Deformation of Beams and Frames

4

The internal forces in beams and frames lead to deformations, and methods for analysis of these deformations are the subject of the present chapter. Typically the satisfactory performance of a structure will impose some limitation on the displacements. However, the deformation of beams also plays another important role, namely in the distribution of the internal forces in statically indeterminate structures. A statically indeterminate structure permits several ways of distributing the internal forces to carry a given load, and the actual distribution depends on the relative stiffness of the individual parts of the structure. These two roles of deformations and displacements are illustrated in Fig. 4.1 showing a horizontal beam supporting a distributed load. As discussed in the previous chapter the load creates a moment in the beam, and this moment in turn leads to curvature of the beam, resulting in transverse displacement. Satisfactory performance of the beam may impose a limit on the transverse displacement w. It is seen from the figure that the transverse displacements lead to rotation of the ends of the beam. If the beam were part of a larger structure, the connection to this structure via the ends of the beam would have to account for the rotation of the beam end. The transverse displacements along the beam and the rotation of the beam ends are parts of the same problem studied in this chapter. As indicated in the figure the deformation depends on the load p, the stiffness represented by the elastic modulus E, the beam length ℓ, the cross-section represented by the section height h, and possibly other parameters as well.

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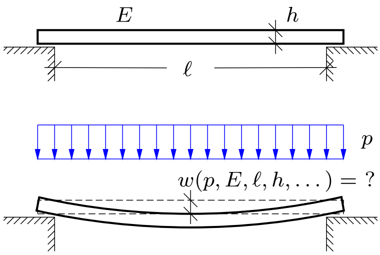


Fig. 4.1: Transverse deformation of simply supported beam.

The main mechanism in the deformation of beams is curvature generated by a bending moment M. The basic mechanism was identified and described quite clearly by Robert Hooke (1678). A bending moment will compress one side and extend the other as seen in the deformed beam in Fig. 4.1. For elastic beams the extension and compression are governed by the elastic relation between stress and strain from Section 2.4. A modern presentation of Hooke's theory of beam bending is presented in Section 4.1. In most cases of structural analysis the displacements are quite small, and a linearized form of the theory based on the undeformed geometry of the structure can then be used. The linear form of the theory is developed in the form of differential equations in Section 4.2.

In addition to bending, beams may also deform due to extension and shear, generated by the normal force N and the shear force Q, respectively. The extension due to a normal force has already been treated in Section 2.4 in connection with truss structures. The effect of deformation due to a shear force Q is treated in Section 4.3, where it is demonstrated, that this effect will often be negligible for slender beams.

In principle, the displacements of elastic beam and frame structures can be determined by solving the corresponding differential equations. However, for frames this would involve special transition conditions at joints, and even for beams the integration of the differential equations may be quite laborious for non-trivial load distributions and non-homogeneous beams. Therefore the principle of virtual work plays an indispensable role in the analysis of displacements of structures. In Section 4.4 the principle of virtual work – encountered in a limited form in connection with bars and trusses in Section 2.4.3 – is extended to beams and frames.

4.1 Bending of elastic beams

The theory for bending of elastic beams is most easily developed by first considering the case of homogeneous bending of a beam, i.e. the bending of a beam by applying moments of equal magnitude but opposite orientation at the two ends of a homogeneous beam. This simple set-up serves to identify the mechanism of bending in a precise way. The theory is then linearized corresponding to 'small displacements' and extended to nonhomogeneous bending.

4.1.1 Homogeneous bending

Figure 4.2a illustrates a straight simply supported homogeneous beam with a cross-section that is symmetric with respect to the plane of the figure. The beam is loaded by moments of equal magnitude but opposite orientation at the beam ends. The two moments are in equilibrium, and there is therefore no reactions. The upper side of the beam is compressed and the lower extended, and the beam therefore deforms as shown in Fig. 4.2b. Somewhere between the upper and lower part of the beam there must be a plane where the beam retains its original length. This is the so-called neutral plane. The axis of the beam is taken as the intersection of the neutral plane and the plane of symmetry, i.e. the plane of the figure, and s denotes the arc-length along this axis. When introducing a section at an arbitrary point s, equilibrium implies that the internal moment is equal to the imposed moment M, and thus

$$M(s) \equiv M \tag{4.1}$$

for any value of s. All cross-sections have the same internal moment and thereby the same state of deformation. Therefore the beam axis is bent into a circle. The center C and the radius R of the circle are indicated in the figure.

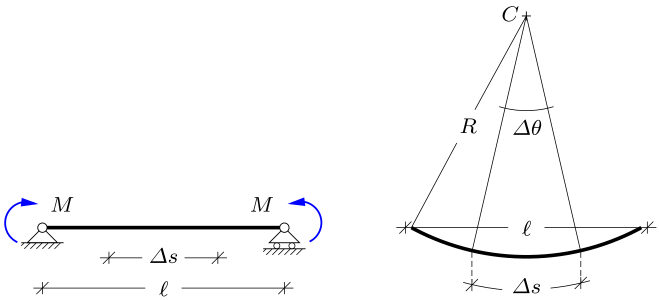


Fig. 4.2: Homogeneous bending of beam.

In connection with bending of beams it is convenient to describe the deformation by the curvature $\kappa = 1/R$, and in order to obtain a quantitative theory the relation between the bending moment M and the curvature κ must be established. Figure 4.3a shows a slice of the beam with initial length Δs. The neutral axis retains its original length, and in the deformed state

$$\kappa = \frac{1}{R} = \frac{\Delta\theta}{\Delta s}, \tag{4.2}$$

where $\Delta\theta$ is the angle between the two cross-sections. Now, consider a 'fiber' located at the distance z below the neutral axis. The initial length of this fiber is Δs. After deformation it is bent into a circle with radius $R + z$ and corresponds to the center-angle $\Delta\theta$. Thus the length of this fiber after deformation is

$$\Delta s_* = (R + z)\,\Delta\theta = (R + z)\,\kappa\,\Delta s, \tag{4.3}$$

where the angle $\Delta\theta$ was substituted from (4.2). The elongation corresponds to the normal strain

$$\varepsilon = \frac{\Delta s_* - \Delta s}{\Delta s} = \kappa\, z. \tag{4.4}$$

Thus, the normal strain is proportional to the curvature κ and to the distance z from the neutral axis.

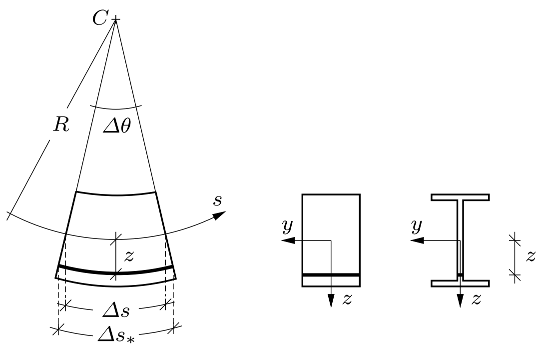


Fig. 4.3: Extension of fiber in beam bending.

If the material is linear elastic with modulus of elasticity E the normal stress follows from the strain relation (4.4)

$$\sigma = E\,\varepsilon = E\,\kappa\, z. \tag{4.5}$$

The stress σ and the modulus of elasticity are expressions of force per unit area, typically expressed in terms of Pascal, [Pa] = [N/m^2]. This expression for the normal stress leads to a condition that determines the location of the neutral axis and an expression for the bending moment in terms of the curvature κ.

In the present problem the loading is assumed to be pure bending. Thus, there is no normal force in the beam, and it follows from the stress relation (4.5) that this corresponds to

$$N = \int_A \sigma \, dA = \kappa \int_A E \, z \, dA = 0 . \tag{4.6}$$

If this condition is not satisfied the origin of the z-axis is not on the neutral axis. If the section is symmetric about the plane of bending but does not have up-down symmetry, the location of the neutral axis is easily determined by introducing a preliminary transverse coordinate z'. The coordinate z, centered at the neutral axis is then determined from the condition (4.6), when substituting $z' = z + z_0$ instead of z. A general analysis of non-symmetric and inhomogeneous cross-sections is presented in Chapter 10.

The bending moment generated by the normal stress σ from (4.5) is determined by adding the contributions from all infinitesimal areas dA. The force on the infinitesimal area is σdA, and the corresponding moment then is $z(\sigma dA)$, when z is the distance to the neutral axis. This gives the bending moment as

$$M = \int_A z \, \sigma \, dA = \kappa \int_A z^2 \, E \, dA . \tag{4.7}$$

It is seen from this relation that the contribution from the area element dA is weighted by the elastic stiffness modulus E. It is not a severe complication to include variable elasticity modulus in the cross section integral, but in order to illustrate the basic theory with minimal complication here, the modulus of elasticity is here assumed constant over the cross-section, and the extension to non-homogeneous material properties is postponed to the discussion of general cross-section analysis in Chapter 10. Thus, for a constant modulus of elasticity the elastic bending relation takes the form

$$M = E \, I_z \, \kappa . \tag{4.8}$$

In this relation EI_z is the bending stiffness of the beam. It consists of the product of the modulus of elasticity E and the geometric quantity

$$I_z = \int_A z^2 \, dA , \tag{4.9}$$

constituting the moment of inertia about the neutral axis. The moment of inertia of the cross-section depends on the shape as well as the size of the cross-section as illustrated in the following example.

Example 4.1. Moment of inertia for rectangle and I-profile. Figure 4.4a shows the distribution of the coordinate z over the cross-section of a rectangular beam of height h and width b. The neutral axis is along the horizontal axis of symmetry, and the bending moment of inertia of the rectangular cross-section then is

$$I_z = \int_A z^2 \, dA = b \int_{-h/2}^{h/2} z^2 \, dz = \tfrac{1}{12} h^3 b = \tfrac{1}{12} h^2 A .$$

The rectangular beam has the largest bending stiffness, when oriented such that $h \geq b$, as normally seen in structures. The result for the rectangular cross section can be used to determine the moment of inertia for various types of cross sections composed of rectangular parts, as demonstrated in Exercise 4.1 for a thin-walled box section.

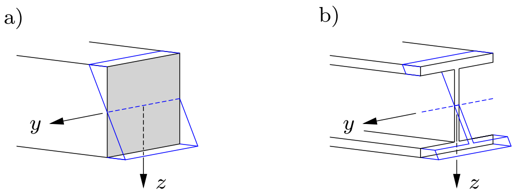


Fig. 4.4: Plane bending deformation: a) Rectangular beam, b) I-section beam.

Figure 4.4b shows an I-beam, consisting of two flanges connected by a web. The web is usually thin and contributes only little to the area and bending moment of inertia. It is therefore neglected in the derivations below. The neutral axis is the axis of symmetry, and if the area is assumed to be mainly located in the two flanges of thickness t, the bending moment of inertia is

$$I_z = \int_A z^2\, dA \simeq 2b \int_{(h-t)/2}^{(h+t)/2} z^2\, dz = \frac{b}{12}\Big((h+t)^3 - (h-t)^3\Big) \simeq \tfrac{1}{2}h^2 tb = \tfrac{1}{4}h^2 A\,,$$

where higher order terms in the thickness, i.e. terms containing the powers t^2 or t^3, are omitted because $t \ll h, b$ for thin-walled cross sections. It is seen that in the I-beam a given area provides three times greater bending stiffness than in a rectangular section of the same height. The stress distribution is also more favorable in the I-beam, which is common in structures. □

4.1.2 Linear kinematic relations

For a general plane curve the curvature is defined as the rate of change of the angle of the tangent with respect to a fixed direction, as illustrated in Fig. 4.5. Thus, the curvature is defined by the limit

$$\kappa(s) = \frac{1}{R(s)} = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}\,. \tag{4.10}$$

In linear beam theory the displacements are considered ‘small’ with respect to the dimensions of the structure. This implies that the change of angle is small, and a linearized representation of the curvature can then be used.

Figure 4.6 illustrates the deformed state of an initially straight beam. The x-axis defines the original beam axis, and the transverse displacement $w(x)$ is positive in the direction of the z-axis. The angle θ is defined via the relation

$$\sin\theta = -\frac{dw}{ds}\,. \tag{4.11}$$

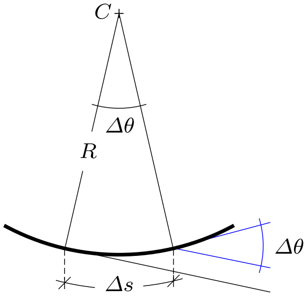


Fig. 4.5: Curvature as change in rotation.

Differentiation of this equation with respect to s gives the following expression for the curvature

$$\kappa = \frac{d\theta}{ds} = -\frac{1}{\cos\theta}\frac{d^2w}{ds^2}. \tag{4.12}$$

When the slope of the deformed beam is limited $dx/ds = \cos\theta \simeq 1$, and the original x-coordinate can be used instead of the arc-length s. This leads to the linearized expressions

$$\theta \simeq -\frac{dw(x)}{dx}, \qquad \kappa \simeq \frac{d\theta(x)}{dx} \simeq -\frac{d^2w(x)}{dx^2}. \tag{4.13}$$

These expressions are used with equality sign in technical 'small displacement' beam theory.

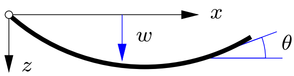


Fig. 4.6: Sign convention of beam deformation.

The kinematic relations (4.13) give approximate definitions of the angle θ to the tangent of the beam axis and the corresponding curvature κ. However, the axial strain from beam bending is defined in terms of the relative rotation of two neighboring cross-sections as illustrated for homogeneous bending in Fig. 4.3. In the case of homogeneous bending the cross-sections remain orthogonal to the beam axis in the deformed state. Thus, the relative rotation of

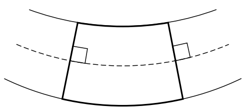


Fig. 4.7: Cross-section perpendicular to deformed beam axis.

neighboring cross-sections is determined by the rotation of the tangent to the beam axis between two sections, as shown in Fig. 4.7. In non-homogeneous bending the change of the moment along the beam generates a shear force, which in turn leads to an inclination of the cross-sections. However, the effect of the deviation of the cross-sections from the orthogonal position is often small, and therefore normal cross-sections are often assumed in the development of technical beam theories. This approach is taken in the following section on the so-called Bernoulli beam theory, while the effect of shear flexibility is addressed in Section 4.3.

