



In statically indeterminate structures the distribution of the internal forces depends on the stiffness of the individual parts of the structure. There are two basically different approaches to this problem. In the force method, described in the previous chapter, a sufficient number of connections in the structure are released and the corresponding local section forces necessary for closing the connection are determined. This method works well for simple structures, formed by linear elastic beams. However, the method requires the determination of the distribution of the internal forces in the statically determinate structure for each of the redundant components obtained by releasing connections, and for large structures this constitutes a considerable task. An alternative approach is to consider the structure as formed by individual beams, connected at nodes. Initially all nodes are considered fixed, and they are then released one at a time. The correct solution is obtained by finding the combination of node displacements, that do not require any additional constraining forces at the nodes. This approach has a number of advantages. First, the release of a constraint only affects beams directly connected to the constraint, and the equations of the method therefore only require a local analysis. Furthermore, this local analysis only involves individual beams, and it can therefore easily be given a general systematic form, suitable for computer analysis. This so-called finite element formulation is quite general and can be developed from the principle of virtual work for plates, shells and solid bodies as well. Here, the main focus is on frame structures, and the development will be based mainly on the equilibrium equations.

The chapter covers the classic deformation method of frames, intended for hand calculation, as well as the finite element method for frame structures. The deformation method is developed in Sections 7.1 and 7.2 for simple plane frames. The stiffness properties of a beam are developed by use of virtual work in the form of a set of basic deformation cases, including the effect of shear flexibility. These deformation cases are then used in the deformation method to determine the effect of sequentially imposing displacements of the constrained nodes. The procedure follows the classic deformation method, in which a manageable size of the problem is obtained by neglecting the effect of axial deformation of the individual beams.

The finite element formulation for elastic frames is obtained by rearranging the procedure of the deformation method into a systematic matrix format. The basic idea, already illustrated for truss structures in Section 2.5, is to represent each beam as an element with a stiffness matrix, including all displacement components at each of its nodes. These elements are then assembled into a frame structure, and the nodal displacements are determined by solution of the corresponding equation system. Two types of beam elements are developed here: beam elements with shear flexibility in Section 7.3.1, and beam-column elements in Section 7.3.2. The beam bending element is the typical element for frame analysis, while the beam-column element enables extension of the column analysis of Chapter 5 to a linearized stability analysis of frames. The finite element formulation has been implemented for plane frames in the MATLAB code MINIFRAME described in Section 7.4.

## 7.1 Stiffness of beams

The basic idea of the deformation method for frames is illustrated in Fig. 7.1. Figure 7.1a shows a T-frame with a distributed load on the part  $BC$ . The supports provide six reaction components. As only three reactions are needed for a structure without hinges, this implies that the frame is three times statically indeterminate. Thus, use of the force method would imply the release of three constraints and the introduction of three unknown force components. In the deformation method the point of view is changed. When neglecting axial deformation of the beams the joint  $B$  is fixed in space, but can rotate as determined by the loading of the frame. The rotation is indicated by the angle

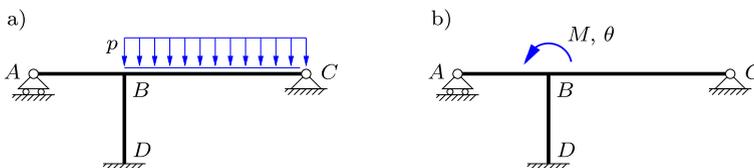


Fig. 7.1: T-frame, a) with distributed load, b) moment and rotation at joint.

$\theta$  in Fig. 7.1b. The idea of the deformation method is to consider the original problem as a superposition of two sub-problems. In the first sub-problem the node is prevented from rotating by imposing a concentrated moment  $M_0$  at the node. Hereby the frame is converted to an assembly of independent beams, all with simple ideal support conditions, either fully fixed or with a simple support permitting rotation. When the moment distribution for the individual beams are known, the magnitude of the moment  $M_0$  required to constrain the rotation of the joint  $B$  can be determined. The second sub-problem consists in imposing a unit rotation of the joint  $B$ . All three beams joined at  $B$  are subjected to the same unit rotation, and the total moment can therefore be obtained as the sum of the end-moments of the individual beams, when subjected to a unit rotation of the end-sections. The real situation is represented by a superposition of these two sub-problems, where the rotation of the node is determined to cancel the total external moment at the joint  $B$ . Thus, a central part of the deformation method is the development of a series of simple deformation load cases in which a unit deformation – here in the form of a rotation – is imposed at one end of a beam. This issue is addressed in the rest of this section, after which the deformation method is developed in Section 7.2.

### 7.1.1 Symmetric and anti-symmetric bending

The stiffness of a beam is characterized by the deformation generated by the application of end loads that are in equilibrium. For a plane beam there are three equilibrium states: constant normal force, constant bending moment, and constant shear force. The latter is accompanied by a bending moment of linear variation. The beam stiffness used in the deformation method for frames is due to bending and shear, while the effect of axial deformation and normal forces are typically neglected. Thus, the relevant stiffness characteristics are described by only two states of deformation. These are conveniently taken as symmetric bending, where there is no shear force, and anti-symmetric bending with a constant shear force. The various special cases needed for use in the deformation method or the finite element method for frames can then subsequently be obtained by linear combination of these two basic cases of deformation. An additional advantage of this simple approach is that it is straight-forward to incorporate the shear flexibility effect without introducing any additional complications in the derivation. However, it is important to note, that most frame structures consist of fairly slender elements with fairly low shear flexibility. In most structures analyzed by the deformation method using hand calculations it is therefore justified to neglect the effect of shear flexibility that will make the calculations more extensive. On the other hand, the finite element formulation for frame structures makes use of a beam element in which the inclusion of the shear flexibility simply consists in a set of appropriate coefficients. Therefore, the shear effect is included

in the derivation of the stiffness properties of beams in the present section, as it serves as basis for both the deformation and the finite element methods.

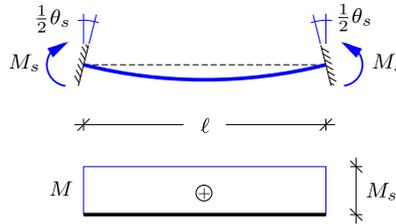


Fig. 7.2: Symmetric bending of beam.

The case of symmetric bending is illustrated in Fig. 7.2. Opposing bending moments of magnitude  $M_s$  are acting at the ends of the beam. The beam is assumed to be symmetric, and the moments then generate opposite rotations  $\pm\frac{1}{2}\theta_a$  at the two end cross-sections of the beam. It is noted that in the present case there is no shear force, and thus the rotation of the cross-section is equal to the slope of the beam axis at the ends. The theory is fairly easily extended to non-symmetric beams, see e.g. Krenk (1994). The rotations are normalized via the factor  $\frac{1}{2}$  in order for the external work to be represented as  $\frac{1}{2}\theta_s M_s + \frac{1}{2}\theta_s M_s = \theta_s M_s$ . When using the principle of virtual work discussed in Section 4.4 with the static field corresponding to the moment distribution  $M(x)$  and the kinematic field consisting of the corresponding curvature distribution  $\kappa(x) = M(x)/EI$ , the virtual work equation takes the form

$$\theta_s M_s = \int_{\ell} \frac{M(x)M(x)}{EI} ds = \frac{\ell}{EI} M_s^2. \tag{7.1}$$

This corresponds to the stiffness relation

$$M_s = \frac{EI}{\ell} \theta_s \tag{7.2}$$

for symmetric bending of the beam.

The case of anti-symmetric bending is illustrated in Fig. 7.3. Here, identical moments  $M_a$  are applied to the ends of the beam. This results in a total external moment of  $2M_a$ , that is counteracted by the shear force of magnitude  $Q = 2M_a/\ell$ . The beam ends do not translate, and the external work is therefore described entirely by the rotation of the to end moments as  $\frac{1}{2}\theta_a M_a + \frac{1}{2}\theta_a M_a = \theta_a M_a$ . In this case the internal work contains contributions from the shear force  $Q(x)$  as well as from the moment  $M(x)$ , whereby the equality of external and internal work takes the form

$$\theta_a M_a = \int_{\ell} \left\{ \frac{M(x)M(x)}{EI} + \frac{Q(x)Q(x)}{GA_s} \right\} ds = \frac{\ell}{3EI} M_a^2 + \frac{\ell}{GA_s} Q^2, \tag{7.3}$$

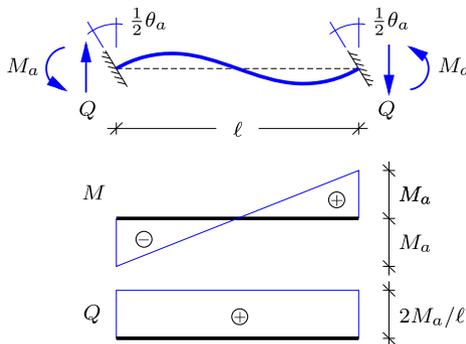


Fig. 7.3: Anti-symmetric bending of beam.

where  $A_s$  is the equivalent shear area of the cross-section. Upon substitution of the shear force  $Q$  in terms of the moment  $M_a$  this provides the anti-symmetric flexibility relation

$$\theta_a = \frac{\ell}{3EI} M_a + \frac{4\ell}{GA_s \ell^2} M_a = \frac{\ell}{3EI} \left( 1 + \frac{12EI}{GA_s \ell^2} \right) M_a. \quad (7.4)$$

The last term represents the additional flexibility introduced by the shear deformation. This shear flexibility effect is conveniently characterized by the shear flexibility parameter

$$\Phi = \frac{12EI}{GA_s \ell^2}. \quad (7.5)$$

When introducing this notation, the stiffness relation becomes

$$M_a = \frac{3EI}{(1 + \Phi)\ell} \theta_a \quad (7.6)$$

for anti-symmetric bending of the beam. The corresponding shear force is

$$Q = \frac{6EI}{(1 + \Phi)\ell^2} \theta_a. \quad (7.7)$$

The subscript on the shear force is left out, as the symmetric part does not contain a shear force.

### 7.1.2 Basic cases of imposed deformation

There are two types of basic load cases for a homogeneous beam  $AB$ , those in which a unit rotation of an end cross-section is imposed shown in Figs. 7.4–7.5, and those in which a relative transverse translation of unit magnitude is imposed as shown in Figs. 7.6–7.7. These figures contain the reaction components, including the effect of shear flexibility. This effect is often included in

the formulation of beam elements. However, in the deformation method this effect is often omitted, and the full set of results and their symmetric forms without the shear flexibility effect is given in Table 7.1 for easy reference.

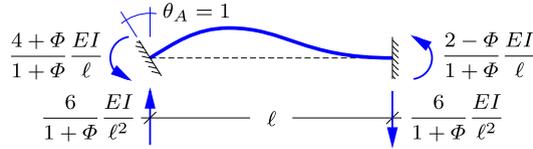


Fig. 7.4: End-section rotation for fixed support.

First the unit rotation load case of Fig. 7.4 is considered. With the sign convention used in the figures, the end moments are

$$M_A = -M_s + M_a, \quad M_B = M_s + M_a. \quad (7.8)$$

The rotation of the end cross-sections are related to their symmetric and anti-symmetric parts as

$$\theta_A = -\frac{1}{2}\theta_s + \frac{1}{2}\theta_a, \quad \theta_B = \frac{1}{2}\theta_s + \frac{1}{2}\theta_a = 0. \quad (7.9)$$

It follows from the sum and difference of these equations that the symmetric and anti-symmetric rotations are

$$\theta_a = -\theta_s = \theta_A = 1. \quad (7.10)$$

The end moments then follow by substitution of the symmetric and antisymmetric moments (7.2) and (7.6), respectively, into (7.8),

$$\begin{aligned} M_A &= \frac{EI}{l} + \frac{3EI}{(1+\Phi)l} = \frac{4+\Phi}{1+\Phi} \frac{EI}{l}, \\ M_B &= -\frac{EI}{l} + \frac{3EI}{(1+\Phi)l} = \frac{2-\Phi}{1+\Phi} \frac{EI}{l}. \end{aligned} \quad (7.11)$$

The shear force  $Q$  follows from (7.7) with  $\theta_a = \theta_A = 1$ ,

$$Q = \frac{6}{1+\Phi} \frac{EI}{l^2}. \quad (7.12)$$

These results are shown in Fig. 7.4. Note, that the shear force follows directly from moment equilibrium as  $Q = (M_A + M_B)/l$ . The reactions of this load case and its symmetric counterpart are given in the first row of Table 7.1. The results of the symmetric load case follows from rotating the original load case by  $180^\circ$  in the plane.

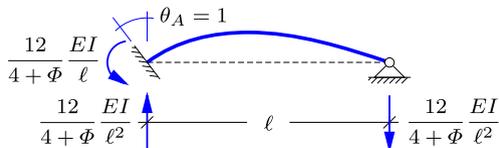


Fig. 7.5: End-section rotation for simple support.

In the practical application of the deformation method it is convenient also to have the corresponding load cases, in which a unit rotation is imposed at the cross-section at one end of the beam, while the other end has a simple support, permitting free rotation. This load cases is shown in Fig. 7.5. The moment vanishes at  $B$ , and thus the moment superposition relations (7.8) here take the form

$$M_A = -M_s + M_a, \quad M_B = M_s + M_a = 0. \tag{7.13}$$

Thus, the symmetric and anti-symmetric parts of the moment are given in terms of  $M_A$  as

$$2M_a = -2M_s = M_A. \tag{7.14}$$

The rotation  $\theta_a$  is now expressed in terms of the moment  $M_A$  by use of the relations (7.2) and (7.6),

$$\theta_A = -\frac{1}{2}\theta_s + \frac{1}{2}\theta_a = -\frac{\ell}{2EI}M_s + \frac{(1 + \Phi)\ell}{6EI}M_a = \frac{4 + \Phi}{12} \frac{\ell}{EI}M_A. \tag{7.15}$$

The stiffness is the inverse relation with  $\theta_A = 1$ ,

$$M_A = \frac{12}{4 + \Phi} \frac{EI}{\ell}, \tag{7.16}$$

and the shear force follows from moment equilibrium as  $Q = M_A/\ell$ , whereby

$$Q = \frac{12}{4 + \Phi} \frac{EI}{\ell^2}. \tag{7.17}$$

These results are shown in Fig. 7.5. The results are included together with those of the symmetric load case in the second row in Table 7.1. The results of the symmetric case again follow from those already derived by a 180° rotation of the beam and its loads in the plane.

The nodes of a frame may rotate and translate. The cases involving imposed rotations have been covered in Figs. 7.4 and 7.5, and the similar cases involving an imposed translation of an end-section are now considered. The first of these, shown in Fig. 7.6, involves a beam  $AB$  in which a unit transverse translation is imposed on the cross-section at  $A$ . Within the degree of approximation involved in the theory of infinitesimal deformation, used here in

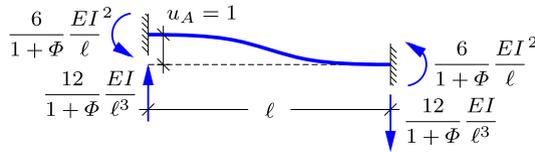


Fig. 7.6: Transverse translation for fixed support.

describing beam deformation, this case corresponds to imposing a clock-wise rotation of  $\frac{1}{2}\theta_a = 1/l$  about  $B$  of the case of anti-symmetric bending, solved previously in connection with Fig. 7.3. The resulting end-moments and shear force have already been derived, and are given explicitly in Fig. 7.6a. The complementary case, in which the end-section at  $B$  is given a unit translation, follows from a simple change in sign and is included for the case without shear flexibility as the third row of Table 7.1. These load cases play an important role in the swaying of frames with fixed supports.

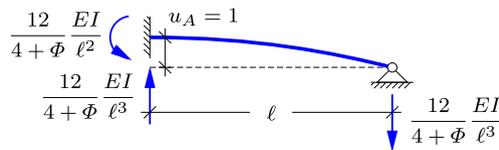


Fig. 7.7: Transverse translation for simple support.

Figure 7.7 shows the similar case of imposed translation, but now on a beam with a simple support at the other end. The results follow directly from those in Fig. 7.5, when the geometry is rotated and the imposed angle scaled by  $\theta = \pm 1/l$ . These results are included without shear flexibility as the last row of Table 7.1.

