearly, $\mathfrak{A}$ is a model of $\mathrm{Th}(\mathfrak{A})$.

Can models be built for any theory? For some theories that cannot be done. If a set of sentences that contains a sentence and its negation it cannot have a model. Such a theory is *inconsistent*. It also cannot be done for theories that do not contain direct contradictions, but from which a contradiction can be derived by logical inference. Remarkably, it turns our that nonderivability of a contradiction is the only condition required for existence of a model of a theory. This was proved by Kurt Gödel in 1929. The result is known as Gödel's completeness theorem.

Theorem 11.1 *Let T be a first-order theory from which a contradiction cannot be formally derived. Then, there is a structure $\mathfrak{A}$ in which all sentences of T are true.*

Gödel proved his theorem for a particular deductive proof system, but it is valid for all systems satisfying certain natural conditions. It also has the following powerful corollary known as the compactness theorem.[3]

[1]For a sentence φ to logically follow from a set of sentences, means that there is a formal proof of φ in which sentences form the set are used as premises. Formal rules of proof have to be specified, and this can be done in several ways.

[2]In technical terms, the theory of a structure is axiomatizable if it has either finite, or a recursive (computable) set of axioms.

[3]Gödel proved the compactness theorem for countable languages. The theorem was extended to uncountable languages by Anatoly Maltsev in 1936.

Theorem 11.2 *If every finite set of sentences of a theory T has a model, then T has a model.*

While Theorem 11.1 is in fact many theorems, one theorem for each formal proof system, Theorem 11.2 is just one statement about theories and their models. Both theorems declare that for any set of first-order properties, as long as they don't contradict one another, there is a structure that has all those properties. What you can think of without contradictions, exists. But where and how do those structure exist? Mathematical theorems do not have the power of bringing objects to life. They are all about the realm of mathematical objects. The completeness and the compactness theorems are logical consequences of the axioms of set theory. They state that under certain assumptions certain sets and relations exist, and they have the required properties. In fact, the axioms of ZF alone are not strong enough to prove both theorems. In full generality, for arbitrary languages that may include uncountably many symbols, their proofs require the famous axiom of choice that declares that for every nonempty collection of disjoint sets there is a set that has exactly one element in common with each set in the collection. This seemingly innocuous axiom cannot be derived from the axioms of ZF, and it is necessary for proofs of many standard results in modern mathematics; hence, it is routinely included among the axioms and the resulting theory is abbreviated by ZFC.

Set theory is a commonly accepted formal framework for modern mathematics.[4] If it follows from the axioms of set theory that a certain set exists, then the set considered a bona fide mathematical object. For a working mathematician, it is as good as a circle drawn on paper. But one has to be cautious, since the set-theoretic approach opens up a whole world of fantastic objects that one can study, and that one becomes familiar in the process. Now and then it makes sense to stop and ask, Aren't we overdoing it? Perhaps the axioms we adopted are too strong. They allow to create a whole universe of objects with intriguing properties, but do we really need all of them? The experience of the last 100 years or so seems to indicate that the answer is yes, but still one should be aware that there may be a problem here.

What right do we have to claim that we gain real knowledge by investigating formal consequences of formal axioms? How do we know that the axioms are correct? How do we even know that they do not contradict each other? Early in the twentieth century several influential mathematicians and philosophers did raise such objections, to which David Hilbert gave his famous response: "No one will drive us from the paradise which Cantor created for us." [11] No one has driven us from the paradise. The new infinitistic methods proved to be extremely effective in all areas of mathematics. High levels of set-theoretic abstraction are used to prove results about much more concrete mathematical domains. A good example is Andrew Wiles' proof of Fermat's Last Theorem.[5]

[4]It is not the only formalism that is used in practice. A powerful alternative is category theory that is preferred in certain areas of algebra and geometry.

[5]For a very readable and comprehensive account see Simon Singh's book [31].

As for consistency of the axioms, after more that a century of intense and diverse investigations, no contradiction has ever been found, giving us a lot confidence, nevertheless the foundational debate is not over. One interesting aspect is the formal status of consistency of theories such as ZF or ZFC. If they are indeed consistent, then Hilbert believed that one should be able to provide a rigorous proof of it. Gödel's second incompleteness theorem, proved in 1931, implies that this is not possible. If a formal theory is strong enough, it cannot prove its own consistency, unless it is inconsistent. The qualification "strong enough" covers all formal theories that one might consider as a foundation base for all of mathematics. It seems that philosophical doubts will never be resolved by formal methods, and that the doubts may linger forever.

