5 Column Stability

A central assumption of the theories for trusses, beams and frames developed in the previous chapters is that the equilibrium conditions have been formulated with reference to the original geometry, i.e. neglecting the fact that the structure via its deformation takes on a somewhat modified geometric configuration. The basic problem of a curve with elastic stiffness – the so-called 'elastica' – loaded by a concentrated force at the end was treated by Leonhard Euler (1707–1783), who gave a very extensive and general analysis of this special problem. In many cases the effect of the normal force on the stability of beam and frame structures can be analyzed by using a somewhat simpler theory including small, but finite, displacements of the original configuration.

In this chapter the theory is developed for a beam with a non-trivial normal force. For simplicity – and because this is often the case – the normal force is assumed to be given, or to be a parameter to be determined by the specific problem in question. The key point of the theory is, that when the beam is displaced $w(x)$ in the transverse direction, the normal force acting in the beam is also displaced. When the normal force has a sufficient magnitude, this

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effect becomes important, and in the case of a compressive force the effective stiffness of the structure may be reduced, possibly leading to instability of the structure. In many cases of practical interest the normal force may be deduced from the load, and therefore considered as a known quantity. This simplified stability problem is the subject of the present chapter.

The present chapter is devoted to the basic properties of the 'linearized' stability problem of a single member. First, the simple bending theory of beams is extended to include the effect of a normal force in Section 5.1, and it is demonstrated that a compressive normal force leads to reduced stiffness. For a sufficiently large normal force there is no stiffness to resist bending and the column becomes unstable and buckles. The magnitude of the load, at which buckling occurs, is called the critical load, and Section 5.2 considers the stability problem and the associated critical load and buckling shape for ideal columns. In practice, columns are not ideally straight and the load is not only axial. Buckling of real columns is therefore a gradual process, in which the displacements increase and eventually become virtually unbounded. The magnitude of the displacements before reaching the critical load is determined by imperfections in the initial geometry and by bending loads, causing initial curvature. A column design procedure based on a combination of material strength and the influence of initial imperfections is developed in Section 5.3. In the specific column problems treated in this chapter the shear deformation is of minor importance, and the column theory is therefore developed as a generalization of Bernoulli beam theory without including shear flexibility. The effects of normal force and shear flexibility can be combined in a convenient approximate way as discussed in Chapter 7.

The focus in this chapter is on the development of the basic principles in the context of the single structural element – the column. Many structures contain beams carrying a substantial normal force. These members are called beam-columns, and they combine the properties of a beam with the particular features of a column. In beam-columns the effect of the normal force is to change the deformation stiffness. A convenient way of representing this effect, suitable for numerical analysis of beams and frames, is presented in Chapter 7.

5.1 Beam with normal force

The basis for the theory of elastic beam-columns is the equilibrium equations, formulated for the deformed state of the beam. Figure 5.1 shows the deformed state of a beam-column with distributed transverse load $p(x)$. The displacements, and in particular the rotations due to the displacement gradients, are assumed to be small, and thus no distinction will be made between the length increment ds along the beam axis in the deformed state and its projection dx on the line of the initial beam axis. The figure shows a slice of thickness $ds \simeq dx$, and the forces and moments acting on it. The internal force vector on a section of the beam has the components N and Q in the axial and transverse directions, respectively. Note, that in the present formulation of a linearized beam-column theory the normal force N is taken as the component along the direction of the original beam axis, and the shear force Q is in a direction normal to this. In the linearized theory the normal force N is treated as prescribed or as an unknown parameter, and the axial equilibrium and extension of the beam is therefore not treated separately.

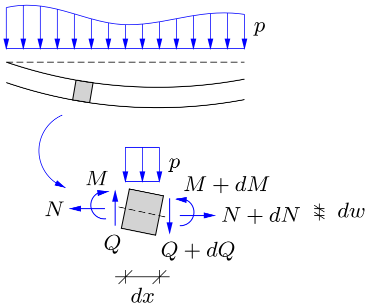


Fig. 5.1: Equilibrium of beam-column in deformed state.

Equilibrium in the transverse direction requires that the projection of all forces on this direction have the sum zero. With the present definition of shear force and normal force, only the shear force Q contributes to the transverse equilibrium, where the sum of internal and external contributions is

$$(Q + dQ) - Q + p\,dx = 0\,. \tag{5.1}$$

The terms $\pm Q$ cancel, and division by dx leads to the differential equation

$$\frac{dQ}{dx} = -p\,. \tag{5.2}$$

The sum of moments must also vanish. In the present case the shear force Q contributes as a force couple with distance dx, and the normal force N contributes as a force couple with distance dw. This gives the moment equilibrium equation

$$(M + dM) - M + N\,dw - Q\,dx = 0\,. \tag{5.3}$$

After cancelation of $\pm M$, division by dx leads to the differential equation

$$\frac{dM}{dx} + N\frac{dw}{dx} = Q\,. \tag{5.4}$$

The two first-order differential equations (5.2) and (5.4) must be satisfied irrespective of the material properties of the beam. The shear force can be eliminated, resulting in the second-order differential equation

$$\frac{d^2 M}{dx^2} + \frac{d}{dx}\Big(N\,\frac{dw}{dx}\Big) + p = 0\,. \tag{5.5}$$

This differential equation contains the moment M, the normal force N and the displacement derivative dw/dx. The dependence on the displacement gradient excludes the possibility of determining the internal forces from statics alone, and it is therefore necessary to express the moment M in terms of the deformation of the beam.

In the linear beam bending theory developed in Chapter 4 the cross-section rotation θ and the curvature of the beam axis κ were introduced as

$$\theta = -\frac{dw}{dx}\,, \qquad \kappa = \frac{d\theta}{dx} = -\frac{d^2 w}{dx^2}\,. \tag{5.6}$$

The relation between the moment M and the curvature κ is not changed by the presence of the normal force, and thus the moment is expressed in terms of the bending stiffness EI_z and the curvature by (4.18),

$$M = EI_z\,\kappa = -\,EI_z\,\frac{d^2 w}{dx^2}\,. \tag{5.7}$$

The difference between the bending theory and the present theory including the displacement of the normal force is found in the relation for the shear force Q. The shear force Q is determined from the equilibrium equation (5.4). When the moment is expressed in terms of the constitutive relation (5.7), the shear force equation takes the form

$$Q = -\,\frac{d}{dx}\Big(EI_z\,\frac{d^2 w}{dx^2}\Big) + N\,\frac{dw}{dx}\,. \tag{5.8}$$

Note the occurrence of the normal force N in the expression for the shear force.

The equilibrium equation follows from substitution of the shear force (5.8) into the transverse equilibrium equation (5.2),

$$\frac{d^2}{dx^2}\Big(EI_z\,\frac{d^2 w}{dx^2}\Big) - \frac{d}{dx}\Big(N\,\frac{dw}{dx}\Big) - p = 0\,. \tag{5.9}$$

For a single beam or column this equation must be solved in connection with two boundary conditions at each end. The kinematic boundary conditions are expressed in terms of the displacement w and the rotation $\theta = -dw/dx$, while static boundary conditions are expressed in terms of the moment M and the shear force Q as given by (5.7) and (5.8).

5.1.1 Stiffness reduction from normal force

An important effect of a normal force in a beam is that it changes the effective stiffness of the beam. This effect is here illustrated by a simple example but is of general nature.

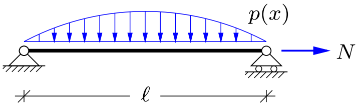


Fig. 5.2: Beam with distributed transverse load and normal force N.

Figure 5.2 shows a simply supported beam-column with normal force N, loaded by a distributed transverse load of intensity

$$p(x) \;=\; p_0 \sin\left(\pi \frac{x}{\ell}\right). \tag{5.10}$$

The simple supports at the ends imply that $w = 0$ and $d^2w/dx^2 = 0$ at $x = 0$ and $x = \ell$. These boundary conditions are satisfied by the function

$$w(x) \;=\; w_c \sin\left(\pi \frac{x}{\ell}\right), \tag{5.11}$$

and the displacement w_c at the center is determined by substitution into the differential equation (5.9),

$$EI_z \left(\frac{\pi}{\ell}\right)^4 w_c + N \left(\frac{\pi}{\ell}\right)^2 w_c - p_0 \;=\; 0\,. \tag{5.12}$$

This determines the center displacement as

$$w_c \;=\; \frac{p_0}{EI_z\Big(\dfrac{\pi}{\ell}\Big)^4 + N\Big(\dfrac{\pi}{\ell}\Big)^2} \;=\; \frac{1}{1 + \dfrac{N}{EI_z}\left(\dfrac{\ell}{\pi}\right)^2}\, \frac{p_0}{EI_z}\left(\frac{\ell}{\pi}\right)^4. \tag{5.13}$$

The last factors represent the displacement in the corresponding beam problem with $N = 0$,

$$w_c^0 \;=\; \frac{p_0}{EI_z}\left(\frac{\ell}{\pi}\right)^4. \tag{5.14}$$

The first factor is an amplification factor, containing the effect of the normal force. It is seen that the amplification becomes infinite at a compressive force $N = -P_E$ of magnitude

$$P_E \;=\; EI_z\left(\frac{\pi}{\ell}\right)^2. \tag{5.15}$$

This particular load is called the Euler load, a reference to the original work of Euler on columns. In column problems it is often convenient to consider compressive axial forces as positive. This is handled by introducing the notation $P = -N$, whereby P denotes an axial force with positive values corresponding to compression.

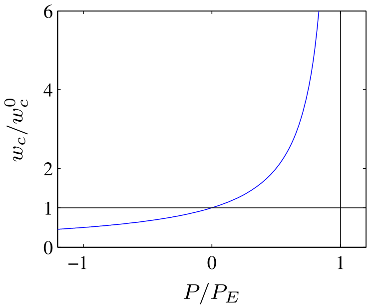


Fig. 5.3: Load-deflection curve for simply supported beam.

In terms of the two characteristic parameters w_c^0 and P_E the center displacement formula takes the form

$$w_c = \frac{w_c^0}{1 - P/P_E}\,. \tag{5.16}$$

This relation is shown graphically in Fig. 5.3. Although the solution is particularly simple in the present case due to the special choice of load distribution, the amplification behavior illustrated in Fig. 5.3 applies to most beam-column problems. It is seen that the application of axial compression ($P > 0$) leads to an increase of the deformations, while axial tension ($P < 0$) reduces the deformations. The effect of axial compression is much more dramatic than axial tension, and for $P = P_E$ the beam-column has lost its stiffness completely, leading to column instability, discussed in the following section.

