The Force Method

6

Statically determinate structures are characterized by having the minimum number of supports and internal connections. While this permits direct analysis based on statics alone, it also entails some disadvantages such as vulnerability if any part of the structure is damaged. In addition statically determinate frames are prone to rather large moments e.g. at mid-span and at corners. This feature of the moment distribution typically leads to rather flexible structures. Thus, there are several reasons why most current structures are statically indeterminate. With proper care statically indeterminate structures can be designed such that the maximum section force is reduced, the structures are less flexible, and vulnerability to failure of individual structural elements may be reduced. A simple but typical example of moment redistribution and increased stiffness is a bridge with a main girder that is continuous over the supports.

The analysis of statically indeterminate structures requires information of the stiffness of the individual structural elements. There are two basically different approaches to the analysis of statically indeterminate structures. In the force method an equivalent statically determinate structure is introduced e.g. by introducing hinges at selected points and removing supports. In the name 'force method' the force refers to any component of the section forces and thereby also includes moments. The analysis then consists in determining the moment pairs necessary for restraining the relative rotation at the hinges

S. Krenk, J. Høgsberg, *Statics and Mechanics of Structures*,
DOI 10.1007/978-94-007-6113-1_6,

and prevent the motion at the temporarily removed supports. This method combines static analysis to determine the section forces in the equivalent statically determinate structure and calculation of the elastic displacements which must be compensated to reestablish the original structure. These subjects have been treated in some detail in Chapters 3 and 4. Thus, the main objective of the present chapter is to present and illustrate the systematic use of these principles to the analysis of statically indeterminate beam and frame structures. As the idea of the force method is to close the auxiliary kinematic discontinuities it is sometimes called the 'method of consistent deformations'.

The force method is rather straightforward to use in hand calculations of small structures with a few degrees of static indeterminacy. However, each step in the analysis involves determination of the internal forces in the full equivalent statically determinate structure. This aspect of the force method makes it less convenient for computer implementation. The stiffness method is based on a different way of thinking. In the stiffness method the structure is characterized by nodes, connected by structural elements. The properties of the structure are related to the structural elements between the nodes, and the unknown parameters of the analysis are the displacements of the nodes. An example of this approach was presented for truss structures in Section 2.5. The stiffness method for frames is presented in Chapter 7, first as a hand calculation method and subsequently generalized to Finite Element format.

6.1 Principle of the force method

The effect of making a structure statically indeterminate is here briefly indicated by the two simply supported two-span beams shown in Fig. 6.1. The beam in Fig. 6.1a has a hinge at the intermediate support. This permits a rotation discontinuity $\Delta\theta_C = \theta_C^+ - \theta_C^-$ between the two beams at C, whereby the bending moment in C vanishes. This provides an additional equilibrium condition $M_C = 0$ required to determine the four reaction forces by pure statics, as discussed in Chapter 1. Thus, the beam structure in Fig. 6.1a is statically determinate.

The beam shown in Fig. 6.1b is continuous at the support C thereby maintaining continuity of the beam rotation at the support ($\Delta\theta_C = 0$). The rotation continuity is the result of a non-vanishing moment at the support ($M_C \neq 0$). As indicated in the figure this leads to a redistribution of the bending moment in the beam, whereby both the maximum moments and the deformations are reduced. However, equilibrium of the beam gives only three equilibrium conditions, one short of the four reaction components, and the moment M_C can therefore not be determined by statics alone. Thus, the beam structure in Fig. 6.1b is statically indeterminate.

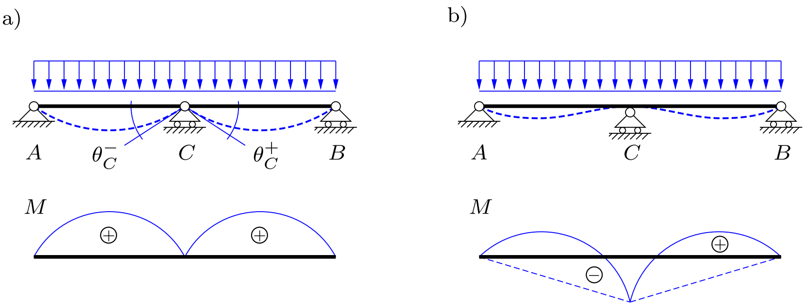


Fig. 6.1: Moment reduction in beam by additional rotation constraint in C.

Statically indeterminate two-span beam

The principle of the force method for solving statically indeterminate structures is demonstrated by the two-span beam shown in Fig. 6.2. The beam structure has four reactions, where the horizontal reaction in A can be found directly by horizontal equilibrium,

$$\rightarrow \qquad R'_A = 0\,.$$

This leaves the three vertical reactions R_A, R_B and R_C as unknown, and since only two equilibrium equations remain the structure is one time statically indeterminate, or it has a degree of indeterminacy equal to 1.

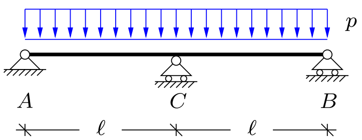


Fig. 6.2: Statically indeterminate two-span beam with distributed load.

The first step is to convert the structure to an equivalent statically determinate structure. This can be done in several ways, as discussed in Section 6.2.1. In the present example the vertical reaction R_C at the center of the beam is considered as an external force acting on the structure. This initially unknown force is denoted as X_1 instead of R_C. With X_1 as an external load the equivalent structure in Fig. 6.3 is statically determinate. The equivalent structure is now analyzed for two load cases: the actual distributed load p, and the new vertical force X_1 acting in C. As the structure is elastic, the principle of superposition applies: Thus the structure can be analyzed for each individual load case, and the result is obtained by superposition of the individual load cases, as explained previously in Section 3.4.1. The value of the auxiliary force X_1 is determined from a condition on the resulting displacement. In the present case this condition is that the structure has zero vertical displacement at C. Thus, the unknown external force X_1 must be determined to satisfy this condition. It is therefore necessary to determine the vertical displacement at C for both the two load cases. This is done by use of the principle of virtually work, developed in detail in Chapter 4.

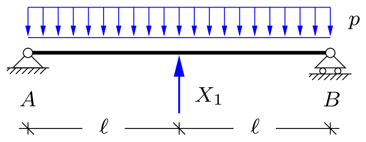


Fig. 6.3: Equivalent statically determinate structure with additional force X_1.

The auxiliary force X_1 represents the force needed to enforce the associated kinematic constraint in the statically determinate structure. As demonstrated in the following it is possible to have multiple auxiliary forces X_j's, and they can represent both external forces and moments as well as internal force and moment couples. The auxiliary forces X_j are denoted as redundant forces or redundant components.

First, the equivalent structure is analyzed for the actual distributed load without the redundant component, i.e. with $X_1 = 0$. This load case is shown in Fig. 6.4a. The reactions are

$$R_A^0 = R_B^0 = p\ell \,,$$

and Fig. 6.4a shows the parabolic moment distribution M^0. The index 0 is used to refer to the load case with the actual external load, in the absence of any redundant force components. The displacement of C from the distributed load is determined by the virtual work equation described in Section 4.4 using the test load $X_1 = 1$. This load case and the corresponding moment distribution $M^1(x)$ are shown in Fig. 6.4b. The transverse displacement is typically dominated by the contribution from beam bending, and only this contribution is included in the following. The transverse displacement at C from the distributed load then follows from the virtual work equation (4.43) as

$$\xi_{10} = \int_0^\ell \frac{M^1 M^0}{EI}\, dx = \frac{2}{EI}\,\frac{\ell}{12}\left(\frac{1}{2}p\ell^2\right)\left(-\frac{5}{2}\ell\right) = -\frac{5}{24}\,\frac{p\ell^4}{EI} \,.$$

The displacement is denoted ξ_{10}, where the first subscript indicates that this displacement corresponds to the redundant force X_1 in location and direction, while the second subscript 0 refers to the load as being the actual uniformly distributed load. The integral is evaluated by using row five in Table 4.1.

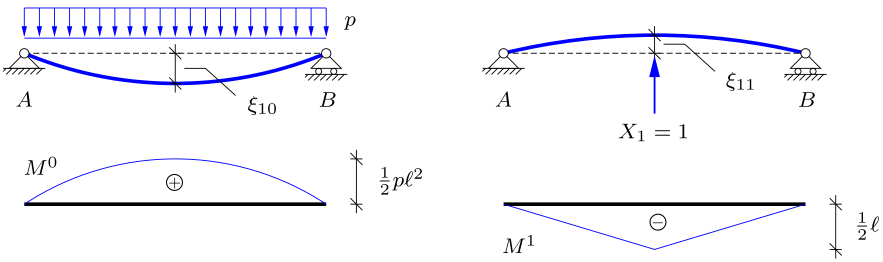


Fig. 6.4: Moment in statically determinate beam: a) Actual load, b) test load $X_1 = 1$.

The second load case corresponds to a unit magnitude of the redundant force, $X_1 = 1$. This load case is shown in Fig. 6.4b with reactions

$$R_A^1 = R_B^1 = -\tfrac{1}{2}$$

and moment distribution $M^1(x)$. The vertical displacement at C from this unit load is determined by the virtual work equation (4.43) as

$$\xi_{11} = \int_0^\ell \frac{M^1 M^1}{EI}\, dx = \frac{2}{EI}\,\frac{\ell}{3}\left(\frac{1}{2}\,\ell\right)^2 = \frac{1}{6}\,\frac{\ell^3}{EI}\,.$$

The first subscript in ξ_{11} refers to the resulting displacement component and is therefore unchanged, while the second subscript is now 1, referring to the load case $X_1 = 1$. The index system is explained in greater detail in Section 6.2.3, where the general procedure for the force method is presented.

The total displacement ξ_1 in C is determined by superposition of the displacements from the two load cases, giving the condition of zero displacement at C as

$$\xi_1 = \xi_{10} + \xi_{11} X_1 = 0\,.$$

The initially unknown redundant component X_1 follows from this condition as

$$X_1 = -\frac{\xi_{10}}{\xi_{11}} = \tfrac{5}{4}\, p\,\ell\,.$$

Note, that X_1 has the dimension of force. The absolute value EI of the beam stiffness does not enter into the expression, while the fact that the two parts of the beam has the same stiffness in this case has been accounted for in the calculation of the displacement ξ_{10} and the coefficient ξ_{11}.

It if often convenient to evaluate the reactions before proceeding to the calculation of the internal forces. In this example X_1 represents the vertical reaction R_C, which therefore is determined directly as

$$R_C = X_1 = \tfrac{5}{4}\,p\ell\,.$$

The two remaining reactions are determined subsequently by equilibrium or by superposition. The latter approach gives

$$R_A = R_A^0 + R_A^1 X_1 = p\ell + (-\tfrac{1}{2})\tfrac{5}{4}\,p\ell = \tfrac{3}{8}\,p\ell\,, \qquad R_B = R_A\,.$$

Vertical equilibrium confirms that $R_A + R_B + R_C = 2p\ell$.

The moment distributions of the individual load cases are always available from the analysis since they have been used in the virtual work equation to determine the equation coefficients. Thus, the resulting moment distribution can be determined by superposition. Alternatively, the moment distribution can be found by a standard section analysis, as the reactions have just been been obtained. In this example the moment distribution is determined by both methods.

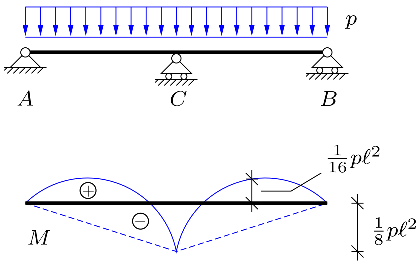


Fig. 6.5: Moment distribution.