Just to give a flavor of the intensity of the foundational debate in the first half of the twentieth century, let me finish this section with a quote about Cantor's paradise from a famous philosopher:

> I would say. 'I wouldn't dream of trying to drive anyone out of this paradise.' I would try to do something quite different: I would try to show you that this is not a paradise—so that you'll leave of your own accord. I would say, 'You are welcome to this; just look about you.' [38]

11.2 New Structures from Old

The compactness theorem is one of the most powerful and frequently used results in modern mathematical logic. In this section, we will examine some of its applications. We will show that every structure with an infinite domain can be extended to a very similar structure with a larger domain. The notion of similarity that we use here will be defined in terms of first-order logic.

We can always expand a given structure to a larger one by adding new elements to its domain and by extending the relations of the structure to the new elements. That is easy, but we also want construct extensions in a way that preserves the character of the structure. If our structure is a graph, we want the extension to be a graph as well. This is not hard to do. However, if it is a field, and we want its extension to be a field as well it is a bit harder to do. It turns our that using the compactness theorem we can make all such extensions and much more without much effort. The theorem gives us a recipe for constructing *elementary* extensions that are defined below. The definition refers to types. See Definition 9.3.

Definition 11.1 Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for the same first-order language, with domains A and B respectively. If A is a subset of B, then we say that $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$ if the type of every tuple of elements of A is the same in $\mathfrak{A}$ and in $\mathfrak{B}$.

It follows directly from the definition that if $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$, then the theory of $\mathfrak{A}$ must be the same as the theory of $\mathfrak{B}$, and that is because the type of every tuple of elements of a structure along with formulas with free variables also contains all sentences that are true in the structure. This remark will provide us with many examples of extensions that are not elementary. In fact, in the process of extending the number structures we have already encountered them. We started with the domain of natural numbers $\mathbb{N}$ and extended it to the integers $\mathbb{Z}$, then to the rational numbers $\mathbb{Q}$, and finally to the real numbers $\mathbb{R}$. In the first-order language of addition and multiplication, none of those extensions are elementary. In fact, the theories of those structures are all different. This is precisely why we introduced those structures in the first place. In each larger structure we can solve some equations that have no solutions the smaller ones. Here are examples of first-order sentences that differentiate between them.

- $\forall x \exists y [x + y = 0]$ is false in $(\mathbb{N}, +, \cdot)$ and true in $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, and $(\mathbb{R}, +, \cdot)$.
- $\forall x [(x = 0) \vee \exists y (x \cdot y = 1)]$ is false in $(\mathbb{Z}, +, \cdot)$ and true in $(\mathbb{Q}, +, \cdot)$ and $(\mathbb{R}, +, \cdot)$.
- $\exists x [x \cdot x = 2]$ is false in the $(\mathbb{Q}, +, \cdot)$ and true in $(\mathbb{R}, +, \cdot)$.[6]

Let us now see an example of an extension that is not elementary, but in which the structure and its extension share the same theory. Let $\mathbb{N}^+$ be the set of natural numbers without 0. Let $\mathfrak{A}$ be the ordered structure $(\mathbb{N}^+, <)$, and let $\mathfrak{B}$ be $(\mathbb{N}, <)$. $\mathfrak{B}$ is an extension of $\mathfrak{A}$, and it is an isomorphic copy of $\mathfrak{A}$. Both ordered sets look exactly the same. The function $f : \mathbb{N}^+ \longrightarrow \mathbb{N}$, defined by $f(n) = n - 1$, is an order preserving one-to-one correspondence between the two domains. Because the structures are isomorphic, their theories are the same, but the extension is not elementary. For example, 1 is the least element in $\mathfrak{A}$, and this property is first-order expressible, so it belongs to the type of 1. In $\mathfrak{B}$, 1 is no longer the least element, so the type of 1 has changed.

The following theorem is a consequence of the compactness theorem and implies that elementary extensions exist in abundance.

Theorem 11.3 *Every structure with an infinite domain has a proper*[7] *elementary extension.*

A proof of Theorem 11.3 is given in Appendix A.4. In the rest of this section we will discuss some powerful consequences.