5.2 Stability of the ideal column

In the previous section it was found that for a sufficiently large compressive axial force a simply supported beam could obtain arbitrarily large transverse deformations, even for a very small transverse load. This axial load, often called the Euler load, can be identified directly, without applying a transverse load. The idea is to consider an ideally straight column as shown in Fig. 5.4a. A compressive axial load $P = -N > 0$ is then applied, Fig. 5.4b. Hereby the column becomes slightly shorter, but in most cases of practical interest, this shortening is negligible. The main point is, that because the column is ideally straight and there is no transverse load, it will remain straight under a limited axial load. If the axial load is increased, a magnitude P_E is reached, at which two solutions exist: a straight configuration, and a buckled form as shown in Fig. 5.4c. This problem has the form of an eigenvalue problem, and the associated critical load P_E is found as an eigenvalue.

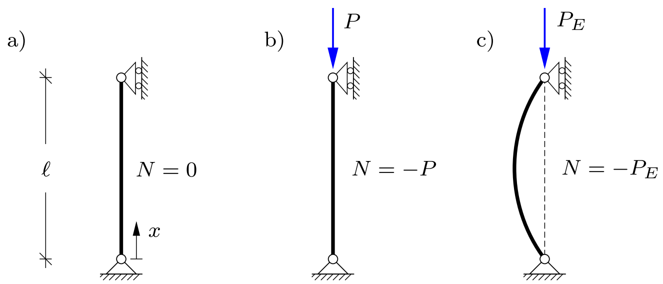


Fig. 5.4: Euler column.

Consider a column of length ℓ and constant bending stiffness EI_z. There is no transverse load, and the differential equation (5.9) then takes the homogeneous form

$$\frac{d^4w}{dx^4} + \frac{P}{EI_z}\frac{d^2w}{dx^2} = 0\,. \tag{5.17}$$

In this equation the axial force P only appears via the coefficient to the second term. This coefficient has the dimension [length^{-2}], and it is therefore advantageous to introduce the parameter k with dimension [length^{-1}], defined by

$$k^2 = \frac{P}{EI_z}\,. \tag{5.18}$$

It is noted that the direct interpretation of k as real-valued assumes a compression force, $P \geq 0$. If the effect of a tension force on the beam bending problem is to be investigated, a modified notation can be used, or the results based on the present k-parameter can be translated into real-valued form.

The differential equation now takes the normalized form

$$\frac{d^4w}{dx^4} + k^2\frac{d^2w}{dx^2} = 0\,. \tag{5.19}$$

The general solution to this homogeneous 4'th order differential equation is

$$w(x) = C_1 + C_2\,kx + C_3\cos(kx) + C_4\sin(kx)\,. \tag{5.20}$$

In this form the coordinate x only appears in the non-dimensional combination kx. The moment follows from (5.7) as

$$\frac{M(x)}{EI_z} = -\frac{d^2w}{dx^2} = C_3\,k^2\cos(kx) + C_4\,k^2\sin(kx)\,, \tag{5.21}$$

and the shear force from (5.8),

$$\frac{Q(x)}{EI_z} = -\frac{d^3w}{dx^3} - k^2\frac{dw}{dx} = -C_2\,k^3\,. \tag{5.22}$$

These relations are used to formulate static boundary conditions. First the general solution is obtained for the simply supported ideally straight column – the so-called Euler column. This solution is used as a reference, and the influence of alternative support conditions is illustrated by examples.

The Euler column

The boundary conditions of the Euler column shown in Fig. 5.4 are

$$w(0) = w(\ell) = 0\,, \qquad M(0) = M(\ell) = 0\,. \tag{5.23}$$

Both the differential equation (5.19) and the boundary conditions (5.23) are homogeneous. Thus, the solution will be $w(x) \equiv 0$, except for particular values k_n of the parameter k that permit a nontrivial solution. These values k_n are the eigenvalues, and to each eigenvalue corresponds an eigenfunction $w_n(x)$. The eigenfunctions describe the buckled shape of the column and are often called the buckling modes.

The boundary conditions at the end $x = 0$ give the equations

$$\begin{aligned} w(0) &= C_1 + C_3 = 0\,, \\ w''(0) &= \quad -k^2\,C_3 = 0\,, \end{aligned} \tag{5.24}$$

where the notation $w'' = d^2w/dx^2$ has been used for the second derivative. These equations determine the parameters $C_1 = C_3 = 0$. The boundary conditions at $x = \ell$ then give

$$\begin{aligned} w(\ell) &= k\ell\,C_2 + \sin(k\ell)\,C_4 = 0\,, \\ w''(\ell) &= \qquad -k^2\sin(k\ell)\,C_4 = 0\,. \end{aligned} \tag{5.25}$$

These equations imply that

$$k\ell\,C_2 = 0\,, \qquad k^2\sin(k\ell)\,C_4 = 0\,. \tag{5.26}$$

A nontrivial solution requires $k\ell > 0$, and thus the first equation gives $C_2 = 0$. This leaves the final equation (5.26b). Naturally this equation can be satisfied by $C_4 = 0$, but this would reduce the solution to $w(x) \equiv 0$.

A nontrivial solution with $C_4 \neq 0$ is found by selecting the parameter k such that

$$\sin(k\ell) = 0\,. \tag{5.27}$$

This equation has the positive roots

$$k\ell = \pi, 2\pi, 3\pi, \cdots \quad \text{or} \quad k_n = n\,\frac{\pi}{\ell}\,, \; n = 1, 2, 3, \cdots \tag{5.28}$$

These roots correspond to the axial loads

$$P_n = EI_z\, k_n^2 = n^2\Big(\frac{\pi}{\ell}\Big)^2 EI_z\,, \quad n = 1, 2, 3, \cdots \tag{5.29}$$

The smallest of these loads is called the Euler load,

$$P_E = EI_z \Big(\frac{\pi}{\ell}\Big)^2. \tag{5.30}$$

At this load the column can buckle into a non-straight mode of deformation, given by the transverse displacement

$$w_E(x) = C \sin\Big(\pi\frac{x}{\ell}\Big)\,, \tag{5.31}$$

illustrated in Fig. 5.4c.

The transverse displacement of a beam-column with a transverse load will grow towards infinity, as the axial compression force P approaches the Euler load P_E as demonstrated for a special case in Section 5.1.1. Thus, it appears that a beam-column gradually looses its bending stiffness with increasing normal compression.

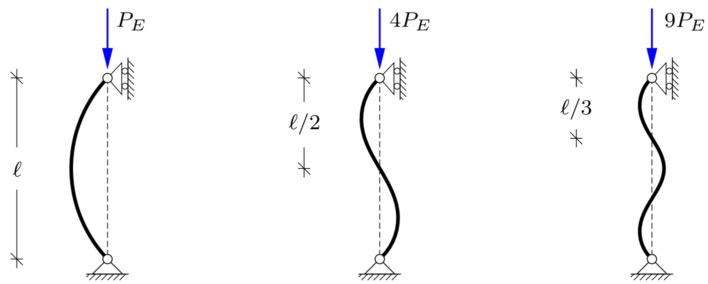


Fig. 5.5: First three buckling forms of the Euler Column.

In general the buckling modes of the Euler column and the corresponding buckling loads are given by

$$\left.\begin{aligned} P_n &= n^2\Big(\frac{\pi}{\ell}\Big)^2 EI_z \\ w_n(x) &= C_n \sin\Big(n\pi\frac{x}{\ell}\Big) \end{aligned}\right\} \quad n = 1, 2, 3, \cdots \tag{5.32}$$

The first three buckling modes are shown in Fig. 5.5. In practice it will be difficult to increase the load beyond the smallest buckling load P_E, if the column is only supported at the ends. However, the higher buckling modes correspond to the buckling modes of columns with equally spaced intermediate supports.

The column theory expressed by the linear differential equation (5.9) is only approximate. In its derivation it was assumed that the rotations are 'small', and that the length along the deformed beam can be represented by its projection, $ds \simeq dx$. These approximations reduce the problem to the form of a linear eigenvalue problem, but also limit the scope of the solution to the onset of instability, where the deformation is small. Thus, the theory is useful in establishing a reference value, such as P_E, for the onset of instability, while description of the development of the load and displacements after the onset of instability requires a non-linear theory, see e.g. Dym (1974) or Bažant and Cedolin (2010).

Example 5.1. Built-in column. Figure 5.6 shows a column of length ℓ with one fixed end, supporting an axial compression force P at the free end. The general solution is given by (5.20), where the arbitrary constants and the parameter k are to be determined by the boundary conditions as in the case of the simply supported column treated above.

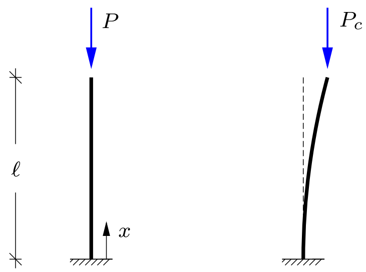


Fig. 5.6: Buckling of built-in column.

In the present problem the boundary conditions at the fixed end are

$$w(0) \;=\; C_1 + \;\; C_3 = 0\,,$$
$$w'(0) = k\,C_2 + k\,C_4 = 0\,.$$

These equations give $C_3 = -C_1$, $C_4 = -C_2$, and the solution (5.20) reduces to the form

$$w(x) \;=\; C_1[\,1-\cos(kx)\,] \,+\, C_2[\,kx - \sin(kx)\,]\,.$$

The boundary conditions at the top of the column are $M(\ell) = 0$ and $Q(\ell) = 0$. By (5.22) the condition $Q(\ell) = 0$ gives $C_2 = 0$, and by (5.21) the condition $M(\ell) = 0$ then is

$$w''(\ell) \;=\; k^2\,\cos(k\ell)\,C_1 \;=\; 0\,.$$

From this equation the eigenvalues are determined as

$$k\ell = \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \tfrac{5}{2}\pi, \cdots \quad \text{or} \quad k_n = n\Big(\frac{\pi}{2\ell}\Big), \; n = 1, 3, 5, \cdots$$

corresponding to the axial loads

$$P_n = EI_z\, k_n^2 = n^2\Big(\frac{\pi}{2\ell}\Big)^2 EI_z\,, \quad n = 1, 3, 5, \cdots$$

and the buckling modes

$$w_n(x) = C_n[\,1 - \cos(k_n x)\,]\,, \quad k_n = n\Big(\frac{\pi}{2\ell}\Big), \; n = 1, 3, 5, \cdots$$

Note, that these modes correspond to the symmetric buckling modes of a simply supported Euler column of length 2ℓ, obtained by extending the actual column symmetrically below the fixed support. □

Example 5.2. Column combining a fixed and a simple support. In Fig. 5.7 a simple support has been added to the column of Example 5.1. This does not change the solution procedure, but the result can no longer be given explicitly. The general solution is given by (5.20), and after imposing the boundary conditions at the fixed end as in Example 5.1 the solution takes the form

$$w(x) = C_1[\,1 - \cos(kx)\,] + C_2[\,kx - \sin(kx)\,]\,.$$

The boundary conditions at the top of the column are $w(\ell) = 0$ and $M(\ell) = 0$, whereby

$$\begin{aligned} w(\ell) &= [\,1 - \cos(k\ell)\,]\,C_1 + [\,k\ell - \sin(k\ell)\,]\,C_2 = 0\,, \\ w''(\ell) &= \quad k^2 \cos(k\ell)\,C_1 + \quad k^2 \sin(k\ell)\,C_2 = 0\,. \end{aligned}$$

A nontrivial solution to this pair of equations can only be obtained, if the determinant of the equation system vanishes, i.e. if

$$[\,1 - \cos(k\ell)\,]\sin(k\ell) - [\,k\ell - \sin(k\ell)\,]\cos(k\ell) = 0\,.$$

This equation can be reformulated as

$$\tan(k\ell) = k\ell\,.$$

This is a transcendental equation. The left and right hand sides are shown in Fig. 5.8.