The moment distributions M^0 and M^1 shown in Fig. 6.4 are parabolic and piecewise linear, respectively. With the origin of the x-coordinate at A the expressions in terms of x are

$$M^0(x) = \tfrac{1}{2}p\,x(2\ell - x)\,, \qquad M^1(x) = \begin{cases} -\tfrac{1}{2}\,x & \text{for } x < \ell\,, \\ -\tfrac{1}{2}(2\ell - x) & \text{for } x > \ell\,. \end{cases}$$

The expression for the resulting moment is then found by superposition as

$$M(x) = M^0(x) + M^1(x)X_1 = \begin{cases} \tfrac{1}{8}p\,x(3\ell - 4x) & \text{for } x < \ell\,, \\ \tfrac{1}{8}p(2\ell - x)(4x - 5\ell) & \text{for } x > \ell\,. \end{cases}$$

The total moment distribution is shown in Fig. 6.5. The numerically largest moment $M_{\max} = -M(\ell) = \frac{1}{8}p\ell$ is located in C, while the moment at the center of the two spans is $M(\frac{1}{2}\ell) = \frac{1}{16}p\ell$. The local maximum of the moment in each of the spans occurs where the shear force is zero: For the left span this gives

$$Q(x_{\max}) = R_A - p\,x_{\max} = 0 \qquad \Rightarrow \qquad x_{\max} = \tfrac{3}{8}\ell\,,$$

with local maximum $M(x_{\max}) = \frac{9}{128}p\ell^2 < M_{\max}$.

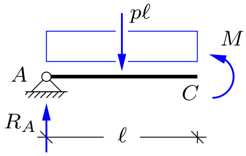


Fig. 6.6: Moment equilibrium in C.

The moment at C can also be determined by section force analysis, as shown in Fig. 6.6, where moment equilibrium for the section at C gives

$$\overset{\frown}{C} \qquad M(\ell) + \tfrac{1}{2}\ell\,p\ell - \ell\,R_A = 0 \quad \Rightarrow \quad M(\ell) = \tfrac{3}{8}\,p\ell^2 - \tfrac{1}{2}\,p\ell^2 = -\tfrac{1}{8}\,p\ell\,.$$

This corresponds to the previously found value and defines the dashed lines in the moment distribution in Fig. 6.5. The final moment distribution is then obtained by superimposing the parabola with maximum value $\frac{1}{8}p\ell^2$, following the procedure presented in Section 3.4.2.

6.2 The general force method

This section presents the general form and a step-by-step procedure of the force method. First the construction of an equivalent statically determinate structure is addressed, with emphasis on the relation between the released kinematic constraints and the corresponding redundant components. The principle of the method is then illustrated for a two times statically indeterminate structure, where coupling effects appear and the solution is obtained by matrix calculus. Finally, the general procedure of the force method for an n times statically indeterminate structure is summarized.

6.2.1 Released structure

For a structure that is n times statically indeterminate, the basis of the force method is the release of n kinematic constraints, whereby the structure becomes statically determinate with n conjugate redundant static components, generating n additional load cases. In the previous example a vertical reaction force was replaced by a vertical force, but in general the kinematic constraints can be of both external or internal type.

External releases

The degree of statically indeterminacy is often found by simply counting unknown reaction components and comparing this to the number of independent equilibrium equations. Therefore, it is often straight forward to simply replace the necessary number of reactions by redundant components X_j. Figure 6.7a shows a fixed support, which represents a reaction moment and two reaction forces. Figures 6.7b-d show the three situations in which the rotation, the vertical displacement and the horizontal displacement have been released, whereby the reaction moment, the vertical reaction force and the horizontal reaction force, respectively, are introduced as a redundant static component X_j.

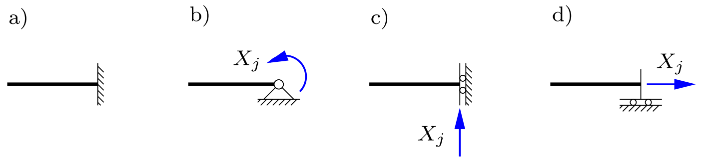


Fig. 6.7: Releasing support conditions.

Figure 6.8a shows a two-span beam with a fixed support in A and simple supports with horizontal rollers in B and C. There are five reactions and only three equilibrium equations, whereby the degree of indeterminacy is 2. The horizontal reaction can be obtained directly from horizontal equilibrium as $R'_A = 0$. Thus, the two redundant components can be chosen among the remaining four reactions. The force method is based on the determination of displacements (or rotations) by the virtual work equation, and therefore simple moment distributions are computationally advantageous. For the beam in Fig. 6.8a this can be obtained by replacing the reactions in B and C by redundant forces, whereby the structure is transformed into a cantilever beam, or by replacing the reaction moment in A and the reaction force in C, resulting in the simply supported beam shown in Fig. 6.8b.

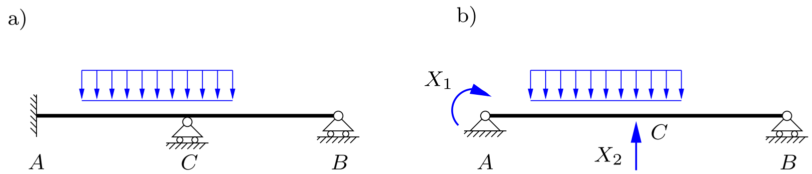


Fig. 6.8: Two-span beam with reactions as redundant components X_1 and X_2.

Internal releases

The equivalent statically determinate structure can also be constructed by releasing internal kinematic constraints and thereby introducing conjugate internal force or moment couples as redundant components. As illustrated in Fig. 6.9a the bending moment, normal force and shear force are the three available internal forces. Equilibrium in a section requires that internal forces always appear in opposing pairs, and thus redundant components following internal releases are introduced as moment and/or force couples. This is indicated in Figs. 6.9b–d, where X_j represents the moment, the shear force and the normal force, respectively.

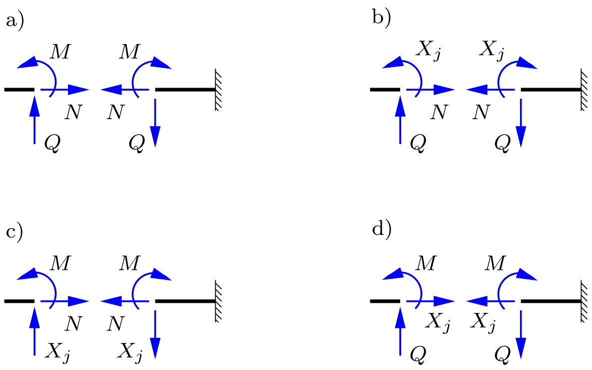


Fig. 6.9: Internal forces as redundant components.

The force method relies on the determination of displacements (or rotations) based on virtual work, where the influence of normal and shear forces is typically omitted. This implies that simple moment distributions often result in a simple analysis, and as demonstrated in the examples of this chapter the moment distributions can often be simplified by using internal moments as redundant components. Hereby, the moment distributions are often localized to limited regions with simple parabolic or linear shapes that are easily integrated in connection with the virtual work equation. Removing the internal bending moment capacity of the beam corresponds to introducing a hinge, as shown in Fig. 6.10, where the redundant component X_j is then added as a pair of opposing moments.

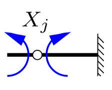


Fig. 6.10: Internal moment couple as redundant force.

Figure 6.11a shows a two span beam with both fixed and simple supports. Previously in Fig. 6.8 the equivalent statically determinant structure was constructed by replacing the reaction moment in A and the reaction force in C by redundant components X_1 and X_2. Instead of the reaction in C, the bending moment in C can be used as redundant component as illustrated in Fig. 6.11b. The hinge is placed in the beam in C and the redundant component X_2 is then introduced as a pair of opposing moments. It is important to note that the pair of moments X_2 cancel in the equilibrium of the full structure. However, the internal moment in C is now $M_C = X_2$, and thus X_2 contributes to the internal moment distribution.

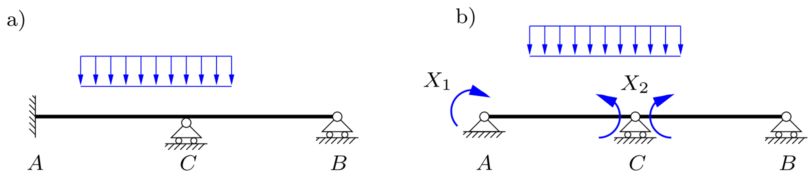


Fig. 6.11: Two-span beam with internal moment in C as redundant force.

Kinematic determinacy of released structure

As discussed above the first step in the force method is to construct an equivalent statically determinate structure and introduce the corresponding conjugate forces/moments as redundant components. In particular, when replacing internal force couples by redundant components it is important that the structure remains kinematically determinate in the sense that no mechanisms are created. For the two-span beam in Fig. 6.11 the reaction moment in A and the internal moment in C have been chosen as redundant components. This structure is statically and kinematically determinate and thus contain no kinematic mechanisms. Another hypothetical choice of the redundant components is shown in Fig. 6.12, where X_1 is the bending moment in C and X_2 is the reaction force in B. As indicated in the figure this choice creates a mechanism and can therefore not be used for the static analysis.

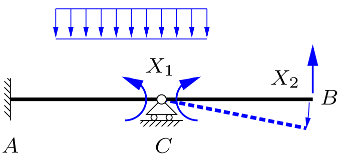


Fig. 6.12: Unstable two-span beam with mechanism in C.

6.2.2 The basic steps

The basic steps of the force method are illustrated here by the two-span beam in Fig. 6.13. This beam is two times statically indeterminate, and thus an equivalent statically determinate structure is obtained by releasing two suitable kinematic constraints.

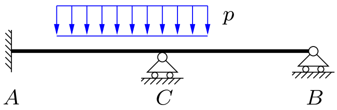


Fig. 6.13: Two-span beam with degree 2 of statically indeterminacy.

The steps of the force method for the two times statically indeterminate two-span beam are as follows:

1) Construct an equivalent statically determinate and stable structure with redundant static components X_1 and X_2.
2) Determine the displacements ξ_{10} and ξ_{20} in the equivalent structure from the external load.
3) Determine the displacements ξ_{11} and ξ_{21} from the load $X_1 = 1$, and the displacements ξ_{12} and ξ_{22} from the load $X_2 = 1$ on the equivalent structure.
4) Form the kinematic constraint equations and determine the redundant components X_1 and X_2.
5) Determine the reactions by superposition or by statics.
6) Determine the distribution of the internal forces by superposition or by equilibrium conditions.

The individual steps of the force method are explained in the following with emphasis on procedure and solution technique.

1) Statically determinate structure and redundant components. The kinematic constraints against rotation in A and vertical displacement in C are released, and the corresponding redundant components are $X_1 = M_A$ and $X_2 = R_C$, as illustrated in Fig. 6.14.

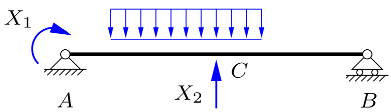


Fig. 6.14: Statically determinate structure with redundant components.

2) Displacements from external load. First the displacements of the equivalent structure with the external loading, but without redundant forces, are determined as illustrated in Fig. 6.15, showing the reactions R_A^0 and R_B^0 and the internal moment $M^0(x)$. The figure also shows the rotation ξ_{10} and the displacement ξ_{20} conjugate to the redundant static components X_1 and X_2. X_1 is a clockwise moment and thus ξ_{10} is positive as a clockwise rotation, while X_2 is an upward force, whereby ξ_{20} is positive if representing an upward displacement.

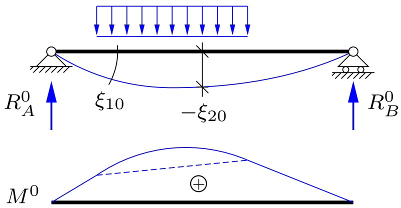


Fig. 6.15: Moment distribution $M^0(x)$ for external load.