Example 4.2. Linearization error for homogeneous bending. According to the exact beam theory a beam with constant internal moment M bends into a circular shape with radius

$$R = \frac{1}{\kappa} = \frac{EI}{M},$$

as illustrated in Fig. 4.2. The angle θ between the cross-sections at the supports follows from trigonometry as

$$\sin(\tfrac{1}{2}\theta) = \frac{\ell}{2R} = \tfrac{1}{2}\kappa\ell .$$

The angle θ may be expanded in a Taylor series as

$$\tfrac{1}{2}\theta = \arcsin(\tfrac{1}{2}\kappa\ell) = (\tfrac{1}{2}\kappa\ell) + \tfrac{1}{6}(\tfrac{1}{2}\kappa\ell)^3 + \cdots ,$$

showing that for small curvature $\theta \simeq \kappa\ell$.

The displacement w in the middle follows from the circular deformed shape as

$$w = R\big[1 - \cos(\tfrac{1}{2}\theta)\big] = \tfrac{1}{2}R(\tfrac{1}{2}\theta)^2\big[1 - \tfrac{1}{12}(\tfrac{1}{2}\theta)^2 + \cdots\big],$$

where the Taylor expansion of the cosine function has been introduced. Substitution of the expansion for $\frac{1}{2}\theta$ in terms of $(\frac{1}{2}\kappa\ell)$ gives

$$w = \tfrac{1}{8}\kappa\ell^2\big[1 + \tfrac{1}{4}(\tfrac{1}{2}\kappa\ell)^2 + \cdots\big].$$

The first term is the result from a linearized analysis,

$$w_{\rm lin} = \tfrac{1}{8}\kappa\ell^2 = \frac{\ell^2 M}{8EI}.$$

This is the result that would be obtained from the approximate theory based on the linearized rotation and curvature relations (4.13).

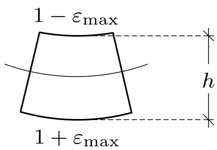


Fig. 4.8: Bending of symmetric unit beam element.

The relative error is represented by the second term in the square brackets, $\frac{1}{4}(\frac{1}{2}\kappa\ell)^2$. This term can be related directly to the local deformation of the beam. The top of the beam is in compression and the bottom in tension as illustrated in Fig. 4.8. The curvature can be expressed in terms of the beam height and the maximum strain by use of the relation (4.4). The maximum strain $\varepsilon_{\max}$ occurs at the distance $\frac{1}{2}h$ from the beam axis, and thus $\kappa = 2\varepsilon_{\max}/h$. The leading relative error term can then be expressed as

$$\frac{w - w_{\text{lin}}}{w_{\text{lin}}} \simeq \left(\varepsilon_{\max}\frac{\ell}{2h}\right)^2$$

Thus, for a limited maximal strain in the material – e.g. $\varepsilon_{\max} \simeq 0.002$ – the error in the linearized theory will only be important for very slender beams. □

4.2 Bernoulli beam theory

A full beam theory must deal with the statics and kinematics of the beam. The statics deals with internal forces and their equilibrium with the imposed loads. Kinematics is the description of displacements and the related measures of deformation such as the curvature. The deformation generated by the internal forces depends on the mechanical behavior of the material of the beam, e.g. linear elasticity. Finally, the beam must be adequately supported to permit transfer of the loads to the supports. This section presents these four aspects in the context of the so-called Bernoulli beam theory, which is a technical theory of elastic beams in which shear deformations are neglected. The extension to shear-flexible beams is discussed in Section 4.3. In the linear theory of beams the normal force and the associated axial deformation is uncoupled from the transverse displacements generated by bending. The normal force and axial deformation has been dealt with in connection with bars, and the presentation here is therefore limited to the beam bending problem in terms of transverse displacements and bending moments and shear forces, as illustrated in Fig. 4.9.

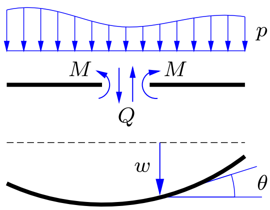


Fig. 4.9: Static and kinematic parameters in Bernoulli beam theory.

Statics

The statics of the Bernoulli beam theory consists of the equilibrium equations for the shear force $Q(x)$ and the internal bending moment $M(x)$. These equilibrium equations were treated in detail in Section 3.3.1 and are just summarized here. The shear force must secure vertical equilibrium, whereby

$$\frac{dQ(x)}{dx} = -p(x)\,. \tag{4.14}$$

Similarly the moment must secure equilibrium for rotation of a slice of the beam, leading to

$$\frac{dM(x)}{dx} = Q(x)\,. \tag{4.15}$$

Bernoulli beam theory is based on the simplifying assumption that the shear force $Q(x)$ does not contribute directly to the deformation of the beam. It may therefore sometimes be of interest to eliminate the shear force from the equilibrium equations to obtain a direct relation between the internal moment and the external load,

$$\frac{d^2M(x)}{dx^2} + p(x) = 0\,. \tag{4.16}$$

For statically determinate beams the boundary conditions permit solution of these differential equations for the statics without consideration of the kinematic relations. However, in general a full description of the beam also requires determination of the kinematic variables, i.e. displacement $w(x)$, rotation $\theta(x)$ and curvature $\kappa(x)$.

Kinematics

In the Bernoulli beam theory the kinematics, i.e. the displacement and deformation, of the beam is described completely by the transverse displacement $w(x)$. Within the approximation of the Bernoulli beam theory the rotation of the cross section is equal to the rotation of the tangent of the beam axis, given by the linearized expression

$$\theta(x) = -\frac{dw(x)}{dx}\,. \tag{4.17}$$

The curvature is the rate of change of the rotation angle $\theta(x)$, and within the linearized theory this is

$$\kappa(x) = \frac{d\theta(x)}{dx} = -\frac{d^2w(x)}{dx^2}\,. \tag{4.18}$$

The angle θ may appear in the support conditions, and the curvature $\kappa(x)$ is related to the internal moment $M(x)$.

Elasticity

In the Bernoulli beam theory the deformation from the shear force is neglected, leaving only the curvature generated by the bending moment. For linear material behavior, such as linear elasticity, this implies proportionality between the curvature $\kappa(x)$ and the bending moment $M(x)$. It was demonstrated by the case of homogeneous bending in Section 4.1 that this relation has the form

$$M(x) \;=\; EI_z\,\kappa(x) \;=\; -\,EI_z\,\frac{d^2w(x)}{dx^2}\,. \tag{4.19}$$

Relations between the static quantities and the deformation measures are often called constitutive relations. In this relation the bending stiffness EI_z consists of a material parameter, representing the elastic modulus E of the beam material, and a geometric parameter I_z, which for a homogeneous material distribution is the moment of inertia about the neutral axis. The more general case of non-homogeneous material distribution over the beam cross-section is treated in Chapter 10.

Support conditions

The beam must be supported by at least the number of constraints that prevent free motion. However, many structures have more supports than strictly needed to fix the structure in the plane or in space. Extra supports and extra connections between structural elements typically increase the stiffness of the structure, while rendering the structure statically indeterminate.

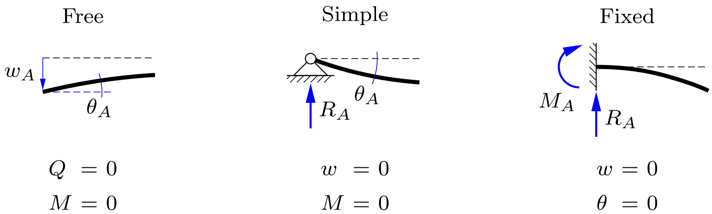


Fig. 4.10: Support conditions for Bernoulli beams.

Three typical support conditions are illustrated in Fig. 4.10 with reference to the left end A of a beam. Additional supports have been discussed previously in connection with Fig. 1.20. The first support condition in Fig. 4.10 is a free end, at which the shear force and the moment vanish, $Q = 0$ and $M = 0$. In the case of a statically determinate beam $Q(x)$ and $M(x)$ are the unknown functions, and this support condition is expressed directly in terms of these functions at the point A. For a statically indeterminate beam the problem is formulated in terms of the transverse displacement $w(x)$, and the boundary conditions must therefore be expressed in terms of the transverse displacement. The expression for the moment follows directly from the constitutive relation (4.19) as

$$M_A \;=\; -\,EI_z\,\frac{d^2w}{dx^2}\bigg|_A\,. \tag{4.20}$$

The expression for the shear force follows from differentiation of the moment equation according to (4.15),

$$Q_A = -\frac{d}{dx}\Big(EI_z\,\frac{d^2w}{dx^2}\Big)_A . \tag{4.21}$$

For a concentrated load at A the moment and/or the shear force are simply defined by the concentrated external load.

Figure 4.10 also illustrates the simple support, in which the transverse displacement vanishes, $w_A = 0$, together with the moment, $M_A = 0$. For problems formulated in terms of the transverse displacement $w(x)$, the moment is expressed by (4.20). Finally, the figure also illustrates the fixed end, representing constraint of the transverse displacement, $w_A = 0$, and constraint of the rotation, $\theta_A = 0$.

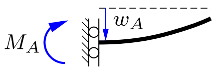


Fig. 4.11: Fixed support with vertical rollers.

The supports in Fig. 4.10 can be combined, as demonstrated by the fixed support with vertical rollers in Fig. 4.11. The vertical rollers represent the unconstrained transverse displacement of the free end in Fig. 4.10a and the constrained rotation of the fixed support in Fig. 4.10c.

4.2.1 Statically determinate beams

In a statically determinate beam the support conditions permit that the distribution of the internal forces $Q(x)$ and $M(x)$ can be determined without reference to the kinematics of the beam. Thus, a statically determinate beam is typically analyzed by first determining $Q(x)$ and $M(x)$ – either from the differential relations (4.14)–(4.16) or by the direct equilibrium based methods developed in Chapter 3. Once the statics part of the problem has been solved, the displacement $w(x)$ can be determined by integration of the moment relation (4.19). The principle may be described as solution of the second order static equation

$$\frac{d^2M(x)}{dx^2} + p(x) = 0\,, \tag{4.22}$$

followed by solution of the second order kinematic equation

$$\frac{d^2w(x)}{dx^2} + \frac{M(x)}{EI_z} = 0\,, \tag{4.23}$$

although the static solution is often obtained by a static procedure based on sections. In particular concentrated or discontinuous loads may introduce complications for a purely mathematical integration. The analysis of statically determinate beams is illustrated by the following examples.

Example 4.3. Simply supported beam with end moment. Figure 4.12 shows a simply supported beam of length ℓ with a moment M_B acting at the right support in B. There is no load acting along the beam, and the moment distribution is therefore linear as shown in Fig. 4.12. The expression for the moment is

$$M(x) \,=\, M_B\, \frac{x}{\ell}\,.$$

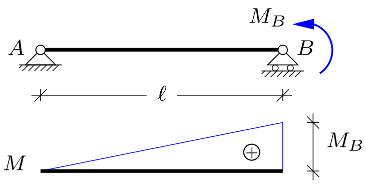


Fig. 4.12: End moment on simply supported beam.

The curvature is given by the constitutive relation (4.23) as

$$\frac{d^2w(x)}{dx^2} \,=\, -\,\frac{M(x)}{EI_z} \,=\, -\,\frac{M_B}{EI_z}\,\frac{x}{\ell}\,.$$

The displacement $w(x)$ then follows by double integration as

$$w \,=\, -\,\frac{1}{6}\,\frac{M_B\ell^2}{EI_z}\left(\frac{x}{\ell}\right)^3 \,+\, C_0 \,+\, C_1 x\,,$$

where C_0 and C_1 are the two arbitrary constants of integration. It is seen that a linear moment distribution implies a cubic solution for the displacement. The two constants are determined by the two kinematic boundary conditions for the simple supports in A and B,

$$w(0) \,=\, w(\ell) \,=\, 0\,.$$

The first condition $w(0) = 0$ directly yields that $C_0 = 0$. Hereafter, the second condition $w(\ell) = 0$ gives

$$-\,\frac{1}{6}\,\frac{M_B\ell^2}{EI_z} \,+\, C_1\ell \,=\, 0 \qquad \Rightarrow \qquad C_1 \,=\, \frac{1}{6}\,\frac{M_B\ell}{EI_z}\,.$$

Now that the two arbitrary constants have been determined the final expression for the displacement can be written as

$$w(x) \,=\, \frac{1}{6}\,\frac{M_B\ell^2}{EI_z}\,\frac{x}{\ell}\left[1 - \left(\frac{x}{\ell}\right)^2\right].$$

The first factor gives the magnitude of the displacement, while the second factor represents the distribution. The rotation is obtained from the derivative of the displacement

$$\theta(x) \,=\, -\,\frac{dw(x)}{dx} \,=\, -\,\frac{1}{6}\,\frac{M_B\ell}{EI_z}\left[1 \,-\, 3\left(\frac{x}{\ell}\right)^2\right].$$

This gives the rotation at the beam ends as

$$\theta_A = \theta(0) = -\frac{1}{6}\frac{M_B\ell}{EI_z} \qquad , \qquad \theta_B = \theta(\ell) = \frac{1}{3}\frac{M_B\ell}{EI_z}.$$

It is noted that the magnitude of the rotation θ_B at the end with the applied moment is the double of the rotation at the other end.

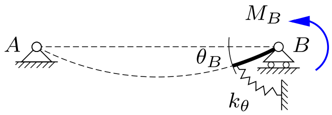


Fig. 4.13: Equivalent rotation spring with stiffness k_θ.

The expression for the rotation θ_B can be written as $M_B = k_\theta\theta_B$, where k_θ is an equivalent rotation spring stiffness parameter given by

$$k_\theta = \frac{3EI_z}{\ell},$$

as illustrated in Fig. 4.13. The rotational stiffness k_θ has the dimension of a moment with unit [N m], and is seen to be inversely proportional to the beam length ℓ.