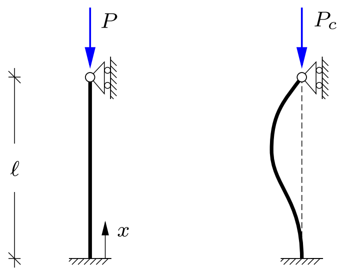


Fig. 5.7: Column with fixed and simple supports.

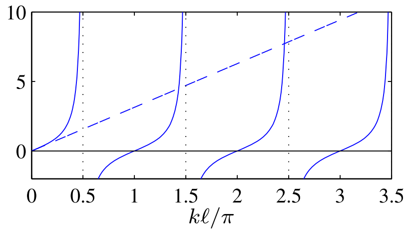


Fig. 5.8: Stability equation: $\tan(k\ell)$ (–) and $k\ell$ (- -).

The roots of the transcendental stability equation are given by the abscissae $k_1\ell, k_2\ell, \cdots$ of the points of intersection. These abscissae can be found by iteration, starting from the values at which $\tan(k\ell)$ has a vertical asymptote,

$$k_n^{(0)}\ell \;=\; (n+\tfrac{1}{2})\,\pi \qquad , \qquad n = 1, 2, 3, \cdots$$

Note, that the root $k_0\ell = 0$, corresponding to $k_0^0 = \frac{1}{2}\pi$ is without interest. The iteration procedure can be formulated as

$$k_n^{(i+1)}\ell \;=\; n\,\pi \;+\; \tan^{-1}(k_n^i\ell)\,.$$

The two first steps and the final value of the parameter $k_n\ell$ are given below.

n	1	2	3	4
$k_n^{(0)}\ell$	4.7124	7.8540	10.9956	14.1372
$k_n^{(1)}\ell$	4.5033	7.7273	10.9049	14.0665
$k_n\,\ell$	4.4934	7.7253	10.9041	14.0662
P_n/P_E	2.0457	6.0468	12.0471	20.0472

The table also gives the buckling loads, conveniently determined by

$$\frac{P_n}{P_E} \;=\; \frac{\ell^2}{\pi^2}\,\frac{P_n}{EI_z} \;=\; \Big(\frac{k_n\ell}{\pi}\Big)^2\,.$$

The parenthesis is the ratio of the non-dimensional parameter $k_n\ell$ of the actual column to its value π for the first buckling mode of the Euler column. This is a convenient form, as $k_n\ell$ is the unknown iteration parameter.

The table illustrates that, apart from the first two roots, the remaining roots are given to within 1 pct. by the formula $k_n\ell \simeq (n+\frac{1}{2}),\;\; n = 3, 4, \cdots$, used as the start value in the iteration. However, in a technical context it is often the first root that is of interest.

The buckling modes are determined by the values $k_n\ell$ that have just been determined. It follows from the boundary condition $w''(\ell) = 0$ that

$$C_1 \;=\; -\,\tan(k\ell)\,C_2 \;=\; -\,k\ell\,C_2\,,$$

where the last relation follows from the determinant equation. When this relation is used to eliminate C_1 the buckling modes can be written as

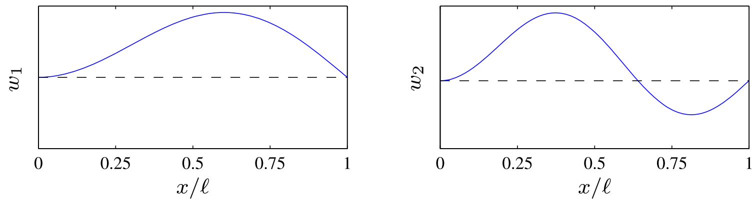


Fig. 5.9: First and second buckling modes of fixed-simple support column.

$$w_n(x) = C_n\Big[k_n\ell\big(\cos(k_nx) - 1\big) + k_nx - \sin(k_nx)\Big],$$

where C_2 is replaced by C_n, indicating that the solution contains a single unknown scaling factor for each mode n. The first buckling mode corresponding to $k_1\ell = 4.49$ is shown in Fig. 5.9a. The displacement is zero at the supports, and the slope is zero at the fixed support to the left in the figure. The maximum of the buckling mode occurs at $x_{\max} = 0.6\,\ell$. Figure 5.9b shows the second buckling mode, associated with $k_2\ell = 7.73$. This mode satisfies the same homogeneous boundary conditions and has an additional zero crossing. In practice this buckling mode will only occur, if the column is supported at the point of the zero crossing. However, an internal support in the neighborhood of this point will lead to a slight modification of buckling load and buckling form. □

Example 5.3. Instability of column supported beam. Figure 5.10a shows a beam BCD of length $3a$ with a uniformly distributed load of intensity p. The beam is simply supported at B and supported at C by the column CA. The column is simply supported at A and is connected to the beam through a hinge in C, whereby no moment is transferred between beam and column.

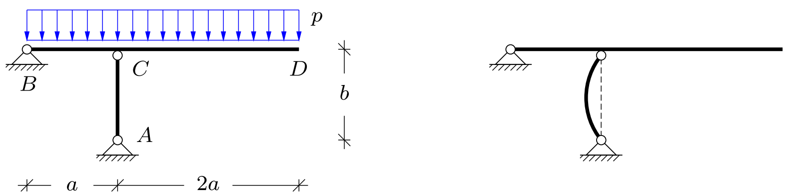


Fig. 5.10: Beam with supporting column.

Moment about C for the column gives zero horizontal reaction in A, and horizontal projection then gives zero horizontal reaction in B. The vertical reactions in A and B are found by moment for the entire structure about B and A, respectively,

$$R_A = \tfrac{9}{2}ap, \qquad R_B = -\tfrac{3}{2}ap.$$

This implies that the normal force in the column is

$$N_{AC} = -\tfrac{9}{2}ap.$$

The minus shows that the column is in compression. The column AC is an Euler column of length b, with zero moment at both ends. Thus the critical magnitude p_c of the intensity of the distributed load is reached, when the magnitude of the normal force is equal to the Euler force, i.e. $|N_{AC}| = P_E$. This gives

$$\tfrac{9}{2}ap \;=\; EI_z\left(\frac{\pi}{b}\right)^2 \qquad \Rightarrow \qquad p_c \;=\; \frac{2\pi^2 EI_z}{9ab^2}\,.$$

The buckling mode is a sine half-wave, as shown in Fig. 5.10b. □

Often a column will be rigidly connected to a support or structure in a way that imposes an elastic partial constraint against rotation. The elastic constraint gives a restraining moment to the column. This constraint will increase the critical column load and form an intermediate state between the free hinge and a full constraint. The general problem is dealt with in detail in Chapter 7, while the simpler problem of a spring supported column is dealt with in the following example.

Example 5.4. Simply supported column with rotation spring. Figure 5.11 shows a column of length ℓ with simple supports. At the support B the column is also supported by a rotation spring with stiffness parameter $k_\theta = M_B/\theta_B$. The critical load P_c depends on the spring stiffness, and by increasing the spring stiffness the critical load increases from the Euler value P_E to that of the column with one hinged and one fixed support, i.e. approximately $2P_E$.

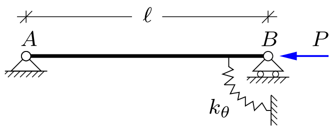


Fig. 5.11: Simply supported column with rotation spring.

The expression for the transverse displacement $w(x)$ is given in (5.20),

$$w(x) \;=\; C_1 \;+\; C_2\,kx \;+\; C_3\,\cos(kx) \;+\; C_4\,\sin(kx)\,.$$

The boundary conditions in $x=0$ are $w(0)=0$ and $M(0)=0$, and imply that $C_1 = C_3 = 0$. The full solution can then be reduced to

$$w(x) \;=\; C_2\,kx \;+\; C_4\,\sin(kx)\,.$$

The third boundary condition is $w(\ell)=0$, which leads to

$$w(\ell) \;=\; C_2\,k\ell \;+\; C_4\,\sin(k\ell) \;=\; 0\,.$$

The final boundary condition represents the balance between the internal moment at the right support and the moment introduced by the rotation spring, i.e. $M(\ell) = -k_\theta\theta(\ell) = k_\theta w'(\ell)$. When introducing the moment relation $M(\ell) = -EI_z w''(\ell)$ this boundary condition can be written as

$$EI_z w''(\ell) \;+\; k_\theta w'(\ell) \;=\; \Big(EI_z k^2\sin(k\ell) + k_\theta k\cos(k\ell)\Big)C_4 \;+\; k_\theta k C_2 \;=\; 0\,.$$

The boundary conditions at $x=\ell$ are conveniently written on matrix form

$$\begin{bmatrix} k\ell & \sin(k\ell) \\ \alpha & \alpha\cos(k\ell) - k\ell\sin(k\ell) \end{bmatrix}\begin{bmatrix} C_2 \\ C_4 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

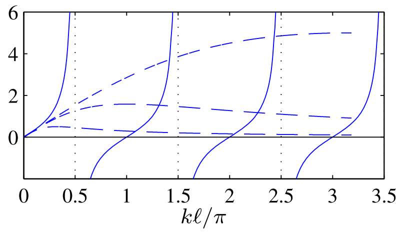


Fig. 5.12: Stability equation: $\tan(k\ell)$ (–) and $k\ell/(1+(k\ell)^2/\alpha)$ (- -).

where $\alpha = k_\theta \ell / EI_z$ is a non-dimensional rotation spring stiffness. The matrix equation is homogeneous and non-trivial solutions requires that the determinant of the matrix is zero. This gives the characteristic equation

$$k\ell\Big(\alpha\cos(k\ell) - k\ell\sin(k\ell)\Big) \;-\; \alpha\sin(k\ell) \;=\; 0\,,$$

which can be reduced to the following stability equation

$$\tan(k\ell) \;=\; \frac{k\ell}{1+(k\ell)^2/\alpha}\,.$$

For $\alpha \to 0$ the second term in the denominator tends to infinity and thus the right hand side vanishes, whereby $\tan(k\ell) = 0$ and $k\ell = n\pi$. This corresponds to the solution for the Euler column as expected. On the other hand $\alpha \to \infty$ implies that $\tan(k\ell) = k\ell$, which agrees with the stability equation found for the column in Example 5.2 with a fixed and a simple support. The intermediate behavior of the stability equation can be illustrated by plotting the two components of the equation. In Fig. 5.12 the solid curve is the tangent function while the dashed curves are the right hand side of the stability equation for $\alpha = 1$, 10 and 100, respectively. The solutions are represented by the intersections of the curves. □

5.2.1 Equivalent column length

The general solution to the homogeneous differential equation for a straight column was given in (5.20) as

$$w(x) \;=\; C_1 \;+\; C_2\,kx \;+\; C_3\,\cos(kx) \;+\; C_4\,\sin(kx)\,. \tag{5.33}$$

This solution consists of a linear part, represented by the two first terms, and a trigonometric part, represented by the two last terms. The critical load, at which instability occurs, is determined by the curvature of the buckled shape. The linear part of the solution does not contribute to the curvature, which is determined by the parameter k^2. In fact, it follows from the definition (5.18) of the parameter k that the critical load is determined by

$$P_c \;=\; k_c^2\,EI_z\,, \tag{5.34}$$

where k_c denotes the value associated with the critical load P_c, corresponding to instability.