The rotation ξ_{10} in A and the displacement ξ_{20} in C are determined by the virtual work equations (4.43) and (4.45), including only the bending moment contribution,

$$\xi_{10} = \int_0^\ell \frac{M^1 M^0}{EI}\, dx\,, \qquad \xi_{20} = \int_0^\ell \frac{M^2 M^0}{EI}\, dx\,.$$

The corresponding two test load cases are $X_1 = 1$ and $X_2 = 1$ with moment distributions $M^1(x)$ and $M^2(x)$, respectively, shown in Fig. 6.16.

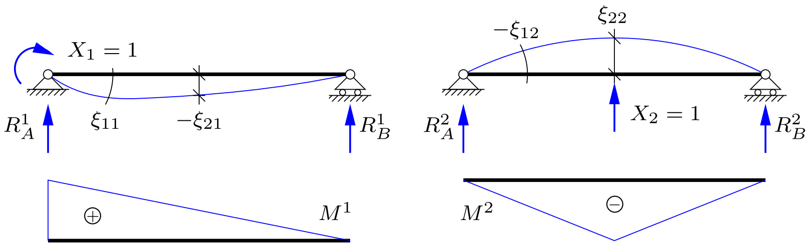


Fig. 6.16: Moment distributions M^1 and M^2 for redundant load cases.

3) Displacements from redundant components. The two remaining load case are for $p = 0$ and the combinations $X_1 = 1$, $X_2 = 0$ and $X_1 = 0$, $X_2 = 1$, respectively. They correspond to the test loads shown in Fig. 6.16, where the displacements and rotations ξ_{11}, ξ_{12}, ξ_{21} and ξ_{22} from the unit loads are also indicated.

In the load case $X_1 = 1$ (with $X_2 = 0$) shown in Fig. 6.16a the rotation ξ_{11} in A and the displacement ξ_{21} in C are determined by the virtual work equations as

$$\xi_{11} = \int_0^\ell \frac{M^1 M^1}{EI}\, dx\,, \qquad \xi_{21} = \int_0^\ell \frac{M^2 M^1}{EI}\, dx\,.$$

The first subscript denotes the number of the displacement/rotation, while the second subscript refers to the index of the redundant static component X_1. In the remaining load case $X_2 = 1$ the rotation ξ_{12} in A and the displacement ξ_{22} in C are

$$\xi_{12} = \int_0^\ell \frac{M^1 M^2}{EI}\, dx = \xi_{21}\,, \qquad \xi_{22} = \int_0^\ell \frac{M^2 M^2}{EI}\, dx\,.$$

It is important to note that because all the loads $X_j = 1$ are normalized to unity the dimensions of ξ_{ij} do not correspond to actual displacements or rotations. Instead they have dimensions of displacement or rotation per unit force or moment, corresponding to flexibility (inverse of stiffness). Thus, ξ_{ij}'s are referred to as *flexibility coefficients*, and the diagonal flexibility coefficients ξ_{jj} are positive, because they are given as the integral of $M^j M^j / EI \geq 0$.

4) Redundant components. The redundant components X_1 and X_2 can now be determined by the kinematic constraints that have been released in the original structure to obtain the equivalent statically determinate structure. In the present case the rotation ξ_1 in A and the transverse displacement ξ_2 in C can be determined by superposition of the three load cases,

$$\xi_1 = \xi_{10} + \xi_{11} X_1 + \xi_{12} X_2\,,$$
$$\xi_2 = \xi_{20} + \xi_{21} X_1 + \xi_{22} X_2\,,$$

where the flexibility coefficients are multiplied by the respective redundant components because they have been computed for unit loads. In the case of multiple equations it is often convenient to express these in matrix form,

$$\begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} - \begin{bmatrix} \xi_{10} \\ \xi_{20} \end{bmatrix},$$

where the flexibility coefficients are assembled in the flexibility matrix on the left side. For fixed supports $\xi_1 = 0$ and $\xi_2 = 0$, and the redundant components can be found by solving the matrix equation

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = - \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix}^{-1} \begin{bmatrix} \xi_{10} \\ \xi_{20} \end{bmatrix}.$$

For a 2×2 matrix the inverse is given explicitly as

$$\begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix}^{-1} = \frac{1}{\xi_{11}\xi_{22} - \xi_{21}\xi_{12}} \begin{bmatrix} \xi_{22} & -\xi_{12} \\ -\xi_{21} & \xi_{11} \end{bmatrix},$$

and the two redundant forces are given by the expression

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -\frac{1}{\xi_{11}\xi_{22} - \xi_{21}\xi_{12}} \begin{bmatrix} \xi_{22} & -\xi_{12} \\ -\xi_{21} & \xi_{11} \end{bmatrix} \begin{bmatrix} \xi_{10} \\ \xi_{20} \end{bmatrix}.$$

For systems with more than two equations it might be necessary to solve these numerically.

5) Reactions. In the present example the redundant components have replaced the reaction moment in A and the vertical reaction force in C, which are therefore directly given as

$$M_A = X_1 \quad , \quad R_C = X_2 \, .$$

As indicated in Figs. 6.15 and 6.16 it is convenient to determine the reactions for each load case during the analysis. Thus, the remaining reactions at this stage often can be found by superposition,

$$\begin{bmatrix} R_A \\ R_B \end{bmatrix} = \begin{bmatrix} R_A^0 \\ R_B^0 \end{bmatrix} + \begin{bmatrix} R_A^1 & R_A^2 \\ R_B^1 & R_B^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} .$$

Alternatively, since X_1 and X_2 are now known the remaining reactions can also be found by statics. The choice of procedure depends on the specific type of problem.

6) Internal forces. The moment distributions of the individual load cases have already been determined and the resulting moment distribution can be found by superposition as

$$M(x) = M^0(x) + M^1(x)X_1 + M^2(x)X_2 \, ,$$

or by a standard static analysis once the reactions have been determined. In the present case it might be easiest to use a direct static analysis, as the moment distribution is fully determined, once the section moments have been determined at C and at the two ends of the interval of the distributed load.

6.2.3 Summary of the force method

The following summary of the force method represents the general case for an n times statically indeterminate structure.

1) *Statically determinate structure and redundant component(s).* Release n kinematic constraints to obtain a statically and kinematically determinate structure. Introduce n redundant components X_j as conjugate forces to the released constraints ξ_j.

2) *Displacements from external load.* The displacements (or rotations) of the statically determinate structure at the released constraints from the external load are found by the virtual work equation

$$\xi_{j0} = \int_L \frac{M^j M^0}{EI} ds\,, \qquad j = 1 \ldots n\,,$$

where M^0 and M^j are the moment distributions for the external load and the test load $X_j = 1$, respectively.

3) *Flexibility coefficients.* For load case j the only load acting on the structure is the unit redundant component $X_j = 1$. The displacements/rotations at the released constraints are found for the individual load cases by the virtual work equation

$$\xi_{ij} = \int_L \frac{M^i M^j}{EI} ds\,, \qquad i, j = 1 \ldots n\,,$$

where M^i and M^j are the moment distributions for the test loads $X_i = 1$ and $X_j = 1$, respectively. The virtual work integral implies symmetry

$$\xi_{ij} = \xi_{ji}\,,$$

and thus only half of the coupling coefficients need to be computed directly. The displacements/rotations ξ_{ij} are determined for unit loads and thus represent *flexibility coefficients.*

4) *Redundant components.* The total displacements/rotations ξ_j can be found by superposition of the individual load cases as

$$\begin{aligned} \xi_1 &= \xi_{10} + \xi_{11} X_1 + \cdots + \xi_{1n} X_n\,, \\ \xi_2 &= \xi_{20} + \xi_{21} X_1 + \cdots + \xi_{2n} X_n\,, \\ &\vdots \qquad\qquad\qquad\quad \vdots \\ \xi_n &= \xi_{n0} + \xi_{11} X_1 + \cdots + \xi_{nn} X_n\,. \end{aligned}$$

These n equations are solved with respect to the n redundant components $X_1 \ldots X_n$. When releasing kinematic constraints, they are reimposed by the conditions $\xi_j = 0$, while $\xi_j \neq 0$ can be used to impose a finite displacement/rotation.

5) Reactions. Reactions that have been replaced by redundant forces are directly available, while the remaining reactions can be determined by superposition,

$$R_A = R_A^0 + R_A^1 X_1 + \ldots + R_A^n X_n \,.$$

When $X_1 \ldots X_n$ have been determined, the remaining reactions can also be found by static equilibrium.

6) Internal forces. The internal forces can be found by superposition of the individual load cases, e.g. the moment distribution

$$M = M^0 + M^1 X_1 + \ldots + M^n X_n \,,$$

and similarly for the normal force N and shear force Q. As the reactions have been determined, the internal forces can also be determined by static analysis as described in Chapter 3.

6.3 Application of the Force Method

In this section the procedure of the force method is illustrated by simple examples involving statically indeterminate beam structures. The extension to analysis of frame structures is considered in Section 6.4.

Example 6.1. Two-span beam – revisited. The two-span beam in Fig. 6.17 corresponds to the introductory example in Section 6.1, and it is revisited here to illustrate the use of an internal moment as redundant component.

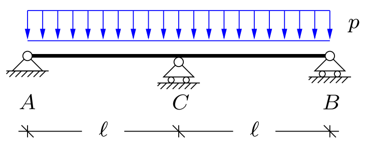


Fig. 6.17: Statically indeterminate two-span beam with distributed load.

The beam is one time statically indeterminate, and the rotation continuity in C is released by the introduction of an internal hinge as indicated in Fig. 6.18. The internal moment at C is the conjugate force to the released rotation, and M_C therefore acts as redundant component X_1. Note, that because the redundant component X_1 represents the internal moment it is introduced as a self-equilibrating moment pair in Fig. 6.18. The direction of X_1 is chosen in agreement with the sign convention for internal moments.

The actual load case with $X_1 = 0$ is shown in Fig. 6.19a. The corresponding moment M^0 vanishes in C due to the internal hinge, and the moment distribution is composed of two separate parabolic distributions with local maximum $\frac{1}{8}p\ell^2$. The rotation discontinuity ξ_{10} at C is determined by the virtual work equation, where the virtual moment distribution

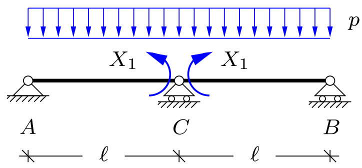


Fig. 6.18: Statically determinate beam with internal moment in C as redundant component.

$M^1(x)$ is obtained for $X_1 = 1$ as shown in Fig. 6.19b. It is seen that both $M^0(x)$ and $M^1(x)$ are symmetric with respect to the center of the beam, whereby

$$\xi_{10} = 2\int_0^\ell \frac{M^1 M^0}{EI}\,dx = \frac{2}{EI}\,\frac{\ell}{3}\Big(\frac{1}{8}\,p\ell^2\Big) = \frac{1}{12}\,\frac{p\ell^3}{EI}\,.$$

The integral is evaluated by row four in Table 4.1, and the above expression is non-dimensional, because ξ_{10} represents a rotation.

