Supplementary Lecture E

Final State Versus Empty Stack

It doesn't matter whether we take NPDAs to accept by empty stack or by final state; the two methods of acceptance are equivalent in the sense that each type of machine can simulate the other. Given an arbitrary NPDA M that accepts by final state or empty stack, we will show how to construct an equivalent NPDA M' with a single accept state for which acceptance by empty stack and by final state coincide.

The construction of M' differs slightly, depending on whether M accepts by final state or by empty stack, but there is enough in common between the two constructions that we will do them together, pointing out where they differ.

We have not discussed deterministic PDAs yet—we will do so in Supplementary Lecture F—but for future reference, the construction we are about to give can be made to preserve determinism.

Let

$$M = (Q,\ \Sigma,\ \Gamma,\ \delta,\ s,\ \perp,\ F)$$

be an NPDA that accepts by empty stack or by final state. Let u, t be two new states not in Q, and let $\perp\!\!\!\perp$ be a new stack symbol not in Γ. Define

$$G \stackrel{\text{def}}{=} \begin{cases} Q & \text{if } M \text{ accepts by empty stack,} \\ F & \text{if } M \text{ accepts by final state;} \end{cases}$$

D. C. Kozen, *Automata and Computability*

$$\Delta \stackrel{\text{def}}{=} \begin{cases} \{\perp\!\!\!\perp\} & \text{if } M \text{ accepts by empty stack,} \\ \Gamma \cup \{\perp\!\!\!\perp\} & \text{if } M \text{ accepts by final state.} \end{cases}$$

Consider the NPDA

$$M' = (Q \cup \{u,t\},\ \Sigma,\ \Gamma \cup \{\perp\!\!\!\perp\},\ \delta',\ u,\ \perp\!\!\!\perp,\ \{t\}),$$

where δ' contains all the transitions of δ, as well as the transitions

$$((u,\epsilon,\perp\!\!\!\perp),\ (s,\perp\perp\!\!\!\perp)), \tag{E.1}$$

$$((q,\epsilon,A),\ (t,A)), \quad q \in G, \quad A \in \Delta, \tag{E.2}$$

$$((t,\epsilon,A),\ (t,\epsilon)), \quad A \in \Gamma \cup \{\perp\!\!\!\perp\}. \tag{E.3}$$

Thus the new automaton M' has a new start state u, a new initial stack symbol $\perp\!\!\!\perp$, and a new single final state t. In the first step, by transition (E.1), it pushes the old initial stack symbol $\perp$ on top of $\perp\!\!\!\perp$, then enters the old start state s. It can then run as M would, since it contains all the transitions of M. At some point it might enter state t according to (E.2). Once it enters state t, it can dump everything off its stack using transitions (E.3). Moreover, this is the *only* way it can empty its stack, since it cannot pop $\perp\!\!\!\perp$ except in state t. Thus acceptance by empty stack and by final state coincide for M'.

Now we show that $L(M') = L(M)$. Suppose first that M accepts by empty stack. If M accepts x, then

$$(s,x,\perp) \xrightarrow[M]{n} (q,\epsilon,\epsilon)$$

for some n. But then

$$(u,x,\perp\!\!\!\perp) \xrightarrow[M']{1} (s,x,\perp\perp\!\!\!\perp) \xrightarrow[M']{n} (q,\epsilon,\perp\!\!\!\perp) \xrightarrow[M']{1} (t,\epsilon,\perp\!\!\!\perp) \xrightarrow[M']{1} (t,\epsilon,\epsilon).$$

Now suppose M accepts by final state. If M accepts x, then

$$(s,x,\perp) \xrightarrow[M]{n} (q,\epsilon,\gamma), \quad q \in F.$$

Then

$$(u,x,\perp\!\!\!\perp) \xrightarrow[M']{1} (s,x,\perp\perp\!\!\!\perp) \xrightarrow[M']{n} (q,\epsilon,\gamma\perp\!\!\!\perp) \xrightarrow[M']{1} (t,\epsilon,\gamma\perp\!\!\!\perp) \xrightarrow[M']{*} (t,\epsilon,\epsilon).$$

Thus in either case, M' accepts x. Since x was arbitrary, $L(M) \subseteq L(M')$.

Conversely, suppose M' accepts x. Then

$$(u,x,\perp\!\!\!\perp) \xrightarrow[M']{1} (s,x,\perp\perp\!\!\!\perp) \xrightarrow[M']{n} (q,y,\gamma\perp\!\!\!\perp) \xrightarrow[M']{1} (t,y,\gamma\perp\!\!\!\perp) \xrightarrow[M']{*} (t,\epsilon,\epsilon)$$

for some $q \in G$. But $y = \epsilon$, since M' cannot read any input symbols once it enters state t; therefore,

$$(s,x,\perp) \xrightarrow[M]{n} (q,\epsilon,\gamma). \tag{E.4}$$

Now let's consider the definitions of G and Δ and transitions (E.2) governing the first move into state t, and ask how the transition

$$(q, \epsilon, \gamma\perp\!\!\!\perp) \xrightarrow[M']{1} (t, \epsilon, \gamma\perp\!\!\!\perp)$$

could come about. If M accepts by empty stack, then we must have $\gamma = \epsilon$. On the other hand, if M accepts by final state, then we must have $q \in F$. In either case, (E.4) says that M accepts x. Since x was arbitrary, $L(M') \subseteq L(M)$.