The critical load of the Euler column, i.e. the lowest instability load for a simply supported ideal column, was given by (5.30) as

$$P_E = \left(\frac{\pi}{\ell}\right)^2 EI_z \,. \tag{5.35}$$

This formula gives the critical load P_E in terms of the bending stiffness EI_z and the length ℓ between the supports of the column. The Euler column has simple supports at both ends, and the column length ℓ is therefore also the length of a trigonometric half-wave, spanning between the supports. This length is also characterized as the length between the inflection points of the buckling form, where an inflection point is defined as a point where the curvature changes sign. It is this property that leads to the role of ℓ in the Euler column formula (5.35).

For general support conditions the parameter k_c is characterized by the length between the inflection points – i.e. a trigonometric half-wave of the buckling form. This length is called the effective column length and is denoted ℓ_e. It follows from its definition as the half-wave length of the column solution that it is related to the parameter k_c as $k_c = \pi/\ell_e$, whereby the trigonometric part of the solution is of the form $\sin(k_c x) = \sin(\pi x/\ell_c)$. Hereby the general critical load formula (5.34) takes the form

$$P_c = \left(\frac{\pi}{\ell_e}\right)^2 EI_z \,. \tag{5.36}$$

Thus, the concept of an equivalent column length ℓ_e translates the Euler column formula into a general format.

The role of the effective column length can be further illustrated by combining the general formula (5.36) for the critical load with the Euler formula (5.35),

$$\frac{P_c}{P_E} = \left(\frac{\ell}{\ell_e}\right)^2. \tag{5.37}$$

This formula illustrates the influence of the boundary conditions in changing the critical load by changing the effective column length via the support conditions.

The importance of the concept of the equivalent column length ℓ_e is two-fold: it gives a compact form of the stability load by generalizing the Euler formula, and it provides a visual interpretation of the column length for columns with general support conditions. This latter property is illustrated in Fig. 5.13 showing the lowest buckling mode for the support conditions treated above. All columns are shown for the same actual length ℓ. The corresponding effective column length ℓ_e is indicated to the right in each of the sub-figures. It is remarkable that even for the column combining a fixed

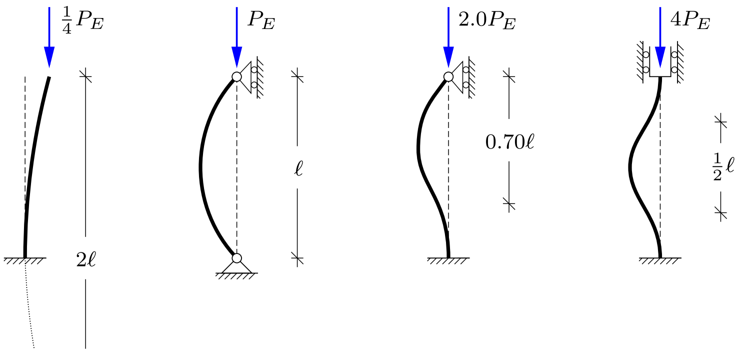


Fig. 5.13: Buckling load and effective column length.

and a hinged support the effective length $\ell_e \simeq 0.7$ can be estimated with fair accuracy from visual inspection of a sketch of the buckled shape.

5.2.2 Buckling direction and intermediate supports

Columns may have cross-sections with different properties with regard to bending and buckling in the transverse y- and z-direction. The problem is illustrated in Fig. 5.14 for a rectangular cross-section with dimensions $a \times b$ with a in the y-direction and b in the z-direction as shown.

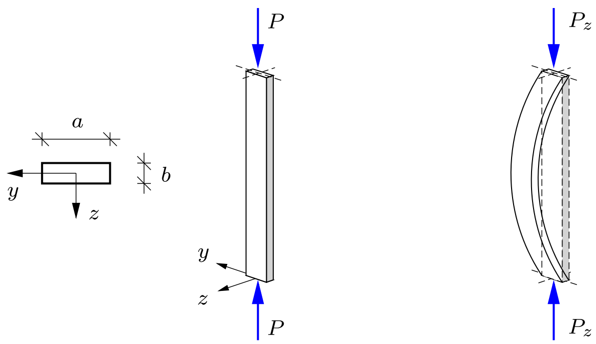


Fig. 5.14: Buckling about the weakest axis.

The area of the cross-section is $A = ab$, and the moments of inertia follow from Example 4.1 as

$$I_y = \int y^2\, dA = \tfrac{1}{12}a^2 A\,, \quad I_z = \int z^2\, dA = \tfrac{1}{12}b^2 A\,. \tag{5.38}$$

The column is simply supported at both ends with respect to displacements in both the y- and the z-direction. Thus, the two corresponding buckling loads are

$$P_E^y = \left(\frac{\pi}{\ell}\right)^2 EI_y = \frac{\pi^2}{12}\frac{a^2}{\ell^2}EA\,, \quad P_E^z = \left(\frac{\pi}{\ell}\right)^2 EI_z = \frac{\pi^2}{12}\frac{b^2}{\ell^2}EA\,. \tag{5.39}$$

Clearly, for $a > b$ the bending stiffness in the y-direction is larger, and accordingly this buckling load is larger, $P_E^y > P_E^z$. The argument depends on the relative magnitude of the moments of inertia I_y and I_z, and is not restricted to rectangular sections, but merely assumes symmetry axes.

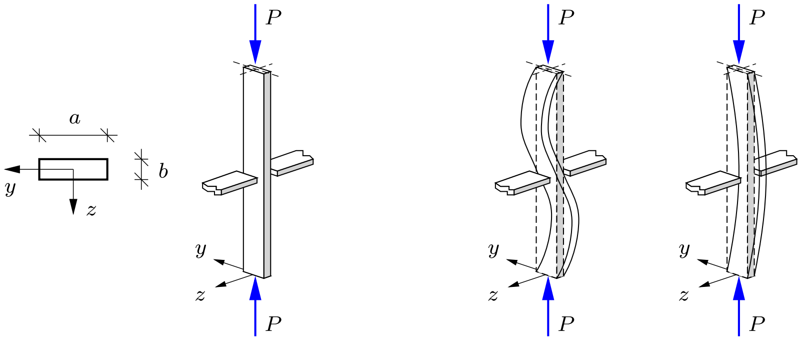


Fig. 5.15: Intermediate support in the z-direction.

Columns are often used as the vertical members of plane frames. This gives the possibility of supporting the column at intermediate locations against motion in the plane of the frame as illustrated in Fig. 5.15, showing an intermediate support at the mid-point providing a constraint against transverse motion in the z-direction. This support increases the buckling load P_E^z associated with buckling in the z-direction. A balanced design can be obtained as illustrated in the following example.

Example 5.5. Column with intermediate support. Consider a column of length ℓ with simple supports in both transverse directions at both ends. The column forms part of a plane frame and is supported against transverse displacement at its mid-point in the z-direction as shown in Fig. 5.15. This support reduces the effective column length for buckling in the z-direction to $\ell_e = \frac{1}{2}\ell$. The cross-section is a rectangle with dimensions a and b in the y- and the z-direction, respectively. The critical loads for buckling in the two directions follow from (5.39), when accounting for the reduced effective column length $\ell_e = \frac{1}{2}\ell$ for buckling in the z-direction,

$$P_E^y = \left(\frac{\pi}{\ell}\right)^2 EI_y = \frac{\pi^2}{12}\frac{a^2}{\ell^2}EA\,, \quad P_E^z = \left(\frac{\pi}{\ell_e}\right)^2 EI_z = \frac{\pi^2}{3}\frac{b^2}{\ell^2}EA\,.$$

The two buckling loads will be identical, $P_E^y = P_E^z$, provided the cross-section dimensions satisfy $a = 2b$. This corresponds closely to building practice for simple wooden frames, where the transverse cross-section dimension is often the double of the in-plane dimension. □

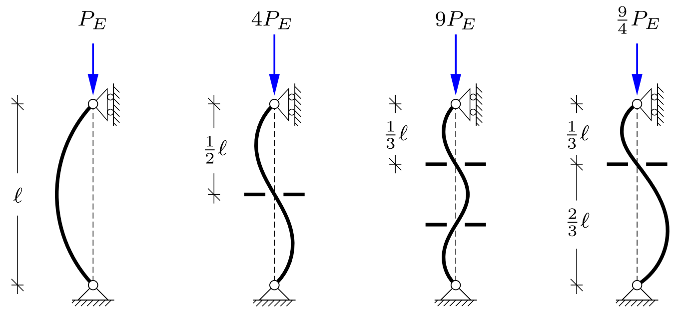


Fig. 5.16: Conservative effective column length estimates.

The effective column length is reduced by the introduction of intermediate supports. For the simply supported columns illustrated in Fig. 5.16 the effective column length is simply $\ell_e = \ell/n$, when the column is divided into n equal parts. If the parts are not of equal length the shorter parts constrain the larger parts. A conservative estimate of the effective column length is $\ell_e \simeq \ell_{\max}$, where $\ell_{\max}$ is the longest distance between neighboring supports or points of inflection.

5.3 Design of columns

The primary goal of column design is to provide sufficient resistance to withstand the load, here primarily in the form of a compressive axial load P. While the previous sections have concentrated on determination of the critical load P_c for an ideal column under various support conditions, design of columns must account for the effect of other factors such as the strength of the material and typical imperfections in column shape. Two extreme cases are illustrated in Fig. 5.17. The left figure shows buckling of a long slender simply supported column at the critical load P_E corresponding to elastic instability. The right figure shows the opposite scenario, in which the capacity of a short column is determined solely by the material strength represented by the yield load P_y.