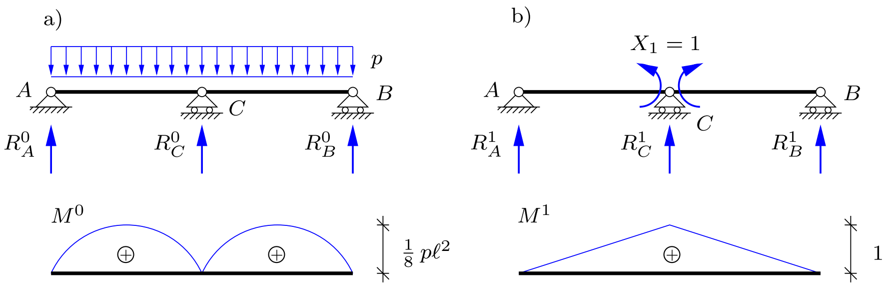


Fig. 6.19: Moment distribution.

The reactions are determined for each of the two load cases, and as shown in Fig. 6.20 the reaction in A is found by moment equilibrium about the section in C. For the actual load case p in Fig. 6.20a the moment in C vanishes due to the hinge, giving the reaction

$$\curvearrowright C \qquad R_A^0\ell - \tfrac{1}{2}\ell\,p\ell = 0 \quad\Rightarrow\quad R_A^0 = \tfrac{1}{2}\,p\ell\,.$$

For the load case $X_1 = 1$ in Fig. 6.20b the internal moment in C is unity, which gives

$$\curvearrowright C \qquad R_A^1\ell - 1 = 0 \quad\Rightarrow\quad R_A^1 = \frac{1}{\ell}\,.$$

The reactions in B can be found by the similar method for the right half of the beam, and the reactions in C are finally found by moment about A for the entire beam. This gives the following reaction components for the two load cases:

$$R_B^0 = R_A^0 = \tfrac{1}{2}\,p\ell\ ,\ R_C^0 = \ p\ell\,,$$

$$R_B^1 = R_A^1 = \ \frac{1}{\ell}\ \ ,\ R_C^1 = -\,\frac{2}{\ell}\,.$$

It is seen that vertical equilibrium is satisfied for both load cases: $R_A^0 + R_B^0 + R_C^0 = 2p\ell$ and $R_A^1 + R_B^1 + R_C^1 = 0$.

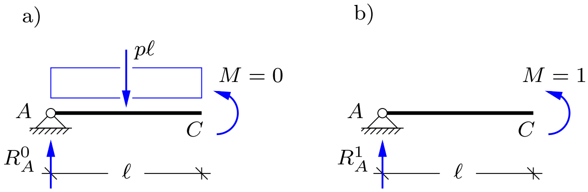


Fig. 6.20: Moment about section in C.

The flexibility coefficient ξ_{11} is governed by the load case in Fig. 6.19b, which only contains the unit load case $X_1 = 1$. The flexibility coefficient is found by the virtual work equation

$$\xi_{11} = 2\int_0^{\ell} \frac{M^1 M^1}{EI}\, dx = \frac{2}{3}\frac{\ell}{EI}\,.$$

It is seen that $\xi_{11} > 0$ and has dimensions [moment]$^{-1}$, which corresponds to [rotation/moment] and thereby rotation flexibility.

The resulting rotation discontinuity at C is found by superposition of the two load cases considered above,

$$\xi_1 = \xi_{10} + \xi_{11} X_1\,.$$

The latter term is scaled by X_1 because ξ_{11} represents the moment flexibility determined for $X_1 = 1$. Continuity of the rotation at C is imposed by the condition $\xi_1 = 0$, and the redundant component is then determined from the above expression as

$$X_1 = -\frac{\xi_{10}}{\xi_{11}} = -\tfrac{1}{8}p\ell^2\,.$$

This corresponds to the internal moment at C found previously in Fig. 6.5.

The reactions of the individual load cases are shown in Fig. 6.19, and superposition is therefore conveniently used in this example. For the reaction in A this gives

$$R_A = R_A^0 + R_A^1 X_1 = \tfrac{1}{2}p\ell - \frac{1}{\ell}\,\tfrac{1}{8}p\ell^2 = \tfrac{3}{8}p\ell\,,$$

while the remaining reactions are determined similarly as

$$R_B = R_A = \tfrac{3}{8}p\ell\,, \qquad R_C = \tfrac{5}{4}p\ell\,.$$

It is seen that the reactions correspond to those obtained previously, satisfying vertical equilibrium: $R_A + R_B + R_C = 2p\ell$.

Finally, the section force distributions are determined, and the redundant component directly gives the internal moment $M_C = X_1$. Thus, the moment distribution is obtained by superimposing the moment parabola with local maximum $\frac{1}{8}p\ell^2$ to the linear curves connecting the moments in A, B and C as shown in Fig. 6.21a. The shear forces in A and B are in equilibrium with the corresponding reactions, giving

$$Q_A = R_A = \tfrac{3}{8}p\ell \quad , \quad Q_B = -R_B = -\tfrac{3}{8}p\ell\,.$$

The shear force at C is found by placing a section at $C-$ immediately to the left of C, and then taking vertical equilibrium as illustrated in Fig. 6.21b,

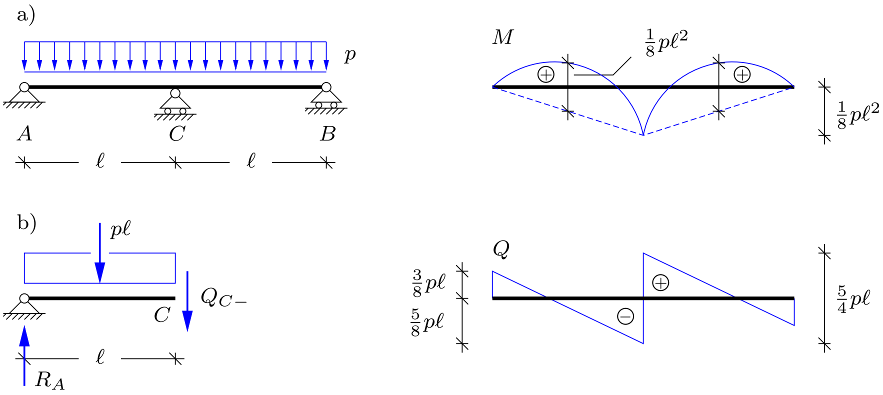


Fig. 6.21: Moment and shear force distribution.

$$\downarrow \qquad Q_{C-} + p\ell - R_A = 0 \qquad \Rightarrow \qquad Q_{C-} = -\tfrac{5}{8}\,p\ell\,.$$

Finally, the reaction R_C produces a discontinuity of $\frac{5}{4}p\ell$ in the shear force in C, and the resulting distribution of the shear force can be determined as shown in Fig. 6.21. □

Example 6.2. Two-span beam with fixed support. Figure 6.22 shows a two-span beam with a fixed support in A and simple supports in B and C. This structure was considered previously in Section 6.2.2 in a general context, while in this example the beam is loaded by a uniformly distributed load with intensity p.

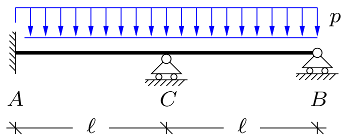


Fig. 6.22: Two-span beam with fixed support at A.

The structure is two times statically indeterminate and an equivalent statically determinate structure similar to that in Example 6.1 is obtained by releasing the rotation in A and the rotation continuity in C. Hereby, X_1 represents the reaction moment in A, while X_2 is the internal moment in C. Figure 6.23 shows the equivalent statically determinate beam structure with the redundant components.

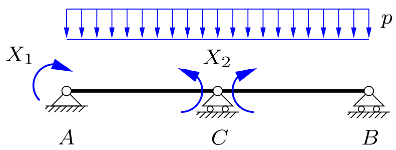


Fig. 6.23: Statically determinate structure and redundant components.

In the first load case only the distributed load is taken into account, while $X_1 = X_2 = 0$. Hereby, the system corresponds to the equivalent structure in Example 6.1, whereby moment distribution $M^0(x)$ and reactions R_A^0, R_B^0 and R_C^0 are directly available from Fig. 6.19a. Furthermore, the moment distribution $M^2(x)$ for the load case only containing $X_2 = 1$ corresponds to the moment distribution M^1 presented in Fig. 6.19b. This leaves $X_1 = 1$ as the only load case not already considered in Example 6.1. The moment distribution M^1 for the load case only containing $X_1 = 1$ is shown in Fig. 6.24, where the two remaining moment distributions $M^0(x)$ and $M^2(x)$ are also shown for convenience.

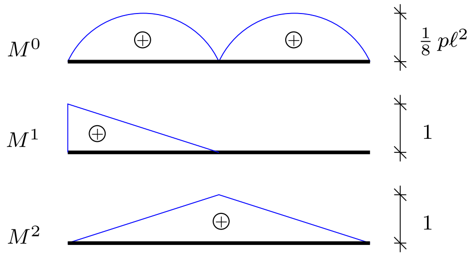


Fig. 6.24: Summary of moment distributions.

The rotation in A for the actual load case is found by the virtual work equation

$$\xi_{10} = \int_0^\ell \frac{M^1 M^0}{EI}\,dx = \frac{\ell}{3EI}\left(\frac{1}{8}p\ell^2\right) = \frac{1}{24}\frac{p\ell^3}{EI},$$

where only the left part of the beam contributes to the integral because $M^1(x) \equiv 0$ on the right part. The integral is solved by using row four of Table 4.1, where a linear curve is combined with a parabola. Similarly the change in rotation in C is found by virtual work, where a triangle and a parabola are combined for both parts of the two-span beam,

$$\xi_{20} = 2\,\xi_{10} = \frac{1}{12}\frac{p\ell^3}{EI}.$$

Figure 6.25 indicates the symmetric deformation form of the equivalent statically determinate structure, where the magnitude of the three rotations are equal, i.e. $\xi_{10} = \xi_{20}^- = \xi_{20}^+$. This verifies the relation $\xi_{20} = 2\xi_{10}$ used above.

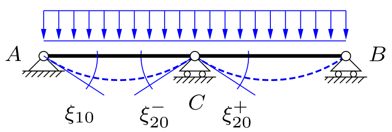


Fig. 6.25: Deformation of statically determinate structure.

The flexibility coefficients ξ_{11}, ξ_{21} and ξ_{12}, ξ_{22}, with $\xi_{12} = \xi_{21}$, represent the rotations in A and C for the two load cases $X_1 = 1$ and $X_2 = 1$, respectively. When combining the moment distributions $M^1(x)$ and $M^2(x)$ in the virtual work equation, the flexibility coefficients are determined as

$$\xi_{11} = \frac{1}{3}\frac{\ell}{EI}, \quad \xi_{22} = 2\xi_{11} = \frac{2}{3}\frac{\ell}{EI}, \quad \xi_{12} = \xi_{21} = \frac{1}{6}\frac{\ell}{EI}.$$

The dimension is the reciprocal of a moment, and thereby rotation flexibility.

Rotation at A and rotation discontinuity at C are constrained in the actual structure in Fig. 6.22, implying that $\xi_1 = 0$ and $\xi_2 = 0$. This gives two equations for X_1 and X_2. In matrix form these equations are

$$\begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = - \begin{bmatrix} \xi_{10} \\ \xi_{20} \end{bmatrix},$$

and as demonstrated in Section 6.2.2 this determines the redundant components as

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = - \frac{1}{\xi_{11}\xi_{22} - \xi_{21}\xi_{12}} \begin{bmatrix} \xi_{22} & -\xi_{12} \\ -\xi_{21} & \xi_{11} \end{bmatrix} \begin{bmatrix} \xi_{10} \\ \xi_{20} \end{bmatrix} = - \begin{bmatrix} 2 \\ 3 \end{bmatrix} \frac{p\ell^2}{28}.$$

It is seen that the magnitude of the moment is larger in C than in A, and that both moments are negative. This is in agreement with the sign convention chosen in Fig. 6.23, where X_1 and X_2 must be negative to prevent rotation in A and relative rotation at C.

