Chapter 2
Basic Concepts

2.1 A Few Words About Mathematics in General

In former times, mathematics was defined as the science concerned with numbers and figures. (This is reflected in the title of the classic book by Hans Rademacher and Otto Toeplitz, *Von Zahlen und Figuren*, literally *On Numbers and Figures* [6].) Nowadays, however, such a definition will not do, for modern algebra deals with abstract structures instead of numbers, and some branches of geometry study objects that barely resemble any figure in the plane or in space. Other branches of mathematics, including analysis, discrete mathematics, and probability theory, also study objects that we would not call numbers or figures. All we can say about the objects studied in mathematics is that generally, they are abstractions from the real world (but not always). In the end, mathematics should be defined not by the objects we study, but how we study them. We can say that mathematics is the science that establishes absolute and irrefutable facts about abstract objects. In mathematics, these truths are called **theorems**, and the logical arguments showing that they are irrefutable are called **proofs**. The method (or language) of proofs is *mathematical logic*.

2.2 Basic Concepts in Logic

Mathematical logic works with *statements*. Statements are declarative sentences that are either *true* or *false*. (That is, we cannot call the wish "If I were a swift cloud to fly with thee" a statement, nor the command "to thine own self be true.") We can link statements together using *logical operators* to yield new statements. The logical operators are *conjunction* (and), *disjunction* (or), *negation* (not), *implication* (if, then), and *equivalence* (if and only if).

M. Laczkovich, V.T. Sós, *Real Analysis*, Undergraduate Texts
in Mathematics, DOI 10.1007/978-1-4939-2766-1_2

Logical Operators

Conjunction joins two statements together to assert that both are true. That is, the conjunction of statements A and B is the statement "A and B," denoted by $A \wedge B$. The statement $A \wedge B$ is true if both A and B are true. It is false otherwise.

Disjunction asserts that at least one of two statements is true. That is, the disjunction of statements A and B is the statement "A or B," denoted by $A \vee B$. The statement $A \vee B$ is true if at least one of A and B is true, while if both A and B are false, then so is $A \vee B$.

It is important to note that in everyday life, the word "or" can be used with several different meanings:

(i) At least one of two statements is true (**inclusive or**).
(ii) Exactly one of two statements is true (**complementary or**).
(iii) At most one of two statements is true (**exclusive or**).

For example, if Alicia says, "I go to the theater or the opera every week," then she is probably using the inclusive or, since there might be weeks when she goes to both. On the other hand, if a young man talking to his girlfriend says, "Today let's go to the movies or go shopping," then he probably means that they will do exactly one of those two activities, but not both, so he is using the complementary or. Finally, if at a family dinner, a father is lecturing his son by saying, "At the table, one either eats or reads," then he means that his son should do at most one of those things at a time, so he is using the exclusive or.

Let us note that in mathematics, the logical operator *or* is, unless stated otherwise, meant as an *inclusive or*.

Negation is exactly what it sounds like. The negation of statement A is the statement "A is false," or in other words, "A is not true." We can express this by saying "not A," and we denote it by $\overline{A}$. The statement $\overline{A}$ is true exactly when A is false.

We need to distinguish between the concepts of a statement that negates A and a statement that contradicts A. We say that a statement B contradicts A if both A and B cannot be true simultaneously. The negation of a statement A is also a contradiction of A, but the converse is not necessarily true: a contradiction is not necessarily a negation. If we let A denote the statement "the real number x is positive," and we let B denote "the real number x is negative," then B contradicts A, but B is not the negation of A, since the correct negation is "the real number x is negative or zero." Similarly, the statement "these letters are black" does not have the contradiction "these letters are white" as its negation, for while it contradicts our original statement, the correct negation would be "these letters are not black" (since there are many other colors besides white that they could be). *The negation of statement A comprises all cases in which A is not true.*

It is easy to check that if A and B are arbitrary statements, then the identities $\overline{A \wedge B} = \overline{A} \vee \overline{B}$ and $\overline{A \vee B} = \overline{A} \wedge \overline{B}$ hold.

Implications help us express when one statement follows from another. We can express this by the statement "if A is true, then so is B," or "if A, then B" for short.