Most real columns are in a parameter interval somewhere between the two extreme cases shown in the figure, and it is important to develop a column design procedure that provides a smooth transition between the two extremes of ideal elastic buckling and pure material strength. This is done in the following by combining three steps. The first, dealt with in Section 5.3.1, identifies a suitable characterization of column length, whereby the capacity of 'long' columns is dominated by instability, while the capacity of 'short' columns is dominated by material strength. The following Section 5.3.2 describes the effect of the fact that real columns are not ideally straight. The deviation from ideal straightness is called the geometric imperfection of the column, and

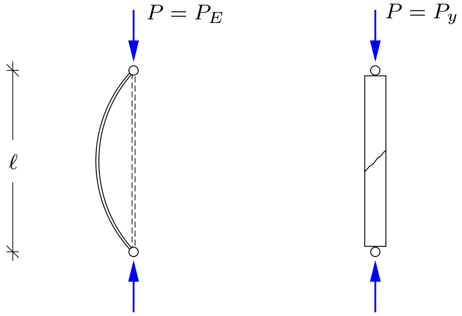


Fig. 5.17: Collapse mechanisms for long and short columns, respectively.

the imperfections lead to a change from buckling at a precise critical load, e.g. P_E, to a more gradual deformation of the column. During this process stresses develop in the column cross-sections as described in Section 5.3.3. Finally, these aspects are combined into a direct column design procedure in Section 5.3.4.

5.3.1 Column length and slenderness

The purpose of classifying a column as 'long' or 'short' is to indicate whether its capacity is primarily governed by stability or strength considerations. Thus, it is clear that the length characterization of a column can not just be a geometric measure, e.g. of actual length relative to a characteristic transverse dimension of the cross-section, but must include some reference to stability and strength parameters for the column.

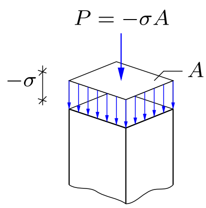


Fig. 5.18: Homogeneous distribution of normal stress σ.

Figure 5.18 illustrates a uniform stress distribution over the cross-section of a short column, acted on by a central compressive force P, as shown in Fig. 5.17b. The stresses are uniformly distributed over the cross-section area A, whereby the constant stress intensity σ is given as

$$\sigma = -\frac{P}{A}\,. \tag{5.40}$$

Note the sign convention with P positive in compression and σ positive in tension.

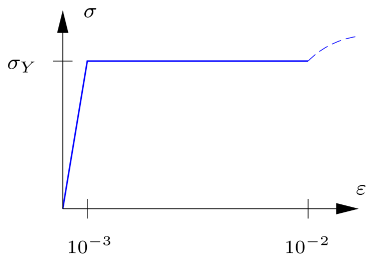


Fig. 5.19: Idealized uniaxial stress-strain curve.

The material can only withstand normal stresses up to a certain magnitude, here denoted σ_Y, where the subscript Y denotes the yield stress. The concept of material yield refers to a behavior observed in most metals, particularly steel. This behavior is illustrated for steel in Fig. 5.19 showing the relation between the normal stress σ and the corresponding longitudinal strain ε in a uniaxial tension or compression test. For limited strain, typically $\varepsilon \lesssim 0.002$, the stress and strain are proportional, corresponding to the linear elastic relation

$$\sigma = E\varepsilon \,. \tag{5.41}$$

When the strain exceeds the elastic limit, the stress remains at the yield stress, i.e. $|\sigma| \leq \sigma_Y$.

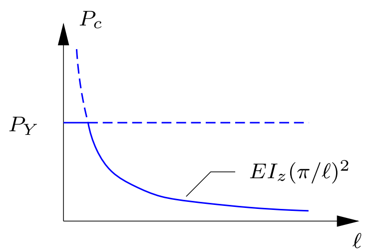


Fig. 5.20: Collapse curve for column.

The collapse of a short column is associated with yielding, and the maximum axial load is determined by the material yield stress as

$$P_Y = \sigma_Y A \,. \tag{5.42}$$

This strength is independent of the equivalent column length ℓ_e, when this length is sufficiently short. This is indicated by a horizontal line in Fig. 5.20, which shows the critical load P_c as function of the equivalent column length ℓ_e. On the other hand, the capacity of an ideal elastic column is given by the buckling formula in (5.36),

$$P_{el} = \left(\frac{\pi}{\ell_e}\right)^2 EI_z \,. \tag{5.43}$$

The subscript *el* refers to *elastic* column instability, as treated in Section 5.2. For an Euler column $\ell_e = \ell$, and P_{el} in (5.43) recovers the Euler load P_E. It follows from (5.43) that the stability load P_{el} decreases with ℓ_e^{-2}. This is also indicated in Fig. 5.20. The two idealized criteria in terms of strength P_Y and elastic instability P_{el}, respectively, define an intersection point, which is used to define the transition from a short to a long column.

Column slenderness

In the further discussion of column capacity it is convenient to shift focus from the axial force P to the normal stress σ. At this point there is no bending, and thus the stress σ is simply a normalized form of the axial force as given by (5.40). When introducing this normalization in the buckling formula (5.43), the following expression is obtained for the mean compressive stress σ_{el} at elastic column instability,

$$\sigma_{el} = \frac{P_{el}}{A} = \frac{EI_z}{A}\left(\frac{\pi}{\ell_e}\right)^2. \tag{5.44}$$

In this expression the influence of the cross-section is represented by the ratio I_z/A. This ratio has the dimension [length^2] and defines a length r_z, called the radius of gyration, as

$$r_z^2 = \frac{I_z}{A} \,. \tag{5.45}$$

The radius of gyration is a characteristic distance from the neutral axis of the cross-section. If the area A was split into two parts and these were concentrated at $\pm r_z$, this equivalent cross-section would have the same bending stiffness as the original.

Example 5.6. Radius of gyration for rectangle and I-profile. The bending of a beam with rectangular cross-section of height h and width b is illustrated in Fig. 4.4a. The moment of inertia was calculated in Example 4.1 as $I_z = \frac{1}{12}h^2 A$. The corresponding radius of gyration then follows from (5.45) as

$$r_z = \sqrt{\frac{I_z}{A}} = \frac{h}{2\sqrt{3}} \simeq 0.289h \,.$$

Similarly, the moment of inertia of an I-section of height h shown in Fig. 4.4b with flange width b and flange thickness t was calculated in Example 4.1 as $I_z = \frac{1}{4}h^2 A$, when neglecting the contribution from the web. For this cross-section the radius of gyration is

$$r_z = \sqrt{\frac{I_z}{A}} = \tfrac{1}{2}h \,,$$

corresponding to the fact that the flanges already have concentrated the cross-section area at the distances $\pm\frac{1}{2}h$ from the neutral axis. □

The compression stress σ_{el} at elastic instability can now be expressed by introducing the substitution $I_z = r_z^2 A$ into (5.44),

$$\sigma_{el} = E\left(\frac{\pi\, r_z}{\ell_e}\right)^2 = E\left(\frac{\pi}{\lambda}\right)^2, \tag{5.46}$$

where the non-dimensional slenderness parameter

$$\lambda = \frac{\ell_e}{r_z} \tag{5.47}$$

has been introduced. The slenderness parameter λ is a purely geometric quantity, expressing column length relative to the cross-section radius of gyration r_z.

The formula gives the instability stress σ_{el} in terms of the elastic modulus E and the slenderness parameter λ. It is often more convenient to express σ_{el} relative to the yield stress σ_Y. This ratio is given by

$$\frac{\sigma_{el}}{\sigma_Y} = \frac{E}{\sigma_Y}\left(\frac{\pi}{\lambda}\right)^2 = \frac{1}{\lambda_r^2}, \tag{5.48}$$

where the last equality defines the *relative* slenderness as

$$\lambda_r = \frac{\lambda}{\pi}\sqrt{\frac{\sigma_Y}{E}} = \frac{\ell_e}{\pi r_z}\sqrt{\frac{\sigma_Y}{E}}. \tag{5.49}$$

It follows from the equations in (5.48) that the intersection point between the strength and the stability curves in Fig. 5.20 is characterized by $\sigma_{el} = \sigma_Y$, corresponding to $\lambda_r = 1$.

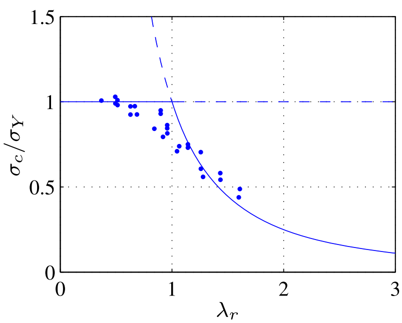


Fig. 5.21: Critical stress as function of relative slenderness.

Figure 5.21 shows the critical column stress in normalized form σ_c/σ_Y plotted against the relative slenderness λ_r. The figure also shows some experimental results for steel columns from Dowling et al. (1988). It is seen that there is a gradual transition from the material strength criterion $\sigma_c \simeq \sigma_Y$ for $\lambda \ll 1$ to the elastic stability criterion $\sigma_c \simeq \sigma_{el}$ for $\lambda \gg 1$. The graph illustrates that the relative slenderness λ_r is an important parameter for characterizing columns, and that a reduction of the idealized column capacity, as material yield *or* instability occurs, must be developed for the transition range around $\lambda_r = 1$. It follows from (5.49) that the transition occurs at the column length

$$\ell_* = \pi \, r_z \sqrt{E/\sigma_Y} \, . \tag{5.50}$$

This formula is convenient for checking, whether a column is 'short' or 'long'.

Example 5.7. Transition length for steel columns. For a steel column the elastic modulus is typically $E = 210\,\text{GPa}$ and for a yield stress $\sigma_Y = 0.3\,\text{GPa}$ this gives the transition length

$$\ell_* = 83.12 \, r_z \, .$$

Consider a rectangular cross-section with height h and width b. It was found in Example 5.6 that buckling in the z-direction has a radius of gyration $r_z = h/(2\sqrt{3})$. For an I-section with height h and flange width b and thickness t the radius of gyration was found as $r_z = \frac{1}{2}h$, corresponding to all the material of the cross-section being concentrated at $\pm\frac{1}{2}h$. Substitution of r_z into the expression for the transition length gives

$$\frac{\ell_*}{h} = \begin{cases} 24.0 & \text{, rectangle} \, , \\ 41.6 & \text{, I-section} \, . \end{cases}$$

This indicates that the transition from a short to a long column occurs for significantly longer columns for the I-section than for the rectangular cross-section. □

5.3.2 Geometric imperfections

Real columns are not ideally straight, and as it turns out the effect of deviations from the ideal straightness provides an explanation of the reduction of column capacity for relative slenderness in the transition interval as illustrated by the data points in Fig. 5.21.