The reactions are now determined, and horizontal equilibrium directly gives $R'_A = 0$. Furthermore, the reaction moment in A is represented directly by the first redundant component,

$$M_A = X_1 = -\tfrac{1}{14}\, p\ell^2 .$$

This leaves the vertical reactions in A, B and C. In this example the reactions from the individual load cases have not been determined, and the remaining reactions are therefore found by static equilibrium.

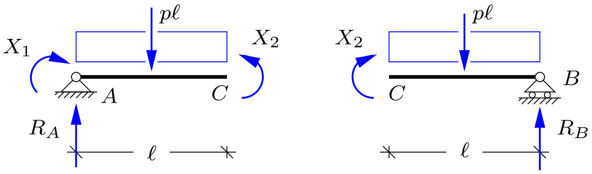


Fig. 6.26: Reactions by moment equilibrium.

Figure 6.26 shows the left and right part of the beam when placing a section in C. The purpose is to use moment equilibrium around C, and the shear forces at $C\pm$ are therefore not shown. For the left part AC moment about C gives the reaction in A,

$$\overset{\frown}{C} \qquad R_A\ell + X_1 - X_2 - \tfrac{1}{2}\ell\, p\ell = 0 \quad \Rightarrow \quad R_A = \tfrac{13}{28}\, p\ell .$$

Similarly, moment about C for the right part CB gives the reaction in B,

$$\overset{\frown}{C} \qquad R_B\ell - X_2 - \tfrac{1}{2}\ell\, p\ell = 0 \quad \Rightarrow \quad R_B = \tfrac{11}{28}\, p\ell .$$

Finally, the vertical reaction in C is obtained by moment equilibrium for the entire structure with respect to B,

$$\overset{\frown}{B} \qquad R_C\ell + R_A 2\ell + X_1 - 2p\ell^2 = 0 \quad \Rightarrow \quad R_C = \tfrac{8}{7}\, p\ell .$$

Note, that X_2 represents a self-equilibrating internal moment pair without contribution to global equilibrium of the structure.

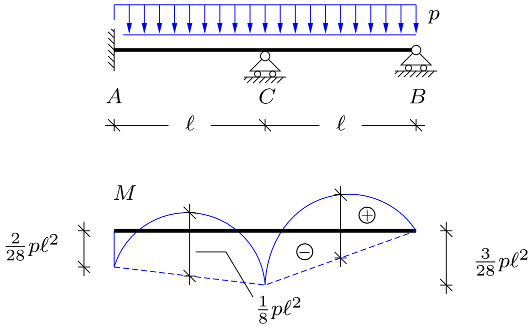


Fig. 6.27: Distribution of moment.

The moment distribution is governed by the internal moments $M_A = X_1$, $M_B = 0$ and $M_C = X_2$. These key values are plotted in the moment diagram in Fig. 6.27 and connected by (dashed) lines. The final moment distribution is then obtained by superimposing the parabolas from the distributed load. This example illustrates that the construction of the moment distribution is often simplified when the redundant components are chosen as suitable reaction moments and/or internal moments. □

Example 6.3. Two-span beam with local forces. Structures are often exposed to various types of loading conditions, e.g. wind excitation or snow loading, and the computational cost may be significantly reduced when results from previous static analyzes of the structure are reused in the analysis of a new load case. With respect to the force method the present example illustrates that only the displacements/rotations ξ_{j0} need to be re-calculated when changing the loading conditions, while the load-independent flexibility parameters ξ_{ij} may be carried on from a previous analysis.

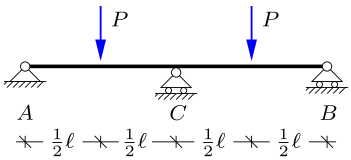


Fig. 6.28: Statically indeterminate two-span beam with local forces P.

Reconsider the simply supported two-span beam with distributed load in Example 6.1. Figure 6.28 shows the same beam structure, but loaded by two local forces P acting at the center of the two spans. The reuse of the previous results from Example 6.1 requires using the same equivalent statically determinate structure, and as shown in Fig. 6.29 the redundant component X_1 again represents the internal moment in C.

Figure 6.30 shows both the moment distribution $M^0(x)$ due to the actual forces P and the moment distribution $M^1(x)$ for $X_1 = 1$. Note, that $M^1(x)$ has previously been obtained in Example 6.1 and is therefore taken directly from Fig. 6.19. The rotation discontinuity at C is obtained by the virtual work equation,

$$\xi_{10} = \int_A^B \frac{M^1 M^0}{EI}\, dx = \frac{\ell}{3EI}\left(\frac{1}{4}P\ell\right)\frac{1}{2} + \frac{\ell}{6EI}\left(\frac{1}{4}P\ell\right)2 = \frac{1}{8}\,\frac{P\ell^2}{EI}\,,$$

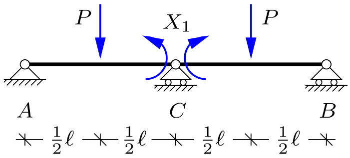


Fig. 6.29: Statically determinate structure and redundant component.

while the flexibility coefficient ξ_{11} can be taken directly from Example 6.1,

$$\xi_{11} = \frac{2}{3}\frac{\ell}{EI}.$$

In particular for structures that are several times statically indeterminate the reuse of the flexibility coefficients might save a significant amount of computational effort because the flexibility matrix need only be inverted once.

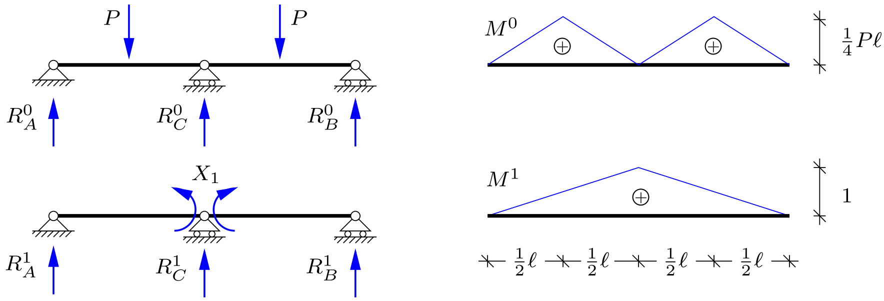


Fig. 6.30: Distribution of moment for external load and unit redundant moment.

In the original structure the rotation discontinuity at C is constrained, $\xi_1 = 0$, and this determines the redundant component

$$X_1 = -\frac{\xi_{10}}{\xi_{11}} = -\tfrac{3}{16}P\ell.$$

The reactions in Fig. 6.30a from the external loading are found as

$$R_A^0 = R_B^0 = \tfrac{1}{2}P, \qquad R_C^0 = P.$$

The reactions in Fig. 6.30b for the load case $X_1 = 1$ are taken from Example 6.1,

$$R_B^1 = R_A^1 = \frac{1}{\ell}, \qquad R_C^1 = -\frac{2}{\ell}.$$

The resulting reactions can now be determined by superposition,

$$R_B = R_A = \tfrac{5}{16}P, \qquad R_C = \tfrac{11}{8}P.$$

Note, that the reaction in C carries a larger part of the total vertical load compared to the case with the distributed load in Example 6.1.

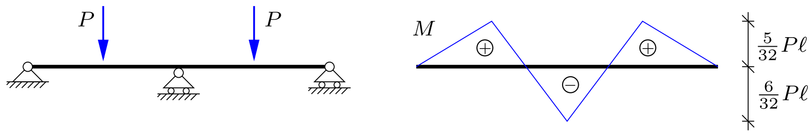


Fig. 6.31: Moment distribution.

The redundant component X_1 gives the internal moment in C directly, while the moment vanishes at the hinges in A and B. Thus, the moment distribution is fully determined once the moment M_P at the location of P is obtained. Superposition of the values obtained from Fig. 6.30 gives

$$M_P = M(\tfrac{1}{2}\ell) = M^0(\tfrac{1}{2}\ell) + M^1(\tfrac{1}{2}\ell)X_1 = \tfrac{1}{4}P\ell + \tfrac{1}{2}(-\tfrac{3}{16}P\ell) = \tfrac{5}{32}P\ell\,.$$

The final moment distribution is then determined by connecting the moment values at the points with transverse forces by straight lines as shown in Fig. 6.31. It is seen that the numerically largest moment $M_{\max} = \frac{6}{32}P\ell$ occurs in C. □

6.4 The force method for frame structures

Modern plane frame structures are typically designed with some kind of static indeterminacy arising from the use of more than the minimum three support components. The additional reaction components are deliberately introduced to obtain a stiffer structure, and to obtain some degree of balance between positive and negative moments, whereby the necessary moment capacity of the members of the frame is reduced. A simple illustration is provided by the rectangular frame shown in Fig. 6.32. The horizontal beam CD carries

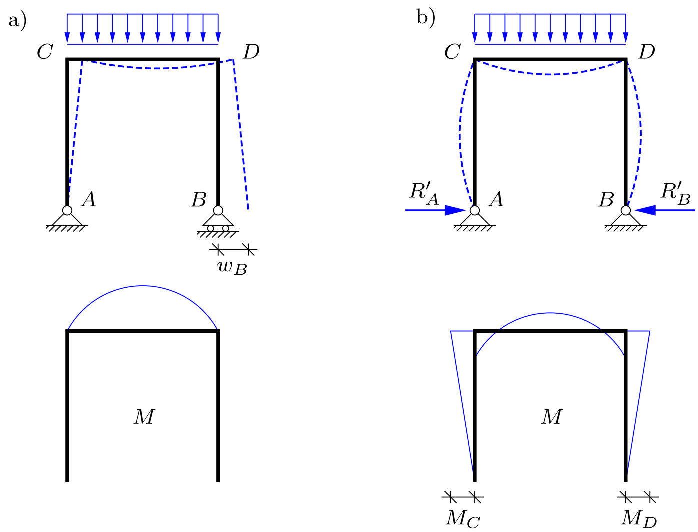


Fig. 6.32: Moment reduction in frame by additional support.

a uniformly distributed vertical load with intensity p. The figure shows two different sets of support conditions. In Fig. 6.32a the frame has a simple fixed support at A and a simple support on horizontal rollers at B. The reactions consist of two equal vertical forces at A and B, each carrying half of the load, while the horizontal reaction component at A vanishes. The horizontal beam CD curves upward, while the columns AC and BD remain straight. As a result there is outward motion w_B of the support B, and the moment distribution along CD is parabolic with zero moment at the frame corners C and D. Figure 6.32b shows a modified version of the frame, in which the support at B is replaced by a fixed simple support. This introduces horizontal reactions that prevent the outward motion of the supports and introduce a negative moment at the frame corners C and D, whereby the maximum moment in the horizontal beam CD is reduced. As a result this form of the frame is stiffer, and has a smaller maximum moment.

6.4.1 Simply supported frames

Simply supported plane frames are often one time statically indeterminate. It is possible to use one of the reactions as redundant component, but it may be advantageous to choose the internal moment at a suitable joint in the frame as X_1. This often leads to simple moment curves and reaction forces. Figure 6.33 shows some useful locations of X_1 for typical frame structures.

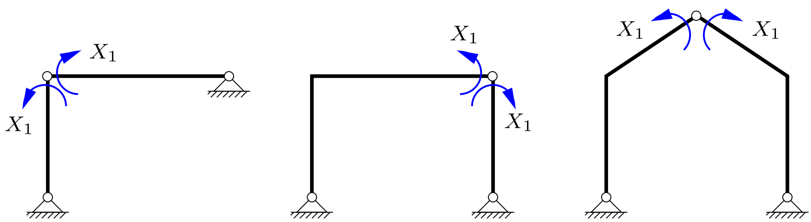


Fig. 6.33: Simply supported frames with the redundant component at a joint.