We denote this by $A \Rightarrow B$. We also say "A implies B." The statement $A \Rightarrow B$ then means that in every case in which A is true, B is true. We can easily see that $A \Rightarrow B$ shares the same value as $\overline{A} \vee B$ in the sense that these statements are true exactly in the same cases. (Think through that the statement "if the victim was not at home Friday night, then his radio was playing" means exactly the same thing as "the victim was at home Friday night or his radio was playing (or both).")

The statement $A \Rightarrow B$ (or $\overline{A} \vee B$) is true in every case in which B is true, and also in every case in which A is false. The only case in which $A \Rightarrow B$ is false is that A is true and B is false. We can convince ourselves that $A \Rightarrow B$ is true whenever A is false with the following example. If we promise a friend "if the weather is nice tomorrow, we will go on a hike," then there is only one case in which we break our promise: if tomorrow it is nice out but we don't go on a hike. In all the other cases—including whatever happens if the weather is not nice tomorrow—we have kept our promise.

In mathematics, just as we saw with the use of the term "or," the "if, then" construction does not always correspond to its use in everyday language. Generally, when we say "if A then B," we mean that there is a cause-and-effect relationship between A and B. And therefore, a statement of the form "if A then B" in which A and B are unrelated sounds nonsensical or comical ("if it's Tuesday, this must be Belgium"). In mathematical logic, we do not concern ourselves with the (philosophical) cause; we care only for the truth values of our statements. The implication "if it is Tuesday, then this is Belgium" is true exactly when the statement "it is not Tuesday or this is Belgium" is true, which depends only on whether it is Tuesday and whether we are in Belgium. If it is Tuesday and this is not Belgium, then the implication is false, and in every other case it is true.

We can express the $A \Rightarrow B$ implication in words as "B follows from A," "A is sufficient for B," and "B is necessary for A."

Equivalence expresses the condition of two statements always being true at the same time. The statement "A is equivalent to B," denoted by $A \iff B$, is true if both A and B are true or both A and B are false. If one of A and B is true but the other false, then $A \iff B$ is false. If $A \iff B$, we can say "A if and only if B," or "A is necessary and sufficient for B."

Quantifiers

We call statements that contain variables *predicates*. Their truth or falsity depends on the values we substitute for the variables. For example, the statement "Mr. N is bald" is a predicate with variable N, and whether it is true depends on whom we substitute for N. Similarly, "x is a perfect square" is a predicate that is true if $x = 4$, but false if $x = 5$. Let $A(x)$ be a predicate, where x is the variable. Then there are two ways to form a "normal" statement (fixed to be either true or false) from it. The first of these is "$A(x)$ is true for all x," which we denote by $(\forall x)A(x)$. The symbol $\forall$ is called the **universal quantifier.** It represents an upside-down A as in

"All." The second is "there exists an x such that $A(x)$ is true," which we denote by $(\exists x)A(x)$. Here $\exists$ is called the **existential quantifier.** It represents the E in "Exists" written backward.

For the above examples, $(\exists N)$(Mr. N is bald) and $(\exists x)(x$ is a square) are true, since there are bald men and some numbers are squares. The statements $(\forall N)$(Mr. N is bald) and $(\forall x)(x$ is a square) are false, however, since not all men are bald, and not all numbers are squares.

It is easy to check that for every predicate $A(x)$, we have the identities

$$\overline{(\exists x)A(x)} = (\forall x)\overline{A(x)} \quad \text{and} \quad \overline{(\forall x)A(x)} = (\exists x)\overline{A(x)}.$$

We will often see statements of the form $(\forall x)(A(x) \Rightarrow B(x))$. This statement means that for every x, if $A(x)$ is true, then $B(x)$ is true. More concisely, whenever $A(x)$ is true, $B(x)$ is true. If $A(x)$ is never true, then $(\forall x)(A(x) \Rightarrow B(x))$ is true.

Exercises

2.1. Boys and girls are dancing together at a party. Let $D(G,B)$ denote the statement that girl G danced with boy B at some point during the party. Determine which of the following statements imply another of the statements:

(a) $(\exists G)(\forall B)D(G,B)$;
(b) $(\forall B)(\exists G)D(G,B)$;
(c) $(\exists B)(\forall G)D(G,B)$;
(d) $(\forall G)(\exists B)D(G,B)$;
(e) $(\forall G)(\forall B)D(G,B)$;
(f) $(\exists G)(\exists B)D(G,B)$.