The maximum displacement $w_{\max}$ occurs for $\theta(x) = 0$. It then follows from the expressions for $\theta(x)$ and $w(x)$ that

$$x_{\max} = \frac{\ell}{\sqrt{3}}, \qquad w_{\max} = w(x_{\max}) = \frac{1}{9\sqrt{3}}\frac{M_B\ell^2}{EI_z}.$$

A maximum permissible displacement $w_{\max}$ is hereby translated into a maximum permissible moment. □

Example 4.4. Simply supported beam with distributed load. Figure 4.14 shows a simply supported beam of length ℓ with uniformly distributed load p. It has two static boundary conditions $M_A = M_B = 0$ and the moment distribution can therefore be determined first, either by introducing a section and using equilibrium or by integration of the moment differential equation (4.22),

$$\frac{d^2M(x)}{dx^2} = -p.$$

The load intensity p is constant, and the moment distribution $M(x)$ is therefore parabolic with value zero at A and B due to the static boundary conditions. It is easily seen that the moment distribution that satisfies these conditions has the form

$$M(x) = \tfrac{1}{2}\,p\,x(\ell - x),$$

as already determined directly from statics in Example 3.6.

The transverse displacement $w(x)$ is found from the relation (4.23) between curvature and moment,

$$\frac{d^2w(x)}{dx^2} = -\frac{M(x)}{EI_z} = -\frac{1}{2}\frac{p\ell^2}{EI_z}\frac{x}{\ell}\left(1 - \frac{x}{\ell}\right).$$

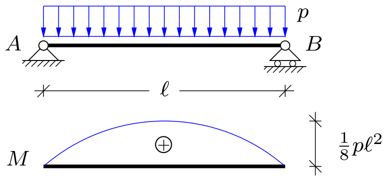


Fig. 4.14: Parabolic moment distribution for simply supported beam.

The displacement $w(x)$ is obtained by double integration as

$$w(x) = -\frac{1}{2}\frac{p\ell^4}{EI_z}\left[\frac{1}{6}\Big(\frac{x}{\ell}\Big)^3 - \frac{1}{12}\Big(\frac{x}{\ell}\Big)^4\right] + C_0 + C_1 x\,,$$

with two arbitrary integration constants C_0 and C_1. The kinematic boundary conditions for the simple supports are

$$w(0) = w(\ell) = 0\,,$$

where the first condition directly implies that $C_0 = 0$. The second condition $w(\ell) = 0$ yields

$$-\frac{1}{24}\frac{p\ell^4}{EI_z} + C_1\ell = 0 \qquad \Rightarrow \qquad C_1 = \frac{1}{24}\frac{p\ell^3}{EI_z}\,,$$

whereby the final expression for the transverse displacement becomes

$$w(x) = \frac{1}{24}\frac{p\,\ell^4}{EI_z}\frac{x}{\ell}\left[1 - 2\Big(\frac{x}{\ell}\Big)^2 + \Big(\frac{x}{\ell}\Big)^3\right].$$

The displacement $w(x)$ is symmetric with respect to the center of the beam, and the maximum displacement therefore occurs at $x = \frac{1}{2}\ell$,

$$w_{\max} = \frac{5}{384}\frac{p\,\ell^4}{EI_z}\,.$$

The rotation follows from the derivative of the displacement $w(x)$ as

$$\theta(x) = -\frac{dw(x)}{dx} = -\frac{1}{24}\frac{p\,\ell^3}{EI_z}\left[1 - 6\Big(\frac{x}{\ell}\Big)^2 + 4\Big(\frac{x}{\ell}\Big)^3\right],$$

and the rotations at the supports are then found to be

$$\theta_B = -\theta_A = \frac{1}{24}\frac{p\,\ell^3}{EI_z}\,.$$

Note, that symmetry implies same magnitude and opposite sign. □

So far the examples have not involved concentrated loads or load discontinuities within the beam span. In the case of a concentrated load on the beam, the moment distribution will be described by different analytical expressions on the two sides of the beam. The solution must therefore be obtained by integration in both intervals and combined by suitable continuity conditions. The following example illustrates the procedure for the case of a concentrated transverse load. It is demonstrated that the discontinuities generated by the concentrated load severely complicates the derivation of the solution.

In practice, problems with concentrated loads are therefore most often solved by using a method based on the principle of virtual work, presented in Section 4.4.

Example 4.5. Simply supported beam with local force. The present example considers a simply supported beam with a concentrated transverse force P acting at distance a from the left support as shown in Fig. 4.15. It is convenient to introduce the notation $a' = \ell - a$ for the distance of the load from the right support, and similarly letting $x' = \ell - x$ denote the coordinate from the right support.

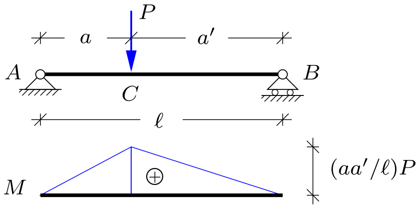


Fig. 4.15: Simply supported beam with concentrated load.

The beam is statically determinate and the moment distribution consists of two linear parts connecting the supports with the maximum moment $M(a) = (aa'/\ell)P$ at $x = a$. The analytical expressions for the moment distribution are,

$$M(x) = \begin{cases} x\,\dfrac{a'}{\ell}\,P\,, & x \le a \\[2ex] x'\,\dfrac{a}{\ell}\,P\,, & x \ge a\,. \end{cases}$$

The curvature relation (4.23) can be expressed in either the variable x or x',

$$\frac{d^2w(x)}{dx^2} = -\frac{M(x)}{EI_z}\,, \qquad \frac{d^2w(x')}{dx'^2} = -\frac{M(x')}{EI_z}\,.$$

Substitution of the moment for the left and the right part of the beam followed by double integration gives

$$w(x) = \begin{cases} -\dfrac{x^3a'}{6\ell}\,\dfrac{P}{EI_z} + C_0 + C_1x\,, & x \le a \\[2ex] -\dfrac{x'^3a}{6\ell}\,\dfrac{P}{EI_z} + D_0 + D_1x'\,, & x \ge a\,. \end{cases}$$

It is seen that the solution contains two arbitrary integration constants for each of the two integration intervals, making a total of four. The boundary conditions $w(x=0) = 0$ and $w(x=\ell) = w(x'=0) = 0$ determine the constants $C_0 = D_0 = 0$.

The remaining two constants C_1 and D_1 are to be determined from the conditions that $w(x)$ and $dw(x)/dx$ are continuous at $x = a$, corresponding to

$$w(a_-) = w(a_+)\,, \qquad \frac{dw}{dx}\bigg|_{a-} = -\frac{dw}{dx'}\bigg|_{a+}\,.$$

The minus in the equation for the derivatives is due to the change of variable from x to x'. The displacement continuity yields the equation

$$-\frac{a^3a'}{6\ell}\frac{P}{EI_z} + C_1\,a \;=\; -\frac{a'^3a}{6\ell}\frac{P}{EI_z} + D_1\,a'\,,$$

while continuity of the derivatives leads to

$$-3\frac{a^2a'}{6\ell}\frac{P}{EI_z} + C_1 \;=\; 3\frac{a'^2a}{6\ell}\frac{P}{EI_z} - D_1\,.$$

The constant D_1 can be eliminated between the two equations by multiplying the second equation by a' and then taking the sum,

$$-(a^2+3aa')\frac{aa'}{6\ell}\frac{P}{EI_z} \;+\; (a+a')C_1 \;=\; 2a'^2\,\frac{a'a}{6\ell}\frac{P}{EI_z}\,.$$

When introducing the identity $a+a'=\ell$ it is found that

$$C_1 \;=\; (a^2+3aa'+2a'^2)\,\frac{a'a}{6\ell^2}\frac{P}{EI_z}\,.$$

This expression can be reduced further by introducing $a=\ell-a'$ in the parenthesis, whereby

$$C_1 \;=\; (\ell+a')\,\frac{a'a}{6\ell}\frac{P}{EI_z} \;=\; (\ell^2-a'^2)\,\frac{a'}{6\ell}\frac{P}{EI_z} \;=\; \left[1-\left(\frac{a'}{\ell}\right)^2\right]\frac{P\,\ell\,a'}{6EI_z}\,.$$

The formulation is symmetric, and the expression for D_1 can therefore be found by interchanging a' and a in the expression for C_1,

$$D_1 \;=\; \left[1-\left(\frac{a}{\ell}\right)^2\right]\frac{P\,\ell\,a}{6EI_z}\,.$$

The final solution is obtained by substitution of the constants C_1 and D_1 into the expressions for the left and right intervals, respectively,

$$w(x) \;=\; \begin{cases} \dfrac{P\ell a'x}{6EI_z}\left[1-\left(\dfrac{a'}{\ell}\right)^2-\left(\dfrac{x}{\ell}\right)^2\right], & x\le a \\[2ex] \dfrac{P\ell a x'}{6EI_z}\left[1-\left(\dfrac{a}{\ell}\right)^2-\left(\dfrac{x'}{\ell}\right)^2\right], & x\ge a\,. \end{cases}$$

The solution is seen to be in a fairly systematic form with full symmetry between marked and unmarked variables. However, it is difficult to use this property to simplify the derivation further than shown here. □

4.2.2 Statically indeterminate beams

In the case of statically indeterminate beams there are more than two kinematic support conditions, and the problem must therefore be solved by use of the kinematic variables. In practice this means formulating the problem in terms of the transverse displacement $w(x)$ right from the start. The transverse displacement defines the moment via (4.23), and the moment must satisfy the equilibrium equation (4.22). Combination of these two conditions gives the following fourth order differential equation for the transverse displacement $w(x)$,

$$\frac{d^2}{dx^2}\Big(EI_z\,\frac{d^2w(x)}{dx^2}\Big)\;-\;p(x)\;=\;0\,. \tag{4.24}$$

Integration of this fourth order differential equation introduces four arbitrary integration constants, that are determined from the four boundary conditions representing the supports. Typical boundary conditions are illustrated in Fig. 4.10. They consist of combinations of relations involving the transverse displacement w and its derivative dw/dx, and the static conditions expressed in terms of the moment and shear force as

$$M(x)\;=\;-\,EI_z\,\frac{d^2w(x)}{dx^2}\,,\qquad Q(x)\;=\;-\,\frac{d}{dx}\Big(EI_z\,\frac{d^2w(x)}{dx^2}\Big)\,. \tag{4.25}$$

The integration is illustrated in the following simple example of a beam with a fixed and a simple support carrying a distributed load of constant intensity. In the case of concentrated loads or non-constant bending stiffness EI_z complications similar to those illustrated in Example 4.5 will occur, and methods presented in Section 4.4, based on the principle of virtual work, may be preferable.

Example 4.6. Cantilever beam with extra end support. Figure 4.16 shows a homogeneous beam AB of length ℓ with a fixed end at A and a simple support at B. There are three reaction components M_A, R_A and R_B associated with the beam bending problem, and the beam is therefore statically indeterminate. The displacement function $w(x)$ is obtained from the differential equation (4.24) with constant load intensity,

$$\frac{d^4w(x)}{dx^4}\;=\;\frac{p}{EI_z}\,.$$

The boundary conditions are

$$w(0)\;=\;0\,,\qquad dw(0)/dx\;=\;0\,,\qquad w(\ell)\;=\;0\,,\qquad d^2w(\ell)/dx^2\;=\;0\,,$$

where the last follows from the moment condition via (4.25a).

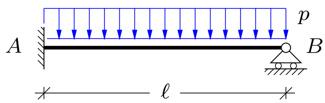


Fig. 4.16: Cantilever beam with uniform load and extra support.

When integrating the differential equation four times the result is the following fourth degree polynomial,

$$w(x)\;=\;C_0\;+\;C_1x\;+\;C_2x^2\;+\;C_3x^3\;+\;\frac{1}{24}\,\frac{p}{EI_z}\,x^4\,.$$

The first four terms containing the integration constants C_0, C_1, C_2 and C_3 represent the solution to the homogeneous equation, while the last term is a particular solution that accounts for the distributed load.

It is convenient first to apply the two boundary conditions at $x = 0$, which immediately give $C_0 = C_1 = 0$. Hereby the expression for the displacement is reduced to

$$w(x) \,=\, C_2 x^2 \,+\, C_3 x^3 \,+\, \frac{1}{24}\,\frac{p}{EI_z}\,x^4\,.$$

The third boundary condition $w(\ell) = 0$ gives the equation

$$C_2 \,+\, C_3 \ell \,+\, \frac{1}{24}\,\frac{p}{EI_z}\,\ell^2 \,=\, 0\,.$$

The vanishing moment condition $d^2w(\ell)/dx^2 = 0$, gives the equation

$$C_2 \,+\, 3C_3 \ell \,+\, \frac{1}{4}\,\frac{p}{EI_z}\,\ell^2 \,=\, 0\,.$$

The constant C_2 is eliminated by forming the difference between the two equations, whereby C_3 is determined. Substitution of C_3 into any of the original equations then determines C_2. This leads to

$$C_2 \,=\, \frac{3}{48}\,\frac{p\ell^2}{EI_z}\,, \qquad C_3 \,=\, -\,\frac{5}{48}\,\frac{p\ell}{EI_z}\,.$$

The resulting solution for the displacement can then be expressed in normalized form as

$$w(x) \,=\, \frac{1}{48}\,\frac{p\ell^4}{EI_z}\left(\frac{x}{\ell}\right)^2\left[3 \,-\, 5\,\frac{x}{\ell} \,+\, 2\left(\frac{x}{\ell}\right)^2\right].$$

The moment is given via (4.25) as

$$M(x) \,=\, -EI_z\frac{d^2w}{dx^2} \,=\, -\,\frac{p\,\ell^2}{8}\left[1 \,-\, 5\,\frac{x}{\ell} \,+\, 4\left(\frac{x}{\ell}\right)^2\right],$$

and the moment at the fixed support at $x = 0$ is

$$M_A \,=\, M(0) \,=\, -\,\tfrac{1}{8}\,p\,\ell^2\,.$$

The moment $M(x)$ is illustrated in Fig. 4.17, normalized by the magnitude of the negative moment $M = M_A$ at the fixed support.