Example 6.4. Angle frame with distributed load. This example considers the simply supported angle frame ACB shown in Fig. 6.34a. A distributed vertical load with intensity p acts on the horizontal beam CB. The frame is one time statically indeterminate and as discussed above it is convenient to construct an equivalent statically determinate structure by releasing the rotation continuity at a suitable joint. Figure 6.34b shows a hinge introduced at the corner C, whereby the redundant component X_1 represents the internal moment M_C.

The moment distributions and the reactions are now determined for the actual load p without X_1, and for the unit load $X_1 = 1$ without p. Figure 6.35 shows these two load cases and the corresponding moment distributions. The rotation discontinuity at the joint C is found by the virtual work equation,

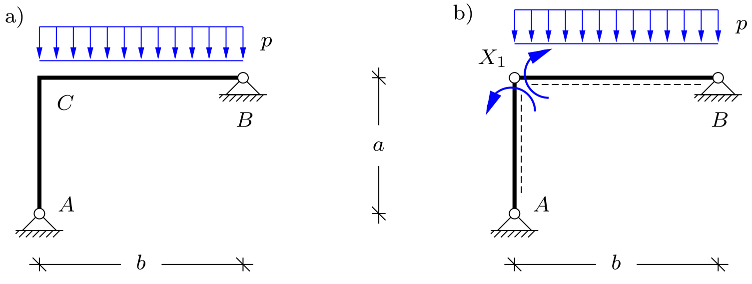


Fig. 6.34: Angle frame with distributed load.

$$\xi_{10} = \int_A^B \frac{M^0 M^1}{EI}\, ds = \frac{b}{3EI}\left(\frac{1}{8}\, pb^2\right) = \frac{1}{24}\,\frac{pb^3}{EI}\,,$$

where only the horizontal beam CB contributes to the integral, because $M^0(s) \equiv 0$ in the vertical beam AC. The integral is evaluated by row four in Table 4.1. It is observed that the rotation discontinuity ξ_{10} is positive, implying that the right angle of the joint in C will 'close' slightly. The corresponding flexibility coefficient is found by the integral

$$\xi_{11} = \int_A^B \frac{M_1 M_1}{EI}\, ds = \left(\frac{a}{3} + \frac{b}{3}\right)\frac{1}{EI} = \frac{1}{3}\,\frac{a+b}{EI}\,,$$

following from the first row in Table 4.1.

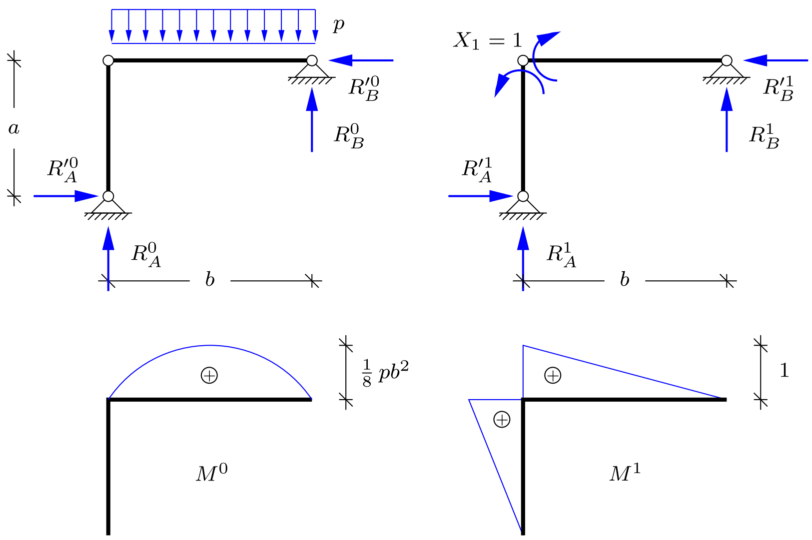


Fig. 6.35: Moment distribution and reactions for load cases.

In the actual frame structure the right angle of the corner at C is maintained during deformation of the frame, whereby $\xi_1 = \xi_{10} + \xi_{11} X_1 = 0$. This gives the redundant component

$$X_1 = -\frac{\xi_{10}}{\xi_{11}} = -\frac{1}{8}\frac{pb^3}{a+b},$$

representing the internal bending moment at C.

The reactions for the individual load cases are determined in accordance with the sign convention in Fig. 6.35. The reactions for the actual load p are

$$R_A^0 = R_B^0 = \tfrac{1}{2}\,pb\,, \qquad R_A'^0 = R_B'^0 = 0\,.$$

For the load case $X_1 = 1$ the reactions are

$$R_A^1 = -R_B^1 = -\frac{1}{b}\,, \qquad R_A'^1 = R_B'^1 = -\frac{1}{a}\,,$$

and the final reactions can now be found by superposition of the individual load cases. For the vertical reaction in A this gives

$$R_A = R_A^0 + R_A^1 X_1 = \frac{pb}{2} + \Big(-\frac{1}{b}\Big)\Big(\frac{1}{8}\frac{pb^3}{a+b}\Big) = \frac{pb}{8}\frac{4a+5b}{a+b}\,.$$

The three remaining reaction forces can similarly be found by superposition as

$$R_B = \frac{pb}{8}\frac{4a+3b}{a+b}\,, \qquad R_A' = R_B' = \frac{1}{8}\frac{pb^3}{a(a+b)}\,.$$

It is seen that the horizontal and the vertical equilibrium are both satisfied.

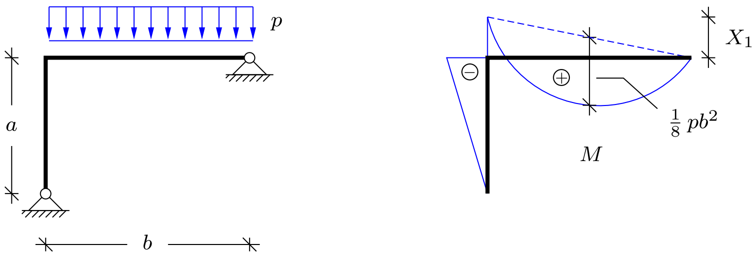


Fig. 6.36: Moment distribution.

The moment distribution is determined via the internal moment at the joint C,

$$M_C = X_1 = -\frac{1}{8}\frac{pb^3}{a+b}\,.$$

The moment distribution shown in Fig. 6.36 is linear in the vertical beam AC, while it is parabolic in the horizontal beam. The parabolic part with local maximum $\frac{1}{8}pb^2$ is superimposed on the (dashed) linear curve connecting the moment values in C and B. □

Example 6.5. Simply supported frame with combined load. Figure 6.37 shows a simply supported frame with distributed load p on CD and a vertical tip force P at E. The sign convention of the internal forces is defined by the dashed line, indicating the lower side of the individual beams. It is assumed that $P = \frac{1}{2}pa$, corresponding to half of the resultant of the distributed load.

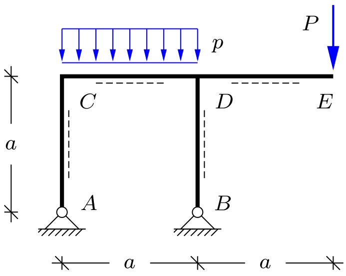


Fig. 6.37: Simply supported frame with distributed load p and local tip force P.

The frame has four reaction force components, while only three independent equilibrium equations are available. Thus, the frame structure is one time statically indeterminate. When choosing the redundant component as an internal moment, special care should be taken if placing a hinge in D, as the structure should remain kinematically determinate without mechanisms. Three possible choices are presented in Fig. 6.38. It is seen that the locations of the hinge in Figs. 6.38a,c imply that the horizontal reactions are zero in the load case $X_1 = 0$, whereby the corresponding moments in the vertical beams vanish. This might simplify the analysis and in the present example the internal moment pair at the joint C is chosen as the redundant component as shown in Fig. 6.39.

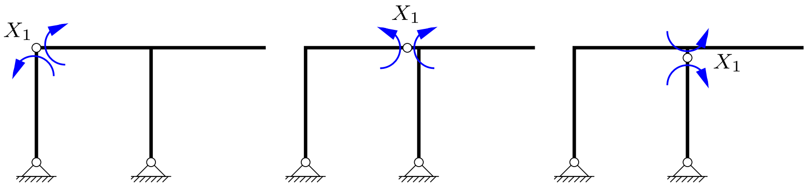


Fig. 6.38: Possible choices of the redundant component X_1.

The external load case is composed of two individual loads: the distributed load p on CD and the vertical tip force P at E. From a computational point of view it is convenient to determine the separate contributions to the virtual work equation from the two individual loads as shown in Fig. 6.40. The individual moment distributions for the two load components are also shown in the figure. The moment distribution M_a^0 due to the distributed load is parabolic on CD, while the moment distribution M_b^0 due to the tip force is piecewise linear. The rotation discontinuity at C is found by the equation of virtual work, where the contributions from the two load cases are added,

$$\xi_{10} = \int \frac{M_a^0 M^1}{EI}\, ds + \int \frac{M_b^0 M^1}{EI}\, ds\,.$$

The reactions for the load case $X_1 = 1$ are determined as follows. The horizontal reactions must be opposite and of equal magnitude to conserve horizontal equilibrium. Thus, they do not contribute to the global moment equilibrium about C, which determines $R_B = 0$. Vertical force equilibrium then gives $R_A = 0$. Finally, the horizontal component $R_A'^1$ is determined by moment equilibrium about the hinge at C of the left part of the frame,

$$\overset{\frown}{C} \qquad R_A'^1 a + 1 = 0 \quad \Rightarrow \quad R_A'^1 = -\frac{1}{a}\,.$$

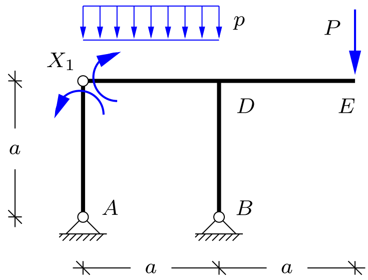


Fig. 6.39: Redundant component X_1 as moment in C.

The horizontal reaction in B then follows from horizontal equilibrium as $R'^1_B = R'^1_A = -1/a$. The moment distribution is linear over the individual beams and is determined from sections just below C and D. The moment distribution $M^1(s)$ is shown in Fig. 6.41.

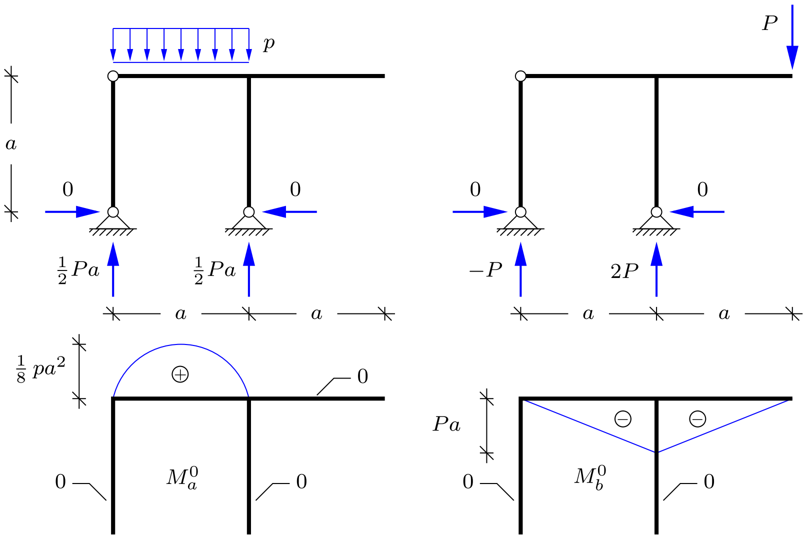


Fig. 6.40: Moment distributions from external load.