Let us observe that the assumption that the domain of the structure in Theorem 11.3 is infinite is essential. If the domain of $\mathfrak{A}$ has a finite number of elements, for example 100, this fact can be expressed by a single first-order sentence. If $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$, then the same sentence must be true in $\mathfrak{B}$, hence the domain of $\mathfrak{B}$ must have 100 elements as well. Moreover, if $\mathfrak{A}$ and $\mathfrak{B}$ are structures with finite domains and $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$, not only their domains of the structures must be of the same size, they must be isomorphic. This is an interesting and not entirely obvious fact. The proof of it is given in Appendix A.3.

[6] The parameters 0, 1, and 2 can be eliminated from these formal statements, since they are definable in each structure.

[7] Each structure is considered an extension of itself and it is an elementary extension. A proper extension is an extension that adds new elements to the domain of the extended structure.

In Theorem 11.3 we talk about structures with infinite domains rather then infinite structures. The reason is that a structure on a finite domain can be infinite, because it can have infinitely many relations. We have seen examples of such structures in Chap. 7.

Every structure with an infinite domain has a proper elementary extension, and that extension has an infinite domain, so it has a proper elementary extension, and so on. The process of creating elementary extensions can go on forever. And "forever" here means forever in the set-theoretic sense. We do not do, or create anything really. We use a metaphorical language to describe what set theory allows us to define formally, and to bring set-theoretic objects into being in this sense. Often, the number of steps in set-theoretic constructions is measured not by natural numbers, or rather not only by natural numbers. After the steps one, two, three, ..., comes the first limit step ω, and then ω plus one, ω plus two, and so on. The counting goes on into the transfinite, and the numbers that are used to count the steps, are *ordinal numbers*. The domain of each structure has its set-theoretic size that is its cardinality.

Theorem 11.3 has a sharper version that says that every structure $\mathfrak{A}$ with an infinite domain has a proper elementary extension with a domain of the same cardinality as the domain of $\mathfrak{A}$. In a proper extension, the domain gets enlarged, but set-theoretic size of the domain does not have to increase. However, iterating extensions sufficiently long, we can obtain elementary extensions of a given structure that are of arbitrarily large cardinality. For every structure there is a much larger structure that looks very much like it. Let us see how it works in an example.

The ordered set of the rational numbers $(\mathbb{Q}, <)$ is a dense linear ordering without endpoints, and this fact is expressible by a single first-order sentence in the language with the relation symbol $<$. Let $(D, <)$ be a proper elementary extension of $(\mathbb{Q}, <)$. Since the theory of $(D, <)$ is the same as the theory of $(\mathbb{Q}, <)$, the sentence expressing that the ordering is dense and has no endpoints is also true in $(D, <)$; hence $(D, <)$ is a dense linear ordering without endpoints. Since $\mathbb{Q}$ is countable we can assume that D is countable as well. Any two countable densely ordered sets without endpoints are isomorphic. This means that $(D, <)$ is an isomorphic copy of $(\mathbb{Q}, <)$. It looks exactly the same. So in this case, even though Theorem 11.3 allows us to construct an extension with a larger domain,[8] we are not really getting a genuinely new structure. However, if we keep making the extensions long enough, if we "go through" uncountably many steps, we will have added uncountably many new elements to the domain, and this will give us densely ordered set of uncountable size. By itself, this is not a great achievement. $(\mathbb{R}, <)$ is also an uncountable dense linear ordering without endpoints, but what is interesting here is not only that the process can be continued forever, creating dense linear orderings of arbitrarily large sizes. Constructions of long chains of elementary extensions can be carried out in many substantially different ways, resulting in many similar but non-isomorphic structures. For example, we can start with $(\mathbb{Q}, <)$ and proceed in such a way that the

[8]Larger in the sense of containment of sets, not their cardinalities.

new elements are always larger than all old elements. If this is done over continuum many steps,[9] then the resulting structure is a densely ordered set of cardinality continuum that is not isomorphic to $(\mathbb{R}, <)$.

11.2.1 Twin Primes

We will finish this chapter with an application of the compactness theorem to a number-theoretic problem.