2.2. Regarding this same dance party, determine which of the following statements imply another:

(a) $\overline{(\exists G)(\forall B)D(G,B)}$;
(b) $(\forall B)(\exists G)\overline{D(G,B)}$;
(c) $(\forall G)(\exists B)\overline{D(G,B)}$;
(d) $(\forall G)(\forall B)\overline{D(G,B)}$.

2.3. How many subsets H does the set $\{1,2,\dots,n\}$ have such that the statement $(\forall x)(x \in H \Rightarrow x+1 \notin H)$ is true? (S)

2.4. How many subsets H does the set $\{1,2,\dots,n\}$ have such that the statement $(\forall x)([(x \in H) \wedge (x+1 \in H)] \Rightarrow x+2 \in H)$ is true?

2.3 Proof Techniques

Proof by Contradiction

Arguing by contradiction proceeds by supposing that the negation of the statement we want to prove is true and then deducing a **contradiction**; that is, we show that some (not necessarily the original) statement and its negation both must hold. Since this is clearly impossible, our original assumption must have been false, so the statement we wanted to prove is true.

Let us see a simple example. It is clear that we cannot place nine nonattacking rooks on a chessboard, that is, nine rooks such that no rook is attacking another rook. Surely, no matter how we place nine rooks, there will be two of them in the same row, and those two rooks are therefore attacking each other. This argument can be viewed as a simple form of proof by contradiction: we suppose that our statement is false (that we can place nine nonattacking rooks on the chessboard), and in this case, we contradict our original statement.

A less straightforward example follows. Cut off two opposing corner squares of a chessboard. Show that one cannot cover the rest of the board using 31 dominoes, each of which covers two adjacent squares. For suppose that such a covering exists. Since each domino covers one black and one white square, we have 31 black and 31 white squares covered. However, two opposing corner squares are of the same color, so the numbers of black and white squares remaining are different: there are 30 of one and 32 of the other. This is a contradiction, so such a covering cannot exist. (In this argument, the statement that we showed to be true along with its negation is the following: the number of covered white squares is equal to the number of covered black squares.)

Perhaps the most classical example of proof by contradiction is the following.

Theorem 2.1. *$\sqrt{2}$ is irrational (that is, it cannot be written as a quotient of two integers).*

We give two proofs.

Proof I. Suppose that $\sqrt{2} = p/q$, where p and q are positive integers, and suppose q is the smallest such denominator. Then $2q^2 = p^2$, so p^2 is even. This means that p must also be even, so let $p = 2p_1$. Then $2q^2 = (2p_1)^2 = 4p_1^2$, so $q^2 = 2p_1^2$, whence q is also even. If we let $q = 2q_1$, then $\sqrt{2} = p/q = p_1/q_1$. Since $q_1 < q$, this contradicts the minimality of q. □

We shall use the symbol □ to indicate the end of a proof. In the above proof, the □ actually stands for "This contradiction can only arise from a faulty assumption. Thus our assumption that $\sqrt{2}$ is rational is false. We conclude that $\sqrt{2}$ is irrational."

Proof II. Suppose again that $\sqrt{2} = p/q$, where p and q are positive integers, and q is the smallest such denominator. Then

$$\frac{2q-p}{p-q} = \frac{2-(p/q)}{(p/q)-1} = \frac{2-\sqrt{2}}{\sqrt{2}-1} = \sqrt{2}.$$

Since $2p-q$ and $p-q$ are integers and $0 < p-q < q$, we have contradicted the minimality of q again. □

The above examples use a proof by contradiction to show the impossibility or nonexistence of something. Many times, however, we need to show the existence of something using a proof by contradiction. We give two examples.

Theorem 2.2. *If a country has only finitely many cities, and every city has at least two roads leading out of it, then we can make a round trip between some cities.*

Proof. Suppose that we cannot make any round trips. Let C_1 be a city, and suppose we start travelling from C_1 along a road. If we arrive at the city C_2, then we can keep going to another city C_3, as there are at least two roads going from C_2. Also the city C_3 is different than C_1, since otherwise we would have made a round trip already, which we assumed is impossible. We can keep travelling from C_3 to a new city C_4, as there are at least two roads leading out from C_3. This C_4 is different from all the previous cities. Indeed if for example $C_4 = C_2$, then $C_2, C_3, C_4 = C_2$ would be a round trip. From C_4 we can go on to a new city C_5, and then to C_6 and so on continuing indefinitely. However, this is a contradiction, as according to our assumption there are only finitely many cities in the country. □