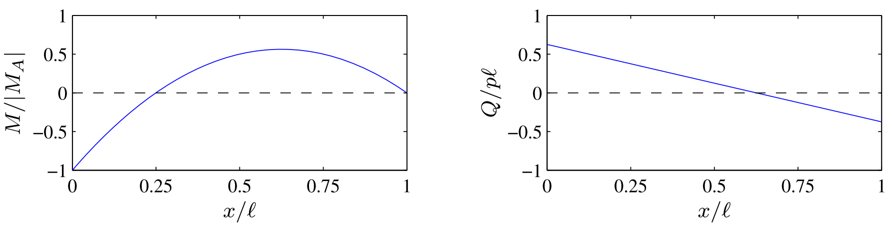


Fig. 4.17: Moment and shear force distribution.

The shear force is obtained as the derivative of the moment,

$$Q(x) = \frac{dM(x)}{dx} = -\frac{p\,\ell}{8}\Big[-5 + 8\Big(\frac{x}{\ell}\Big)\Big].$$

A local maximum moment occurs where $Q(x) = 0$. It is found that

$$x_{\max} = \tfrac{5}{8}\ell, \qquad M_{\max} = M(x_{\max}) = \tfrac{9}{128}\,p\,\ell^2 .$$

Note, that the magnitude of the moment at the fixed end is larger, since $|M_{\max}/M_A| = \frac{9}{16} < 1$. The shear force also determines the vertical reactions

$$R_A = Q(0) = \tfrac{5}{8}p\,\ell, \qquad R_B = -Q(\ell) = \tfrac{3}{8}p\,\ell .$$

It is observed that the sum of the vertical reactions is equal to the imposed load, $R_A + R_B = p\,\ell$. It is also observed that the extra stiffness provided by the fixed support increases the vertical reaction R_A relative to the value $\frac{1}{2}p\,\ell$ in a similar simply supported beam. □

4.3 Shear flexible beams

The basic assumption of the Bernoulli beam theory is that the only deformation mechanism is curvature. This mechanism was identified with reference to constant bending moment, as illustrated in Fig. 4.3. In this particular case the rotation of the cross-section is identical to the rotation of the tangent of the beam axis, and thus sections initially orthogonal to the beam axis will remain orthogonal to the beam axis in the deformed state. However, for non-homogeneous bending this is an approximation, in which the effect of deformation generated by the shear force is neglected. Shear deformation can be illustrated by placing a paperback book on a desk and pushing the top cover towards the front of the book. All pages retain their original size but slide a little bit in the direction of the push. Thereby the originally vertical lines along the sides of the book form an angle with vertical – the shear angle. Figure 4.18 illustrates the bending and the shear deformation modes for a beam.

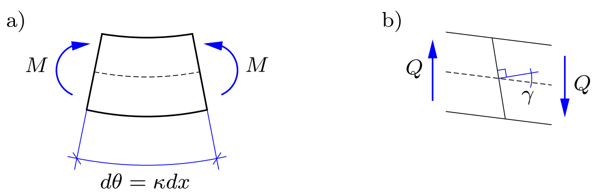


Fig. 4.18: a) Bending and b) shear deformation mechanisms.

Shear stresses and strains

Before addressing shear deformation of beams a few basic concepts in relation to shear stress and shear strain are briefly introduced. The basic shear mechanism is illustrated in Fig. 4.19 for a cube of unit side length. The top and bottom faces are loaded by forces of magnitude τ in their own plane towards the right and the left, respectively. The forces are in-plane and normalized per unit area, and therefore called shear stresses. Equilibrium requires that the left and right faces are loaded with a downward and an upward shear stress of the same magnitude to prevent rotation of the cube. This is the basic shear load. The shear load will rotate the sides of the cube as shown in the figure as γ_1 and γ_2. The change of the angle between the faces of the cube is the shear strain

$$\gamma = \gamma_1 + \gamma_2 \,. \tag{4.26}$$

The shear strain is a change of angle *between* the faces, and is independent of any overall rotation of the cube.

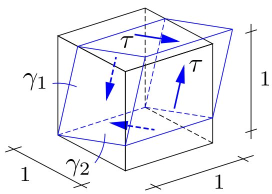


Fig. 4.19: Unit cube with shear stress τ and shear strain γ.

In a linear elastic material the shear stress τ is proportional to the shear strain γ,

$$\tau = G\gamma \,. \tag{4.27}$$

The coefficient G is a material parameter, called the shear modulus. It is similar to the modulus of elasticity and also has the dimension of force per area. For an isotropic material – that is a material in which the properties are independent of the direction in the material – there is a connection between the modulus of elasticity E and the shear modulus G. The relation implies that

$$\tfrac{1}{3}E < G < \tfrac{1}{2}E \,. \tag{4.28}$$

A typical value is $G \simeq 0.4\,E$. In composite materials the effective shear modulus may be much lower, see e.g. Jones (1999). A more detailed discussion of general states of stress and strain is given in Chapters 8–9.

Equations of shear flexible beams

In a theory for beams with shear deformation it is important to make a distinction between the rotation of the cross-section $\theta(x)$ and the rotation of the tangent of the beam axis $-dw(x)/dx$. In bending, fibers parallel to the beam axis change length due to a change of the cross-section rotation $\theta(x)$ along the beam. The change of angle between neighboring cross-sections is $d\theta = \kappa\, dx$. Thus, the parameter κ associated with beam bending is the change in cross-section rotation per unit length along the beam. The shear mechanism accounts for the fact that a shear force Q in the beam will introduce shear strains as illustrated in Fig. 4.19, and the 'average' shear strain γ appears as an angle between the beam axis tangent and the cross-section normal as shown in Fig. 4.18b. Thus, the kinematic relations for a beam theory including shear deformations consist of a definition of κ in terms of the angle θ and a definition of the angle θ in terms of the transverse displacement w and the shear strain γ. The shear mechanism in Fig. 4.18b shows that the rotation of a cross-section is $-dw/dx$ due to rotation of the beam axis plus an additional rotation γ due to shear straining,

$$\theta(x) = -\frac{dw(x)}{dx} + \gamma(x)\,. \tag{4.29}$$

It is seen that for $\gamma \equiv 0$ the cross-section rotation angle θ is defined solely by the derivative of the beam axis dw/dx as in Bernoulli beam theory.

The theory of beams with shear flexibility – often called Timoshenko beam theory – has the same basic ingredients as the Bernouilli theory for beams without shear flexibility. There are two internal forces, the shear force $Q(x)$ and the internal moment $M(x)$. They must satisfy transverse and rotation equilibrium, whereby

$$\frac{dQ(x)}{dx} = -p(x)\,, \qquad \frac{dM(x)}{dx} = Q(x)\,. \tag{4.30}$$

The kinematics of the beam introduces two measures of deformation, the curvature $\kappa(x)$ and the shear strain $\gamma(x)$. They are defined in terms of the transverse displacement $w(x)$ and the cross-section rotation $\theta(x)$ as

$$\kappa(x) = \frac{d\theta(x)}{dx}\,, \qquad \gamma(x) = \frac{dw(x)}{dx} + \theta(x)\,. \tag{4.31}$$

The internal forces M and Q and the deformation measures κ and γ are connected by the constitutive relations that describe the mechanical behavior of the material of the beam.

In elastic beams the generalized strains κ and γ are proportional to the moment M and the shear force Q, respectively, as expressed by the relations

$$M(x) = EI_z\,\kappa(x)\,, \qquad Q(x) = GA_z\,\gamma(x)\,, \tag{4.32}$$

where EI_z and GA_z are the bending and shear stiffness, respectively. The bending stiffness is expressed in terms of the elastic modulus E and the moment of inertia I_z of the cross-section about its neutral axis. The shear stiffness is expressed in terms of the elastic shear modulus G and an equivalent 'shear area' of the cross-section A_z. Shear stresses are non-uniformly distributed over the cross section, and this implies that the shear area A_z is smaller than the full cross-section area A. For a rectangular cross-section $A_z = \frac{5}{6}A$, and for an I-section the shear area A_z is approximately equal to the area of the web. The analysis of the shear stress distribution and the associated cross-section stiffness is treated in Chapter 11.

The boundary conditions of a beam are expressed in terms of the kinematic parameters w and θ, or the internal forces Q and M. When shear flexibility is included, it is often most convenient to integrate the basic equilibrium and kinematic relations directly. This procedure is illustrated in the following two examples.

Example 4.7. Shear flexible cantilever. The effect of shear flexibility is illustrated by the cantilever beam loaded with a concentrated force P at the end as shown in Fig. 4.20. The cantilever is statically determinate, and the internal moment and shear force,

$$M(x) = (x-\ell)\,P\,, \qquad Q(x) = P\,,$$

are shown below the beam in the figure.

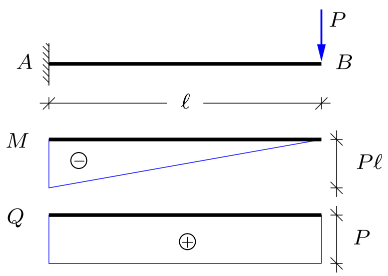


Fig. 4.20: Cantilever with shear flexibility.

The bending relation for the cross-sections rotation $\theta(x)$,

$$\frac{d\theta}{dx} = \frac{M(x)}{EI_z} = \frac{P}{EI_z}\,(x-\ell)$$

is integrated to give

$$\theta = \frac{P}{EI_z}\left(\tfrac{1}{2}x^2 - x\ell\right) + C_1\,.$$

The arbitrary constant C_1 is the value of θ at the support, and thus $C_1 = 0$.

The transverse displacement $w(x)$ is now determined by integrating the shear strain relation

$$\frac{dw}{dx} = -\theta + \gamma = \frac{P}{EI_z}\left(x\ell - \tfrac{1}{2}x^2\right) + \frac{P}{GA_z},$$

where the shear strain γ has been expressed in terms of the shear force P. Integration gives

$$w = \frac{P}{EI_z}\left(\tfrac{1}{2}x^2\ell - \tfrac{1}{6}x^3\right) + \frac{P}{GA_z}\,x + C_2\,.$$

The arbitrary constant C_2 is the displacement w at the support, whereby $C_2 = 0$. Thus, the displacement of a shear flexible cantilever is

$$w = \frac{P\ell^3}{6EI_z}\left(\frac{x}{\ell}\right)^2\left[3 - \frac{x}{\ell}\right] + \frac{P\ell}{GA_z}\,\frac{x}{\ell}\,.$$

It is seen that the rotation of the cross-sections $\theta(x)$ is independent of the shear flexibility, while the displacement $w(x)$ consists of two additive contributions, a contribution from bending flexibility identical to that for Bernoulli beams, and a contribution from shear deformation. This additive form of the displacement remains valid for other statically determinate load cases. For most long and slender beams the displacement contribution from shear flexibility is fairly small. This is discussed further in Section 4.4.2. □

In the deformation and finite element methods distributed loads are included via their equivalent concentrated loads on the nodes. These concentrated loads correspond to the reactions at the ends of a rigidly supported beam, and it is therefore of particular interest to investigate the influence of shear flexibility on the reactions of a rigidly supported beam.

Example 4.8. Shear flexible beam with fixed ends. Figure 4.21 shows a rigidly supported homogeneous beam with uniformly distributed load p. In this case it is convenient to use a coordinate x with origin at the center of the beam. The solution is obtained by sequential integration of the basic relations.

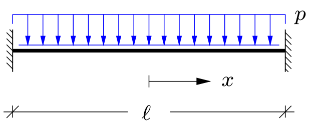


Fig. 4.21: Fixed beam with uniformly distributed load.

The shear force $Q(x)$ is obtained by integration of the transverse equilibrium equation,

$$\frac{dQ}{dx} = -p \qquad \Rightarrow \qquad Q(x) = -p\,x\,,$$

where the symmetry condition $Q(0) = 0$ has been used to eliminate an arbitrary constant. Integration of the moment relation then gives

$$\frac{dM}{dx} = Q = -p\,x \qquad \Rightarrow \qquad M(x) = -\tfrac{1}{2}p\,x^2 + M_0\,,$$

where the arbitrary constant M_0 is the moment at the center of the beam. The bending relation then gives

$$\frac{d\theta}{dx} = \frac{M}{EI_z} = \frac{1}{EI_z}\big(-\tfrac{1}{2}p\,x^2 + M_0\big) \quad \Rightarrow \quad \theta(x) = \frac{1}{EI_z}\big(-\tfrac{1}{6}p\,x^3 + M_0 x\big)\,,$$

where the symmetry condition $\theta(0) = 0$ has been used to eliminate an arbitrary constant. The moment M_0 is determined by the boundary condition $\theta(\frac{1}{2}\ell) = 0$, whereby $M_0 = \frac{1}{24}p\ell^2$. The center moment M_0, and thereby the full distribution of internal forces, is seen to be independent of shear flexibility. The internal moment distribution is illustrated in Fig. 4.22. It is observed that the moment at the supports is $M_{A,B} = -\frac{1}{12}p\ell^2$, i.e. negative and of double magnitude compared to that at the center.

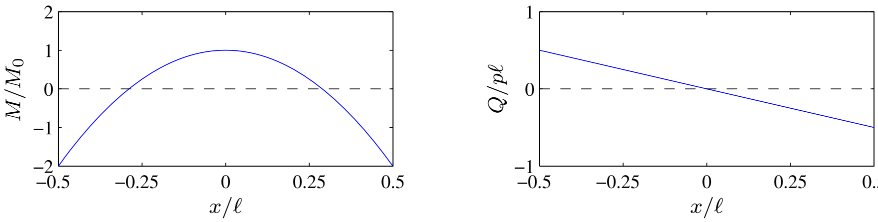


Fig. 4.22: Moment and shear force distribution.