Figure 5.22a shows an imperfect simply supported column, where the (small) deviation from a straight line in the unloaded state is described by the function $w_0(x)$. The length of the column is ℓ and the constant bending stiffness is EI_z. When loaded by an axial compression force P, as shown in Fig. 5.22b, an *additional* transverse displacement $w(x)$ occurs. The moment is determined by this additional displacement as

$$M = -EI_z \, \frac{d^2 w}{dx^2} \, . \tag{5.51}$$

The effect of the normal force on the equilibrium is through the *total* displacement $w(x) + w_0(x)$, and thus the equilibrium equation takes the form

$$\frac{d^2}{dx^2}\Big(EI_z\,\frac{d^2w}{dx^2}\Big) \;-\; \frac{d}{dx}\Big(N\,\frac{dw + dw_0}{dx}\Big) \;=\; 0\,. \tag{5.52}$$

For constant axial force $N = -P$ and bending stiffness EI_z this equation can be written in normalized form as

$$\frac{d^4w}{dx^4} \;+\; k^2\,\frac{d^2w}{dx^2} \;=\; -k^2\,\frac{d^2w_0}{dx^2}\,, \tag{5.53}$$

where the parameter $k^2 = P/EI_z$ has been introduced in (5.18). The boundary conditions for the simply supported column are $w(0) = w(\ell) = 0$ and $M(0) = M(\ell) = 0$.

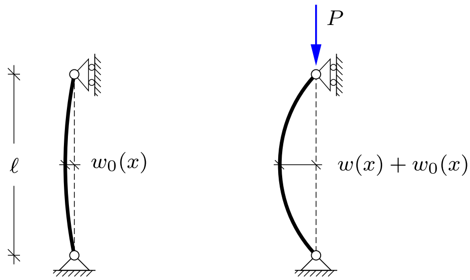


Fig. 5.22: Column with initial imperfection $w_0(x)$.

It is seen that the initial imperfection $w_0(x)$ introduces a non-homogeneous term in the column equation, and thus the imperfect column problem is not an eigenvalue problem. The complete solution to the non-homogeneous differential equation (5.53) is obtained by representing $w(x)$ as a series expansion in terms of the eigenfunctions of the corresponding homogeneous equation. For the present problem the corresponding eigenvalue problem was solved in Section 5.2 and the solution given in (5.32). The series representation of the displacement $w(x)$ then takes the form

$$w(x) \;=\; \sum_{n=1}^{\infty} w_n\,\sin(k_n x)\,, \qquad k_n \;=\; n\,\frac{\pi}{\ell}\,. \tag{5.54}$$

Substitution of this expansion into the non-homogeneous equilibrium equation (5.53) gives

$$\sum_{n=1}^{\infty} \big(k_n^4 - k^2 k_n^2\big)\,w_n \sin(k_n x) \;=\; -\,k^2\,\frac{d^2w_0}{dx^2}\,. \tag{5.55}$$

The unknown displacement coefficients w_n are determined by representing the initial imperfection $w_0(x)$ as a series similar to the expansion (5.54) for $w(x)$,

$$w_0(x) = \sum_{n=1}^{\infty} w_n^0 \sin(k_n x), \qquad k_n = n\frac{\pi}{\ell}. \tag{5.56}$$

The coefficients w_n^0 in this expansion are determined by use of the orthogonality relation for the sine function as

$$w_n^0 = \frac{2}{\ell}\int_0^\ell w_0(x)\sin(k_n x)\,dx, \quad n = 1, 2, \cdots \tag{5.57}$$

When the series expansion (5.56) for $w_0(x)$ is substituted into (5.55), the displacement coefficients are found to be

$$w_n = \frac{w_n^0}{k_n^2/k^2 - 1} = \frac{w_n^0}{P_n/P - 1}, \quad n = 1, 2, \cdots \tag{5.58}$$

This result shows that for increasing tension, $P \to -\infty$, the column becomes increasingly straight, $w_n \to -w_0^n$. Conversely, if an applied compression force $P > 0$ approaches any of the buckling loads P_n the corresponding component of the initial imperfection is amplified. In principle infinite amplification is obtained for $P = P_n$.

The present analysis has been applied to the special case of a simply supported column. However, the method of expanding both the unknown displacement function $w(x)$ and the initial imperfection function $w_0(x)$ in terms of the buckling modes of the corresponding homogeneous problem corresponding to an ideal straight column is also valid for other boundary conditions, and in each case the buckling modes also satisfy a suitable orthogonality relation. This leads to the general conclusion, that the application of a central axial compression load P on an imperfect column will lead to amplification of the contributions of the different buckling mode components in the initial imperfection function $w_0(x)$. In practice the imperfection component corresponding to the first buckling mode will most often dominate the deformation.

Example 5.8. Column with initial curvature. Consider a special case of the imperfect simply supported column shown in Fig. 5.22, in which the imperfection consists of a single sine half-wave of amplitude $w_1^0 = e$. It follows from (5.58) that an axial load P gives the transverse displacement

$$w(x) = \frac{e}{P_E/P - 1}\sin\left(\pi\frac{x}{\ell}\right),$$

where the Euler load is given by (5.30) as $P_E = EI_z(\pi/\ell)^2$. The total displacement for an axial compression force P then is

$$w(x) + w_0(x) = \left(\frac{e}{P_E/P - 1} + e\right)\sin\left(\pi\frac{x}{\ell}\right) = \frac{w_0(x)}{1 - P/P_E}.$$

This result shows that the initial displacement $w_0(x)$ is amplified by the same amplification factor for all points on the column. The amplification of the displacements is illustrated in Fig. 5.23 showing the increase of the center displacement for an initial center eccentricity e. It is noted that the amplification factor here is the same as that determined in Section 5.1.1 for the amplification of the displacement from a transverse load.

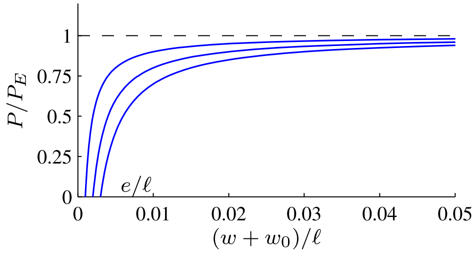


Fig. 5.23: Displacement amplification from $e/\ell = 0.001,\ 0.002,\ 0.003$.

The moment can be found either by the second derivative from (5.51) or from equilibrium of part of the column defined by a section at the position x,

$$M = (w + w_0)\, P = = \frac{w_0(x)\, P}{1 - P/P_E}\,.$$

Thus, the moment exhibits the same amplification as shown in Fig. 5.23. □

5.3.3 Stresses in column cross-sections

The strength of a column depends on the magnitude of the stresses that develop in it. The individual cross-section supports a compressive force P from the imposed axial load and a bending moment M from the transverse displacement $w_{tot} = w + w_0$ of the beam cross-section. The corresponding stress distribution is illustrated in Fig. 5.24. The normal force P produces a uniformly distributed state of compressive stress given by (5.40),

$$\sigma = -\frac{P}{A}\,. \tag{5.59}$$

The bending moment M produces a stress distribution with linear variation over the cross-section, found by combining the strain relation (4.5) with the elastic curvature relation (4.8),

$$\sigma = \frac{M}{I_z}\, z\,. \tag{5.60}$$

The total stress in the cross-section is obtained by addition of these two contributions as illustrated in Fig. 5.24,

$$\sigma = -\frac{P}{A} + \frac{M}{I_z}\, z\,. \tag{5.61}$$

It is seen that the combination of a compressive force and a moment leads to a stress distribution with large compressive stresses at one side of the cross-section and a moderate stress level in either compression or tension at the other side.

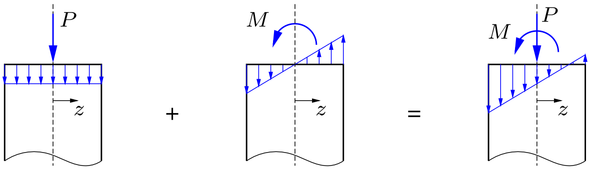


Fig. 5.24: Combination of normal stress from normal force and moment.

In the present context the moment is caused by eccentricity of the axial compressive force P, and it is of interest to investigate the influence of this eccentricity on the stress distribution. If the force P acts close to the neutral line it will generate compressive stresses over the full cross-section. However, for larger eccentricity of the force tension stresses will occur at the side opposite to the eccentric force. The area, within which a normal compressive force will not produce tension in the cross-section, is called the kernel of the cross-section.

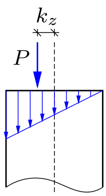


Fig. 5.25: Offset of force P by kernel radius k_z.

The limiting case, where the axial stress vanish at the side of the cross-section, is shown for a cross-section that is symmetric about the plane of the paper in Fig. 5.25. The figure illustrates the case in which the eccentric force acting at $z = -k_z$ gives zero axial stress at the opposite side of the cross-section, defined by $z = z_{\max}$. In this case the moment is $M = k_z P$, and the stress distribution formula (5.61) then gives the condition

$$0 = -\frac{P}{A} + \frac{k_z P}{I_z} z_{\max} \,. \tag{5.62}$$

This condition determines the so-called kernel radius,

$$k_z = \frac{I_z}{A\, z_{\max}} = \frac{r_z^2}{z_{\max}} \,, \tag{5.63}$$

in the symmetry plane of a cross-section.

As demonstrated in Example 4.1 the moment of inertia is $I_z = \frac{1}{4}h^2A$ for an idealized I-section of height h, in which the influence of the web is neglected. This means that the kernel radius in the web-direction is

$$k_z = \frac{r_z^2}{z_{\max}} \simeq \frac{h^2}{4}\frac{2}{h} = \tfrac{1}{2}h\,. \tag{5.64}$$

Thus, the kernel of the I-section contains the full height h of the section.

Kernel area of a rectangle

The kernel of a cross-section is an area within the cross-section, here illustrated by the rectangle with height h and width b in Fig. 5.26. The planes of symmetry contain the y- and z-axis. When the height of the section is along the z-axis the kernel radius along the two axis are

$$k_y = \frac{r_y^2}{y_{\max}} = \frac{b^2}{12}\frac{2}{b} = \tfrac{1}{6}b\,, \qquad k_z = \frac{r_z^2}{z_{\max}} = \frac{h^2}{12}\frac{2}{h} = \tfrac{1}{6}h\,. \tag{5.65}$$

Thus, for a rectangle the kernel extends along the central third of the cross-section width along the axes as illustrated in Fig. 5.26.

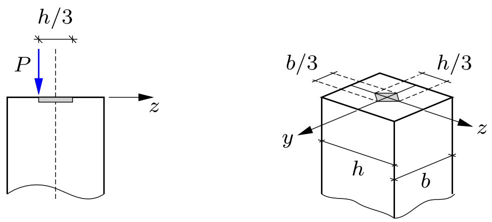


Fig. 5.26: Kernel area for rectangular cross-section.