The two integrals in the virtual work equation are evaluated by the integral formulae in Table 4.1,

$$\xi_{10} \;=\; \frac{a}{3EI}\left(\tfrac{1}{8}\,pa^2\right)(1+1) \;+\; \frac{1}{EI}\,\tfrac{1}{2}\,a(-Pa) \;=\; \frac{1}{12}\,\frac{a^2}{EI}\,(PA-6p)\,.$$

When introducing the relation $P = \frac{1}{2}pa$ the final expression becomes

$$\xi_{10} \;=\; -\,\frac{1}{6}\,\frac{pa^3}{EI}\,.$$

It is seen that the major part of the rotation discontinuity in the equivalent structure is due to the tip force P. The flexibility coefficient follows from the moment distribution in Fig. 6.41 as

$$\xi_{11} = \int_A^B \frac{M_1 M_1}{EI}\, ds = \frac{5}{3}\frac{a}{EI},$$

and the redundant component is then obtained as

$$X_1 = -\frac{\xi_{10}}{\xi_{11}} = \tfrac{1}{10}\, pa^2,$$

where the relation $P = \frac{1}{2}pa$ has been introduced.

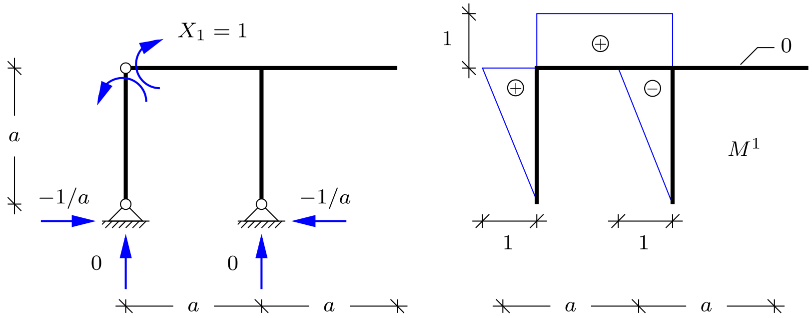


Fig. 6.41: Moment distributions from $X_1 = 1$.

The reactions have been determined for the three individual load cases in Figs. 6.40 and 6.41, and the resulting reactions can thus be determined by superposition. For $P = \frac{1}{2}pa$ the vertical reaction forces are

$$R_A = \tfrac{1}{2}pa - P = 0, \qquad R_B = \tfrac{1}{2}pa + 2P = \tfrac{3}{2}pa,$$

while the horizontal reactions are

$$R'_A = R'_B = \Big(-\frac{1}{a}\Big)X_1 = -\tfrac{1}{10}\, pa^2 .$$

It is found that both horizontal and vertical force equilibrium are satisfied.

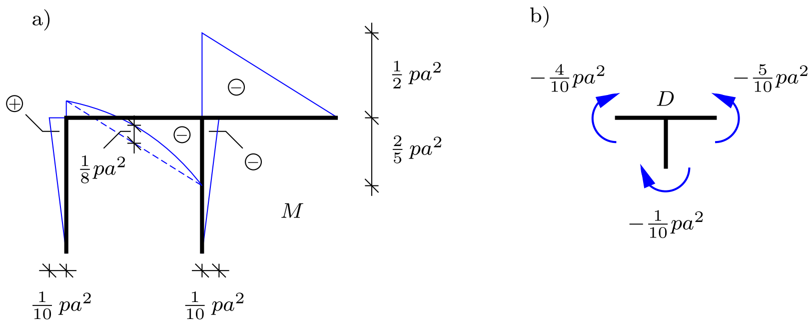


Fig. 6.42: a) Moment distribution, b) Equilibrium in D.

The moment distribution is obtained by determining the moment in C and D, whereafter the parabolic distribution from the distributed load is superimposed on CD. The moment at the left joint C is equal to the redundant component,

$$M_C = X_1 = \tfrac{1}{10}\,pa^2 .$$

At the joint D three internal moments occur – one for each beam attached to D. These moments are determined by superposition of the three load cases,

$$\begin{aligned} M_{DC} &= -Pa + X_1 &&= -\tfrac{2}{5}\,pa^2 ,\\ M_{DE} &= -Pa &&= -\tfrac{1}{2}\,pa^2 ,\\ M_{DB} &= -X_1 &&= -\tfrac{1}{10}\,pa^2 , \end{aligned}$$

where $P = \frac{1}{2}pa$ has again been introduced. The moment distribution is shown in Fig. 6.42a, where it is seen that the largest moment occurs in the cantilever beam DE at D,

$$M_{\max} = -M_{DE} = \tfrac{1}{2}\,pa^2 .$$

In joints connecting more than two elements it is a good idea to check moment equilibrium. In Fig. 6.42b the joint has been isolated and the three internal moments are applied. It is found that moment equilibrium is satisfied,

$$\overset{\frown}{C} \qquad -\tfrac{1}{10}\,pa^2 - \tfrac{4}{10}\,pa^2 - \left(-\tfrac{5}{10}\,pa^2\right) = 0 .$$

Similarly, horizontal and vertical force equilibrium could be checked at the joint. □

6.4.2 Frames with fixed supports

As demonstrated in the previous examples, simply supported frames are typically one time statically indeterminate and therefore conveniently analyzed by the force method. However, for frames with fixed supports, such as in Fig. 6.43, use of the force method requires three redundant components and thereby solution of three coupled equations. This is typically very cumbersome by analytical means, and some form of system reduction technique would therefore be of interest.

Symmetry conditions

Plane frame structures are often symmetric. This property can sometimes be used to separate the system into two parts, each with a reduced number of redundant components. Figure 6.43 shows a symmetric frame structure with a vertical force acting on the left inclined beam. Obviously the load is non-symmetric with respect to the line of symmetry of the frame. However, the load can be decomposed into a symmetric and an anti-symmetric part. These corresponding load cases are illustrated in Fig. 6.43 and summarized as follows:

- symmetric structure with symmetric load
- symmetric structure with anti-symmetric load

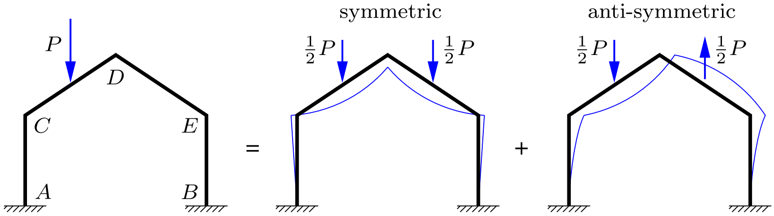


Fig. 6.43: Separation into symmetric and anti-symmetric load cases.

The deformation forms of the frame structure for each of the generic load cases are also indicated in Fig. 6.43. For the symmetric load the structure deforms symmetrically with respect to the line of symmetry. Thus, at the section in D along the line of symmetry of the structure the force component along the line of symmetry (in this case vertical) must vanish. The force component R'_D perpendicular to the line of symmetry (in this case horizontal) restrains D against motion in that direction, while the moment M_D restrains D against rotation. Typically, these two components shown in Fig. 6.44a do not vanish. For the ani-symmetric loading the frame deforms anti-symmetrically, and the force component perpendicular to the line of symmetry (in this case horizontal) and the moment therefore vanish. The remaining force component R_D along the line of symmetry (in this case vertical) restrains D against vertical motion and therefore does not vanish. This case is shown in Fig. 6.44b. These observations for symmetric plane frames can be generalized in the following symmetry conditions.

In a section located on the line of symmetry of a symmetric plane structure:

- the resulting internal force parallel to the line of symmetry vanishes for symmetric loading.
- the moment and the internal force perpendicular to the line of symmetry vanish for anti-symmetric loading.

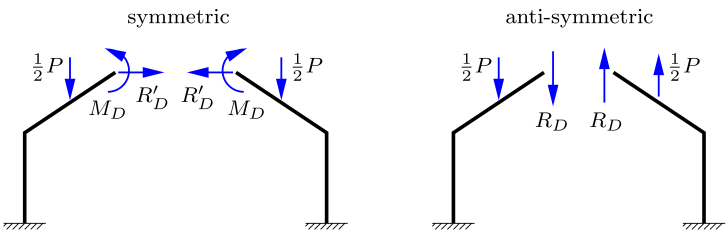


Fig. 6.44: Internal force components for symmetric and anti-symmetric loading.

It is observed that in Fig. 6.44 the moment M_D corresponds to the internal moment in D. However, due to the inclination of the beams CD and DE the resulting section forces R_D and R'_D are not equal to the shear force and the normal force.

When applying the symmetry conditions for the symmetric and the anti-symmetric load cases the degree of statically indeterminacy can be reduced. For the structure in Fig. 6.44 the left part of the symmetric load case contains five unknown reactions, three at A and two at D, and this part is two times statically indeterminant. For the anti-symmetric load case the degree of indeterminacy is reduced to one, with three unknown reactions at A and only a single unknown force component at D. Note, that similar symmetry conditions are also available for anti-symmetric structures.

Example 6.6. Symmetric frame with anti-symmetric load. In this example the use of symmetry conditions is illustrated for the symmetric rectangular frame shown in Fig. 6.45, loaded by horizontal forces P at the joints C and D. Thus, the loading is anti-symmetric with respect to the line of symmetry of the frame. The frame is symmetric and exposed to an anti-symmetric load. Therefore, the section at E only transmits a force component along the line of symmetry, and the left half of the frame is one time statically indeterminate. In this example the vertical force component in E is used as the redundant component X_1, as shown in Fig. 6.45b.

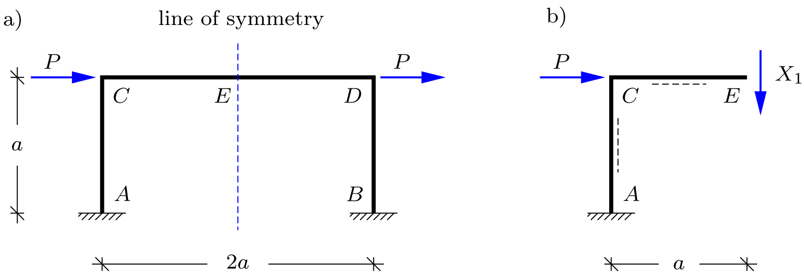


Fig. 6.45: Rectangular frame with fixed supports and anti-symmetric loading.