Finally, the displacement field follows from integration of the shear relation, with $\gamma = Q/GA_z$. This gives

$$\frac{dw}{dx} = -\theta + \gamma = -\frac{p}{24EI_z}\big(x\ell^2 - 4x^3\big) - \frac{p\,x}{GA_z}\,,$$

from which

$$w(x) = -\frac{p}{24EI_z}\big(\tfrac{1}{2}x^2\ell^2 - x^4\big) - \frac{p\,x^2}{2GA_z} + w_0\,.$$

The arbitrary constant w_0, representing the displacement at the center, is determined from the boundary condition $w(\frac{1}{2}\ell) = 0$,

$$w_0 = \frac{p\,\ell^4}{384EI_z} + \frac{p\,\ell^2}{8GA_z}\,.$$

Substitution of this gives the displacement field

$$w(x) = \frac{p\,\ell^4}{384EI_z}\left[1 - 2\Big(\frac{2x}{\ell}\Big)^2 + \Big(\frac{2x}{\ell}\Big)^4\right] + \frac{p\,\ell^2}{8GA_z}\left[1 - \Big(\frac{2x}{\ell}\Big)^2\right].$$

Also in this case the displacement field is the sum of a bending and a shear deformation contribution. At the center the relative magnitude of the shear contribution is $w_G/w_E = 48EI_z/(GA_z\ell^2)$, and thus the shear contribution to the deformation is determined by the non-dimensional parameter $EI_z/(GA_z\ell^2)$. □

It is seen from these two examples that a full analysis of a beam with shear flexibility becomes quite extensive, when applying a direct integration approach. It is demonstrated in the following section that the shear flexibility effect can be included rather easily when using virtual work principles as basis of the analysis. That approach is also used when developing a simple beam element with shear flexibility in Chapter 7.

4.4 Virtual work and displacements of beams

The equilibrium equations for beams and frames constitute conditions by which each part of the beam is in local equilibrium. This equilibrium implies that if the beam or frame is subjected to a hypothetical small displacement the associated work must vanish. The principle of virtual work was derived for truss structures in Section 2.4.3, and it was demonstrated that it could be used to determine displacements of elastic truss structures from an expression of internal work. Here the same ideas are extended to beams and frames. The derivations are described in detail for plane beams and frames, but are easily generalized to three-dimensional beams and frames. First, the equation of virtual work is derived for a beam. The virtual work equation is an identity by which the external virtual work performed by the loads on the beam is identical to the internal virtual work formed by the section forces via the virtual deformation of the beam. For elastic beams this results in an expression for local displacements in terms of internal work. This result constitutes an important alternative to calculation of beam displacements by direct integration of the differential equations. After presenting the results for single beams a simple generalization to frames is presented.

4.4.1 Principle of virtual work

The principle of virtual work will be derived for plane beams and frames, and in order to obtain full generality the formulation includes the normal force N, the shear force Q and the bending moment M. The beams can then later be joined to form frames.

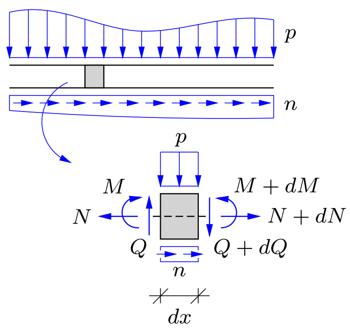


Fig. 4.23: Equilibrium of beam with section forces N, Q and M.

Figure 4.23 shows a plane beam with normal force $N(x)$, shear force $Q(x)$ and moment $M(x)$. The beam is loaded by an axial load $n(x)$, a transverse load $p(x)$ and a distributed moment load $m(x)$. Typically, a distributed moment $m(x)$ will occur, if the axial load $n(x)$ is not acting along the beam axis but at some transverse distance. The equilibrium equations are obtained by considering a small slice of the beam of thickness dx as in Section 3.3.1. The resulting three equilibrium equations are

$$\frac{dN}{dx} + n = 0 , \qquad \frac{dQ}{dx} + p = 0 , \qquad \frac{dM}{dx} - Q + m = 0 . \tag{4.33}$$

Section force distributions $N(x)$, $Q(x)$ and $M(x)$ that satisfy the equilibrium conditions and appropriate static boundary conditions are said to satisfy the static conditions. The loads are here represented by continuous load densities. Concentrated loads can be considered as local load densities of very high intensity over a very short length of the beam.

The beam kinematics is expressed in terms of an axial displacement $u(x)$, a transverse displacement $w(x)$, and the cross-section rotation $\theta(x)$. Together the three functions are called the generalized displacements of the beam, or simply the displacements. The displacements of a beam typically lead to deformation, described in terms of the axial strain $\varepsilon(x)$, the shear strain $\gamma(x)$, and the curvature $\kappa(x)$. These kinematic quantities describing the deformation are called the generalized strains. As discussed previously they are defined as

$$\varepsilon = \frac{du}{dx} , \qquad \gamma = \frac{dw}{dx} + \theta , \qquad \kappa = \frac{d\theta}{dx} . \tag{4.34}$$

These relations between the (generalized) displacements and the (generalized) strains are called the kinematic conditions of the beam.

The idea of virtual work is to consider a beam in equilibrium, i.e. a beam that satisfies the equilibrium equations (4.33). The beam is then subjected to a hypothetical displacement described by the virtual displacements $\delta u(x)$, $\delta w(x)$ and $\delta\theta(x)$. A small unbalance in the axial equilibrium condition would lead to virtual work through the virtual axial displacement $\delta u(x)$, and similarly for virtual transverse motion and rotation. The total virtual work over the whole beam is expressed by integration over the length of the beam. This virtual work is expressed as

$$\int_0^\ell \Big\{ \delta u\Big(\frac{dN}{dx} + n\Big) + \delta w\Big(\frac{dQ}{dx} + p\Big) + \delta\theta\Big(\frac{dM}{dx} - Q + m\Big)\Big\} dx = 0 . \tag{4.35}$$

For a beam in equilibrium each of the expressions within the parentheses vanishes identically, and thus the integral must be equal to zero for any choice of the virtual displacements.

Each of the parentheses contains a derivative of a generalized section force and a term representing a load intensity. The products containing derivatives of the generalized section forces can be reformulated by use of integration by parts. This procedure generates terms at the interval end-points, and derivatives of the virtual displacements. When arranged with the external load terms first, the result takes the form

$$
\begin{aligned}
&\int_0^\ell \big(\delta u\, n \,+\, \delta w\, p \,+\, \delta\theta\, m\big) dx \,+\, \Big[\, \delta u\, N \,+\, \delta w\, Q \,+\, \delta\theta\, M \,\Big]_0^\ell \\
&\quad - \int_0^\ell \Big(\frac{d(\delta u)}{dx}\, N \,+\, \Big(\frac{d(\delta w)}{dx} + \delta\theta \Big)\, Q \,+\, \frac{d(\delta\theta)}{dx}\, M \Big) dx \,=\, 0\,.
\end{aligned}
\tag{4.36}
$$

In this formula the first integral represents the virtual work of the applied load along the beam, and the terms in the square brackets represent the virtual work of the forces and moments acting at the ends of the beam. This is illustrated in Fig. 4.24.

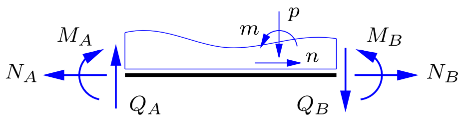


Fig. 4.24: Components of external virtual work.

The kinematic quantities in the last integral are seen to correspond to the virtual strains defined via the kinematic relations (4.34). When the notation for the corresponding virtual strains is introduced the virtual work equation takes the form

$$
\begin{aligned}
&\Big[\, \delta u\, N \,+\, \delta w\, Q \,+\, \delta\theta\, M \,\Big]_0^\ell \,+\, \int_0^\ell \big(\delta u\, n \,+\, \delta w\, p \,+\, \delta\theta\, m\big) dx \\
&\qquad = \int_0^\ell \big(\delta\varepsilon\, N \,+\, \delta\gamma\, Q \,+\, \delta\kappa\, M\big) dx\,.
\end{aligned}
\tag{4.37}
$$

In this equation the terms on the left side represent the external virtual work, i.e. the virtual work of the loads acting on the beam,

$$
\begin{aligned}
\delta V_{\rm ex} \,=\, &\Big[\, \delta u\, N \,+\, \delta w\, Q \,+\, \delta\theta\, M \,\Big]_0^\ell \\
&+ \int_0^\ell \big(\delta u\, n \,+\, \delta w\, p \,+\, \delta\theta\, m\big) dx\,.
\end{aligned}
\tag{4.38}
$$

The integral on the right side of the virtual work equation represents the internal virtual work, i.e. the work of the internal forces through the virtual strains.

$$
\delta V_{\rm in} \,=\, \int_0^\ell \big(\delta\varepsilon\, N \,+\, \delta\gamma\, Q \,+\, \delta\kappa\, M\big) dx\,.
\tag{4.39}
$$

The internal work of the beam is associated with three deformation mechanisms – extension, shear and bending – illustrated in Fig. 4.25.

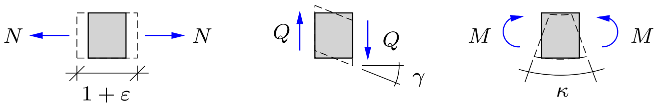


Fig. 4.25: Axial, shear and bending deformation of a beam.

With these definitions the virtual work equation (4.37) is an expression of the identity of external and internal virtual work,

$$\delta V_{\rm ex} = \delta V_{\rm in} \, . \tag{4.40}$$

This virtual work equation serves several useful purposes. It is based on the equilibrium equations (4.33) for the beam. By the reformulation via integration by parts it uniquely defines the virtual strains that correspond to the internal forces. Thus, the kinematic relations (4.34) actually *follow* from the assumption of the existence of a virtual work equation, and need not be defined independently. By making suitable assumptions about material behavior and deformation characteristics the virtual work equation serves to determine deformation and stiffness properties of beams.

In particular, the virtual work equation for Bernoulli beams follows as a special case of vanishing shear flexibility. In the case of Bernoulli beams the shear strain vanishes identically, $\gamma \equiv 0$, and the shear force term disappears from the internal work. This special formulation is often used in practice, because the effect of shear flexibility deformation is insignificant in many structures. However, as demonstrated in the following, it is often fairly simple to include the shear flexibility effect in virtual work based calculations, if needed.

4.4.2 Displacements in elastic beams

The virtual work equation is an identity of the internal work and the external work for a combination of an actual static state with given loads and a virtual displacement field. For an elastic beam it is of interest to reverse the roles such that the virtual displacement field represents the actual displacement field of interest. On the other hand, the static field is selected to serve a specific diagnostic role, e.g. focusing on the force/displacement component at a specific point. A special case of this has been considered in Section 2.4.3 for truss structures, in which bending and shear do not occur.

The general procedure is illustrated by a specific example in Fig. 4.26, showing a simply supported elastic beam with a load distribution $p(x)$. The cor-

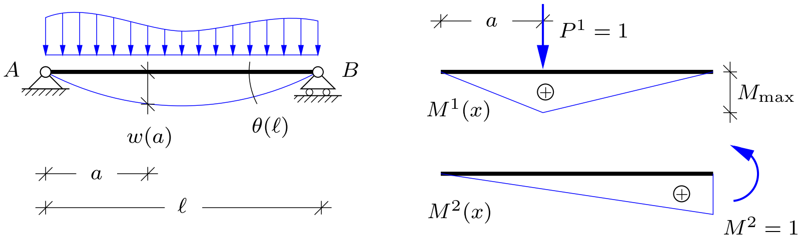


Fig. 4.26: Actual load P and unit test loads P^1 and M^2.

responding displacements are $u^0(x)$, $w^0(x)$ and $\theta^0(x)$. The superscript 0 indicates that these distributions correspond to the actual situation. These displacements define a deformation field described by an axial strain $\varepsilon^0(x)$, a shear strain $\gamma^0(x)$ and a curvature $\kappa^0(x)$, given via the kinematic relations (4.34). The beam is now assumed to be linear elastic. This implies linear constitutive relations between the internal forces and the strains for the three deformation mechanisms illustrated in Fig. 4.25,

$$\varepsilon^0 = \frac{N^0}{EA}, \qquad \gamma^0 = \frac{Q^0}{GA_z}, \qquad \kappa^0 = \frac{M^0}{EI_z}. \tag{4.41}$$

The extension stiffness EA combines the elastic modulus E with the area of the beam, the shear stiffness GA_z combines the shear modulus G with a modified 'shear area' of the beam cross-section, and finally the bending stiffness EI_z combines the elastic modulus with the moment of inertia about the neutral axis of the cross-section. The elastic relations (4.41) imply that the actual beam kinematics can be characterized by the internal forces $N^0(x)$, $Q^0(x)$ and $M^0(x)$ corresponding to the actual loads.

The static components of the virtual work equation are taken to represent a suitable test case of particular interest. Two cases are illustrated in Fig. 4.26b: a concentrated unit transverse force $P^1 = 1$ at the distance a from the left end, and a concentrated unit counterclockwise moment $M^2 = 1$ at the right support, respectively.

The external work for test case 1 consists of the single term $w^0 P^1 = w^0$, corresponding to the actual displacement of the beam in the direction of the test force. According to the virtual work equation (4.40) this is equal to the internal virtual work, expressed as

$$w^0 = w^0\, P^1 = \int_0^\ell \Big(\varepsilon^0(x)\, N^1(x) + \gamma^0(x)\, Q^1(x) + \kappa^0(x)\, M^1(x)\Big) dx. \tag{4.42}$$

When the actual strains are expressed via the elastic relations (4.41), the virtual work equation takes the form

$$w^0 = \int_0^\ell \Big(\frac{N^0(x)\, N^1(x)}{EA} + \frac{Q^0(x)\, Q^1(x)}{GA_z} + \frac{M^0(x)\, M^1(x)}{EI_z} \Big) dx. \qquad (4.43)$$

This relation permits calculation of a displacement component w^0 by introducing a unit test load $P^1 = 1$ at the point of the desired displacement and then evaluating the appropriate integrals involving the distribution of internal forces from the actual load, and from the test load.