A pair of *twin primes* is a pair of prime numbers that differ by 2. Here are the first six twin prime pairs: (3,5), (11,13), (17,19), (29,31), (41, 43), (59,61). As we go up the ladder of natural numbers we sporadically encounter twin primes. Since there are infinitely many prime numbers, one can ask whether there are also infinitely many twin primes. The Twin Primes Conjecture says that there are, but despite serious efforts, it has not been confirmed.[10]

Here is how one could try to approach the conjecture using model theory. Let $P(x)$ be a first-order formula defining the set of prime numbers in $(\mathbb{N}, +, \cdot)$. The sentence

$$\forall x \exists y [x < y \land P(y) \land P(y + 2)]$$

formally expresses the Twin Primes Conjecture. Let us call this sentence TPC.[11]

The compactness theorem not only tells us that $(\mathbb{N}, +, \cdot)$ has a proper elementary extension, it also can be used to show that there is a great variety of such extensions. Since all those extensions are elementary, each has an unbounded set of elements with the property expressed by the formula $P(x)$. In each extension all old prime numbers in $\mathbb{N}$ still have property $P(x)$, but there are also some new elements c for which $P(c)$ holds. Let us call them *nonstandard primes*. Suppose now that in some extension $(\mathbb{N}^*, +, \cdot)$, we can find a nonstandard prime c such that $c + 2$ is also a prime. Then for every (standard) natural number n the following sentence is true in $(\mathbb{N}^*, +, \cdot)$

$$\exists y [n < y \land P(y) \land P(y + 2)].$$

This sentence is true in $(\mathbb{N}^*, +, \cdot)$, because c is such a y. But this sentence is not referring to any new element in the extension, hence—because the extension is

[9] Continuum is the cardinal number of the set of real numbers $\mathbb{R}$.

[10] According to Wikipedia, the current largest twin prime pair known is $2996863034895 \cdot 2^{1290000} \pm 1$, with 388,342 decimal digits. It was discovered in September 2016.

[11] Some small alterations are needed to make TPC comply with the first-order formalism. To express TPC as a first-order sentence in the language of $+$ and $\cdot$, one has to replace $<$ by its definition in $(\mathbb{N}, +, \cdot)$, the expression $P(y + 2)$ can be written as $\forall z[(z = y + 2) \implies P(z)]$, and the reference to 2 can be eliminated with the help of its definition in $(\mathbb{N}, +, \cdot)$.

elementary—it must be also true in $(\mathbb{N}, +, \cdot)$. It follows that for any natural number n there are twin primes in $\mathbb{N}$ that are bigger than n, and that means that there are infinitely many (standard) twin primes.

By considering elementary extensions, the task of proving that there are infinitely many twin primes, got reduced to the task of finding one nonstandard model with just one pair of nonstandard twin primes. One could hope to solve an outstanding problem in number theory by constructing a particular structure. Unfortunately, it is not as easy as it seems. Recent work in number theory suggests that while the conjecture is likely to be true, any proof of it will be very difficult. If the conjecture is in fact true, then in any elementary extension of $(\mathbb{N}, +, \cdot)$ there are infinitely many nonstandard twin primes, but most likely the only way to prove it is by actually proving the conjecture first.

Exercises

Exercise 11.1 *Write the axiom of choice as a first-order statement using only the membership relation symbol* $\in$.

Exercise 11.2 *Suppose that* $\mathfrak{B}$ *is an elementary extension of* $\mathfrak{A}$. *Show that every relation of* $\mathfrak{B}$ *restricted to the domain of* $\mathfrak{A}$ *is one of the relations of* $\mathfrak{A}$. *Hint: This follows directly from Definition* 11.1.

Exercise 11.3 ** Prove Cantor's theorem: if* $(D, <)$ *and* $(E, <)$ *are countable dense linearly ordered sets without endpoints, then they are isomorphic. Hint: Assume that* $D = \{d_1, d_2, \dots\}$ *and* $E = \{e_1, e_2, \dots\}$. *Then, in a step-by-step fashion, construct a one-to-one correspondence between D and E preserving the ordering of the sets.*

Exercise 11.4 *Show that there are infinitely many types of pairs of elements in* $(\mathbb{Z}, <)$. *Hint: For integer numbers* a *and* b, *with* $a < b$, *consider the distance between* a *and* b, *i.e. the number of integers* x *such that* $a < x + 1 < b$. *Show that for each natural number* n, *the relation "the distance between* a *and* b *is* n*" is definable in* $(\mathbb{Z}, <)$.

Exercise 11.5 *Show that if* a, b, c, *and* d *are rational numbers, and* $a < b$ *and* $c < d$, *then the types of* (a, b) *and* (c, d) *in* $(\mathbb{Q}, <)$ *are equal. Hint: Find a symmetry* $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ *such that* $f(a) = c$ *and* $f(b) = d$.

Exercise 11.6 ** Show that there are nonisomorphic dense linear orderings of power continuum. Hint: A construction is outlined in Sect.* 11.2.