The shape of the kernel is determined by combining the effect of moments about the y- and the z-axis of the cross-section. It follows from superposition that the normal stress formula (5.61) generalizes to

$$\sigma = -\frac{P}{A} + \frac{M_y}{I_y}\,y + \frac{M_z}{I_z}\,z\,. \tag{5.66}$$

The moments are $M_y = \pm k_y P$ and $M_z = \pm k_z P$, and the kernel condition then generalizes to

$$0 = -\frac{P}{A} \pm \frac{k_y P}{I_y}\,y_{\max} \pm \frac{k_z P}{I_z}\,z_{\max}\,. \tag{5.67}$$

After division by P/A, this condition takes the form

$$\pm\,\frac{y_{\max}}{r_y^2}\,k_y \pm \frac{z_{\max}}{r_z^2}\,k_z = 1\,. \tag{5.68}$$

This is a set of four equations for the lines interpolating the point (k_y, k_z) on the kernel boundary between the four points on the axes, that have already been determined. The result is the hatched area shown in Fig. 5.26.

5.3.4 Perry-Robertson's column design criterion

The capacity of columns depends on several factors. The importance of material strength σ_Y and the bending stiffness EI_z have been analyzed in Section 5.3.1, but geometric imperfections, non-ideal material behavior and residual stresses left from the fabrication process are also important. Several of these factors depend on the type of column and the fabrication process, e.g. the residual stresses left by hot rolling as discussed in considerable detail by Beedle and Tall (1960). The non-uniform axial stress distribution implies that yield of the section occurs in a more gradual fashion than shown in the idealized stress-strain relation in Fig. 5.19.

It turns out that it is possible to combine the influence of stiffness EI_z, strength σ_Y and an imperfection parameter e to generate a family of curves for capacity of columns that represent experimental results well over the whole range of column slenderness. The central result is the Perry-Robertson column capacity formula. The idea of the Perry-Robertson column capacity formula is that the column has an initial deflection represented by the eccentricity parameter e, and that the capacity of the column is exhausted when the maximum stress in the most severely loaded cross-section reaches the ultimate stress represented by σ_Y. A historical account of the Perry-Robertson formula has been given by Heyman (1998).

The assumption is that the maximum compressive stress reaches σ_Y. For a column with positive transverse displacement this stress will occur at the negative side of the section at $z = -z_{\max}$. Thus, it follows from (5.61) that the maximum compressive stress is given by

$$\sigma_Y = \frac{P}{A} + \frac{M}{I_z}\,z_{\max}\,. \tag{5.69}$$

In this formula the following three substitutions are made:

a) The load is represented by the critical stress $\sigma_c = P/A$.
b) The factor $z_{\max}$ is expressed by the kernel radius $k_z = I_z/Az_{\max}$.
c) The moment is introduced via its amplified value $M = eP/(1 - P/P_{el})$, as determined in Example 5.8 (with P_E replaced by P_{el}).

With these substitutions the condition (5.69) takes the form

$$\sigma_Y = \sigma_c + \frac{e}{k_z} \frac{\sigma_c}{1 - \sigma_c/\sigma_{el}} . \tag{5.70}$$

Multiplication of this equation with the denominator gives the product format

$$(\sigma_Y - \sigma_c)(\sigma_{el} - \sigma_c) = \frac{e}{k_z} \sigma_{el} \sigma_c . \tag{5.71}$$

In this format it is seen that, if there is no eccentricity, i.e. for $e = 0$, the equation is simply a product of the strength and the elastic stability criteria, while $e > 0$ leads to an interpolation between these two criteria. Figure 5.27 illustrates the influence of imperfections by showing the critical stress for $e/k_z = 0.00$, 0.02 and 0.06.

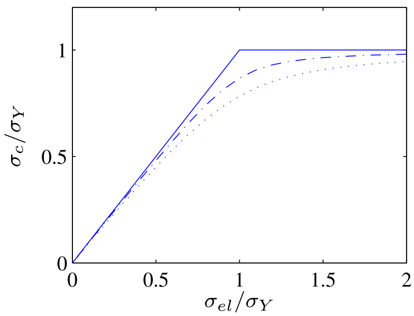


Fig. 5.27: Critical stress for $e/k_z = 0.00$ (—), 0.02 (- -) and 0.06 ($\cdots$).

The explicit solution to the criterion (5.71) is found by expressing the mean stress σ_{el} at elastic instability in terms of the relative slenderness, defined in (5.48) by

$$\frac{\sigma_{el}}{\sigma_Y} = \frac{1}{\lambda_r^2} . \tag{5.72}$$

This is substituted into the Perry-Robertson equation (5.71), which then takes the following form of a quadratic equation,

$$\lambda_r^2 \left(\frac{\sigma_c}{\sigma_Y}\right)^2 - \left(\lambda_r^2 + 1 + \frac{e}{k_z}\right) \frac{\sigma_c}{\sigma_Y} + 1 = 0 . \tag{5.73}$$

The solution to this equation is conveniently expressed as

$$\frac{\sigma_c}{\sigma_Y} = \beta - \sqrt{\beta^2 - \lambda_r^{-2}} , \tag{5.74}$$

where the non-dimensional parameter β has been defined as

$$\beta = \frac{1}{2\lambda_r^2}\Big(\lambda_r^2 + 1 + \frac{e}{k_z}\Big). \tag{5.75}$$

These relations determine the critical stress σ_c in terms of the yield stress σ_Y, the relative slenderness λ_r, and the relative imperfection e/k_z. The relative slenderness follows by (5.72) from the elastic instability stress σ_{el} and the yield stress σ_Y as

$$\lambda_r = \sqrt{\sigma_Y/\sigma_{el}}\,, \tag{5.76}$$

while the magnitude of the imperfection parameter depends on a number of factors.

In practice, the imperfection depends on the length of the column, and structural codes often assume a direct relation between e/k_z and the relative slenderness λ_r. The simplest relation is linear proportionality,

$$\frac{e}{k_z} = \alpha\,\lambda_r\,. \tag{5.77}$$

The non-dimensional parameter α represents the total effect of imperfections, residual stresses etc. According to the theory α is determined by

$$\alpha = \frac{e}{k_z}\,\frac{1}{\lambda_r} = \pi\,\frac{e}{\ell_e}\,\frac{r_z}{k_z}\sqrt{\frac{E}{\sigma_Y}}\,. \tag{5.78}$$

If the material is linear up to the yield limit, the corresponding yield strain is $\varepsilon_Y = \sigma_Y/E$. For steel the yield strain is around $\varepsilon_Y \simeq 0.002$. A typical imperfection of a steel column is $e \simeq 0.001\ell$, see e.g. Timoshenko and Gere (2009). For steel columns a representative value of α is then of the order

$$\alpha \simeq \pi\,\frac{0.001}{\sqrt{0.002}}\,\frac{r_z}{k_z} = 0.070\,\frac{r_z}{k_z} = \begin{cases} 0.070 & \text{I-section} \\ 0.122 & \text{rectangle} \end{cases}, \tag{5.79}$$

with the kernel radii from (5.63) and (5.65), and the gyration radii from Example 5.6. It is seen that among the other factors the parameter α also

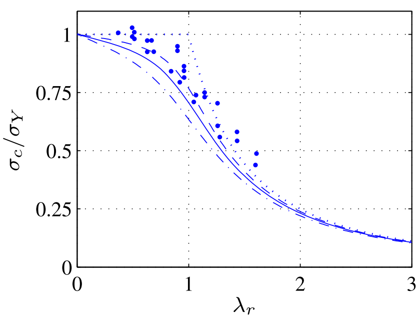


Fig. 5.28: Critical stress as function of relative slenderness.

depends on the cross-section shape. Figure 5.28 shows the critical stress for the imperfection e/k_z given by (5.77) with $\alpha = 0.07$, 0.122 and 0.21.

Simulated column capacity results, obtained by statistical combination of material properties and imperfections, demonstrate that a curve through a lower fractile – e.g. 5 pct. – is similar in shape to the curves generated by the Perry-Robertson formula, see e.g. Chen and Han (1985). This supports a design procedure in which the critical design stress is obtained by increasing the parameter α beyond its representative mean value.

Example 5.9. Critical stress for column with tubular cross-section. This example illustrates the design procedure for the simply supported column shown in Fig. 5.29. The cross-section is tubular with radius a and wall thickness t, whereby $A = 2\pi a t$ and $I_z = \pi t a^3$. The wall thickness is $t = a/10$, and the length is $\ell = 100a$. The imperfection e is represented via (5.77) with $\alpha = 0.2$. The material is steel with elastic modulus $E = 210\,\text{GPa}$ and yield stress $\sigma_Y = 300\,\text{MPa}$.

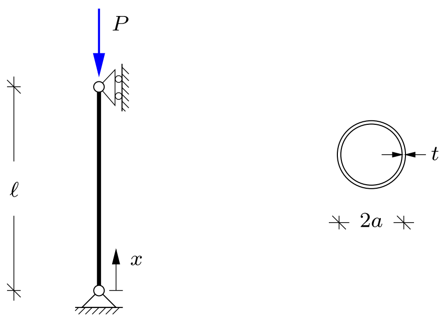


Fig. 5.29: Euler column with tubular cross-section.

First the instability stress is calculated from (5.44),

$$\sigma_{el} = P_E/A = \tfrac{1}{2}(\pi a/\ell)^2 E = 106.6\,\text{MPa}\,.$$

The relative slenderness then follows from (5.76),

$$\lambda_r = \sqrt{\sigma_Y/\sigma_{el}} = \sqrt{300/106.6} = 1.70\,.$$

As $\lambda_r > 1$, this is a 'long' column. The parameter β is now calculated from (5.75) after substituting e/k_z from (5.77) with $\alpha = 0.2$,

$$\beta = \frac{1}{2\lambda_r^2}\Big(\lambda_r^2 + 1 + \alpha\lambda_r\Big) = 0.73\,.$$

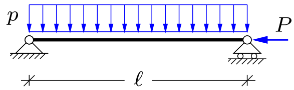
The relative magnitude of the critical stress is then found from (5.74)

$$\frac{\sigma_c}{\sigma_Y} = \beta - \sqrt{\beta^2 - \lambda_r^{-2}} = 0.296 \quad\Rightarrow\quad \sigma_c = 88.8\,\text{MPa}\,.$$

This is a substantial reduction relative to the yield stress because of the length of the column. The magnitude of the critical stress relative to the instability stress is $\sigma_c/\sigma_{el} = 0.86$, where the reduction shows the effect of the imperfection. □

5.4 Exercises

Exercise 5.1. The figure shows a simply supported beam of length ℓ with constant bending stiffness EI_z. It is loaded by a uniformly distributed transverse load with intensity p and a horizontal compression force P at the right support, which produces a constant normal force $N = -P$.



a) Set up the differential equation for the transverse displacement $w(x)$, and find the solution with four arbitrary constants.

b) Use the four boundary conditions to determine the expression for the displacement $w(x)$.