Theorem 2.3. *There are infinitely many prime numbers.*

Proof. Suppose there are only finitely many primes; call them $p_1, \ldots, p_n$. Consider the number $N = p_1 \cdots p_n + 1$. It is well known—and not hard to prove—that N (like every integer greater than 1) has a prime divisor. Let p be any of the prime divisors of N. Then p is different from all of $p_1, \ldots, p_n$, since otherwise, both N and $N-1$ would be divisible by p, which is impossible. But by our assumption $p_1, \ldots, p_n$ are all the prime numbers, and this is a contradiction. □

Proof by Induction

Another important proof technique is known as **mathematical induction**. Induction arguments prove infinitely many statements at once. In its simplest form, we prove the statements $A_1, A_2, A_3, \ldots$ in two steps: First we show that A_1 is true, and then we show that A_{n+1} follows from A_n for every n. (This second step, namely $A_n \Rightarrow A_{n+1}$, is called the **inductive step**, and A_n is called the **induction hypothesis**.)

These two steps truly prove all of the statements A_n. Indeed, the first statement, A_1, is proved directly. Since we have shown that A_n implies A_{n+1} for each n, it follows that in particular, A_1 implies A_2. That is, since A_1 is true, A_2 must be true as well. But by the inductive step, the truth of A_3 follows from the truth of A_2, and thus A_3 is true. Then from $A_3 \Rightarrow A_4$, we get A_4, and so on.

Let us see a simple example.

Theorem 2.4. *For every positive integer n, $2^n > n$.*

Proof. The statement A_n that we are trying to prove for all positive integers n is $2^n > n$. The statement A_1, namely $2^1 > 1$, is obviously true. Given the induction hypothesis $2^n > n$, it follows that $2^{n+1} = 2 \cdot 2^n = 2^n + 2^n \geq 2^n + 1 > n+1$. This proves the theorem. $\square$

It often happens that the index of the first statement is not 1, in which case we need to alter our argument slightly. We could, for example have stated the previous theorem for every nonnegative integer n, since $2^0 > 0$ is true, and the inductive step works for $n \geq 0$.

Here is another example: if we want to show that $2^n > n^2$ for $n \geq 5$, then in the first step we have to check this assertion for $n = 5$ $(32 > 25)$, and then show, as before, in the inductive step that if the statement holds for n, then it holds for $n+1$.

Now as an application of the method, we prove an important inequality (from which Theorem 2.4 immediately follows).

Theorem 2.5 (Bernoulli's[1] Inequality). *If $a \geq -1$, then for every positive integer n, we have*

$$(1+a)^n \geq 1+na.$$

We have equality if and only if $n = 1$ or $a = 0$.

Proof. If $n = 1$, then the statement is clearly true. We want to show that if the statement holds for n, then it is true for $n+1$. If $a \geq -1$, then $1+a \geq 0$, so

$$\begin{aligned}(1+a)^{n+1} &= (1+a)^n(1+a) \geq (1+na)(1+a) = 1+(n+1)a+na^2 \\ &\geq 1+(n+1)a.\end{aligned}$$

The above argument also shows that $(1+a)^{n+1} = 1+(n+1)a$ can happen only if $na^2 = 0$, that is, if $a = 0$. $\square$

There are times when this simple form of induction does not lead to a proof. To illustrate this, we examine the **Fibonacci[2] numbers**, which are defined as follows. Let $u_0 = 0$ and $u_1 = 1$. We define $u_2, u_3, \ldots$ to be the sum of the two previous (already defined) numbers. That is, $u_2 = 0+1 = 1$, $u_3 = 1+1 = 2$, $u_4 = 1+2 = 3$, $u_5 = 2+3 = 5$, and so on. Clearly, the Fibonacci numbers are increasing, that is, $u_0 \leq u_1 \leq u_2 \leq \cdots$. Using induction, we now show that $u_n < 2^n$ for each n. First of all, $u_0 = 0 < 1 = 2^0$. If $u_n < 2^n$ is true, then

$$u_{n+1} = u_n + u_{n-1} \leq u_n + u_n < 2^n + 2^n = 2^{n+1},$$

which proves the statement.

---

[1] Jacob Bernoulli (1654–1705), Swiss mathematician.

[2] Fibonacci (Leonardo of Pisa) (c. 1170–1240), Italian mathematician.