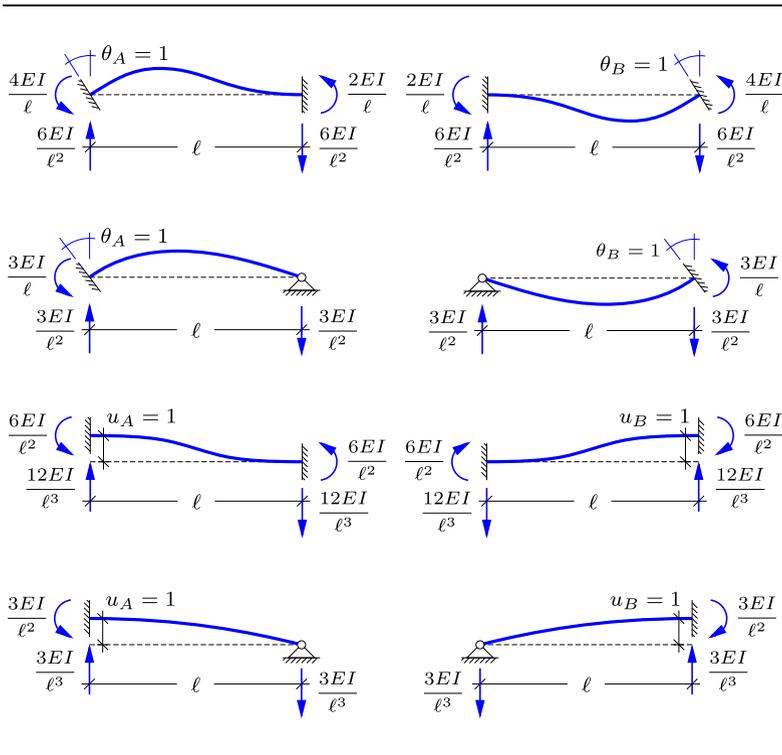
The load cases describing the internal forces generated by a unit displacement serve to determine how a statically indeterminate frame distributes the load to the supports. In most cases the shear flexibility effect, represented by  $\Phi$ , can be neglected, and the formulae can be used with  $\Phi = 0$ . A simple example of load distribution is given below.

**Example 7.1. Load distribution in angle frame.**

Figure 7.8 shows an angle frame  $ABC$  in which the individual beam members  $AB$  and  $BC$  are of length  $a$  and with bending stiffness  $EI$ . The frame has a fixed support at  $A$  and a simple support at  $C$ . The load consists of an external moment  $M_0$  applied at the corner  $B$ , and the issue is, how the moment is distributed to the two supports. The effect of shear deformation is neglected, corresponding to  $\Phi = 0$  in both beams.

The problem is solved by imposing a rotation of magnitude  $\theta_0$  of the joint  $B$ , and determining the corresponding internal forces by use of the unit displacement load cases determined

Table 7.1: Constraining forces on deformed beams.



above. Rotation of the joint  $B$  by  $\theta_0$  corresponds to rotating one end cross-section of both the beams  $AB$  and  $BC$ . The beam  $AB$  then corresponds to the load case in Fig. 7.4, while the beam  $BC$  corresponds to the load case in Fig. 7.5. The moments and transverse forces corresponding to these load cases are shown in Fig. 7.9.

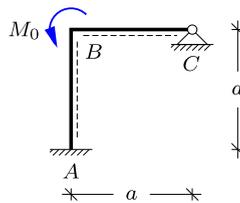


Fig. 7.8: Angle frame with external corner moment  $M_0$ .

The moment  $M_0$  imposed at the joint  $B$  corresponds to the sum of the moments transferred to the two beams at  $B$ , and thus it follows from the figure that

$$M_0 = \frac{4EI}{a}\theta_0 + \frac{3EI}{a}\theta_0 = \frac{7EI}{a}\theta_0,$$

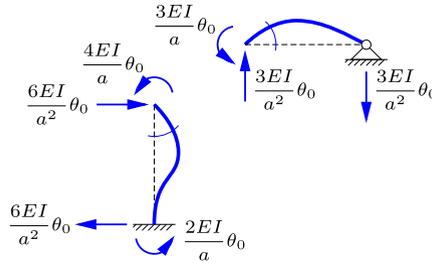


Fig. 7.9: Individual beams with common corner rotation  $\theta_0$ .

where the smaller moment comes from the beam  $BC$  with the more flexible support. This equation determines the rotation angle as

$$\theta_0 = \frac{a}{7EI} M_0.$$

With this value of the rotation the normalized internal force diagram in Fig. 7.9 can be evaluated in terms of actual magnitudes, shown in Fig. 7.10a. The moments at  $B$  are

$$M_{BA} = \frac{4EI}{a} \theta_0 = \frac{4}{7} M_0, \quad M_{BC} = \frac{3EI}{a} \theta_0 = \frac{3}{7} M_0$$

where  $M_{BA}$  is the moment at  $B$  in the beam  $BA$ , and  $M_{BC}$  is the moment at  $B$  in the beam  $BC$ .

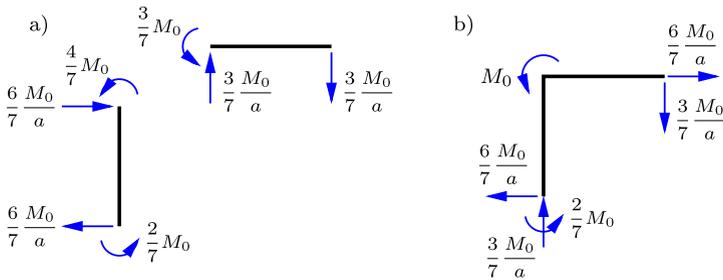


Fig. 7.10: a) End loads on individual beams, b) Assembled frame with reactions.

The internal forces in Fig. 7.10a are taken directly from the basic unit deformation load cases, and therefore do not contain normal forces components. The normal force in the beam  $AB$  must provide the upward transverse force  $\frac{3}{7} M_0/a$ , and thus must carry a compressive force of this magnitude. Similarly, the normal force in  $BC$  must produce the transverse force  $\frac{6}{7} M_0/a$  in the vertical beam  $AB$  as indicated in the figure. After determination of the normal forces the reactions on the assembled frame can be shown in Fig. 7.10b.

It is seen that the total loads on the frame, including reactions, consist of the external moment  $M_0$ , a reaction moment  $\frac{2}{7} M_0$ , and two force couples of  $\pm \frac{6}{7} M_0/a$  and  $\pm \frac{3}{7} M_0/a$ , respectively. A total moment balance gives

$$M_0 + \frac{2}{7} M_0 - \frac{6}{7} \frac{M_0}{a} a - \frac{3}{7} \frac{M_0}{a} a = 0,$$

thus demonstrating moment equilibrium.

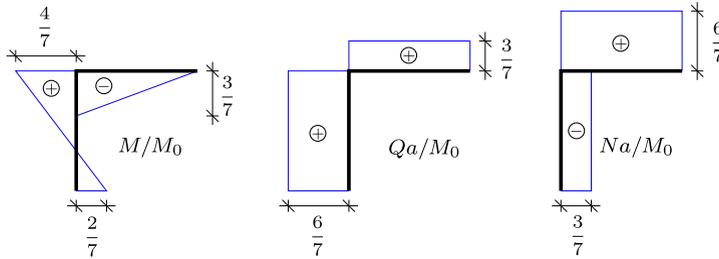


Fig. 7.11: Internal forces in angle frame.

Once the reactions have been determined, the internal force distributions can be generated as shown in Fig. 7.11. The moment and shear force distributions also follow directly from the diagram of the individual beams in Fig. 7.10a. It is seen that the larger part of the corner moment  $M_0$  is taken in the beam  $AB$  with the fixed support.  $\square$

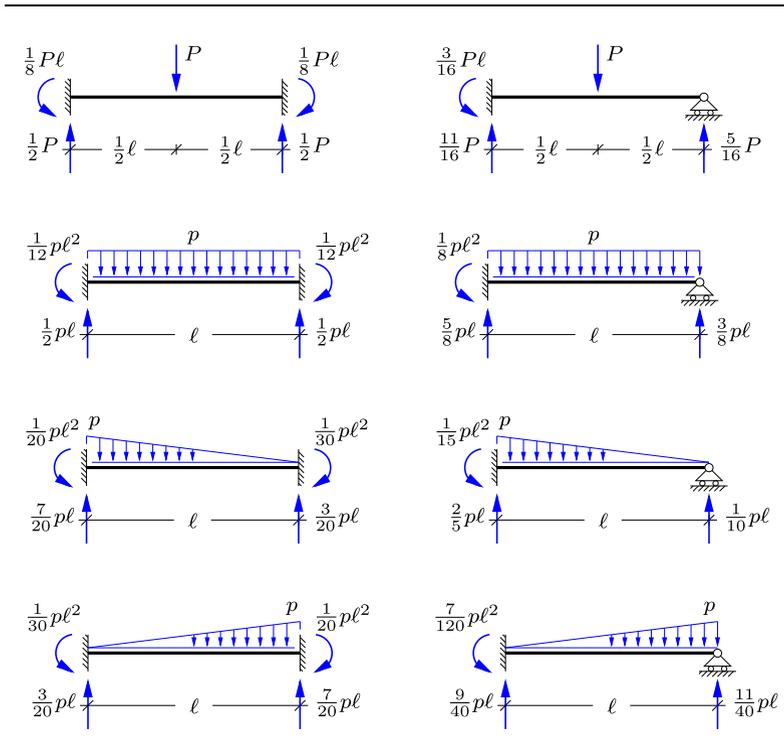
### 7.1.3 Loads on constrained beams

If all loads act directly on the nodes of the structure, the reactions and the distribution of the internal forces can be determined solely by use of the stiffness properties of the individual beam elements, derived above. However, most frame structures carry loads acting on the individual structural members, and therefore an additional step is needed, in which the local loads on the beams are distributed to the nodes. This distribution of the loads is accomplished by first considering the loaded beam as fully constrained. This requires suitable constraining forces and moments, that are later released in the form of loads at the nodes of the frame structure. Thus, the magnitude of the forces necessary to constrain the ends of a loaded beam must be known.

The constraining forces/moments are shown for several simple load cases in Table 7.2. The fully constrained beams are shown in the left column with the corresponding load cases for a beam with a simple support at the right end of the beam are shown in the right column. It is seen, that when the right support is changed from rigid to simple the moment and reaction force at the left end are increased. Conversely, at the end of the beam permitted to rotate, the reaction force is reduced.

The constraining forces shown in Table 7.2 have been calculated without including the effect of shear flexibility. The fully symmetric cases a) and c) are independent of the shear flexibility parameter  $\Phi$ , and the other cases only exhibit a small redistribution of the reactions, resulting from the lack of symmetry. The approximate nature of the loads specified in most situations hardly justifies the additional complications of including the dependence of  $\Phi$  in the following calculations.

Table 7.2: Constraining forces on loaded beams.



## 7.2 Deformation method for frames

In this section the deformation method for beams and frames is developed in a systematic way, first carefully considering two specific structures and then summarizing the general procedure. The first structure is a two-span beam, solved by introducing a single constraint. The second structure generalizes the procedure by considering a frame with two constraints. These two cases serve to introduce the procedure as well as the notation in a specific context, and subsequently the general procedure and notation are summarized in a concise form.

### Two-span beam with a single constraint

The simplest case of the deformation method, in which only a single constraint is needed, is illustrated in Fig. 7.12 showing a homogeneous beam with bending stiffness  $EI$  that is continuous over the two spans  $AB$  and  $BC$ , each of length  $\ell$ . The beam is fixed at  $A$  and supported by simple supports on

horizontal rollers at  $B$  and  $C$ . The load consists of a vertical load  $P$  applied to the center of the span  $BC$ .

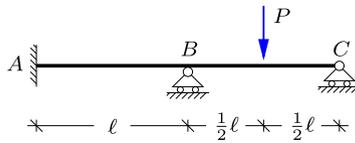


Fig. 7.12: Two-span beam with concentrated load.

The structure is twice statically indeterminate, and solution by the force method would then require the release of two constraints, e.g. the vertical reactions in  $B$  and  $C$ . In the deformation method the first step is to introduce constraints to reduce the full structure to a number of beams with fixed or simply supported ends. In the present case this is attained by constraining the rotation of node  $B$ . The rotation at  $B$  is prevented by introducing a constraining moment  $Z_{10}$  at  $B$  as illustrated in Fig. 7.13. Hereby the structure is reduced to a beam  $AB$  fixed at both ends, and a beam  $BC$  with a transverse force at the center, fixed at  $B$  and simply supported at  $C$ . The first subscript on  $Z_{10}$  identifies the degree of freedom, while the second subscript 0 identifies  $Z_{10}$  as the moment constraining the corresponding degree of freedom, when the structure is acted upon by the external load.

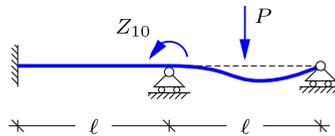


Fig. 7.13: Constraining moment on two-span beam.

The constraining moment  $Z_{10}$  must have a magnitude corresponding to constraining both the beam  $AB$  and the beam  $BC$ . In the present case the beam  $AB$  is unloaded, and thus requires no constraining moment, while the constraining moment of the beam  $BC$  follows from the figure to the right in the first row of Table. 7.2,

$$Z_{10} = M_{BA}^0 + M_{BC}^0 = 0 + \frac{3}{16}P\ell = \frac{3}{16}P\ell.$$

In the present notation the superscript 0 on  $M_{BA}^0$  indicates a constraining moment, and the subscript  $BA$  identifies the location as node  $B$  of the beam  $BA$ . The constraining moment  $Z_{10}$ , as well as its contributing parts, are considered positive when acting in the counter-clockwise direction.

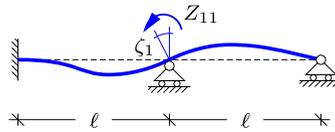


Fig. 7.14: Imposed unit rotation on two-span beam.

In the actual structure the node  $B$  can rotate, and this rotation is denoted by  $\zeta_1$ . Figure 7.14 illustrates an imposed unit rotation  $\zeta_1 = 1$  at node  $B$  and the corresponding moment  $Z_{11}$ . The first subscript 1 indicates that  $Z_{11}$  is a moment contribution, while the second subscript defines this as a contribution from a unit rotation associated with  $\zeta_1$ . The forces and moments in the beams  $AB$  and  $BC$  associated with a unit rotation at node  $B$  have been derived as deformation load cases in Section 7.1.2, and are listed to the right in the first row and the to the left in the second row of Table 7.1. When using the results for the end moments, the moment necessary to impose a unit rotation at  $B$  is found as

$$Z_{11} = M_{BA}^1 + M_{BC}^1 = 4\frac{EI}{\ell} + 3\frac{EI}{\ell} = 7\frac{EI}{\ell}.$$

In this relation the superscript is changed to 1 to indicate that this moment is associated with a unit deformation of  $\zeta_1$ .

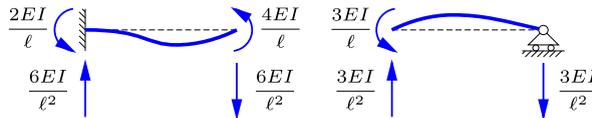


Fig. 7.15: Constraining forces and moments for imposed unit rotation  $\zeta_1 = 1$ .

In the actual structure there is no imposed external moment at  $B$ . This provides the following equation,

$$Z_1 = Z_{10} + Z_{11}\zeta_1 = 0.$$

This is an equation for the initially unknown rotation  $\zeta_1$  with the solution

$$\zeta_1 = -\frac{Z_{10}}{Z_{11}} = -\frac{3}{112} \frac{P\ell^2}{EI}.$$

The rotation  $\zeta_1$  of node  $B$  is negative, indicating a clockwise rotation.

Internal forces and reactions can now be evaluated by considering the full solution as the superposition of the constrained case of Fig. 7.13 and the unit rotation case from Fig. 7.15, multiplied by the parameter  $\zeta_1$ . In practice, it is often most convenient to calculate the reactions from this superposition

principle, and then to evaluate the section force distributions from an ordinary static analysis, based directly on the loads and reactions. This is the procedure shown here.