The rotation of a cross-section in the beam can be determined by introducing a unit test moment. Figure 4.26b illustrates the application of a unit test moment $M^2 = 1$ at the right support of the beam. The corresponding external work is $\theta^0 M^2 = \theta^0$, and the equation of virtual work takes the form

$$\theta^0 = \theta^0 M^2 = \int_0^\ell \Big(\varepsilon^0(x)\, N^2(x) + \gamma^0(x)\, Q^2(x) + \kappa^0(x)\, M^2(x) \Big) dx. \qquad (4.44)$$

When the actual strains are represented in terms of the internal forces by use of the elastic relations (4.41), the following expression for the cross-section rotation is obtained,

$$\theta^0 = \int_0^\ell \Big(\frac{N^0(x)\, N^2(x)}{EA} + \frac{Q^0(x)\, Q^2(x)}{GA_z} + \frac{M^0(x)\, M^2(x)}{EI_z} \Big) dx. \qquad (4.45)$$

It is observed that the only difference between the procedures for computing a displacement and a cross-section rotation is the choice of test load. A displacement is determined from the internal forces generated by a unit test force in the direction of the desired displacement, while a cross-section rotation is determined from the internal forces generated by application of a unit moment at the cross-section.

Calculation of a displacement component by the virtual work equation implies three steps:

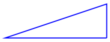
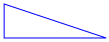
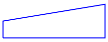
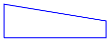
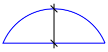
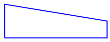
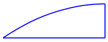
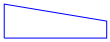
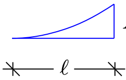
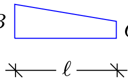
i) calculation of the internal forces $N^0(x)$, $Q^0(x)$ and $M^0(x)$ from the actual loads,

ii) calculation of the internal forces $N^j(x)$, $Q^j(x)$ and $M^j(x)$ from the unit test load in case j,

iii) evaluation of the integrals defining the internal work.

The two first tasks have been dealt with in considerable detail in the previous chapters. In particular it was found in Chapter 3 that distributed loads lead to linear variation of $N(x)$ and $Q(x)$ while the internal moment $M(x)$ has parabolic variation. For these types of distributions of the internal forces the integrals can be obtained in terms of the principal measures of the corresponding internal force distributions. A collection of basic results are summarized in Table 4.1. The use of the table is illustrated in the following examples.

Table 4.1: Product integrals for simple shapes.

$\int_0^\ell f(x)\, g(x)\, dx$	$f(x)$	$g(x)$
$\frac{\ell}{3} A\, B$	 A	 B
$\frac{\ell}{6} A\, B$	 A	B 
$\frac{\ell}{6} \Big(2AC + 2BD + AD + BC\Big)$	A  B	C  D
$\frac{\ell}{3} A \Big(B + C\Big)$	 A	B  C
$\frac{\ell}{12} A \Big(3B + 5C\Big)$	 A	B  C
$\frac{\ell}{12} A \Big(B + 3C\Big)$	 A	B  C

Example 4.9. Displacement of cantilever with tip load. Figure 4.27a shows a cantilever with a transverse force P at the tip. The goal is to calculate the transverse displacement $w(x)$ as a function of x and to evaluate the influence of shear flexibility on the solution. The similar problem was considered by integration of the differential equations in Example 4.7.

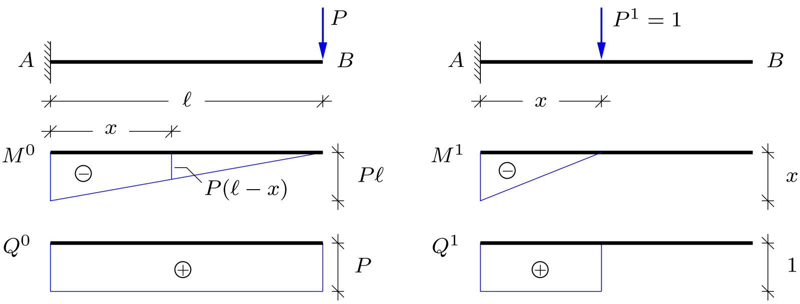


Fig. 4.27: Displacement of cantilever: Actual and test load distributions.

The normal force vanishes identically along the beam, $N^0 \equiv 0$. The distribution of the shear force $Q^0(s)$ and the moment $M^0(s)$ is shown in the lower part of Fig. 4.27a. A vertical test load in the form of a transverse unit force $P^1 = 1$ at the distance x from the support is shown in Fig. 4.27b together with the corresponding shear force and moment distributions $Q^1(s)$ and $M^1(s)$. The displacement $w(x)$ under the test load is then determined by the virtual work relation (4.43) in the form

$$w(x) = \int_0^x \Big(\frac{M^0(s)}{EI_z} M^1(s) + \frac{Q^0(s)}{GA_z} Q^1(s) \Big) ds .$$

Note, that the integration interval only extends over the interval from the support to the applied test load, as $M_1(s) = 0$ and $Q_1(s) = 0$ for $s > x$.

The integral is computed by the integral formulas in Table 4.1. The moment integral is computed from the linear distributions in the third row, where it is indicated in the figure that the actual moment at the location of the test force is $M^0(x) = P(\ell - x)$. The shear forces are constant. Thus, the integral is

$$w(x) = \frac{1}{EI_z} \frac{x}{6} \Big[2 P\ell x + P(\ell - x) x \Big] + \frac{1}{GA_z} x P .$$

After a slight rearrangement this takes the normalized form

$$w(x) = \frac{1}{6} \frac{P\ell^3}{EI_z} \Big(\frac{x}{\ell} \Big)^2 \Big[3 - \frac{x}{\ell} \Big] + \frac{P\ell}{GA_z} \frac{x}{\ell} .$$

This result agrees with the result obtained by solving the differential equation in Example 4.1. The rotation $\theta(x)$ can be determined similarly by applying a local moment $M^1 = 1$ at x. This problem is considered in the exercises. □

The importance of including shear flexibility in the beam theory can be estimated by comparing bending and shear contributions, w^E and w^G, to the displacement at the tip of the beam. As seen from the solution

$$w^E = \frac{P\ell^3}{3EI_z} , \qquad w^G = \frac{P\ell}{GA_z} .$$

Thus, the relative importance of the shear contribution in the present case is determined by the ratio

$$\frac{w^G}{w^E} = \frac{3EI_z}{GA_z\ell^2} \sim \frac{3}{10} \frac{E}{G} \frac{h^2}{\ell^2} , \qquad (4.46)$$

where the last expression is representative of a rectangular cross-section with height h. For isotropic beams $G \simeq 0.4E$, whereby the two first factors combine to a number around 0.75, and thus the relative magnitude of the shear flexibility contribution is approximately $(h/\ell)^2$. For slender beams this is small and the shear flexibility effect will be negligible. However, for short beams and for composite beams where the shear stiffness may be much smaller than the axial stiffness, $G \ll E$, the shear flexibility effect may be significant.

Example 4.10. Cantilever with discontinuous bending stiffness. The virtual work is efficient when evaluating the influence of local changes in the stiffness of the structural elements. Figure. 4.28 shows a cantilever of length ℓ with a transverse force P acting at the tip. The moment increases linearly towards the support, and a smaller cross-section on the exterior part of the beam may therefore be sufficient from a strength perspective. The present example investigates the influence of reducing the bending stiffness from EI_z to $\frac{1}{2}EI_z$ in the exterior half of the beam. The beam is considered as slender, and the effect of shear deformation is therefore neglected.

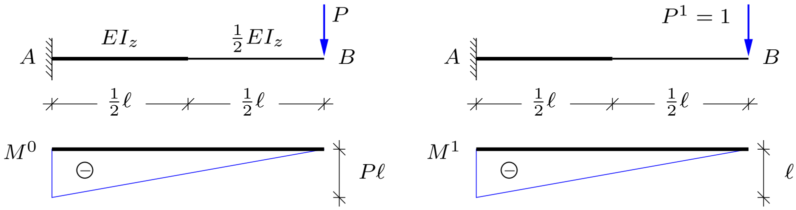


Fig. 4.28: Tip displacement of cantilever: Actual and test load distributions.

The tip displacement w_B is found by the virtual work equation (4.43), and the integral is divided into two parts due to the change in bending stiffness,

$$w_B = \frac{1}{EI_z}\int_0^{\ell/2} M^0 M^1 dx + \frac{2}{EI_z}\int_{\ell/2}^{\ell} M^0 M^1 dx.$$

The formula for integrating the product of two triangular densities is considerably simpler than the general formula for densities with linear variation. It is therefore convenient to absorb half of the last integral into the first by extending the interval of integration to the full beam length,

$$w_B = \frac{1}{EI_z}\int_0^{\ell} M^0 M^1 dx + \frac{1}{EI_z}\int_{\ell/2}^{\ell} M^0 M^1 dx.$$

Hereby both integrals relate to triangles with zero value at the right end. They are both computed by the formula in the first row of Table 4.1,

$$w_B = \frac{\ell}{3EI_z}(-P\ell)(-\ell) + \frac{\frac{1}{2}\ell}{3EI_z}(-\tfrac{1}{2}P\ell)(-\tfrac{1}{2}\ell) = (1+\tfrac{1}{8})\frac{P\ell^3}{3EI_z} = \frac{3}{8}\frac{P\ell^3}{EI_z}.$$

The additional displacement is given by the last term, giving a relative increase of $\frac{1}{8}$. The reason for the rather modest increase is that the section was deliberately reduced in the part of the beam that carries only a small moment. □

Example 4.11. Midpoint displacement of simply supported beam. The simply supported beam with uniform load shown in Fig. 4.29 is a common structural element. This example investigates the displacement at the center and the contribution from shear flexibility. The beam is statically determinate and the moment distribution, determined in Example 3.6, is parabolic with a maximum of $M_{\max} = \frac{1}{8}p\ell^2$, as shown in Fig. 4.29a. This is the actual moment distribution, denoted by $M^0(x)$.

The transverse displacement at the middle of the beam is determined by the virtual work equation (4.43). The test load consists of a transverse unit force $P_1 = 1$, acting at the center of the beam as shown in Fig. 4.29b. The center displacement is then expressed as

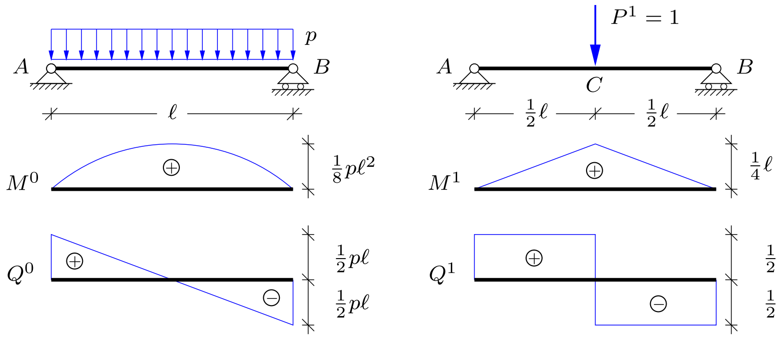


Fig. 4.29: Displacement of midpoint: Actual and test load distributions.

$$w_C = \int_0^\ell \Big(\frac{M^0(x)}{EI_z}\, M^1(x) + \frac{Q^0(x)}{GA_z}\, Q^1(x)\Big) dx$$
$$= 2\int_0^{\ell/2} \Big(\frac{M^0(x)}{EI_z}\, M^1(x) + \frac{Q^0(x)}{GA_z}\, Q^1(x)\Big) dx.$$

The moment and shear force distributions $M^0(x)$, $M^1(x)$ and $Q^0(x)$, $Q^1(x)$ are shown in Fig. 4.29, and the integral can be computed by using the integral formulas in Table 4.1. The moment integral is evaluated by the formula in the fifth row of Table 4.1 and the shear force integral is evaluated as a simple mean value,

$$w_C = 2\,\frac{\ell}{2}\Big(\frac{1}{EI_z}\,\frac{5}{12}\,\frac{p\ell^2}{8}\,\frac{\ell}{4} + \frac{1}{GA_z}\,\frac{1}{2}\,\frac{p\ell}{2}\,\frac{1}{2}\Big) = \frac{5}{384}\,\frac{p\,\ell^4}{EI_z} + \frac{1}{8}\,\frac{p\,\ell^2}{GA_z}\,.$$

The first term is the result obtained in Example 4.4 by integrating the differential equation for a beam without shear flexibility. Also in this case the relative magnitude of the contribution from shear flexibility is given by the parameter $EI_z/(GA_z\ell^2)$ identified in Example 4.9. The conclusion in this case is the same, namely that shear flexibility is mainly important for rather short beams and beams with low shear modulus. □

Example 4.12. Support rotation of simply supported beam. In this example the task is to determine the rotation at the support of the uniformly loaded beam shown in Fig. 4.30a. The test load is a unit moment $M^1 = 1$, applied at the support as shown in Fig. 4.30b.

The rotation follows from the virtual work relation (4.45). It is observed from the figure that the actual shear force distribution $Q^0(x)$ is anti-symmetric, while the shear force distribution $Q^1(x)$ from the test load is symmetric. Thus, the corresponding integral vanishes and there is no contribution from shear flexibility to the rotation at the support. The expression for the rotation therefore takes the simplified form

$$\theta_B = \int_0^\ell \frac{M^0(x)}{EI_z}\, M^1(x)\, dx\,.$$

The bending stiffness EI_z is constant, and the result then follows from the product of a parabolic and a triangular distribution, given in the fourth row in Table 4.1,

$$\theta_B = \frac{1}{EI_z}\int_0^\ell M^0(x)\, M^1(x)\; dx = \frac{1}{EI_z}\,\frac{2\ell}{3}\,\frac{p\,\ell^2}{8}\,\frac{1}{2} = \frac{1}{24}\,\frac{p\,\ell^3}{EI_z}\,.$$

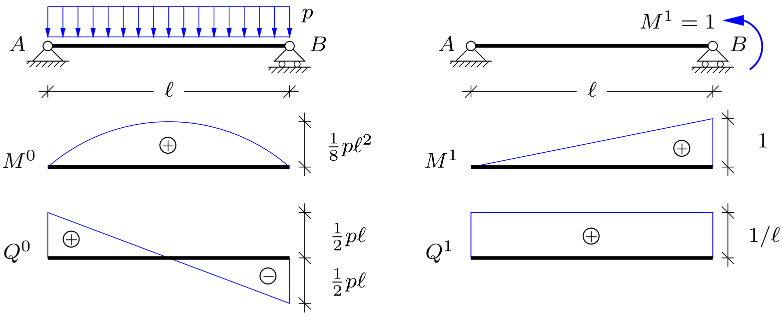


Fig. 4.30: Rotation at support: Actual and test load distributions.