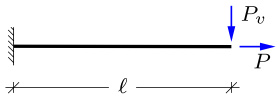
c) Find an expression for the maximum displacement $w_{\text{max}} = w(\frac{1}{2}\ell)$.

d) Find the critical load P_c corresponding to $w_{\text{max}} \to \infty$.

Exercise 5.2. The figure shows a cantilever of length ℓ with constant bending stiffness EI_z. It is loaded by a vertical force P_v and a horizontal force P at the tip. Note that the latter produces a constant positive normal force $N = P$.

a) Set up the differential equation for the transverse displacement $w(x)$, and show that the solution can be written as

$$w(x) = C_1 + C_2 kx + C_3 \cosh(kx) + C_4 \sinh(kx)\,.$$

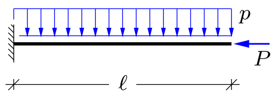


b) Use the four boundary conditions to determine the expression for the displacement $w(x)$.

c) Find the transverse tip displacement w_{tip}, and the tip displacement w^0_{tip} for $P = 0$.

d) Find the magnitude of the axial load P required to reduce the transverse tip deflection to $w_{\text{tip}} = \frac{1}{2} w^0_{\text{tip}}$.

Exercise 5.3. The figure shows a cantilever of length ℓ with constant bending stiffness EI_z. It is loaded by a uniformly distributed transverse load with intensity p and a horizontal force P at the tip, which produces a constant normal force $N = -P$.



a) Set up the differential equation for the transverse displacement $w(x)$, and find the solution with four arbitrary constants.

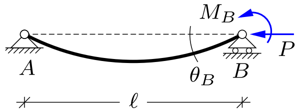
b) Use the four boundary conditions to determine the expression for the displacement $w(x)$.

c) Find an expression for the transverse tip displacement w_{tip} and the moment at the support M_{sup}.

d) Determine the critical load P_c corresponding to $w_{\text{tip}} \to \infty$.

Exercise 5.4. The figure shows a simply supported beam of length ℓ with constant bending stiffness EI_z. At the right support the beam is loaded by a local moment M_B and a horizontal force P. The latter produces the constant normal force $N = -P$.

a) Set up the differential equation for the transverse displacement $w(x)$, and find the solution with four arbitrary constants.



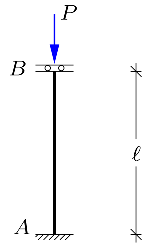
b) Use the four boundary conditions to determine the expression for the displacement $w(x)$.

c) Find the expression for the rotation $\theta(x)$, and use this expression to show that the relation between the support rotation $\theta_B = \theta(\ell)$ and the applied moment M_B can be written as

$$M_B = \frac{(k\ell)^2}{1 - k\ell\cot(k\ell)} \frac{EI_z}{\ell} \theta_B \, .$$

d) The stiffness of the beam with respect to the rotation at the right support can be expressed in non-dimensional form as $M_B\ell/(3EI_z\theta_B)$, which is unity for $P = 0$. Use the solution in c) to find the expression for this rotational stiffness and explain what happens when $P \to P_E$.

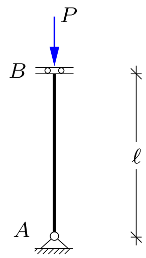
Exercise 5.5. The figure shows a column of length ℓ with a fixed support in A and a fixed support with horizontal rollers at the top in B. The column is loaded by an axial compression force P, producing the constant normal force $N = -P$.



a) Sketch the critical buckling form of the column and estimate the equivalent column length ℓ_e and the critical load ratio P_c/P_E.

b) Use the general solution $w(x)$ with four arbitrary constants and the four boundary conditions to determine P_c/P_E and ℓ_e. Compare with the results in a).

Exercise 5.6. The figure shows a column of length ℓ with a simple support in A and a fixed support with horizontal rollers at the top in B. The column is loaded by an axial compression force P, producing the constant normal force $N = -P$.

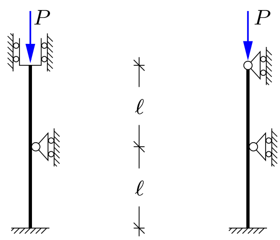


a) Sketch the critical buckling form of the column and estimate the equivalent column length ℓ_e and the critical load ratio P_c/P_E.

b) Use the general solution $w(x)$ with four arbitrary constant and the four boundary conditions to determine P_c/P_E and ℓ_e. Compare with the results in a).

Exercise 5.7. The figure shows ideal columns with intermediate supports. As discussed in Section 5.2.2 an approximative result for the equivalent column length ℓ_e is the longest distance between neighboring support or inflection points.

a) Sketch the buckling form for each of the columns and estimate the equivalent column length. Note that the inflection point for the second column is located below the intermediate support.



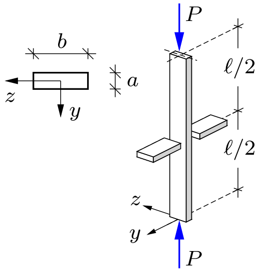
b) Determine the critical buckling load ratio P_c/P_E associated with the estimated column lengths. Compare with $P_c/P_E = 2.0$ and 1.3, respectively, obtained by numerical analysis.

Exercise 5.8. The figure shows an elastic column of length ℓ with rectangular cross-section with dimensions a and b. The column is part of a wall structure containing horizontal beams. This supports the motion in the plane of the wall, while motion is possible in the out-of-plane direction. The cross-section area is $A = ab$. The moment of inertia in the plane of the wall is $I_y = \frac{1}{12}Aa^2$, while it is $I_z = \frac{1}{12}Ab^2$ out of the plane. From a stability point of view it is important that the column has larger stiffness out of the plane direction than in the plane, which implies that $b > a$. The question is how much? The column is assumed to be fully fixed at both ends in both directions.

a) Determine the effective column length ℓ_e^y and the critical load P_c^y corresponding to buckling in the plane of the wall.

b) Determine the effective column length ℓ_e^z and the critical load P_c^z corresponding to buckling out of the plane of the wall.

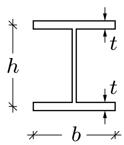
c) Determine the ratio b/a such that $P_c^y = P_c^z$.



Exercise 5.9. Consider a linear elastic column of length ℓ, with simple supports at both ends and in both directions. The cross-section of the column is an I-profile as shown in the figure, with height h, width b and wall thickness t.

a) Determine the two moments of inertia I_y and I_z, neglecting the contribution from the web.

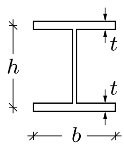
b) Find the ratio h/b so that the Euler load P_E is the same with respect to buckling in the two directions.



Exercise 5.10. The figure shows an I-profile with height h, width b and wall thickness t. The contribution of the web to the bending stiffness is omitted.

a) Make a sketch of the cross-section and indicate the points that limit the kernel with respect to bending in the two directions of symmetry.

b) Find the kernel area by superposition of the stress distributions from bending in the two directions of symmetry.



Exercise 5.11. Consider a simply supported column of length $\ell = 4.00\,\text{m}$, and with quadratic cross-section of dimensions $50\,\text{mm} \times 50\,\text{mm}$. The material is steel with elastic modulus $E = 210\,\text{GPa}$ and yield stress $\sigma_Y = 250\,\text{MPa}$.

a) Find the critical stress σ_{el} corresponding to elastic buckling.

b) Determine the slenderness λ and the relative slenderness λ_r.

c) Find the length of the column $\ell = \ell_*$, which corresponds to the limit between a short and a long column.

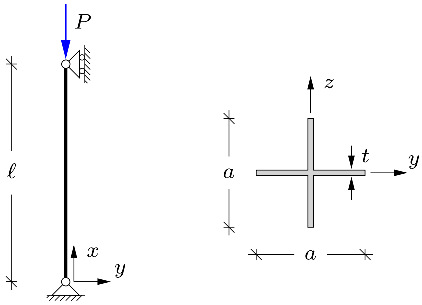
Exercise 5.12. Consider a simply supported column of length $\ell = 4.00\,\text{m}$, and with quadratic cross-section of dimensions $50\,\text{mm} \times 50\,\text{mm}$. The material is steel with elastic modulus $E = 210\,\text{GPa}$ and yield stress $\sigma_Y = 250\,\text{MPa}$. The imperfection of the column is represented by the parameter $\alpha = 0.22$.

a) Determine the relative slenderness λ_r.

b) Determine the ratio σ_c/σ_Y by Perry-Robertson's formula.

c) Determine the stress σ_{el} for the euler column and the ratio σ_c/σ_{el}. Comment on the influence of the imperfection.

Exercise 5.13. The figure shows an Euler column with a cruciform cross-section with width a and wall thickness t. The moment of inertia is $I_z = \frac{1}{12}a^3t$. The length of the column is $\ell = 25a$. The material is steel with elastic modulus $E = 210\,\text{GPa}$ for stresses below the yield stress $\sigma_Y = 235\,\text{MPa}$. The imperfection is given by the parameter $\alpha = 0.1$.

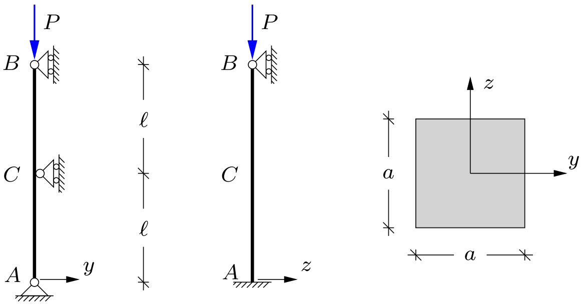


a) Determine the stress σ_{el} corresponding to the Euler load P_E.

b) Find the relative slenderness λ_r and determine if the column is long or short.

c) Determine σ_c/σ_Y by Perry-Robertson's formula, and the ratio σ_c/σ_{el}.

Exercise 5.14. The figure shows a column ACB of length 2ℓ. For deflection in the y-direction the column has a simple support in A and simple supports with vertical rollers in B and C. For deflection in the z-direction the column has a fixed support in A and a simple support with vertical rollers in B. The column is loaded by an axial compression force P in B, whereby the constant normal force is $N = -P$. As shown in the figure to the right the cross section of the column is quadratic with height a and width a. The column is elastic with elastic modulus E for stresses below the yield stress σ_Y. It is assumed that $E/\sigma_Y = 900$.



a) Sketch the buckling form for deflections in the y- and z-direction, respectively. Determine the corresponding equivalent column lengths ℓ_e^y and ℓ_e^z.

b) Determine the critical load P_{el} for the elastic column.

c) Determine the relative slenderness λ_r with respect to the critical buckling form, and find the value of ℓ/a that corresponds to the transition between a short and a long column.