First, it follows directly from the considered load cases that the horizontal reaction force at  $A$  vanishes,  $R'_A = 0$ . The vertical reaction forces are evaluated by the superposition principle. For the vertical reaction in  $A$

$$R_A = R_A^0 + R_A^1 \zeta_1 = 0 + 6 \frac{EI}{\ell^2} \zeta_1 = -\frac{9}{56} P,$$

where all components are positive in the upward direction. It is seen that the reaction  $R_A$  is negative, and thereby downward. It receives no contribution from the constrained load case, because the beam  $AB$  carries no external load. The vertical reaction at  $B$  is

$$R_B = R_B^0 + R_B^1 \zeta_1 = \frac{11}{16} P - 3 \frac{EI}{\ell^2} \zeta_1 = \frac{43}{56} P.$$

Here the reaction  $R_B^1$  corresponding to a unit rotation is a combination of a downward component from  $AB$  and an upward component from  $BC$ . Finally, the reaction at  $C$  follows as

$$R_C = R_C^0 + R_C^1 \zeta_1 = \frac{5}{16} P - 3 \frac{EI}{\ell^2} \zeta_1 = \frac{22}{56} P.$$

The reaction forces are shown in Fig. 7.16, where the arrows indicate the corresponding positive direction.

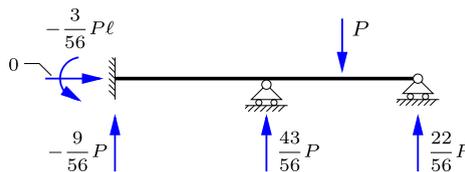


Fig. 7.16: Load and reactions on two-span beam.

It is seen that the sum of the reactions give  $R_A + R_B + R_C = P$ , corresponding to vertical projection equilibrium. However, it follows from the expression for the reactions in terms of  $\zeta_1$  that the sum of the vertical reactions is independent of the value of  $\zeta_1$ , and thus the check does not verify the correctness of the solution, but merely constitutes a useful consistency check.

The reaction moment  $M_A$  can now be determined, either by taking moment equilibrium of the structure including load and reaction forces as shown in Fig. 7.16, or by use of the superposition procedure. The latter gives

$$M_A = M_A^0 + M_A^1 \zeta_1 = 0 + 2 \frac{EI}{\ell} \zeta_1 = -\frac{3}{56} P\ell.$$

It is easily verified that this moment satisfies global moment equilibrium of the structure.

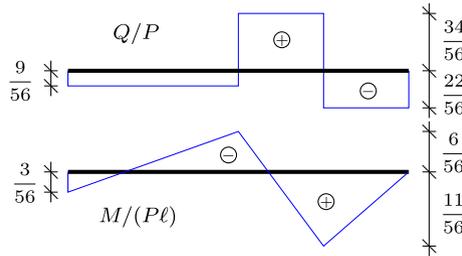


Fig. 7.17: Shear force and moment distributions for on two-span beam.

The section forces follow directly from the structure with the load and reaction components indicated in Fig. 7.16. When the reactions have been determined, it is immaterial for this part of the analysis that the structure is statically indeterminate. The moment and shear force distributions are shown in Fig. 7.17. It is seen that the maximum moment  $M_{\max} = \frac{11}{56} P\ell$  is found under the load. This is a reduction relative to if the load had been carried only by the beam  $BC$  with simple supports, in which case  $M_{\max} = \frac{14}{56} P\ell$ .

**Frame with two constraints**

The procedure and notation of the deformation method is now extended to structures with two constraints by considering the simple frame shown in Fig. 7.18. The frame consists of a horizontal continuous beam  $ABC$ , supported by vertical beams  $BD$  and  $CE$ . For simplicity of the presentation the bending stiffness of all members is  $EI$  and all lengths are  $a$  as indicated in the figure, but these features are not important for the principles of the method.

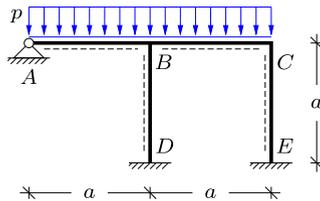


Fig. 7.18: Frame with distributed load.

First, the nodes that should be constrained against motion – rotation and translation – are identified. In the present case the location of all nodes are fixed in space, and only the rotation of the internal nodes  $B$  and  $C$  should be constrained against rotation. Thus, the problem has two degrees of freedom, and the rotations and constraining moments are identified by subscript(s)  $j = 1, 2$ , where  $j = 1$  refers to  $B$ , while  $j = 2$  refers to  $C$ . External moments  $Z_{10}$  and  $Z_{20}$  are then introduced to simultaneously constrain the rotation of the nodes  $B$  and  $C$ , respectively, as shown in Fig. 7.19. The first subscript identifies the node, at which the moment is acting, while the second subscript indicates that the moments are constraining motion from external loads.

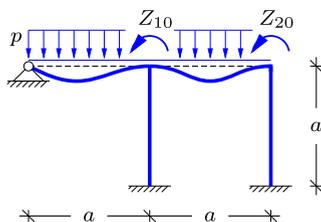


Fig. 7.19: Constraining moments on frame.

The magnitude of the constraining moments  $Z_{10}$  and  $Z_{20}$  are determined from the second row in Table 7.2. For simplicity all external moments introduced in the present analysis are considered positive, when acting in the counter-clockwise direction. The beam  $AB$  is simply supported at  $A$ , and therefore constraining rotation at  $B$  requires the moment  $M_{BA}^0 = -\frac{1}{8}pa^2$ , as shown in the right figure of the second row in Table 7.2. The beam  $BC$  is constrained against rotation at  $D$ , and thus constraining  $B$  in this beam requires the moment  $M_{BC}^0 = \frac{1}{12}pa^2$ , as shown in corresponding right figure in Table 7.2. Note, that the moment  $M_{BA}^0$  acts in the clockwise direction, and the moment  $M_{BC}^0$  in the counter-clockwise direction. The constraining moment  $Z_{10}$  is the sum of these two moments,

$$Z_{10} = M_{BA}^0 + M_{BC}^0 = -\frac{1}{8}pa^2 + \frac{1}{12}pa^2 = -\frac{1}{24}pa^2.$$

The moment needed to prevent rotation at  $C$  is given at the right of the second row in Table 7.2 as  $M_{CB}^0 = -\frac{1}{12}pa^2$ , whereby the constraining moment at  $C$  is

$$Z_{20} = M_{CB}^0 = -\frac{1}{12}pa^2.$$

With these constraining moments all internal nodes are fixed against translation and rotation, and only nodes with simple support can rotate.

Next, the displacement components  $\zeta_1$  and  $\zeta_2$  associated with the constraints  $Z_{10}$  and  $Z_{20}$  are given unit magnitude, one at a time. In the present case  $\zeta_1$  is the rotation of node  $B$ , while  $\zeta_2$  is the rotation of node  $C$ . Figure 7.20a

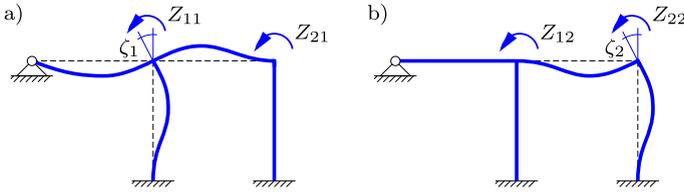


Fig. 7.20: Imposed unit rotations on frame.

shows a unit rotation of node  $B$  corresponding to  $\zeta_1 = 1$ , while the other nodes with the exception of simple supports are constrained. This requires a moment  $Z_{11}$  at node  $B$  and a moment  $Z_{21}$  at node  $D$ . Similarly, a unit rotation of node  $D$  corresponding to  $\zeta_2 = 1$  with the other nodes constrained requires a moment  $Z_{22}$  at node  $D$  and a moment  $Z_{12}$  at node  $C$ , as shown in Fig. 7.20. The moments in the individual beams generated by these imposed rotations are found from the basic cases of imposed deformation presented in Table 7.1.

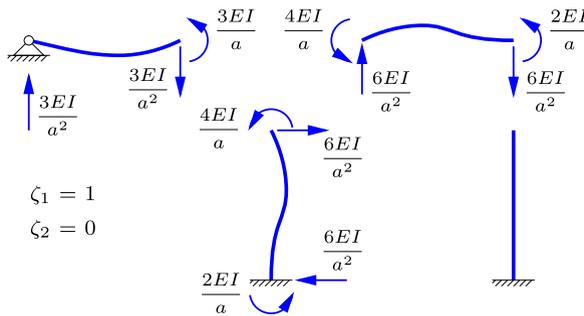


Fig. 7.21: Constraining forces and moments for imposed unit rotation  $\zeta_1 = 1$ .

The end-moments and transverse forces introduced in the individual beams by an imposed unit rotation  $\zeta_1 = 1$  are shown in Fig. 7.21. The beam  $AB$  has an imposed rotation and a simply supported end corresponding to the second row in Table 7.1, while both of the beams  $BC$  and  $BD$  have an imposed rotation and a fully constrained end corresponding to the first row in Table 7.1. The imposed moment  $Z_{11}$  is the sum of all end moments at node  $B$ ,

$$Z_{11} = M_{BA}^1 + M_{BC}^1 + M_{BD}^1 = 3\frac{EI}{a} + 4\frac{EI}{a} + 4\frac{EI}{a} = 11\frac{EI}{a}.$$

The superscript 1 indicates that the moments correspond to the imposed deformation  $\zeta_1 = 1$ . Note, that in  $Z_{11}$  each of the contributing moments is positive, as each beam produces resistance to the imposed rotation. The constraining moment  $Z_{21}$  at node  $D$  comes from the beam  $BC$ ,

$$Z_{21} = M_{CB}^1 = 2\frac{EI}{a}.$$

The remaining end forces and moments are used to determine the reactions and the distribution of internal forces, once  $\zeta_1$  and  $\zeta_2$  have been determined.

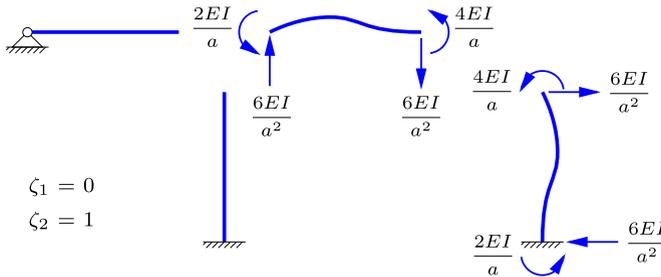


Fig. 7.22: Constraining forces and moments for imposed unit rotation  $\zeta_2 = 1$ .

Figure 7.22 shows the end moments and transverse forces introduced in the individual beams by imposing the unit rotation  $\zeta_2 = 1$  at node  $C$ . The imposed moment  $Z_{22}$  is the sum of all end-moments of node  $C$ ,

$$Z_{22} = M_{CB}^2 + M_{CE}^2 = 4\frac{EI}{a} + 4\frac{EI}{a} = 8\frac{EI}{a}.$$

Finally, the constraining moment  $Z_{12}$  at node  $C$  comes from the beam  $BC$ ,

$$Z_{12} = M_{BC}^2 = 2\frac{EI}{a}.$$

It is observed that  $Z_{12} = Z_{21}$ , corresponding to a symmetric coefficient matrix  $Z_{ij}$ . This is a general property, following from the principle of virtual work as discussed later.

The initially unknown rotations  $\zeta_1$  and  $\zeta_2$  are now determined by considering the actual deformation and internal forces as a superposition of the case of constrained loads, shown in Fig. 7.19, and the cases of imposed rotations  $\zeta_1$  and  $\zeta_2$ , respectively, shown in Fig. 7.20. In the actual state of the frame structure the constraining moments  $Z_1$  and  $Z_2$  at the nodes  $B$  and  $C$  vanish, thus providing the equations

$$Z_1 = Z_{10} + Z_{11}\zeta_1 + Z_{12}\zeta_2 = 0,$$

$$Z_2 = Z_{20} + Z_{21}\zeta_1 + Z_{22}\zeta_2 = 0.$$

This is an equation system of the form

$$\begin{aligned} Z_{11}\zeta_1 + Z_{12}\zeta_2 &= -Z_{10}, \\ Z_{21}\zeta_1 + Z_{22}\zeta_2 &= -Z_{20}, \end{aligned}$$

in which the coefficients  $Z_{ij}$  have just been calculated. Thus, in the present problem the equations are

$$\frac{EI}{a} \begin{bmatrix} 11 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \frac{pa^2}{24} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The solution is obtained by pre-multiplication with the inverse matrix,

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \frac{pa^3}{24EI} \frac{1}{84} \begin{bmatrix} 8 & -2 \\ -2 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{504} \frac{pa^3}{EI} \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Thus, both nodes rotate counter-clockwise, and  $\zeta_2 = 5\zeta_1$ .

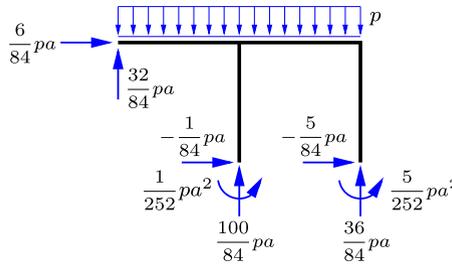


Fig. 7.23: Loads and reactions on frame.

Figure 7.23 shows the loads and reactions on the frame. The reactions are determined as follows. First, the reaction moments are evaluated, as they are determined directly by superposition of the corresponding load cases. Reaction moments are positive in the counter-clockwise direction. At  $A$  there is a simple support, whereby  $M_A = 0$ . At  $D$  the reaction moment is

$$M_D = M_D^0 + M_D^1\zeta_1 + M_D^2\zeta_2 = 0 + 2\frac{EI}{a}\zeta_1 + 0 = \frac{1}{252}pa^2,$$

where the unit rotation moment  $M_D^1$  is given in Fig. 7.21. Similarly, at the support  $E$

$$M_E = M_E^0 + M_E^1\zeta_1 + M_E^2\zeta_2 = 0 + 0 + 2\frac{EI}{a}\zeta_2 = \frac{5}{252}pa^2,$$

where the unit rotation moment  $M_E^2$  is given in Fig. 7.22. Neither of the reaction moments contain a contribution from the constrained frame with external loads, as neither of the adjoining beams are loaded directly.

The reaction force in the direction transverse to the beam also follows directly from the reaction components of the load cases. Horizontal reactions are positive towards the right, and vertical reactions are positive upwards. At the simple support  $A$  the vertical component of the reaction force is

$$R_A = R_A^0 + R_A^1\zeta_1 + R_A^2\zeta_2 = \frac{3}{8}pa + 3\frac{EI}{a^2}\zeta_1 + 0 = \frac{32}{84}pa.$$

The horizontal reaction at  $D$  is determined by the transverse force component in the beam  $DB$  as

$$R'_D = R_D^0 + R_D^1\zeta_1 + R_D^2\zeta_2 = 0 - 6\frac{EI}{a^2}\zeta_1 + 0 = -\frac{1}{84}pa.$$

This component is given entirely in terms of  $\zeta_1$  and the transverse force component for the corresponding unit deformation given in Fig. 7.21. Similarly, the horizontal reaction component at  $E$  is given by the transverse force in the beam  $EC$  as

$$R'_E = R_E^0 + R_E^1\zeta_1 + R_E^2\zeta_2 = 0 + 0 - 6\frac{EI}{a^2}\zeta_2 = -\frac{5}{84}pa.$$

This component is given entirely in terms of  $\zeta_2$  and the transverse force component for the corresponding unit deformation given in Fig. 7.22.

The remaining reactions are normal forces in the corresponding beams and they are therefore not represented explicitly by the load cases. The horizontal reaction component in  $A$  is most easily found by horizontal equilibrium of the full loaded frame,

$$R'_A = -R'_D - R'_E = \frac{6}{84}pa.$$

The vertical reaction  $R_D$  is determined from the transverse forces in  $ABC$  at  $B$ ,

$$R_D = R_D^0 + R_D^1\zeta_1 + R_D^2\zeta_2 = \left(\frac{5}{8} + \frac{1}{2}\right)pa + (6 - 3)\frac{EI}{a^2}\zeta_1 + 6\frac{EI}{a^2}\zeta_2 = \frac{100}{84}pa.$$

Similarly, the vertical reaction  $R_E$  is determined from the transverse forces in  $ABC$  at  $C$ ,

$$R_E = R_E^0 + R_E^1\zeta_1 + R_E^2\zeta_2 = \frac{1}{2}pa - 6\frac{EI}{a^2}\zeta_1 - 6\frac{EI}{a^2}\zeta_2 = \frac{36}{84}pa.$$

It is easily verified that the sum of the vertical reactions equal the load.