Note, that because the parabolic distribution is symmetric, only the symmetric part of the triangular distribution contributes to the integral. The symmetric part of the triangle is a constant intensity of 1/2, and thus the integral represents half the area of the parabolic distribution. Symmetry arguments like this can often be used to reduce the computation of the virtual work integral. □

Example 4.13. Deformation of beam with two load components. Structures are often analyzed for a number of load combinations, and it may be advantageous to treat the effect of simple load components individually, and then combine the results. The problem is illustrated for a simply supported beam extending beyond one of the supports as shown in Fig. 4.31. The load is a combination of a concentrated force P acting at C and a distributed load with intensity p on AB. The present example determines the tip displacement by superposition of the tip displacements for the individual load cases. The effect of shear deformation is neglected.

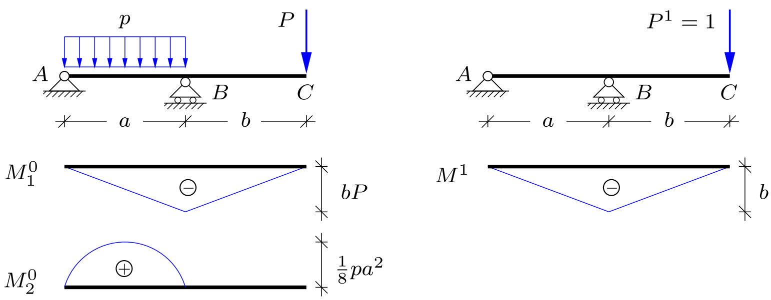


Fig. 4.31: Beam with combined concentrated and distributed load.

The actual moment distribution M^0 consists of the separate moment distributions M_1^0 and M_2^0 for tip load P and for the distributed load p, respectively, as indicated in Fig. 4.31a. The tip displacement for each of the two load cases is determined by the virtual work equation (4.43) by using the moment distribution M^1 from a unit test force $P^1 = 1$ acting at C as illustrated in Fig. 4.31b. The moment curves for the individual load components are seen to be simple shapes, easily integrated by use of the results in Table 4.1. The tip displacement from the concentrated load is obtained from the moment distribution $M_1^0(x)$ as

$$w_{C_1} = \int_{ABC} \frac{M_1^0 M^1}{EI_z} ds = \frac{(a+b)}{3EI_z} (-bP)(-b) = \frac{1}{3} \frac{Pb^2(a+b)}{EI_z},$$

while the tip displacement from the distributed load is obtained from the moment distribution $M_2^0(x)$ as

$$w_{C_2} = \int_{AB} \frac{M_2^0 M^1}{EI_z} ds = \frac{a}{3EI_z} \frac{1}{8} pa^2 (-b) = -\frac{1}{24} \frac{pa^3 b}{EI_z}.$$

The total tip displacement for the load combination is found as the sum of the individual contributions,

$$w_C = w_{C1} + w_{C2} = \frac{b}{3EI_z} \Big(Pb(a+b) - \tfrac{1}{8} pa^3 \Big).$$

Zero tip displacement $w_C = 0$ is obtained for

$$Pb(a+b) - \tfrac{1}{8} pa^3 = 0 \qquad \Rightarrow \qquad P = \frac{1}{8} \frac{pa^3}{b(a+b)}.$$

For spans of same length zero tip displacement corresponds to a distributed load of magnitude $pa = 16P$. □

4.4.3 Virtual work and displacements in frames

The virtual work equation (4.40) for beams can be extended to frame structures, formed by joining beams. The procedure is here illustrated for plane frames, but applies also to three-dimensional frame structures. The frame consists of a number of beams, that each have a set of internal forces that satisfy equilibrium with the external load. The frame is now subjected to a virtual displacement field, by which joints are moved and beams are deformed. In this process the principle of virtual work applies to each of the individual beams, and therefore also to the sum of the external and internal work contributions from all the beams,

$$\sum_{\text{beams}} \delta V_{\text{ex}} = \sum_{\text{beams}} \delta V_{\text{in}}. \tag{4.47}$$

The internal virtual work δV_{in} in each of the beams follows directly from its definition (4.39) in terms of the internal forces and the generalized virtual strains $\delta\varepsilon$, $\delta\gamma$ and $\delta\kappa$ for a single beam.

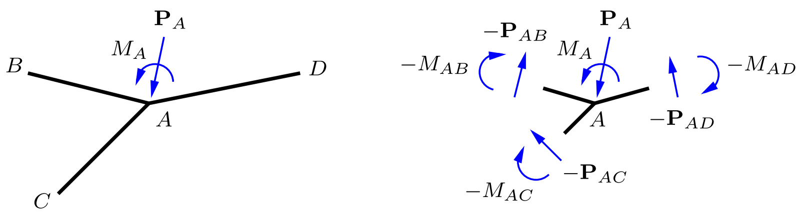


Fig. 4.32: Balance between internal and external forces at joint.

The external virtual work $\delta V_{\rm ex}$ relating to the beams consists of the virtual work of the loads on the beams plus the external virtual work of the section forces at the ends of the beams, illustrated in Fig. 4.24. The virtual work of the loads inside the beams are retained as contributions relating to the individual beam, but the end loads formed by the section forces at the beam ends are reformulated. The principle is illustrated in Fig. 4.32 showing three beams AB, AC and AD joined at A. Equilibrium of the joint implies that the external load on the joint is balanced by the section forces from the beams connected to the joint. The section forces act in the opposite direction of those acting on the beams, and thus the equilibrium conditions for the joint can be written as

$$\mathbf{P}_A - \sum \mathbf{P}_{A*} = \mathbf{0}, \qquad M_A - \sum M_{A*} = 0, \tag{4.48}$$

where the subscript $A*$ denotes any of the connected beams AB, AC, etc. By these equilibrium conditions the sum of all contributions from the beams connected to a joint is equal to the external load applied to the joint. Thus, the contribution from all the forces and moments from beams connected to a joint j can be replaced by $\delta\mathbf{u}_j^T\mathbf{P}_j$ and $\delta\theta_j M_j$. After replacing the contributions from the beam end sections with the contribution from the corresponding joints, the virtual work equation for frames takes the form

$$\begin{aligned}\sum_{\rm joints}\Big[\,\delta\mathbf{u}_j^T\mathbf{P}_j + \delta\theta_j\,M_j\Big] + \sum_{\rm beams}\int_{\ell_i}\big(\delta\mathbf{u}^T\mathbf{p} + \delta\theta\,m\big)ds \\ = \sum_{\rm beams}\int_{\ell_i}\big(\,\delta\varepsilon\,N + \delta\gamma\,Q + \delta\kappa\,M\,\big)ds.\end{aligned} \tag{4.49}$$

The virtual work equation for frames is a straightforward extension of the similar result (4.37) for beams, and its use for calculation of displacements in elastic frames is illustrated in the following examples.

Example 4.14. Displacements in simple frame. Figure 4.33 shows a simple angle frame with a uniformly distributed vertical load with intensity p acting on the horizontal beam CDB. The effect of extension/compression effects due to the normal force on the vertical displacement at the center D is illustrated in the context of slender beams, where it was demonstrated in Example 4.11 that the shear flexibility effect can be neglected.

The vertical displacement is determined by the virtual work equation (4.43) involving an integral over the entire frame,

$$w_D = \int_{ACB}\Big(\frac{M^0M^1}{EI_z} + \frac{N^0N^1}{EA}\Big)ds.$$

The moment distributions are only present over CB and symmetric, while the normal force only contributes over AC. Thus, the virtual work relation can be written as

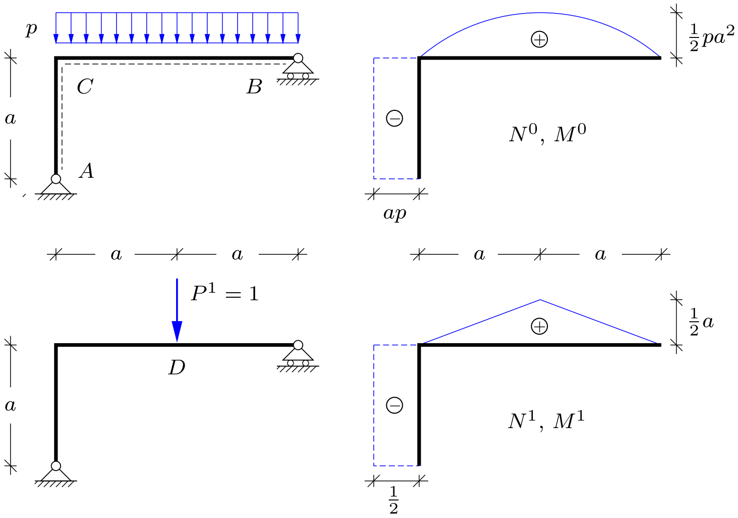


Fig. 4.33: Angle frame: Actual and test load distributions, M (–), N (- -).

$$w_D = \frac{2}{EI_z}\int_{CD} M^0 M^1\, ds + \frac{1}{EA}\int_{AC} N^0 N^1\, ds\,.$$

The first integral involves a half parabola and a triangle, given by the fifth row of Table 4.1, while the normal force distributions are piecewise constant as indicated in Fig. 4.33b by the dashed lines. The integrals then give

$$w_D = \frac{2}{EI_z}\,\frac{5a}{12}\,\frac{pa^2}{2}\,\frac{a}{2} + \frac{a}{EA}\,pa\,\frac{1}{2} = \frac{5}{24}\,\frac{pa^4}{EI_z} + \frac{1}{2}\,\frac{pa^2}{EA}\,.$$

The relative importance of the axial deformation is determined by the parameter combination

$$\frac{EI_z}{EA\ell^2} \sim \frac{1}{12}\,\frac{h^2}{\ell^2}\,.$$

where $\ell = 2a$, and the numerical factor corresponds to a rectangular cross-section. The relative magnitude of the axial deformation is proportional to $(h/\ell)^2$ as for the effect of shear flexibility. Therefore the axial deformation is often neglected in hand-calculation analysis of elastic frames. □

Example 4.15. Horizontal loading of simple frame. In the present example the loading of the simple frame in the previous example is changed, so that a distributed load with intensity p is acting in the horizontal direction on the vertical beam AC. The horizontal displacement w_C of the junction is determined by the virtual work equation. The reactions can for instance be determined by horizontal projection, moment about B and moment about A:

$$R'_A = ap\,, \qquad R_A = -\tfrac{1}{4}ap\,, \qquad R_B = \tfrac{1}{4}ap\,.$$

The moment is zero at the supports in A and B and varies linearly in BC and as a parabola in AC. Thus, the full moment distribution is determined by the moment M_C at the joint C and the local maximum of the moment in AC. By section in BC at C it is found that

$$\curvearrowright \qquad M_C - 2aR_B = 0 \qquad \Rightarrow \qquad M_C = \tfrac{1}{2}pa^2\,.$$

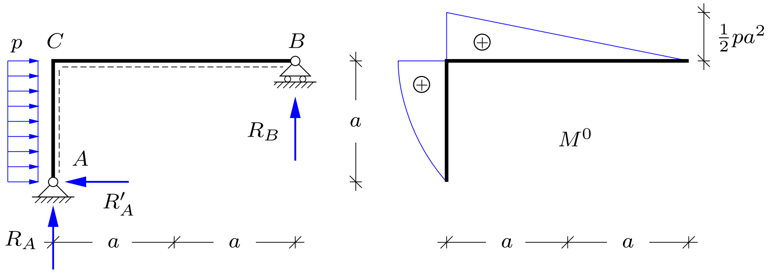


Fig. 4.34: Frame with horizontal distributed load on vertical beam.

The local maximum of the moment in AC is found via the shear force Q. Due to the roller support in B there is no normal force in CB, and thus the shear force in AC vanishes at the joint C. This implies that the moment distribution in AC has the maximum value M_C and zero slope in C. Hereby, the moment distribution M^0 is determined and it is shown in Fig. 4.34b.

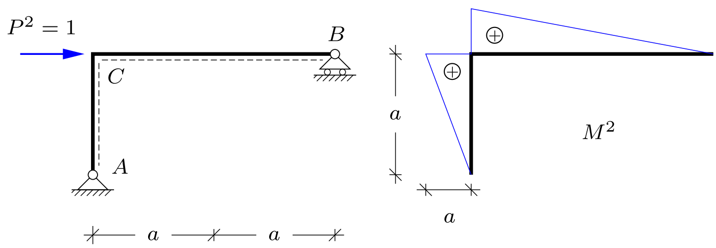


Fig. 4.35: Virtual moment distribution corresponding to the horizontal displacement of C.

The horizontal displacement of the joint C is determined from the extended form of the virtual work relation (4.43) integrating the product of the actual curvature and the moment from the test load in Fig. 4.35 over the whole frame,

$$w_C = \int_{ACB} \kappa^0 M^2 \, ds = \int_{AC} \frac{M^0 M^2}{EI_z} ds + \int_{CB} \frac{M^0 M^2}{EI_z} ds \, .$$

The integral for AC is solved by the integral formula in the fifth row, while the integral for CB is determined by the triangles in the first row of Table 4.1,

$$w_C = \frac{a}{12EI_z} \frac{pa^2}{2} 5\,a + \frac{2a}{3EI_z} \frac{pa^2}{2} a = \frac{13}{24} \frac{pa^4}{EI_z} \, .$$

This displacement is considerably larger than the vertical displacement found in the previous example. This is primarily due to the rolling support in B. □

Example 4.16. Displacement of a cable supported beam. The virtual work equation is also a convenient tool for determining displacements in structures with different member types. Figure 4.36 shows a cable supported beam as found e.g. in canopy roofs and simple bridges. The beam ADC has a fixed simple support at A and is supported by the cable BD at its center. The structure is statically determinate, and the internal hinge at D allows for four support reactions as shown in the figure. The length of the beam ADC is $8a$, and the height AB is $3a$. These simple proportions give the cable length BD as $5a$. The goal is to calculate the vertical displacement of the beam, represented by the vertical displacement of the tip w_C. The calculation is based on the bending deformation of the beam ADC with stiffness EI_z and the extension of the cable BD with stiffness EA. The cable cross-section is typically much smaller than the area of the beam cross-section, and the extension of the cable can therefore provide a considerable contribution to the displacement w_C.