Figure 6.46 shows the left half of the frame with the actual load case P, but without X_1, and the unit load case with $X_1 = 1$ and without P. The figure also shows the corresponding moment distributions. The displacement in E from the external load P is obtained by the equation of virtual work as

$$\xi_{10} = \int_A^C \frac{M^0 M^1}{EI}\, ds = \frac{M^1}{EI} \int_A^C M^0\, ds = \frac{-a}{EI}\, \frac{1}{2}\, a\,(-Pa) = \frac{1}{2}\, \frac{Pa^3}{EI}\,,$$

where only the vertical beam AC contributes because $M^0(s) \equiv 0$ in the horizontal beam. Because $M^1(s)$ is constant on AC it can be taken outside the integral, which then represents the area $\frac{1}{2}a(-Pa)$ underneath the triangular M^1-curve. The flexibility coefficient is obtained by combining $M^1(s)$ with itself in the virtual work equation. This gives

$$\xi_{11} = \int_A^E \frac{M^1 M^1}{EI}\, ds = \frac{(-a)^2}{EI}\, a + \frac{a}{3EI}\,(-a)^2 = \frac{4}{3}\, \frac{a^3}{EI}\,.$$

The kinematic component $\xi_1 = \xi_{10} + \xi_{11} X_1$ associated with the vertical redundant force X_1 represents a discontinuity in the transverse displacement at E. In the actual structure in Fig. 6.45a the left and right part of the structure are rigidly joined at E, whereby the displacement discontinuity $\xi_1 = 0$. This gives a condition for determining the redundant

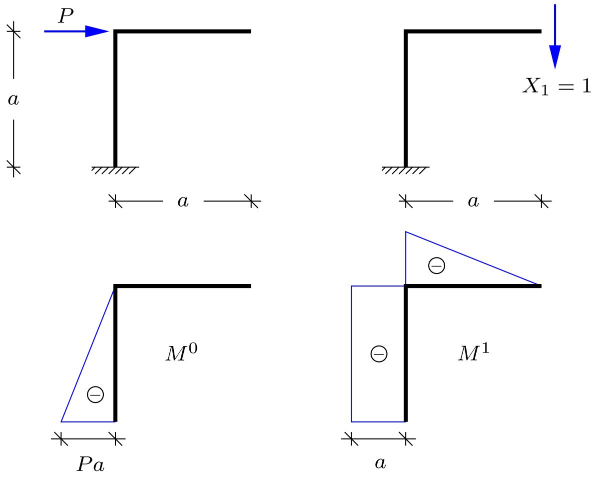


Fig. 6.46: Left half of frame with shear force in E as redundant component.

component,

$$X_1 = -\frac{\xi_{10}}{\xi_{11}} = -\frac{3}{8}P\,.$$

The horizontal beam CED is perpendicular to the line of symmetry, and the redundant component X_1 therefore represents the shear force at E.

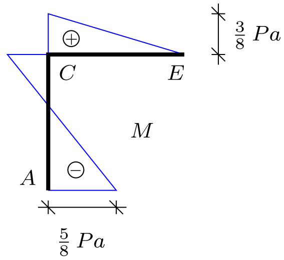


Fig. 6.47: Moment distribution for left half of frame.

The internal moment at A and C are obtained by superposition,

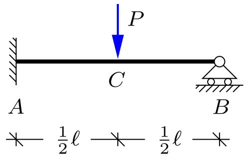
$$M_A = -Pa + (-a)X_1 = -\frac{5}{8}Pa\,, \qquad M_C = (-a)X_1 = \frac{3}{8}Pa\,.$$

Figure 6.47 shows the moment distribution, and it is seen that the maximum moment $M_{\max} = \frac{5}{8}Pa$ occurs at the fixed support at A – and because of symmetry also at B. □

6.5 Exercises

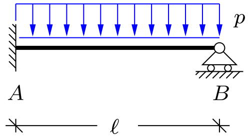
Exercise 6.1. The figure shows a beam of length ℓ with a fixed support in A and a simple support in B. A vertical force P is acting at the center of the beam in C.

a) Determine the degree of statically indeterminacy.
b) Determine the reaction moment in A.
   Hint: Choose this moment as X_1.
c) Determine the moment distribution.
d) Determine the distribution of the shear force.



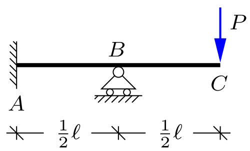
Exercise 6.2. The figure shows a beam of length ℓ with a fixed support in A and a simple support in B. The beam is loaded by a distributed load with constant intensity p. Note that the structure is similar to that in the previous exercise. Furthermore, the solutions to this problem can be found in Example 4.6, where the differential equation has been solved.

a) Use the force method to determine the reaction moment in A.
b) Find the remaining reactions.
c) Determine the moment distribution.
d) Determine the distribution of the shear force.



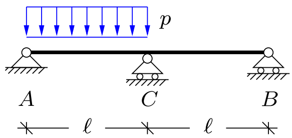
Exercise 6.3. The figure shows a beam of length ℓ with a fixed support in A and a simple support in B. A vertical force P acts at the tip of the beam in C.

a) Use the force method to determine the reaction moment in A.
b) Find the remaining reactions.
c) Determine the distribution of the moment M and the shear force Q.



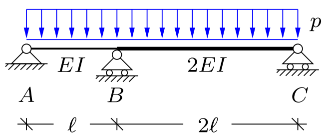
Exercise 6.4. The figure shows a two-span beam similar to that in Examples 6.1 and 6.3, but in this case only with distributed load on the left span AC.

a) Use the force method to find the internal moment in C.
b) Find the reactions.
c) Determine the distribution of the moment M and the shear force Q.

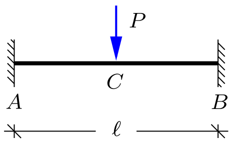


Exercise 6.5. The figure shows a two-span beam loaded by a distributed load with constant intensity p. The left span AB has the length ℓ and bending stiffness EI. The right span has the double length 2ℓ and double bending stiffness $2EI$.

a) Use the force method to determine the bending moment in B.
b) Find the reactions.
c) Determine the distribution of the moment M and the shear force Q.



Exercise 6.6. The figure shows a beam of length ℓ with fixed supports in A and B. A vertical force P acts at the center of the beam at C.



a) Apply symmetry conditions and determine the degree of indeterminacy.

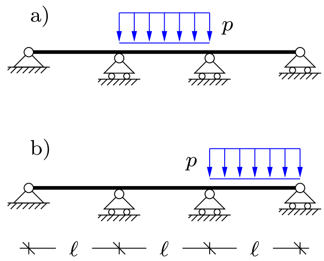
b) Use the force method to determine the reaction moment in A.

c) Find the remaining reactions.

d) Determine the distribution of the moment M and the shear force Q.

e) Replace P by a distributed load with intensity p and repeat b)–d). Note that this problem has been solved in Example 4.8.

Exercise 6.7. The figure shows a three-span beam, where the length of each span is ℓ. The beam is loaded by a distributed load with intensity p on a) the center span, and b) the right span, respectively. Use the force method to analyze the structure for each load case following the outline presented below.

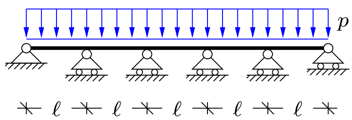


a) Determine internal moments at the two intermediate supports.

b) Find the reactions.

c) Determine the distribution of the moment M.

Exercise 6.8. The figure shows a five-span beam loaded by a distributed load with constant intensity p. The length of each span is ℓ.

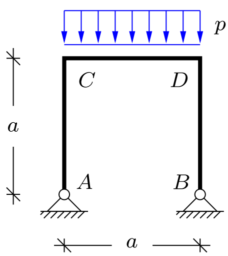


a) Determine the degree of indeterminacy before and after applying symmetry conditions.

b) Use the force method to find the reactions for the left half of the beam.

c) Determine the distribution of the moment M and the shear force Q in the left half of the beam.

Exercise 6.9. The figure shows a simply supported rectangular frame with a distributed load acting in the vertical direction on the horizontal beam CD with constant intensity p.



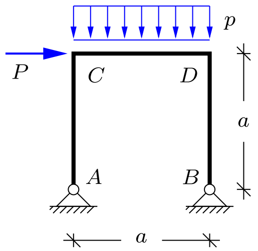
a) Use the force method to determine the internal moment in the upper left joint C.

b) Determine the reactions.

c) Determine the moment distribution and the magnitude and location of the maximum moment.

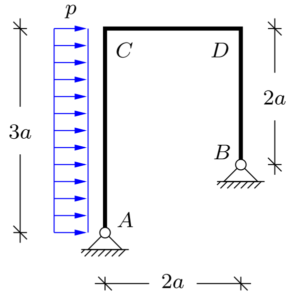
d) Determine the distribution of the normal and shear force.

Exercise 6.10. Reconsider the simple frame with a distributed load of Exercise 6.9. In this exercise a horizontal force P acts at the corner C in addition to the distributed load. Structure and loading are shown in the figure below. It can be assumed that $P = \frac{1}{2}pa$.



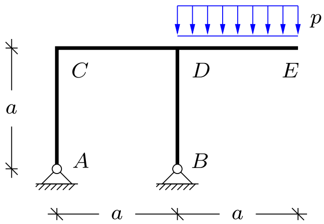
a) Determine the internal moment in the upper left corner C.
b) Determine the reactions.
c) Determine the moment distribution and the magnitude and location of the maximum moment.
d) Determine the distribution of the normal and shear force.

Exercise 6.11. The figure shows a simply supported rectangular frame with a distributed load p acting in the horizontal direction on AC. The length of the left vertical beam AC is $3a$, while the length of the two other beams CD and BD is $2a$.



a) Use the force method to determine the internal moment in the upper left joint C.
b) Determine the reactions.
c) Determine the moment distribution and the magnitude and location of the maximum moment.
d) Determine the distribution of the normal and shear force.

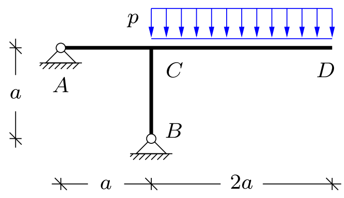
Exercise 6.12. The figure shows a rectangular frame with a cantilever, similar to the frame structure in Example 6.5. In this exercise the loading of the example is replaced by a distributed load with intensity p, acting in the vertical direction on the horizontal cantilever DE. The dimensions of the frame are given in terms of a, as shown in the figure.



a) Use the force method to determine the internal moment in the upper left joint C.
b) Determine the reactions.
c) Determine the moment distribution and the magnitude and location of the maximum moment.
d) Determine the distribution of the normal and shear force.
e) Check equilibrium in the joint D.

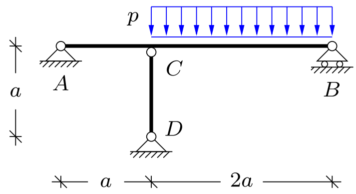
Exercise 6.13. The figure shows a T-frame with simple fixed supports in A and B. The frame is loaded by a distributed load with intensity p, acting in the vertical direction on the horizontal cantilever CD. The dimensions of the frame are given in terms of a as shown in the figure.

a) Find a statically determinate system with redundant component X_1.
b) Determine the reactions.
c) Determine the moment distribution and the magnitude and location of the maximum moment.
d) Check moment equilibrium in the joint C.



Exercise 6.14. The figure shows a simply supported beam AB with an additional vertical support in C by the bar element CD. The dimensions of the structure are given in terms of a, as shown in the figure. The elastic modulus is E for both beam and bar, while the moment of inertia for the beam is I and the cross sectional area of the bar is A.

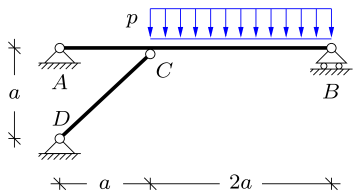
a) Use the force method to determine the bar force N_{CD}.
b) Determine the reactions.
c) Determine the moment distribution and the magnitude and location of the maximum moment.



d) Include the contribution of the normal force of the bar element in the virtual work equation. Redetermine the normal force N_{CD} and compare with the result in a). Assume that the cross sectional area A of the bar relates to the moment of inertia I of the beam by $A = 2000\, I/a^2$.

Exercise 6.15. The figure shows a simply supported beam AB with an additional vertical support in C by the inclined bar element CD. The dimensions of the structure are given in terms of a, as shown in the figure. The bending stiffness of the beam is EI, and contributions to the deformations from shear and normal forces are neglected.

a) Use the force method to determine the normal force N_{CD} in the inclined bar CD.
b) Determine the reactions.
c) Determine the moment distribution and the magnitude and location of the maximum moment.
d) Determine the distribution of normal and shear force.



Exercise 6.16. The figure shows a typical geometry for frames used in barns and warehouses, with simple fixed supports in A and B. The dimensions of the frame are given in terms of a, as shown in the figure. A distributed load is acting in the vertical direction on CD.

a) Use the force method to determine the internal moment in D.

b) Determine the reactions.

c) Determine the moment distribution and the magnitude and location of the maximum moment.