Once the reactions have been determined as illustrated in Fig. 7.23 it is a standard procedure to calculate the associated internal force distributions. The moment distribution in the frame is shown in Fig. 7.24. The basic behavior of the frame is illustrated by the moment distribution over the beam

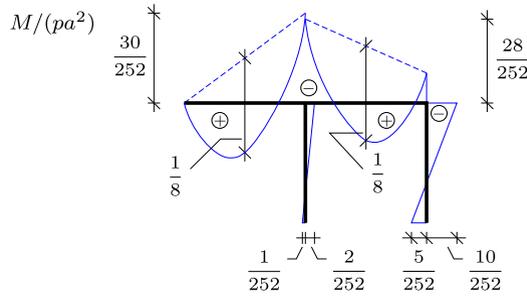


Fig. 7.24: Moment distribution in frame.

*ABC*. If the nodes *B* and *C* were fully constrained the moment curve over *BC* would have been symmetric. However, while the node *B* has a much smaller rotation than node *C* it appears as nearly fixed and thereby retains the rather large moment, while node *C* appears as a flexible support and thereby reduces the moment at *C*.

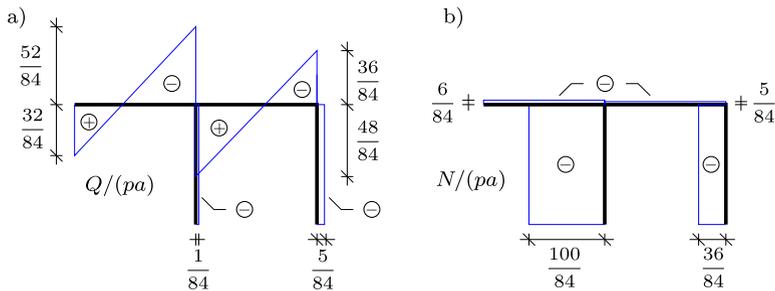


Fig. 7.25: (a) Shear and (b) normal force distribution in frame.

The shear force and normal force distributions are shown in Fig. 7.25a and b, respectively. The figures clearly illustrate that the two legs of the frame primarily act as columns in compression.

**General procedure of the deformation method**

On the basis of the two examples the deformation method for analysis of beam and frame structures can now be described in concise terms. A number of points of the structure are designated as nodes. The initially unconstrained degrees of freedom of these nodes are denoted  $\zeta_1, \dots, \zeta_n$ . These degrees of freedom may include displacements and rotations. The loading is now considered to be applied to a structure in which the motion of the nodes has been constrained by imposing forces and moments  $Z_{10}, \dots, Z_{n0}$  corresponding to the degrees of freedom  $\zeta_1, \dots, \zeta_n$ . The forces/moments  $Z_{10}, \dots, Z_{n0}$  needed to constrain the nodes are obtained by summation of the forces/moments



$$R = R_{i0} + R_{i1}\zeta_1 + \cdots + R_{in}\zeta_n, \quad i = 1, \cdots, n, \quad (7.20)$$

and then to determine the internal force distributions by a static analysis of the structure with the full load and reactions. In this way superposition of internal force distributions over the beams is avoided.

Although in practice the constraint loads  $Z_{i0}$  and  $Z_{ij}$  are often obtained from a table of simple load cases for a single beam, it is instructive to review a procedure for calculation of the coefficients  $Z_{ij}$  by use of the principle of virtual work as described in Section 4.4.2. Consider the case with all nodes constrained, except  $\zeta_i = 1$ . The corresponding loads are  $Z_{i1}, \cdots, Z_{in}$  and the internal force distributions are  $M_i(s)$ ,  $Q_i(s)$  and  $N_i(s)$ . This combination of loads and internal forces is now used as the static field in the principle of virtual work together with the virtual displacement field corresponding to the unit motion  $\zeta_j = 1$ . The corresponding virtual strain field is  $\kappa_j = M_j(s)/EI$ ,  $\gamma_j = Q_j(s)/GA_s$  and  $\varepsilon_j = N_j(s)/EA$ . There are no distributed loads or discontinuities in the virtual displacement field, and it therefore follows from the principle of virtual work in the form (4.43) and (4.45) that the external work  $Z_{ij}\zeta_j = Z_{ij}$  is given as

$$Z_{ij} = \int \left\{ \frac{M_i(s)M_j(s)}{EI} + \frac{Q_i(s)Q_j(s)}{GA_s} + \frac{N_i(s)N_j(s)}{EA} \right\} ds, \quad (7.21)$$

where the internal force fields correspond to the imposed isolated unit displacements  $\zeta_i = 1$  and  $\zeta_j = 1$ , respectively. The integral relation (7.21) implies, that the stiffness coefficients  $Z_{ij}$  satisfy the symmetry relations

$$Z_{ji} = Z_{ij}, \quad i, j = 1, \cdots, n. \quad (7.22)$$

Thus, the equation system (7.19) of the deformation method is symmetric. The contribution from the normal force is usually negligible, and is therefore omitted. For slender beams of isotropic material the shear contribution can also often be neglected, while it may be important for short beams and for composite beams with stiff webs and a core of a more flexible material.

**Example 7.2. Influence of relative stiffness in symmetric frame.** Figure 7.26 shows a symmetric frame  $ABCD$  with fixed supports at  $A$  and  $D$ . The frame supports a uniformly distributed load of intensity  $p$  over  $BC$ . Both frame and load are symmetric, and thus the deformation and the internal forces have symmetry properties. The columns  $AB$  and  $DC$  have bending stiffness  $EI_1$ , while the horizontal beam  $BC$  has bending stiffness  $EI_2$ . The reactions and internal force distribution are analyzed, using a slight modification of the deformation method as described above, and the influence of the relative stiffness of the frame members is illustrated.

The displacements are symmetric, and thus the displacement components to be constrained are the rotation of the corner nodes  $B$  and  $C$ . Due to symmetry the constraining moments are of equal magnitude but opposite orientation, and they can therefore both be represented by the constraining moment  $Z_{10}$  as shown in Fig. 7.27. The magnitude of the constraining

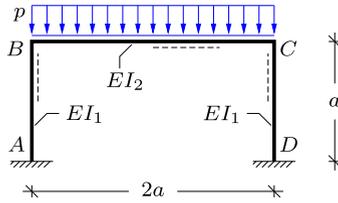


Fig. 7.26: Symmetric frame with column stiffness  $EI_1$  and beam stiffness  $EI_2$ .

moments follow from Table 7.2 as

$$Z_{10} = \frac{1}{12}p(2a)^2 = \frac{1}{3}pa^2.$$

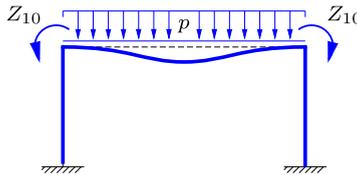


Fig. 7.27: Constraining moments  $Z_{10}$  at corners.

The rotation of the corners are of the same magnitude but opposite orientation, and the unit deformation case is therefore defined as a deformation where both corners rotate simultaneously as illustrated in Fig. 7.28. The magnitude of the rotation is denoted  $\zeta_1$ , and the magnitude of the moments needed to produce a unit rotation is denoted  $Z_{11}$ .

The moments and shear forces in the individual members of the frame corresponding to a unit rotation are shown in Fig. 7.29. The moments and shear forces in the columns correspond to the unit end-rotation case illustrated in Fig. 7.4, while the beam  $BC$  experiences uniform bending as illustrated in Fig. 7.2. The moment  $Z_{11}$  to be applied at both the corners follows from adding the contributions from the column and the beam,

$$Z_{11} = \frac{4EI_1}{a} + \frac{EI_2}{a}.$$

The corner rotation follows from the condition that the total moment applied to each corner must vanish,

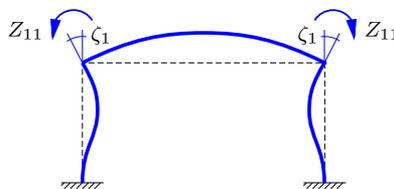


Fig. 7.28: Imposed unit rotation  $\zeta_1 = 1$ .

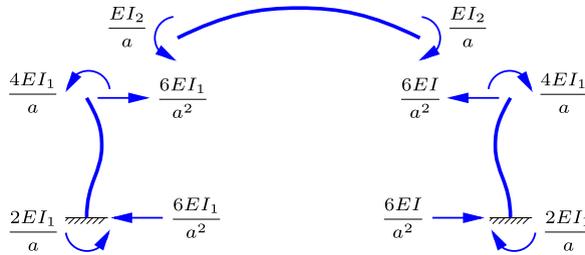


Fig. 7.29: Beam moments and shear forces for  $\zeta_1 = 1$ .

$$Z_1 = Z_{10} + Z_{11}\zeta_1 = 0,$$

whereby

$$\zeta_1 = -\frac{Z_{10}}{Z_{11}} = -\frac{\frac{1}{3}pa^3}{4EI_1 + EI_2}.$$

The transverse reaction forces and the reaction moments now follow from their normalized values given in Fig. 7.29, when multiplied by  $\zeta_1$ . This product involves the relative stiffness of the columns and the beam, conveniently expressed in terms of the parameter

$$\alpha = \frac{4EI_1}{4EI_1 + EI_2}.$$

It is seen that this parameter is the ratio of the moment rotating the top of one of the columns, to the moment needed to rotate the frame corner. Clearly, this ratio has a lower limit of zero for extremely flexible columns and increases towards unity for very stiff columns. The reactions are given in terms of the parameter  $\alpha$  in Fig. 7.30.

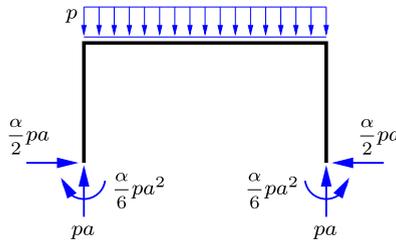


Fig. 7.30: Loads and reactions on frame.

With the load and the reactions given, the moment distribution follows from simple statics. The support moments have the magnitude  $\frac{1}{6}\alpha pa^2$ , while the corner moments are  $-\frac{1}{3}\alpha pa^2$ . The moment distribution and the transverse force in the two columns are proportional to  $\alpha$ , implying that moments in the columns decrease with decreasing relative stiffness of the columns.

In the special case where the bending stiffness of the beam and the columns are identical  $\alpha = \frac{4}{5}$ . In this case the corner moments are  $-\frac{4}{15}pa^2$ , while the moment at the center of the beam is  $\frac{7}{30}pa^2$ , i.e. roughly the same magnitude.  $\square$

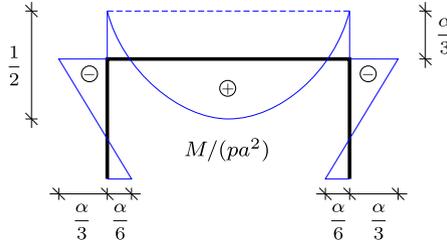


Fig. 7.31: Moment distribution in frame.

**Example 7.3. Swaying of symmetric frame with anti-symmetric load.** Figure 7.32 shows a symmetric frame  $ABCD$  loaded by a horizontal force  $2P$  at the corner  $B$ . The beam  $BC$  is considered inextensible, and thus the load can be considered as distributed equally between the nodes  $B$  and  $C$ . All members have bending stiffness  $EI$ .

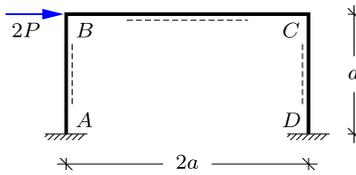


Fig. 7.32: Symmetric frame with horizontal force.

In the present case full constraint of the corner nodes  $B$  and  $C$  requires both a pair of equal horizontal forces  $Z_{10}$  and a pair of moments  $Z_{20}$  as shown in Fig. 7.33. It is clear from the figure, that if the horizontal constraining forces balance the external load, there will be no need for constraining moments at the corners, and thus

$$Z_{10} = -P, \quad Z_{20} = 0.$$

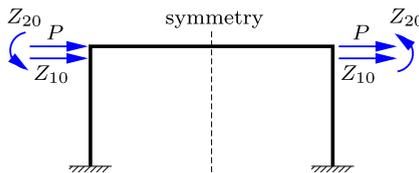


Fig. 7.33: Constraining forces  $Z_{10}$  and moments  $Z_{20}$  at corners.

The deformation modes corresponding to the constraining forces  $Z_{10}$  and moments  $Z_{20}$  are represented by a horizontal translation  $\zeta_1$  and a counter-clockwise rotation  $\zeta_2$  of both the corner nodes  $B$  and  $C$  as shown in Fig. 7.34.

The deformation modes are anti-symmetric, and therefore only the left half of the frame need to be considered when calculating the stiffness coefficients  $Z_{ij}$ . The coefficients  $Z_{11}$  and  $Z_{21}$  corresponding  $\zeta_1 = 1$  can be identified from Fig. 7.35a as the horizontal force

and the moment at node  $B$ . The beam  $BC$  is not deformed in this case, and thus these coefficients follow directly from the column  $AB$  as

$$Z_{11} = 12 \frac{EI}{a^3}, \quad Z_{21} = 6 \frac{EI}{a^2}.$$

Similarly, the coefficients  $Z_{12}$  and  $Z_{22}$  corresponding  $\zeta_2 = 1$  can be identified from Fig. 7.35b as the horizontal force and the total moment at node  $B$ . Note, that the beam  $BC$  is here deformed in anti-symmetric bending, corresponding to a hinge at its center on the line of symmetry and no axial force.

$$Z_{12} = 6 \frac{EI}{a^2}, \quad Z_{22} = 4 \frac{EI}{a} + 3 \frac{EI}{a} = 7 \frac{EI}{a}.$$

Note, that by symmetry  $Z_{12} = Z_{21}$ .

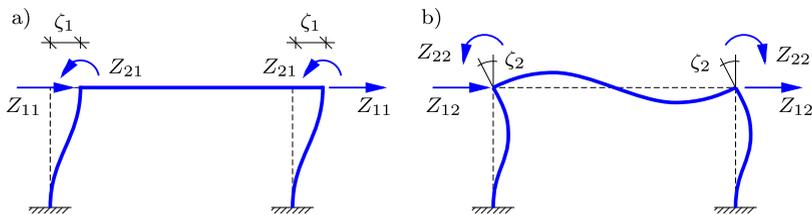


Fig. 7.34: Imposed unit displacements. (a) corner translation  $\zeta_1$ , (b) corner rotation  $\zeta_2$ .

The total imposed horizontal force and moment at each of the corner nodes must vanish, whereby

$$\begin{aligned} Z_1 &= Z_{10} + Z_{11}\zeta_1 + Z_{12}\zeta_2 = 0, \\ Z_2 &= Z_{20} + Z_{21}\zeta_1 + Z_{22}\zeta_2 = 0. \end{aligned}$$

When inserting the  $Z_{ij}$  coefficients just calculated the following equations are obtained,

$$\frac{EI}{a^3} \begin{bmatrix} 12 & 6a \\ 6a & 7a^2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix}.$$

The rotation  $\zeta_2$  follows e.g. by subtracting  $2/a$  times the second equation from the first, and  $\zeta_1$  then follows immediately in terms of  $\zeta_2$  from the second equation. The result is

$$\zeta_1 = \frac{7 Pa^3}{48 EI}, \quad \zeta_2 = -\frac{1 Pa^2}{8 EI}.$$

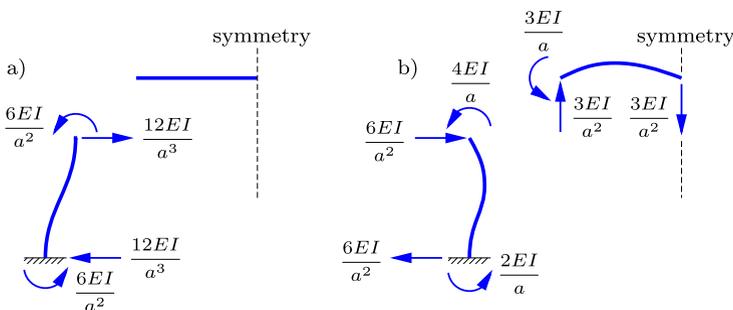


Fig. 7.35: Beam forces and moments for unit displacements: a)  $\zeta_1 = 1$ , b)  $\zeta_2 = 1$ .