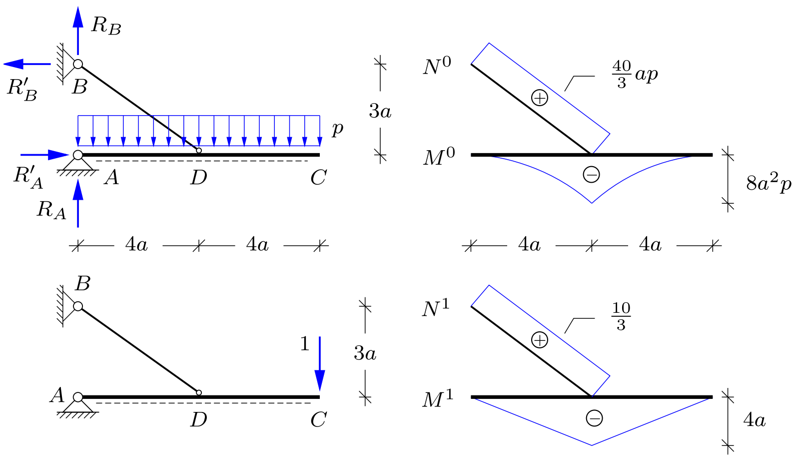


Fig. 4.36: Beam with simple and cable supports.

The horizontal reaction components are of equal magnitude and determined by moment about A or B,

$$R'_A = R'_B = \frac{32}{3}\,ap\,.$$

The vertical reaction component R_B must lead to a total reaction at B in the direction of the cable, whereby the reactions in B are the horizontal and vertical projections of the bar force N_{BD}, which implies the following equilibrium relations

$$R_B = \tfrac{3}{4}R'_B = 8ap\,.$$

The reaction component R_A is determined from moment equilibrium of the beam about the center D, whereby $R_A = 0$. The tension in the cable N^0_{BD} follows e.g. from vertical projection of node B,

$$N^0_{BD} = \tfrac{5}{3}R_B = \tfrac{40}{3}ap\,.$$

The moment curve for the beam ADC is zero at the ends where it has horizontal tangent as the shear force vanishes. The maximum value occurs at the center D, where

$$M_D = 2a\,(4ap) = 8a^2p\,.$$

The moment distribution in the beam and the axial force in the cable are shown in the top right part of Fig. 4.36.

The virtual system corresponding to the vertical displacement w_C of the beam tip is shown in the lower part of Fig. 4.36, indicating a vertical unit force $P^1 = 1$ acting in C. The moment in the beam ADC is zero at the ends, and varies linearly to a moment $M^1_D = 4a$

at the center. The cable force is found e.g. by calculating the horizontal reaction in B by moment for the full structure about A, and then obtaining the cable force via horizontal projection of the forces acting on the node B. This determines the cable force $N^1_{BD} = \frac{10}{3}$.

The vertical displacement w_C is now determined from the virtual work equation including the terms representing bending of the beam ADC and extension of the cable BD. This can be formulated as

$$w_C = \int_{ADC} \frac{M^0(s)M^1(s)}{EI_z}\, ds + \int_{BD} \frac{N^0(s)N^1(s)}{EA}\, ds\,.$$

The moment integral is evaluated by splitting it into equal integrals over AD and DC and then using the last row in Table 4.1. The normal forces are constant and this integral therefore follows immediately from the length $5a$ of BD,

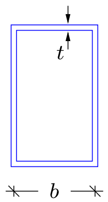
$$w_C = 2\frac{4a}{EI_z}\,\frac{1}{4}(8a^2p)(4a) + \frac{5a}{EA}\,\frac{40ap}{3}\,\frac{10}{3} = 64\frac{a^4p}{EI_z} + \frac{200}{9}\,\frac{a^2p}{EA}\,.$$

The first term is the contribution from beam bending while the second is the effect of cable extension. It is seen that a small cable cross-section area A relative to the beam cross-section may lead to a considerable contribution from extension of the cable.

It is worth noting that the effect of flexibility of e.g. the support B can easily be included in the form of a local spring that would be loaded by the corresponding reaction at B. □

4.5 Exercises

Exercise 4.1. The figure shows a rectangular box section with height h, width b and wall thickness t. As demonstrated in Example 4.1 the moment of inertia of a massive rectangular cross section is $I_z = \frac{1}{12}h^3b$. This result can be used to determine the moment of inertia of the present box section. The cross section is assumed to be thin-walled, whereby $t \ll h$ and b. Note that dimensions h and b are with respect to the centerlines of the individual flanges, which e.g. means that the total height of the box section is $h + t$.

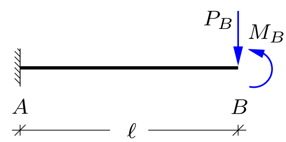


a) Determine the moment of inertia I_z^{out} for a massive rectangular cross section with dimensions corresponding to the outer dimensions of the box section.

b) Determine the moment of inertia I_z^{in} for a massive rectangular cross section with dimensions corresponding to the inner dimensions of the box section.

c) Use the results in a) and b) to determine the moment of inertia I_z of the box section.

Exercise 4.2. The figure shows a cantilever beam of length ℓ with constant bending stiffness EI_z. A concentrated moment M_B and a vertical force P_B act at the tip of the beam in B.



a) Find an expression for the moment $M(x)$.

b) Determine the expression for $w(x)$.

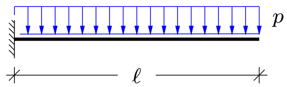
c) Find the displacement $w(\ell)$ and rotation $\theta(\ell)$ at the tip.

Exercise 4.3. The figure shows a cantilever beam of length ℓ with constant bending stiffness EI_z. The beam is loaded by a transverse uniformly distributed load with intensity p.

a) Find an expression for the moment $M(x)$.

b) Determine the expression for $w(x)$.

c) Find the displacement $w(\ell)$ and rotation $\theta(\ell)$ at the tip.



Exercise 4.4. Consider the cantilever beam in Fig. 4.27, but let the force P act at distance a from the left support instead of at the tip. Determine an expression for the displacement $w(x)$ in the two intervals $x < a$ and $x > a$, respectively.

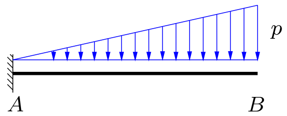
Exercise 4.5. The figure shows a cantilever beam of length ℓ with constant bending stiffness EI_z. The beam is loaded by a transverse distributed load with linearly increasing intensity $p(x) = px/\ell$, whereby p is the tip intensity.

a) Solve the differential equation (4.16) to find the expression for the moment $M(x)$.

b) Determine the expression for the displacement $w(x)$ and the rotation $\theta(x)$.

c) Find the displacement $w(\ell)$ and rotation $\theta(\ell)$ at the tip.

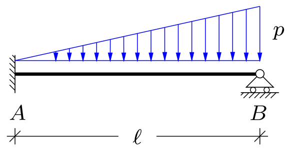
d) Find the magnitude of the reactions.



Exercise 4.6. The figure shows a beam of length ℓ with constant bending stiffness EI_z. The beam is fixed in A and simply supported in B, which makes it statically indeterminate. It is loaded by a transverse distributed load with linearly increasing intensity $p(x) = px/\ell$, where p is the intensity at the simple support.

a) Setup the fourth order differential equation governing the transverse displacement $w(x)$, and find the solution containing four arbitrary constants.

b) Use the four boundary conditions to determine the expression for the displacement $w(x)$ and the rotation $\theta(x)$.



c) Determine the expression for the moment $M(x)$ and the shear force $Q(x)$ and find the magnitude of the reactions.

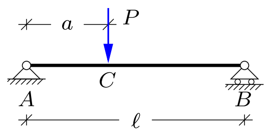
d) Determine the location $x_{\max}$ and the magnitude $M_{\max}$ of the maximum moment.

Exercise 4.7. The figure shows a simply supported beam of length ℓ and with constant bending stiffness EI_z. A concentrated vertical load P acts at distance a from the left support. This problem is solved in Example 4.5 via the differential equation, whereby the beam must be divided into two parts, which complicates the analysis. In this exercise the problem is solved by the principle of virtual work, and hopefully it is observed how comparatively easy the results are obtained.

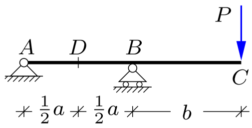
a) Draw the actual moment distribution M^0.

b) Use the virtual work equation to find the transverse displacement w_C at the location of the force.

c) Use the virtual work equation to find the rotations at the supports.



Exercise 4.8. The figure shows a beam of length $a + b$ with a simple fixed support in A and a simple support permitting horizontal motion in B. The beam is linear elastic with constant bending stiffness EI_z, and a vertical force P acts at the tip of the beam in C. Solve the following problems by the principle of virtual work.



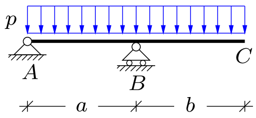
a) Determine the displacement w_C at the tip of the beam.

b) Determine the rotation θ_C at the tip of the beam.

c) Determine the rotation θ_A at the simple support in A.

d) Determine the displacement w_D at the center D of AB.

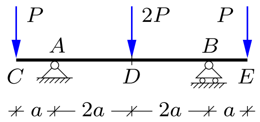
Exercise 4.9. The figure shows a two span simply supported beam similar to that in the previous exercise. The beam is linear elastic with bending stiffness EI_z, and a vertical uniformly distributed load with intensity p acts on both spans of the beam. Solve the following problems by the principle of virtual work.



a) Determine the displacement w_C at the tip of the beam.

b) Determine the rotation θ_A at the simple support in A.

Exercise 4.10. The figure shows a three span simply supported beam with constant bending stiffness EI_z. The two exterior spans AC and BE both have length a, while the inner span ADB has length $4a$. Vertical tip forces P act at C and E, while a force $2P$ acts at the center of the beam in D. Solve the following problems by the principle of virtual work.

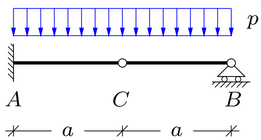


a) Determine the displacement w_C of the tip.

b) Find the rotation θ_A at the left support.

c) Determine the transverse displacement w_D at the center of the beam.

Exercise 4.11. The figure shows a beam of length $2a$ which is fixed in A and simply supported in B. The beam is furthermore hinged at the center C. The beam is linear elastic with constant bending stiffness EI_z, and it is loaded by a uniformly distributed load with intensity p. Solve the following problems by the principle of virtual work.



a) Determine the displacement w_C at the location of the hinge.

b) Determine the rotation θ_B at the simple support.

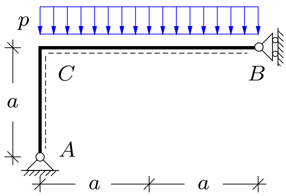
Exercise 4.12. Consider the simple frame in Example 4.15. Determine the rotation θ_C of the joint and sketch the deformation form of the frame.

Exercise 4.13. The figure shows a simple frame similar to that in Example 4.14, with width $2a$ and height a. The supports are fixed simple in A and simple with a vertical roller in B. All beams are linear elastic with constant bending stiffness EI_z. The frame is loaded by a uniformly distributed load with intensity p on the horizontal beam BC. Solve the following problems by the principle of virtual work.

a) Determine the vertical displacement w_B at the support in B.

b) Determine the rotation θ_C at the joint.

c) Determine the rotation θ_A at the fixed simple support in A.

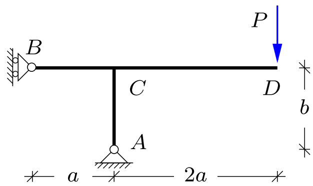


Exercise 4.14. The figure shows a T-frame with width $a + 2a$ and height b. A vertical force P is acting at the tip in D. The distribution of the internal forces has previously been determined in Exercise 3.17. All beams in the frame are linear elastic with constant bending stiffness EI_z. Solve the following problems by the principle of virtual work.

a) Determine the vertical displacement w_D at the location of the force.

b) Determine the rotation θ_C at the joint.

c) Determine the vertical displacement w_B at the left support in B.

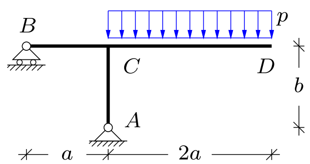


Exercise 4.15. The figure shows a T-frame similar to that in the previous exercise, with width $a+2a$ and height b. Note that the roller in B permits horizontal motion. A uniformly distributed vertical load with intensity p acts on the cantilever part CD. All beams in the frame are linear elastic with constant bending stiffness EI_z. Solve the following problems by the principle of virtual work.

a) Determine the vertical displacement w_D at the tip D.

b) Determine the rotation θ_C at the joint C.

c) Determine the horizontal displacement w_B at the left support in B.

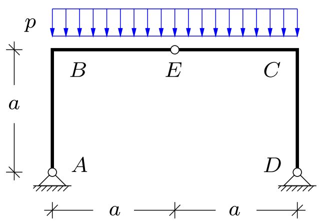


Exercise 4.16. The figure shows a three-hinge frame with an internal hinge placed in E at the center of the horizontal beam BC. The frame is loaded by a uniformly distributed load with intensity p on BC. The distribution of the internal forces has previously been determined in Exercise 3.18. All beams in the frame are linear elastic with constant bending stiffness EI_z. Solve the following problems by the principle of virtual work.

a) Determine the vertical displacement at the hinge in E.

b) Determine the rotation θ_B at the left joint.

c) Determine the rotation θ_A at the left support.



Exercise 4.17. The figure shows a three-hinge frame with an internal hinge placed in E at the center of the horizontal beam BC. The frame is similar to that in the previous exercise, but is now exposed to horizontal forces P acting at the joints B and C, respectively. The distribution of the internal forces has previously been determined in Exercise 3.19.

a) Determine the sum of the horizontal displacements in B and C, and use this result to obtain the individual horizontal displacements in B and C.

b) Determine the rotation θ_B at the left corner B.

c) Determine the rotation θ_A at the left support.