The horizontal reactions  $R'_A = R'_D$  can be calculated from the transverse forces at  $A$  in Fig. 7.35. When considered positive towards the left

$$\leftarrow \quad R'_A = R'_D = 12 \frac{EI}{a^3} \zeta_1 + 6 \frac{EI}{a^2} \zeta_2 = P.$$

Alternatively, this result could have been determined by horizontal projection of forces. Now, this projection serves as a check. The reaction moments  $M_A = M_D$ , considered positive in the counter-clockwise direction, follow from weighted superposition of the moments in Fig. 7.35 as

$$\curvearrowleft \quad M_A = M_D = 6 \frac{EI}{a^2} \zeta_1 + 6 \frac{EI}{a} \zeta_2 = \frac{1}{8} Pa.$$

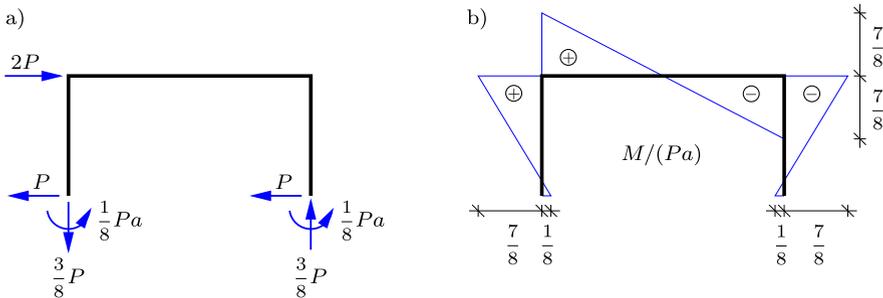


Fig. 7.36: a) Load and reactions, b) Moment distribution.

The vertical reactions  $R_A = -R_D$  can be determined from the transverse force at the center of the beam  $BC$ ,

$$\downarrow \quad R_A = -R_D = -3 \frac{EI}{a^3} \zeta_2 = \frac{3}{8} P.$$

Alternatively, the vertical reactions can be determined from moment equilibrium of the complete frame, including both the reaction moments  $M_A = M_D$ . The load and the reactions on the full frame are shown in Fig. 7.36a, and the corresponding moment distribution in Fig. 7.36b. If the frame had fixed simple supports at  $A$  and  $D$  the horizontal reactions would still be  $R'_A = R'_D = P$ , and the moment at the corners would then be  $M_B = -M_C = Pa$ . The constraint of the supports is seen to reduce the corner moments by  $\frac{1}{8} Pa$ . □

The examples have demonstrated, how the deformation method can be used to determine the internal forces and the displacements in statically indeterminate beam and frame structures. The analysis is systematic and quite straightforward in principle. However, once the constrained degrees of freedom have been found by solving an equation system, the remaining determination of the internal forces is complicated by the fact that axial deformation has been omitted from the analysis, and therefore the normal forces must be determined via a separate static analysis of the structure. In most cases the influence of the axial deformation on the resulting distribution of the internal forces is small, and thus this approximation appears attractive, as it reduces the number of degrees of freedom that needs to be constrained, and

thereby reduces the size of the equation system. When solving the problem numerically the balance changes, and it is advantageous to include the axial deformation, as it leads to a more systematic method for determining the internal forces and displacements of the structure. In that case it is advantageous to formulate the problem within the format of the finite element method, illustrated for truss structures in Section 2.5. The corresponding finite element formulation of frame structures is described in the following two sections – first considering the individual beam element in Section 7.3, and then assembling the elements into a model of a frame structure in Section 7.4.

### 7.3 Beam elements

The typical beam element treated here consists of a straight beam connecting the two nodes  $A$  and  $B$ . The properties of the element are first expressed in a local frame of reference  $\{x', y'\}$  with the element placed along the  $x'$ -axis as shown in Fig. 7.37. The two nodes have three generalized force components,

$$\mathbf{f}'_A = [f_{x'}^A, f_{y'}^A, m^A]^T, \quad \mathbf{f}'_B = [f_{x'}^B, f_{y'}^B, m^B]^T, \quad (7.23)$$

where  $f_{x'}$  is the axial force component,  $f_{y'}$  is the transverse force component, and  $m$  is the moment in the counter-clockwise direction. The corresponding generalized displacement components at  $A$  and  $B$  are shown in Fig. 7.37b,

$$\mathbf{u}'_A = [u_{x'}^A, u_{y'}^A, \theta^A]^T, \quad \mathbf{u}'_B = [u_{x'}^B, u_{y'}^B, \theta^B]^T, \quad (7.24)$$

where  $u_{x'}$  is the axial translation component,  $u_{y'}$  is the transverse translation component, and  $\theta$  is the counter-clockwise rotation.

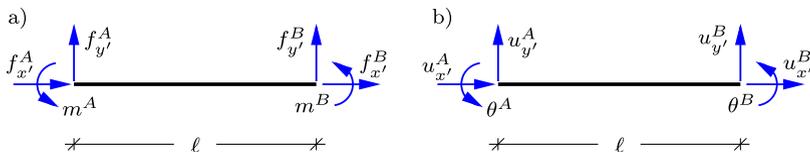


Fig. 7.37: Two-dimensional beam element. a) node forces, b) node displacements.

Each of the generalized displacement components are associated with a displacement mode as illustrated in Fig. 7.38, and the displacement of the beam element is obtained by superposition of these displacement modes.

Traditionally, the nodal forces generated by the nodal displacements are obtained by solving the differential equations for the displacements of the beam for each of the six unit displacement cases shown Fig. 7.38. However, greater generality as well as simplicity is gained by using the flexibility of the de-

formation modes, illustrated for the two bending modes in Section 7.1.1. When using the flexibility formulation, variation of cross-section properties as well as curvature can be accounted for explicitly without obtaining the displacement field, see e.g. Krenk (1994). Here, the presentation is limited to two particular types of straight homogeneous beam elements: the bending element with shear flexibility, and the beam-column element.

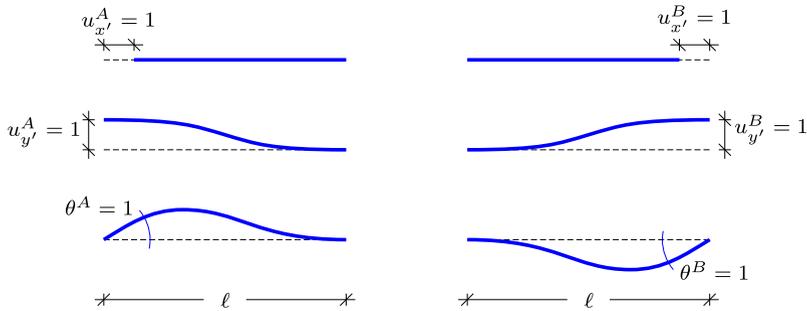


Fig. 7.38: Displacement modes of two-dimensional beam element.

### 7.3.1 Beam bending element

The beam bending element gives a relation between the generalized forces and the generalized moments shown in Fig. 7.37. For the case of a straight homogeneous beam the necessary expressions have already been obtained, when neglecting a possible column effect from the normal force – the normal force relation in connection with bars, and the unit displacement bending cases in Section 7.1. The results are expressed in the form of a relation between the generalized forces of the element, contained in the six-component vector

$$[\mathbf{f}'_A, \mathbf{f}'_B]^T = [f_{x'}^A, f_{y'}^A, m^A, f_{x'}^B, f_{y'}^B, m^B], \tag{7.25}$$

and the six-component generalized displacement vector

$$[\mathbf{u}'_A, \mathbf{u}'_B]^T = [u_{x'}^A, u_{y'}^A, \theta^A, u_{x'}^B, u_{y'}^B, \theta^B]. \tag{7.26}$$

The relation is linear and can be written in the generic block matrix format

$$\begin{bmatrix} \mathbf{f}'_A \\ \mathbf{f}'_B \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{K}'_{AA} & \mathbf{K}'_{AB} \\ \mathbf{K}'_{BA} & \mathbf{K}'_{BB} \end{bmatrix}}_{\mathbf{K}'_{\text{beam}}} \begin{bmatrix} \mathbf{u}'_A \\ \mathbf{u}'_B \end{bmatrix}, \tag{7.27}$$

where  $\mathbf{K}'_{\text{beam}}$  is the stiffness matrix of the beam element, when located in a local frame of reference along the  $x'$ -axis. In compact form the element stiffness matrix relation is

$$\mathbf{f}'_e = \mathbf{K}'_{\text{beam}} \mathbf{u}'_e \quad (7.28)$$

where  $\mathbf{f}'_e = [\mathbf{f}'_A{}^T, \mathbf{f}'_B{}^T]^T$  and  $\mathbf{u}'_e = [\mathbf{u}'_A{}^T, \mathbf{u}'_B{}^T]^T$  contain all generalized force and displacement components of the element.

For unit displacements at node  $A$  the sub-matrix  $\mathbf{K}'_{AA}$  represents the generalized forces at node  $A$ , while the sub-matrix  $\mathbf{K}'_{BA}$  represents the generalized forces at node  $B$ . These are the generalized forces at  $A$  and  $B$ , respectively, corresponding to the three unit deformation cases illustrated in the first column of Fig. 7.38. The first column in these matrices corresponds to the unit deformation  $u'_{x'} = 1$ , which generates a normal force of magnitude  $-EA/\ell$ . The second column corresponds to the unit transverse displacement  $u'_{y'} = 1$ . This case corresponds to Fig. 7.6, where the end forces are indicated. Finally, the generalized forces corresponding to a unit rotation  $\theta^A = 1$  follows from Fig. 7.4. The sub-matrices  $\mathbf{K}'_{AB}$  and  $\mathbf{K}'_{BB}$  similarly represent the generalized forces at  $A$  and  $B$ , respectively, from unit displacements at  $B$ . The stiffness matrix  $\mathbf{K}'_{\text{beam}}$  follows from collecting these generalized forces in matrix format,

$$\mathbf{K}'_{\text{beam}} = \begin{bmatrix} \frac{EA}{\ell} & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 \\ 0 & \frac{12}{1+\Phi} \frac{EI}{\ell^3} & \frac{6}{1+\Phi} \frac{EI}{\ell^2} & 0 & -\frac{12}{1+\Phi} \frac{EI}{\ell^3} & \frac{6}{1+\Phi} \frac{EI}{\ell^2} \\ 0 & \frac{6}{1+\Phi} \frac{EI}{\ell^2} & \frac{4+\Phi}{1+\Phi} \frac{EI}{\ell} & 0 & -\frac{6}{1+\Phi} \frac{EI}{\ell^2} & \frac{2-\Phi}{1+\Phi} \frac{EI}{\ell} \\ -\frac{EA}{\ell} & 0 & 0 & \frac{EA}{\ell} & 0 & 0 \\ 0 & -\frac{12}{1+\Phi} \frac{EI}{\ell^3} & -\frac{6}{1+\Phi} \frac{EI}{\ell^2} & 0 & \frac{12}{1+\Phi} \frac{EI}{\ell^3} & -\frac{6}{1+\Phi} \frac{EI}{\ell^2} \\ 0 & \frac{6}{1+\Phi} \frac{EI}{\ell^2} & \frac{2-\Phi}{1+\Phi} \frac{EI}{\ell} & 0 & -\frac{6}{1+\Phi} \frac{EI}{\ell^2} & \frac{4+\Phi}{1+\Phi} \frac{EI}{\ell} \end{bmatrix}. \quad (7.29)$$

The shear flexibility is represented via the non-dimensional parameter

$$\Phi = \frac{12EI}{GA_s \ell^2} \quad (7.30)$$

introduced in (7.5) in connection with the anti-symmetric bending problem with constant shear force distribution. The classic result for the so-called Euler beam without shear flexibility is obtained as the special case of  $\Phi = 0$ , corresponding to infinite shear stiffness. Clearly, in a computer program it is advantageous to implement the full expression, and obtain the classic theory as a special case.

### 7.3.2 Beam-column element

For frames with slender members it is important to be able to include the effect of reduced flexibility due to normal compression forces. In slender beams the shear flexibility effect is most often negligible, and the beam-column element is therefore derived for a beam without shear flexibility. The basic equations were derived in Section 5.1, and only the main points needed for development of the beam-column element are summarized here. The differential equation for the transverse displacement of a beam in the presence of a normal tension force  $N$  was derived as (5.9). In the present formulation the element is located along the  $x'$ -axis, and the transverse direction is  $y'$  with displacement  $u_{y'}$ . In this notation the bending of a beam without transverse load is governed by the differential equation

$$\frac{d^2}{dx'^2} \left( EI \frac{d^2 u_{y'}}{dx'^2} \right) - \frac{d}{dx'} \left( N \frac{du_{y'}}{dx'} \right) = 0. \tag{7.31}$$

The typical application involves a compressive normal force, denoted by  $P = -N$ . For a homogeneous beam this equation can be expressed as

$$\frac{d^4 u_{y'}}{dx'^4} + k^2 \frac{d^2 u_{y'}}{dx'^2} = 0, \tag{7.32}$$

where the parameter  $k$  has been introduced by the definition

$$k^2 = \frac{P}{EI}. \tag{7.33}$$

Real-valued parameters  $k$  correspond to compression, and the corresponding expressions for a tension force can be obtained by using complex notation, whereby trigonometric functions translate into their corresponding hyperbolic counterpart. However, the linearized form of the present theory only contains  $k^2$ , and these results can therefore be expressed directly in terms of the normal force  $N$ .

The general solution to the homogeneous 4'th order beam-column equation (7.33) is

$$u_{y'}(x') = C_1 + C_2 kx' + C_3 \cos(kx') + C_4 \sin(kx'). \tag{7.34}$$

In the present notation with transverse displacement  $u_{y'}(x')$  the moment follows from (7.34) as

$$\frac{M(x')}{EI} = \frac{d^2 u_{y'}}{dx'^2} = -C_3 k^2 \cos(kx') - C_4 k^2 \sin(kx'). \tag{7.35}$$

The shear force at the end sections will be determined directly from equilibrium of the element, when needed.

### Symmetric bending

As in the case of the beam bending element it is convenient to construct the element stiffness matrix from the symmetric and anti-symmetric deformation modes. The symmetric bending mode is shown in Fig. 7.39 with rotations  $\pm \frac{1}{2}\theta_s$  of the end sections.

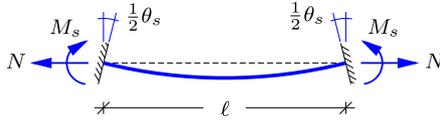


Fig. 7.39: Symmetric bending with normal force.

The  $y'$ -axis of the coordinate system is placed in the central line of symmetry, and thus the solution will be in the form of the symmetric part of (7.34),

$$u_{y'}^s(x') = C_1 + C_3 \cos(kx'). \quad (7.36)$$

The boundary conditions consist of vanishing displacement and prescribed rotation at  $\pm \frac{1}{2}\ell$ ,

$$\begin{aligned} u_{y'}^s\left(\frac{1}{2}\ell\right) &= C_1 + \cos\left(\frac{1}{2}k\ell\right) C_3 = 0, \\ u_{y'}^{s'}\left(\frac{1}{2}\ell\right) &= -k \sin\left(\frac{1}{2}k\ell\right) C_3 = \frac{1}{2}\theta_s. \end{aligned} \quad (7.37)$$

The constant  $C_3$  follows from the second equation as

$$C_3 = -\frac{\frac{1}{2}\theta_s}{k \sin\left(\frac{1}{2}k\ell\right)}. \quad (7.38)$$

The moment at the end-sections then follows from (7.35) in the form

$$M_s = \varphi \frac{EI}{\ell} \theta_s \quad (7.39)$$

with the stiffness coefficient for symmetric bending

$$\varphi(k\ell) = \left(\frac{1}{2}k\ell\right) \cot\left(\frac{1}{2}k\ell\right). \quad (7.40)$$

The case of zero normal force is represented by the limit  $k\ell = 0$ , for which  $\varphi(0) = 1$ . This case recovers the previous result (7.2) for symmetric bending without axial force. The variation of the stiffness coefficient  $\varphi$  is shown as a function of the normal compressive force  $P$  in Fig. 7.40, normalized with respect to the Euler load  $P_E$ . It is seen that the bending stiffness decreases with increasing compression, and turns negative when exceeding the Euler load.

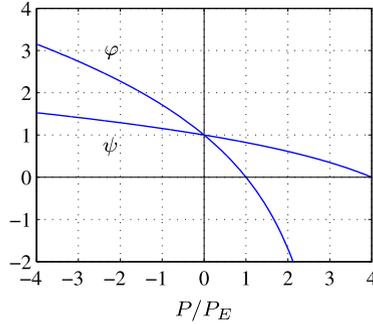


Fig. 7.40: Beam-column bending coefficients.

The shear force in the beam is determined by equilibrium of the full element. The normal forces are co-linear and therefore do not contribute to the moment equilibrium. Thus the shear force vanishes in the case of symmetric bending,  $Q = 0$ .

**Anti-symmetric bending**

The case of anti-symmetric bending in the presence of a normal force is shown in Fig. 7.41 with rotation  $\frac{1}{2}\theta_a$  of both end sections.

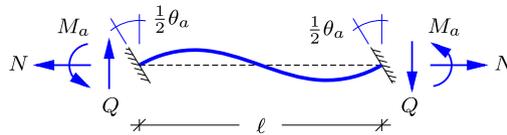


Fig. 7.41: Anti-symmetric bending with normal force.

As in the case of symmetric bending the  $y'$ -axis of the coordinate system is placed in the central line of symmetry, and thus the solution will be in the form of the anti-symmetric part of (7.34),

$$u_{y'}^a(x') = C_2 kx' + C_4 \sin(kx'). \tag{7.41}$$

Also in this case the boundary conditions consist of vanishing displacement and prescribed rotation at  $\pm\frac{1}{2}\ell$ ,

$$\begin{aligned} u_{y'}^a(\frac{1}{2}\ell) &= \frac{1}{2}\ell C_2 + \sin(\frac{1}{2}k\ell) C_4 = 0, \\ u_{y'}^{a'}(\frac{1}{2}\ell) &= k C_2 - k \sin(\frac{1}{2}k\ell) C_4 = \frac{1}{2}\theta_a. \end{aligned} \tag{7.42}$$

The constant  $C_4$  follows from these equations as

$$C_3 = - \frac{\frac{1}{4}\ell\theta_a}{\sin(\frac{1}{2}k\ell) - (\frac{1}{2}k\ell)\cos(\frac{1}{2}k\ell)}. \tag{7.43}$$

The moment at the end sections then follows from (7.35) in the form

$$M_a = \psi \frac{3EI}{\ell} \theta_a \tag{7.44}$$

with the stiffness coefficient for anti-symmetric bending

$$\psi(k\ell) = \frac{\frac{1}{12}(k\ell)^2}{1 - (\frac{1}{2}k\ell)\cot(\frac{1}{2}k\ell)} = \frac{\frac{1}{12}(k\ell)^2}{1 - \varphi}. \tag{7.45}$$

The coefficient is normalized such that the limit of vanishing normal force corresponds to  $\psi(0) = 1$ . This case recovers the previous result (7.6) for anti-symmetric bending without axial force in the case of vanishing shear flexibility,  $\Phi = 0$ . The variation of the coefficient  $\psi$  is shown as a function of the normal compressive force  $P$  in Fig. 7.40. Also in this case the bending stiffness decreases with increasing compression. In this case the stiffness turns negative when exceeding the load  $4P_E$ , corresponding to the stability load of an Euler column with a node in the middle.

The shear force in the beam is determined by equilibrium of the full element. Also in this case the normal forces are co-linear and therefore do not contribute to the moment equilibrium. The shear force then is determined as,  $Q = 2M_a/\ell$ .

**Translation of end-section**

The deformation mode for translation of one of the end-sections is illustrated in Fig. 7.42. The deformation mode is generated by equal moments  $M_t$ , applied at both end-sections. This is similar to the antisymmetric bending shown in Fig. 7.41, but in the present case the normal force  $N$  and the shear force  $Q$  are along a different set of axes. The displacements are assumed ‘small’ and thus the effect of the normal force can be represented via the parameter  $k^2 = -N/EI$ , using the normal force  $N$  shown in Fig. 7.42.

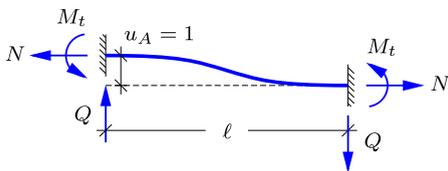


Fig. 7.42: Translation of end section.

The equivalent anti-symmetric rotation angle  $\frac{1}{2}\theta_a$  is identified by considering a counter-clockwise rotation of magnitude  $1/\ell = \frac{1}{2}\theta_a$ . Thus, the moment  $M_t$  needed for the translation  $u_{y'}^A = 1$  follows from (7.44) as

$$M_t = \psi \frac{6EI}{\ell^2}. \quad (7.46)$$

The normal force  $N$  is considered given, and the shear force  $Q$  then follows from moment equilibrium of the element as

$$Q = \frac{2}{\ell}M_t + \frac{1}{\ell}N = 12\psi \frac{EI}{\ell^3} + k^2 \frac{EI}{\ell} = (12\psi - (k\ell)^2) \frac{EI}{\ell^3}. \quad (7.47)$$

When substituting  $\psi$  from (7.45) the expression for the shear force takes the compact form

$$Q = 12\varphi\psi \frac{EI}{\ell^3}. \quad (7.48)$$

Thus, the contribution of the normal force to the moment equilibrium introduces an extra factor  $\varphi$  into the expression for the shear force.

### **Beam-column stiffness matrix**

The  $6 \times 6$  stiffness matrix contains the constraining forces/moments corresponding to the six unit deformation cases shown in Fig. 7.38. Extension generates a normal force of magnitude  $N = EA(u_{x'}^B - u_{x'}^A)$ , and the corresponding nodal forces appear in the first and fourth column and row in the stiffness matrix (7.49). The forces and moments corresponding to the case of a unit translation of node  $A$  with the other degrees of freedom constrained appear as the second column of the stiffness matrix. This case of deformation was illustrated in Fig. 7.42, and the moment and transverse force given in (7.46) and (7.48), respectively. The deformation in which node  $A$  is given a unit rotation,  $\theta^A = 1$ , while the other degrees of freedom are constrained, is obtained by superimposing the symmetric and anti-symmetric cases of deformation with  $\theta_a = -\theta_s = 1$ . The corresponding constraint forces and moments appear as the third column in the stiffness matrix. The last two columns follow from considering the symmetric load cases. In total these six cases of unit deformation defines the stiffness matrix

$$\mathbf{K}'_{\text{beam}} = \begin{bmatrix} \frac{EA}{\ell} & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 \\ 0 & 12\varphi\psi\frac{EI}{\ell^3} & 6\psi\frac{EI}{\ell^2} & 0 & -12\varphi\psi\frac{EI}{\ell^3} & 6\psi\frac{EI}{\ell^2} \\ 0 & 6\psi\frac{EI}{\ell^2} & (3\psi + \varphi)\frac{EI}{\ell} & 0 & -6\psi\frac{EI}{\ell^2} & (3\psi - \varphi)\frac{EI}{\ell} \\ -\frac{EA}{\ell} & 0 & 0 & \frac{EA}{\ell} & 0 & 0 \\ 0 & -12\varphi\psi\frac{EI}{\ell^3} & -6\psi\frac{EI}{\ell^2} & 0 & 12\varphi\psi\frac{EI}{\ell^3} & -6\psi\frac{EI}{\ell^2} \\ 0 & 6\psi\frac{EI}{\ell^2} & (3\psi - \varphi)\frac{EI}{\ell} & 0 & -6\psi\frac{EI}{\ell^2} & (3\psi + \varphi)\frac{EI}{\ell} \end{bmatrix}. \quad (7.49)$$

In the stiffness matrix the entries corresponding to the various forms of bending include the coefficients  $\phi$  and  $\psi$  that depend on the normal force  $N$  as given by (7.40) and (7.45) and illustrated in Fig. 7.40. It is seen from the figure that within the range  $|N| \lesssim P_E$ , the coefficients  $\varphi$  and  $\psi$  are nearly linear functions of the normal force. Thus, there is a substantial range of normal forces, for which a linear approximation constitutes a fair representation of the effect of the normal force. If needed, the linearized approximation can be improved by sub-dividing the elements, whereby the element length  $\ell$  decreases. Hereby the linearized form becomes an attractive option for performing a stability analysis as described in Section 7.4.2.

The linearized form of the stiffness matrix (7.49) is obtained by using a Taylor series expansion of the two functions  $\varphi(k\ell)$  and  $\psi(k\ell)$ . It follows directly from the Taylor expansion of the cot-function that

$$\varphi(k\ell) = \frac{1}{2}k\ell \cot\left(\frac{1}{2}k\ell\right) \simeq 1 - \frac{1}{3}\left(\frac{1}{2}k\ell\right)^2 - \frac{1}{45}\left(\frac{1}{2}k\ell\right)^4, \quad (7.50)$$

and the  $\psi$ -function expansion then follows from the last expression in (7.45) as

$$\psi(k\ell) = \frac{\frac{1}{3}\left(\frac{1}{2}k\ell\right)^2}{1 - \varphi} \simeq 1 - \frac{1}{15}\left(\frac{1}{2}k\ell\right)^2. \quad (7.51)$$

In these relations  $(k\ell)^2 = -N\ell^2/EI$ , whereby the linearized expressions in the normal force  $N$  take the form

$$\varphi \simeq 1 + \frac{1}{12} \frac{N\ell^2}{EI}, \quad \psi \simeq 1 + \frac{1}{60} \frac{N\ell^2}{EI}. \quad (7.52)$$

When using these linearized expressions, the stiffness matrix (7.49) can be written as the sum of two matrices,

$$\mathbf{K}'_{\text{beam}} \simeq \mathbf{K}'_{\text{beam}}{}^c + \mathbf{K}'_{\text{beam}}{}^g, \quad (7.53)$$

where  $\mathbf{K}'_{\text{beam}}{}^c$  is the constitutive stiffness matrix corresponding to  $\varphi = \psi = 1$ ,

$$\mathbf{K}_{\text{beam}}^{c'} = \begin{bmatrix} \frac{EA}{\ell} & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 \\ 0 & 12\frac{EI}{\ell^3} & 6\frac{EI}{\ell^2} & 0 & -12\frac{EI}{\ell^3} & 6\frac{EI}{\ell^2} \\ 0 & 6\frac{EI}{\ell^2} & 4\frac{EI}{\ell} & 0 & -6\frac{EI}{\ell^2} & 2\frac{EI}{\ell} \\ -\frac{EA}{\ell} & 0 & 0 & \frac{EA}{\ell} & 0 & 0 \\ 0 & -12\frac{EI}{\ell^3} & -6\frac{EI}{\ell^2} & 0 & 12\frac{EI}{\ell^3} & -6\frac{EI}{\ell^2} \\ 0 & 6\frac{EI}{\ell^2} & 2\frac{EI}{\ell} & 0 & -6\frac{EI}{\ell^2} & 4\frac{EI}{\ell} \end{bmatrix} \quad (7.54)$$

and  $\mathbf{K}_{\text{beam}}^{g'}$  is the so-called geometric stiffness matrix, corresponding to the linear terms in the normal force  $N$ ,

$$\mathbf{K}_{\text{beam}}^{g'} = \frac{N}{30\ell} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3\ell & 0 & -36 & 3\ell \\ 0 & 3\ell & 4\ell^2 & 0 & -3\ell & -\ell^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3\ell & 0 & 36 & -3\ell \\ 0 & 3\ell & -\ell^2 & 0 & -3\ell & 4\ell^2 \end{bmatrix}. \quad (7.55)$$

It is observed that the constitutive stiffness matrix (7.54) is a special case of the form (7.29) in which the shear flexibility has been omitted. In practice, the geometric stiffness is less sensitive to the details of the shape functions than the constitutive stiffness, as it depends on the first derivatives while the latter depends on the second derivatives. Thus, a convenient option is to use the form (7.29) including shear flexibility for the constitutive stiffness, while retaining the simple form (7.55) for the geometric stiffness.

### 7.3.3 Transformation to global form

The beam elements treated above are located in a local frame of reference  $\{x', y'\}$  with the beam axis along the  $x'$ -axis. In order to use these elements in a model of a structure the corresponding generalized displacement and force components must be transformed into a common global frame of reference  $\{x, y\}$  as illustrated in Fig. 7.43. The generalized displacement components  $\mathbf{u}' = [u_{x'}, u_{y'}, \theta]^T$  in the local frame are related to the corresponding components  $\mathbf{u} = [u_x, u_y, \theta]^T$  in the global frame by the transformation

$$\mathbf{u}' = \mathbf{A} \mathbf{u}, \quad \mathbf{u} = \mathbf{A}^T \mathbf{u}', \quad (7.56)$$

with the component transformation matrix for the generalized displacements at a node given by

$$\mathbf{A} = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.57}$$

In this transformation the displacement components  $[u_x, u_y]$  transform as a vector, while the rotation  $\theta$  is the same in both frames of reference. The transformation between local and global components is identical for the generalized forces.

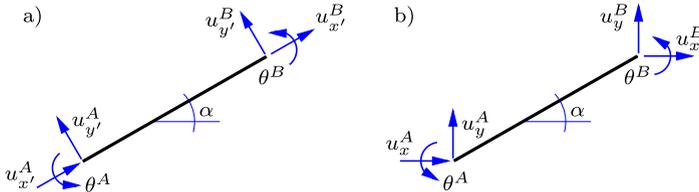


Fig. 7.43: Beam element: a) Local components, b) Global components.

The beam element has two nodes, and the components at each of the nodes must be transformed according to (7.56). It is convenient to combine the transform matrix for the components at a single node into a diagonal block matrix

$$\mathbf{A}_e = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{bmatrix} \tag{7.58}$$

that transforms the components at all element nodes at the same time. In terms of this element transformation matrix the six-components relations for the generalized displacements are

$$\mathbf{u}'_e = \mathbf{A}_e \mathbf{u}_e, \quad \mathbf{u}_e = \mathbf{A}_e^T \mathbf{u}'_e. \tag{7.59}$$

The local form of the element stiffness relation (7.28) is transformed into global component form by pre-multiplication with  $\mathbf{A}_e^T$ , whereby

$$\mathbf{A}_e^T \mathbf{f}'_e = \mathbf{A}_e^T \mathbf{K}'_{\text{beam}} \mathbf{u}'_e = \mathbf{A}_e^T \mathbf{K}'_{\text{beam}} \mathbf{A}_e \mathbf{u}_e. \tag{7.60}$$

The left side is recognized as the global components of the generalized forces via the transform (7.59), whereby the global form of the element stiffness relation takes the form

$$\mathbf{f}_e = \mathbf{K}_{\text{beam}} \mathbf{u}_e \tag{7.61}$$

with the global element stiffness matrix given by

$$\mathbf{K}_{\text{beam}} = \mathbf{A}_e^T \mathbf{K}'_{\text{beam}} \mathbf{A}_e. \tag{7.62}$$

The global form of the stiffness matrix can be assembled into a stiffness matrix for a frame structure as explained in the following section.

### 7.4 Finite element method for frames

The finite element analysis of a frame structure is organized in a way similar to that of a truss structure, described in Section 2.5. Therefore only the main points are discussed here with reference to the frame shown in Fig. 7.44. When using one beam element for each member of the frame, the structural model has five nodes, here ordered sequentially as  $A, B, C, D, E$ . The frame consists of the four elements  $AB, BC, BD$  and  $CE$  with global element stiffness matrices  $\mathbf{K}_{AB}$  etc., obtained as described in the previous section.

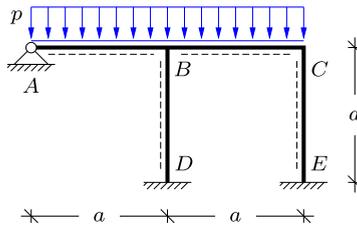


Fig. 7.44: Frame with distributed load.

The stiffness matrix of the structure is formed by including the stiffness contributions from each of the elements. In order to do this all element matrices must first be transformed to a common global frame of reference, and the parts of the stiffness matrix must be associated with the degrees of freedom as organized for the total structure. This procedure, usually called assembling the stiffness matrix, is illustrated in (7.63). Here  $\mathbf{K}^1$  is the stiffness matrix of element  $AB$  denoted as element No. 1. The stiffness matrix consists of four sub-matrices  $\mathbf{K}_{AA}, \mathbf{K}_{AB}, \mathbf{K}_{BA}$  and  $\mathbf{K}_{BB}$  as indicated in (7.28). The first subscript indicates the node of the resulting generalized force, when unit displacements are imposed at the node indicated by the second subscript. The corresponding sub-matrices of element No. 1 are located at the first and second row and column, associated with the element nodes  $A$  and  $B$ . Element No. 2 is associated with nodes  $B$  and  $C$  and the sub-matrices therefore enter the rows and columns 2 and 3.

$$\begin{aligned}
 \mathbf{K}^1 &= \begin{bmatrix} \mathbf{K}_{AA} & \mathbf{K}_{AB} & - & - & - \\ \mathbf{K}_{BA} & \mathbf{K}_{BB} & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix}, & \mathbf{K}^2 &= \begin{bmatrix} - & - & - & - & - \\ - & \mathbf{K}_{BB} & \mathbf{K}_{BC} & - & - \\ - & \mathbf{K}_{CB} & \mathbf{K}_{CC} & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix}, \\
 \mathbf{K}^3 &= \begin{bmatrix} - & - & - & - & - \\ - & \mathbf{K}_{BB} & - & \mathbf{K}_{BD} & - \\ - & - & - & - & - \\ - & \mathbf{K}_{DB} & - & \mathbf{K}_{DD} & - \\ - & - & - & - & - \end{bmatrix}, & \mathbf{K}^4 &= \begin{bmatrix} - & - & - & - & - \\ - & - & - & - & - \\ - & - & \mathbf{K}_{CC} & - & \mathbf{K}_{CE} \\ - & - & - & - & - \\ - & - & \mathbf{K}_{EC} & - & \mathbf{K}_{EE} \end{bmatrix}.
 \end{aligned}
 \tag{7.63}$$

Element No. 3 is the beam/column connecting nodes  $B$  and  $D$ , and in this case the sub-matrices are therefore located in the rows and columns 2 and 4. Similarly for element No. 4 connecting nodes  $C$  and  $E$ . The actual assembling process starts with an empty structure stiffness matrix  $\mathbf{K}$ , and then loops over all beam elements, adding each of the four element sub-matrices directly into its correct global position. There is no need to form the element matrices explicitly in the global format. The association of the two nodes of a beam element with global node numbers is described in the topology matrix  $\mathbf{T}$ .

The resulting equilibrium equations are of the form

$$\mathbf{K} \mathbf{u} = \mathbf{f}, \quad (7.64)$$

where  $\mathbf{K}$  is the global stiffness matrix,  $\mathbf{u}$  is the global displacement vector, and  $\mathbf{f}$  is the global load vector. The loads can either be associated with a node or an element. While nodal loads are entered directly into the global equations, element based loads must be defined in connection with an element, and then translated into equivalent nodal loads. This procedure makes use of the constraint forces illustrated in Table 7.2. Later, when calculating the section forces the contribution from element based loads also need representation of the local variation corresponding to the load distribution within the element.

The support conditions, constraining the displacement at nodes, are implemented as for the case of truss structures in Section 2.5. A simple, but approximate procedure, is to retain the full equation system and then introduce a stiff spring as a diagonal term for each constrained degree of freedom. Alternatively, the constrained degrees of freedom can be eliminated by removing the corresponding rows and columns from the equation system. The reactions can then be recovered from the force components generated by the removed rows of the stiffness matrix, as explained in detail in Section 2.5.2.

### 7.4.1 The *MiniFrame* program

The principles described in the previous sections have been implemented in a small Finite Element program `MINIFRAME` using the high level programming language `MATLAB`. The structure of the program is similar to that of the `MINITRUSS` program described in Section 2.5.3, while element details, external loads and internal forces are new here. The main features of the program and its data structure are explained in relation to the specific frame shown in Fig. 7.44 and already analyzed by the deformation method in Section 7.2.

The program is built as a script file `MiniFrame.m` that serves as a driver that reads a data file and activates subroutines that set up the model, form the global stiffness matrix, apply the load, and solve the constrained equations for the displacement of all nodes of the supported structure. The structure

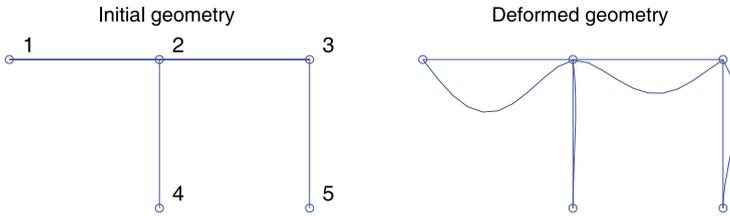


Fig. 7.45: Frame geometry plots: a) Initial, b) Deformed.

and content of the data file corresponding to the frame in Figs. 7.44–7.46 are described in the following.

**Node data.** The frame structure is described in an  $xy$ -coordinate system with the horizontal  $x$ -axis through the column support points  $D$  and  $E$  and the vertical  $y$ -axis vertical through the left support at  $A$ . For the purpose of illustration the horizontal distances are given in terms of  $a$ , and the vertical distances in terms of  $b$ . The node coordinates are given in the array  $X$ , with each node corresponding to one row. The first part of the data file then is

```
% Width 'a' and height 'b' of truss
a = 4.0; b = 4.0;

% Coordinates of nodes X = [x y],
X = [ 0.0  b
      a    b
      2*a  b
      a    0.0
      2*a  0.0 ];
```

The node coordinates  $[x, y]$  are given in the order of the node number, starting with node 1. Thus, the node number is not given explicitly, but implied by the row number in the node coordinate matrix  $X$ .

**Element data.** The beam elements are defined in the topology matrix  $T$ . Each row of this matrix defines an element, by listing its two nodes by their node number, and by giving a third number identifying a set of element properties, given as a row of parameter values in the element properties matrix  $H$ .

```
% Topology matrix T = [node1 node2 propno],
T = [ 1  2  1
      2  3  1
      2  4  2
      3  5  2 ];
```

```
% Element property matrix H = [ E A I G As ],
H = [ 2.1e11  1.00e-3  1.00e-6  8.1e10  0.80e-3
      2.1e11  1.00e-3  1.00e-6  -1.00  -1.00  ];
```

The element properties consist of the elastic modulus  $E$  and the section area  $A$  and moment of inertia  $I$ . If the element includes shear flexibility, the shear modulus  $G$  and the shear area  $A_s$  must also be included in the element data. In classic analysis the shear flexibility effect is often neglected. This corresponds to the condition of infinitely large shear stiffness  $G \cdot A_s$ . In the data file this condition is identified by a non-positive shear parameter, i.e. by  $G \leq 0$  or  $A_s \leq 0$  as illustrated for material No. 2 used for the columns in the sample file. The shear flexibility parameter  $\Phi$  is calculated in the function `kebeam`, when forming the beam element stiffness matrix.

**Loads.** The loads can be given as concentrated loads at the nodes or as distributed transverse loads with uniform distribution within the length of a beam element. The concentrated loads are specified in the load matrix  $P$ . This matrix contains a row for each loaded node. The data row specifies the node number and the generalized force components. In the present example there are no concentrated loads. Thus the following line of code describing a concentrated downward vertical force at node No. 2 is just an illustration.

```
% Prescribed loads P = [ node Px Py M ]
P = [ 2   0.000 -1.00e4 0.000 ];
```

The elements can support a uniform transverse load specified by the load intensity  $p$ . The transverse downward load on the frame in Fig. 7.44 is generated by the input array

```
% Prescribed loads p = [ elno p ]
p = [ 1   2.0e4
      2   2.0e4 ];
```

The load data arrays are processed by the function `febeam` and the specific load components are entered into the global load vector  $f$ . The program `MINIFRAME` accepts imposed nodal displacements as input, and thus it may happen that there are no loads in the form of generalized forces. The code therefore checks for the existence of data arrays  $P$  and  $p$ .

**Support conditions.** The support conditions are given in the constraint matrix  $C$ . The constraint matrix contains a row for each constrained generalized displacement component. In the present example there are 2 constrained displacement components at node 1, and 3 generalized displacement constraints at the nodes 4 and 5.

```
% Constraints C = [ node 'dof' (uc) ]
C = [ 1 1
      1 2
      4 1
      4 2
      4 3
      5 1
      5 2
      5 3 ];
```

The last column is optional. In the case of an imposed displacement it contains the magnitude of the constrained displacement *uc*.

**Graphics.** The MINIFRAME program produces two plots of the structure: a plot of the initial geometry without deformation including node numbers, and another plot of the deformed structure after application of the load, Fig. 7.45. The displacements are scaled to give a visual impression of the deformation, that would typically not be directly visible. The coordinate window used for the plots is controlled via definition of the plot axes, specified in the array

```
% Axes used for geometry plots [Xmin Xmax Ymin Ymax]
PlotAxes = [-0.40*a 2.50*a -0.25*a 1.40*a];
```

The deformed geometry is plotted using the computed nodal displacements, accounting for the difference between cross-section and center line rotation at the element ends by using the shear force and the equivalent shear strain.

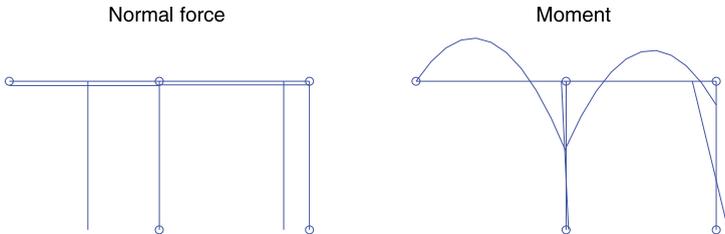


Fig. 7.46: Internal force distributions: a) normal force, b) internal moment.

The MINIFRAME program also provides plots of the internal force and moment distributions within the elements as shown in Fig. 7.46. The internal forces at the ends of the element are calculated from the element stiffness matrix, and the moment is corrected with a parabolic variation in the case of a transverse load on the element.

**Analysis process.** The analysis procedure using the MINIFRAME program is quite similar to that of MINITRUSS. The first step is to read the appropriate data file into memory, either by writing the data file name `DoubleFrame`

in the command window, or by uploading `DoubleFrame.m` to the MATLAB editor and pressing the F5 key from the editor. The data is now available in active memory and the analysis is carried out by activating the script file `MiniFrame.m` from the command window or by pressing the F5 key with `MiniFrame.m` in the editor. A sample of the file `MiniFrame.m` is shown below.

```

% Nodal loads into load vector
if exist('P','var')
    f = loadnode(f,P,dof);
end

% Element loads into load vector
if exist('p','var')
    f = loadelem(f,p,T,X,dof);
end

% Global stiffness matrix
K = kbeam(T,X,H,dof);

% Solve stiffness equation
[u,r,ic] = solveeq(K,f,C,dof);

% Nodal displacements
Un = reshape(u,dof,size(X,1))';

% Element section forces
Se = sbeam(T,X,H,Un,11);
if exist('p','var')
    Se = sebeam(Se,p,T,X);
end

% Element displacements
Ue = uebeam(T,X,H,Un,Se,11);
if exist('p','var')
    Ue = uebeam(Ue,p,T,X,H);
end

```

The program activates the following processes: i) builds up the full load vector  $f$  from nodal loads  $P$  and transverse element loads  $p$ , ii) generates the structure stiffness matrix  $K$ , iii) solves the constrained equation system including the support conditions, iv) presents the generalized node displacements, in the column format  $u$  and in matrix format  $Un$ , and finally v) computes the internal forces  $Se$  and the displacements  $Ue$  in the beam elements.

### 7.4.2 Stability analysis of frames

A compressive normal force in a beam reduces its stiffness as discussed in Chapter 5 in connection with column stability. The same effect may be important in a frame structure, where the stiffness reduction in symmetric and in anti-symmetric bending of an individual member are expressed in terms of

the coefficients  $\varphi$  and  $\psi$ , illustrated in Fig. 7.40. A direct way to include this stiffness reduction is to use the beam-column element stiffness matrix (7.49), whereby the global equilibrium equations take the form

$$\mathbf{K}(N_e) \mathbf{u} = \mathbf{f}, \quad (7.65)$$

where the notation  $\mathbf{K}(N_e)$  indicates that the stiffness matrix depends on the normal forces in the individual elements. In a statically indeterminate structure the internal forces, including the normal force, depend on the stiffness of the elements and thereby the equilibrium equation (7.65) becomes non-linear.

In practice, the problem is often solved without use of non-linear analysis by introducing the linearized form (7.53), in which the stiffness matrix is represented as the sum of the constitutive stiffness matrix  $\mathbf{K}^c$  and the geometric stiffness matrix  $\mathbf{K}^g$ . This gives the global equilibrium equations as

$$[\mathbf{K}^c + \mathbf{K}^g] \mathbf{u} = \mathbf{f}. \quad (7.66)$$

This set of equations is then solved approximately by assuming that the redistribution of the normal forces within the structure is only moderate. The load is then represented in the form  $\alpha \mathbf{f}_0$ , where the scalar variable  $\alpha$  acts as a load factor. The analysis then proceeds in three steps. First the displacements  $\mathbf{u}_0$  are calculated without accounting for geometric stiffness effects by solving the equilibrium equations when including only the constitutive stiffness,

$$\mathbf{K}^c \mathbf{u}_0 = \mathbf{f}_0. \quad (7.67)$$

Then the normal forces in the elements are calculated from the constitutive element stiffness matrices for the displacement field  $\mathbf{u}_0$ . Finally, the critical value of the load factor  $\alpha$ , at which the structure loses its stiffness and buckles is obtained from the eigenvalue problem

$$[\mathbf{K}^c + \alpha \mathbf{K}^g] \mathbf{u} = \mathbf{0}. \quad (7.68)$$

The smallest eigenvalue  $\alpha$  determines the lowest ideal stability load  $\alpha \mathbf{f}_0$  and the corresponding buckling mode  $\mathbf{u}$ , when neglecting deformation before buckling. This is equivalent to the elastic stability load introduced in the design procedure for columns in Section 5.3.

The linearized stability problem is implemented as an extension of the MINIFRAME program called MINIFRAMES. This program first solves the linear initial problem (7.67) with a reference load  $\mathbf{f}_0$  using the procedure described in connection with the MINIFRAME program in the previous section. This solution determines the normal force in each of the elements, and these normal forces are used to form the global geometric stiffness matrix  $\mathbf{K}^g$ . The eigenvalue problem (7.68) is then solved for the load factor  $\alpha$ , which

appears as output together with a plot of the corresponding buckling mode. The linearized stability analysis is illustrated in the following two examples.

**Example 7.4. Number of elements in beam-columns.** The linear eigenvalue problem 7.68 is based on a linearized form of the stiffness matrix. This introduces an approximation regarding the shape of the buckling modes, and it is important to determine the appropriate number of elements in a finite element model for a buckling problem. This is illustrated with reference to a column with a fixed support at one end and a simple support at the other. Shear flexibility is neglected in the present example. The solution of this problem was treated in Example 5.2, where the buckling load was found by an iterative procedure as  $P_c = 2.0457 P_E$ , with  $P_E$  as the Euler buckling load for the similar simply supported column.



Fig. 7.47: Ideal column with fixed/simple support: a) initial geometry, b) buckling mode.

The Finite Element model is illustrated in Figure 7.47 with  $n_{el} = 4$  elements, showing graphs from MINIFRAMES. The left end is fixed, while the right end has a simple support permitting motion in the axial direction as well as rotation. It is seen that the shape of the buckling mode has an inflexion point and this places extra demands on the shape functions in the element model.

$n_{el}$	1	2	3	4
$P/P_c$	1.486	1.026	1.006	1.002
$\ell_e/a_{el}$	0.574	1.381	2.091	2.794

The results are summarized in the table for  $n_{el} = 1, 2, 3, 4$  elements. The second row gives the buckling load  $P$  obtained by the Finite Element model, normalized with the analytically determined critical load  $P_c$ . The third row gives the ratio of the effective column length  $\ell_e$  to the element length  $a_{el}$ . This number indicates the number of elements per effective column length and thus has relevance for general frame structures, where the effective column length of a member is simply the distance between inflexion points of the buckling mode. The results indicate that use of a single element is clearly insufficient, providing less than one element per effective column length and leading to an overestimation of the critical load by about 50 pct. The results indicate that an accurate determination of the buckling load requires 3 elements in the present case, corresponding to two or more elements per effective column length. For a column with two fixed supports this corresponds to a minimum of four elements. Thus, the simplicity of the linearized analysis is attained at the cost of a more detailed element model. □

**Example 7.5. Buckling of angle-frame.** The finite element model of an angle frame with fixed simple supports is shown in Fig. 7.48a. The horizontal and vertical dimension is  $a$ . The load consists of a uniformly distributed downward load of intensity  $p$  on the horizontal part of the frame, i.e. a total load of  $pa$ . The load is characterized by the non-dimensional load factor  $\alpha = \frac{1}{2}pa/P_E$ , where  $P_E$  is the Euler load of a simply supported column with the same properties as the vertical part of the fame. Shear flexibility is neglected, and the effect of axial strain is negligible. The results therefore only depend on the load factor  $\alpha$ .

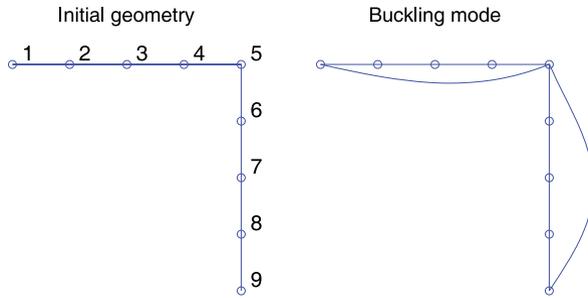


Fig. 7.48: Angle frame with simple supports and uniformly distributed load.

The effect of the number of elements is illustrated via the buckling mode in Fig. 7.48b and the buckling load shown in tabular form for  $n_{el} = 1, 2, 3, 4$  together with the buckling load at convergence, obtained for  $n_{el} \geq 15$ . In principle only the vertical member needs to be subdivided, as the horizontal member is nearly without normal load and the stiffness is therefore correctly represented by the constitutive stiffness matrix.

$n_{el}$	1	2	3	4	$\geq 15$
$P/P_E$	1.617	1.238	1.229	1.227	1.226

The first part of the analysis consists in determination of displacements and internal forces, when neglecting the flexibility effect of the normal force. The moment distribution is shown in Fig. 7.49a. It consists of a linear variation vanishing at the supports and with corner value  $-M_0$ , supplemented by a parabolic variation along the horizontal member. When neglecting the effect of axial strain the corner moment is easily determined by the deformation method as  $M_0 = \frac{1}{16}pa^2$ .

When increasing the load the vertical member becomes increasingly flexible, whereby the magnitude of the corner moment decreases. The rotation of the corner node is constrained by the horizontal member, and thus the vertical member can support more than the Euler load  $P_E$ . When passing  $P_E$  the rotation stiffness becomes negative, and the corner moment changes sign. Thus, the corner moment becomes positive before the buckling load  $p_c$  is reached. The development of the corner moment with the normalized load  $p/p_c$  can be

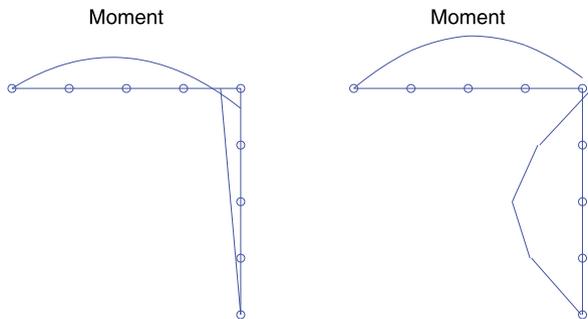


Fig. 7.49: Normalized moment distribution: a)  $p/p_c \approx 0$ , b)  $p/p_c = 0.8$ .

calculated by using the full stiffness matrix  $\mathbf{K}^c + \alpha\mathbf{K}^g$  and is shown in the following table.

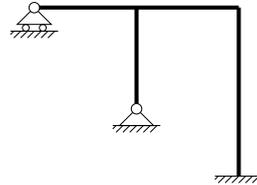
$p/p_c$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$M/M_0$	0.99	0.95	0.90	0.75	0.62	0.43	0.12	-0.52	-2.43

The gradual transition of the corner moment from a negative value, constraining the horizontal beam, to a positive value, constraining the vertical column, is an example of redistribution of the internal forces due to change in stiffness of the various parts of the structure with increasing normal load. This is a non-linear effect, and a better representation can be obtained by recalculating the normal force from the full stiffness matrix in an iterative procedure. □

### 7.5 Exercises

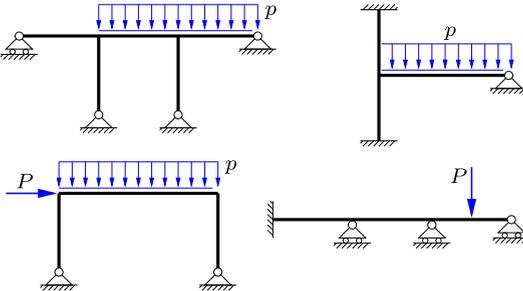
**Exercise 7.1.** The figure shows a statically indeterminate frame structure.

- Identify by how many components the structure is statically indeterminate, and show a set of redundant forces for use with the force method.
- Show the kinematic degrees of freedom to be used with the deformation method.
- Is it advantageous to use the deformation method for this frame?
- Introduce an additional horizontal constraint at the left support, and repeat a)–c).



**Exercise 7.2.** The figure shows four statically indeterminate beam and frame structures, and the following questions are answered for each of the structures.

- Identify by how many components the structure is statically indeterminate, and show a possible set of generalized forces for use with the force method.
- Show the kinematic degrees of freedom to be used with the deformation method.
- Is it advantageous to use the deformation method for this frame.

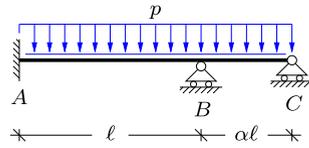


**Exercise 7.3.** Consider the load case Fig. 7.5, in which the cross-section  $A$  of a beam  $AB$  with a simple support at  $B$  is rotated  $\theta_A = 1$ . Determine the rotation  $\theta_B$  at the simple support, including the effect of shear flexibility.

**Exercise 7.4.** The figure shows a continuous beam over two spans with a fixed support in  $A$  and simple supports with horizontal rollers in  $B$  and  $C$ . The length of  $AB$  is  $\ell$ , while the length of  $BC$  is  $\alpha\ell$ . The beam is loaded by a uniformly distributed load with vertical

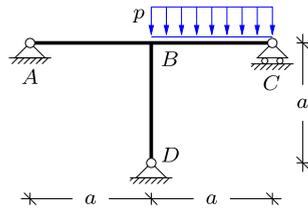
intensity  $p$ . The bending stiffness for all beams is  $EI$  and the influence of shear flexibility is neglected ( $\Phi = 0$ ).

- a) Use the deformation method to determine the rotation at  $B$ .
- b) Determine the reactions and the moment distribution in the frame for the special case  $\alpha = 1$ .
- c) The rotation at  $B$  can be clockwise or counter-clockwise, depending on the magnitude of  $\alpha$ . Determine the sign conditions.



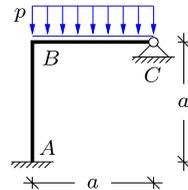
**Exercise 7.5.** The figure shows a T-frame. The three beams of length  $a$  and bending stiffness  $EI$  are joined rigidly at  $B$  and have simple supports at  $A$ ,  $C$  and  $D$ , respectively. The shear flexibility effect is neglected ( $\Phi = 0$ ).

- a) Use the deformation method to determine the rotation at  $B$ .
- b) Determine all reactions on the frame.
- c) Determine the distribution of moment, shear force and normal force in the frame.
- d) Replace the simple supports in  $A$  and  $D$  by fixed supports and repeat the questions in a)–c) for the modified structure.



**Exercise 7.6.** The figure shows an angle frame similar to that in Example 7.1, but now loaded by a distributed load  $p$  on the horizontal beam  $BC$ . Both beams have length  $a$  and bending stiffness  $EI$ .

- a) Use the deformation method to determine the rotation at  $B$ , when assuming  $\Phi = 0$ .
- b) Determine the reactions and the moment distribution in the frame.
- c) Include the effect of shear flexibility,  $\Phi > 0$ , and repeat the analysis in a) and b).
- d) Replace the simple support in  $C$  by a fixed support and repeat the analysis in a) and b).

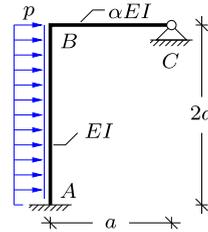


**Exercise 7.7.** Change the load in the angle frame in Exercise 7.6 to a concentrated vertical downward force  $P$  acting at the center of  $BC$ , and repeat the analysis in a) and b) without the influence of shear flexibility.

**Exercise 7.8.** The figure shows an angle frame with a fixed support in  $A$  and a simple support in  $C$ . Length and bending stiffness are  $2a$  and  $EI$  for the vertical beam  $AB$  and  $a$  and  $\alpha EI$  for the horizontal beam  $BC$ . The frame is loaded by a distributed load with

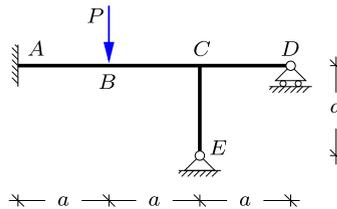
horizontal intensity  $p$  on the vertical beam  $AB$ . The influence of shear flexibility is neglected in the analysis ( $\Phi = 0$ ).

- Use the deformation method to determine the rotation at  $B$  for  $\alpha = \frac{1}{2}$ .
- Determine the reactions and the moment distribution in the frame.
- Determine the distribution of the shear force and the normal force in the frame.
- Determine the rotation at  $B$  for  $\alpha \rightarrow 0$ , and comment on the associated reduction in rotational stiffness.



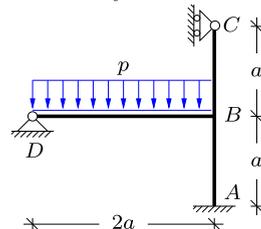
**Exercise 7.9.** The figure shows a T-frame with a fixed support in  $A$  and simple supports in  $D$  and  $E$ . The elements  $AB$ ,  $BC$ ,  $CD$  and  $CE$  all have length  $a$  and bending stiffness  $EI$ . The frame is loaded by a vertical force  $P$  at  $B$ . The influence of shear flexibility is neglected ( $\Phi = 0$ ).

- Use the deformation method to determine the rotation of the joint  $C$ .
- Determine all reactions on the frame.
- Determine the moment distribution.
- Determine the distribution of the shear and normal force.



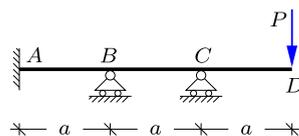
**Exercise 7.10.** The figure shows a frame consisting of a column  $ABC$  supporting a horizontal beam  $DB$  loaded by a uniformly distributed load with intensity  $p$ . The frame has a fixed support in  $A$  and fixed simple support in  $D$  and a simple support with vertical rollers in  $C$ . The beam elements have the lengths  $|AB| = |BC| = a$  and  $|DB| = 2a$ , and bending stiffness  $EI$ . The influence of shear flexibility is neglected in the analysis.

- Use the deformation method to determine the rotation of the joint  $B$ .
- Determine all reactions on the frame.
- Determine the moment distribution in the frame.
- Determine the distribution of shear force and normal force in the frame.



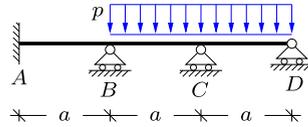
**Exercise 7.11.** The figure shows a beam structure with a fixed support in  $A$ , intermediate transverse supports in  $B$  and  $C$ , and a free end in  $D$  with a transverse tip load  $P$ . The length of each span is  $a$  and the bending stiffness is  $EI$  for the entire beam. The effect of shear flexibility is neglected ( $\Phi = 0$ ).

- Determine the moment  $M_C$  at the intermediate support in  $C$ .
- Use the deformation method to determine the rotation at the intermediate support in  $B$ .
- Determine the reactions.
- Determine the moment and shear force distribution in the beam.



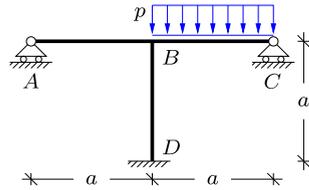
**Exercise 7.12.** The figure shows a beam structure with a fixed support in  $A$  and simple supports with horizontal rollers in  $B$ ,  $C$  and  $D$ . The beam is loaded by a uniformly distributed load  $p$  on the two outer spans  $BCD$ . The length of each span is  $a$  and the bending stiffness is  $EI$  for the entire beam. The effect of shear flexibility is neglected ( $\Phi = 0$ ).

- Use the deformation method to determine the rotations at the intermediate supports in  $B$  and  $C$ .
- Determine the reactions and the moment distribution in the beam.



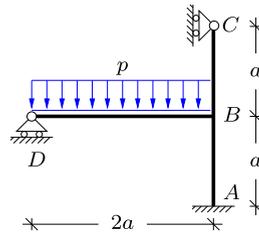
**Exercise 7.13.** The figure shows a T-frame similar to that in Exercise 7.5, where the left simple support now permits horizontal motion.

- Use the deformation method to determine the rotation and horizontal displacement at  $B$ .
- Determine all reactions on the frame.
- Determine the moment distribution in the frame.



**Exercise 7.14.** The figure shows a frame similar to that in Exercise 7.10, where the rollers at the support in  $D$  now permits horizontal motion of beam  $DB$ .

- Use the deformation method to determine the rotation and the horizontal displacement at  $B$ .
- Determine all reactions on the frame.
- Determine the moment distribution in the frame.



**Exercise 7.15.** Consider the angle frame in Example 7.1 with  $E = 210 \cdot 10^9$ ,  $A = 10^{-3}$ ,  $I = 10^{-6}$ ,  $a = 10$  and  $M_0 = 10^3$ .

- Create a data file `AngleFrameM0.m` to be used in `MiniFrame`.
- Determine the rotation at the corner of the frame and compare with  $\theta_0$  in Example 7.1.
- Determine the reactions and the internal moments at the corner, and compare with the results obtained in Example 7.1.
- Plot the distribution of the internal forces  $M$ ,  $Q$  and  $N$  and compare with the diagrams in Fig. 7.11.

**Exercise 7.16.** Consider the angle frame with distributed load in Exercise 7.6. Use the following parameters:  $E = 210 \cdot 10^9$ ,  $A = 10^{-3}$ ,  $I = 10^{-6}$ ,  $a = 10$  and  $p = 100$ .

- Create a data file `AngleFramep.m` to be used in `MiniFrame`.
- Find the magnitude and location of the maximum transverse displacement and the maximum moment.
- Take the influence of shear flexibility into account with  $G = E/2.6$  and  $A_s = 0.8A$ . Compare with the results in b).

**Exercise 7.17.** Consider the frame in Exercise 7.10 and use the following parameters:  $E = 210 \cdot 10^9$ ,  $A = 10^{-3}$ ,  $I = 10^{-6}$ ,  $a = 10$  and  $p = 100$ .

- a) Create a data file `FrameStorey.m` to be used in `MiniFrame`.
- b) Find the reactions on the frame.
- c) Find the magnitude and location of the maximum transverse displacement and the maximum moment.
- d) Permit the simple support to move horizontally and compare with the results in Exercise 7.14 and in c).

**Exercise 7.18.** Consider the T-frame in Exercise 7.5 and use the following parameters:  $E = 210 \cdot 10^9$ ,  $A = 10^{-3}$ ,  $I = 10^{-6}$ ,  $a = 10$  and  $p = 100$ .

- a) Create a data file `TFrame.m` to be used in `MiniFrame`.
- b) Find the reactions on the frame.
- c) Plot the distribution of the section forces and find the maximum moment.

**Exercise 7.19.** Consider the T-frame in Exercise 7.13 and use the following parameters:  $E = 210 \cdot 10^9$ ,  $A = 10^{-3}$ ,  $I = 10^{-6}$ ,  $a = 10$  and  $p = 100$ .

- a) Create a data file `TFrameSway.m` to be used in `MiniFrame`.
- b) Find the reactions on the frame.
- c) Plot the distribution of the section forces and find the maximum moment.
- d) Take the influence of shear flexibility into account with  $G = E/2.6$  and  $A_s = 0.8A$ . Compare with the results in b).