Actually, more than $u_n < 2^n$ is true, namely $u_n < 1.7^n$ holds for all n. However, this cannot be proved using induction in the same form as above, since the most we can deduce from $u_n < 1.7^n$ is $u_{n+1} = u_n + u_{n-1} \leq u_n + u_n < 1.7^n + 1.7^n = 2 \cdot 1.7^n$, which is larger than 1.7^{n+1}. To prove that $u_{n+1} < 1.7^{n+1}$, we need to use not only that $u_n < 1.7^n$, but also that $u_{n-1} < 1.7^{n-1}$. The statement then follows, since

$$\begin{aligned} u_{n+1} &= u_n + u_{n-1} < 1.7^n + 1.7^{n-1} = 1.7 \cdot 1.7^{n-1} + 1.7^{n-1} = 2.7 \cdot 1.7^{n-1} \\ &< 2.89 \cdot 1.7^{n-1} = 1.7^2 \cdot 1.7^{n-1} = 1.7^{n+1}. \end{aligned}$$

The proof then proceeds as follows. First we check that $u_0 < 1.7^0$ and $u_1 < 1.7^1$ hold (both are clear). Suppose $n > 1$, and let us hypothesize that $u_{n-1} < 1.7^{n-1}$ and $u_n < 1.7^n$. Then as the calculations above show, $u_{n+1} < 1.7^{n+1}$ also holds. This proves the inequality for every n. Indeed, let us denote the statement $u_n < 1.7^n$ by A_n. We have shown explicitly that A_0 and A_1 are true. Since we showed $(A_{n-1} \wedge A_n) \Rightarrow A_{n+1}$ for every n, $(A_0 \wedge A_1) \Rightarrow A_2$ is true. Therefore, A_2 is true. Then $(A_1 \wedge A_2) \Rightarrow A_3$ shows that A_3 is true, and so on.

We can use this method of induction to prove relationships between the arithmetic, geometric, and harmonic means. We define the **arithmetic mean** of the numbers $a_1, \ldots, a_n$ to be

$$A = \frac{a_1 + \cdots + a_n}{n}.$$

For nonnegative $a_1, \ldots, a_n$, the **geometric mean** is

$$G = \sqrt[n]{a_1 \cdots a_n},$$

and for positive $a_1, \ldots, a_n$, the **harmonic mean** is

$$H = n \cdot \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} \right)^{-1}.$$

The term "mean" indicates that each of these numbers falls between the largest and the smallest of the a_i. If the largest of the a_i is M and the smallest is m, then $A \geq n \cdot m/n = m$, and similarly, $A \leq M$. If the numbers are not all equal, then $m < A < M$ also holds. This is so because one of the numbers is then larger than m, so $a_1 + \cdots + a_n > n \cdot m$ and $A > m$. We can similarly obtain $A < M$.

The same method can be used to see that for $m > 0$, we have $m \leq G \leq M$ and $m \leq H \leq M$; moreover, if the numbers are not all equal, then the inequalities are strict.

Another important property of the three means above is that if we append the arithmetic mean A to $a_1, \ldots, a_n$, then the resulting extended set of numbers has the same arithmetic mean, namely A. That is,

$$\frac{a_1 + \cdots + a_n + A}{n+1} = \frac{n \cdot A + A}{n+1} = A.$$

The same is true (and can be proved similarly) for the geometric and harmonic means.

Theorem 2.6. *If $a_1, \ldots, a_n$ are arbitrary nonnegative numbers, then*

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

Equality holds if and only if $a_1 = \cdots = a_n$.

Proof. Let $a_1, \ldots, a_n$ be arbitrary nonnegative numbers, and let A and G denote their respective arithmetic and geometric means. Suppose that k of the numbers $a_1, \ldots, a_n$ are different from A. We prove the statement using induction on k. (The number of terms, n, is arbitrary.) If $k = 0$, then the statement clearly holds, since then, all terms are equal to A, so the value of each mean is also A. Since $k = 1$ can never happen (if all terms but one of them are equal to A, then the arithmetic mean cannot be equal to A), the statement holds for $k = 1$ as well.

Now let $k > 1$, and suppose that our statement is true for all sets of numbers such that the number of terms different from A is either $k-1$ or $k-2$. We prove that $G < A$. We may suppose that $a_1 \leq \cdots \leq a_n$, since the order of the numbers plays no role in the statement. Since there are terms not equal to A, we see that $a_1 < A < a_n$. This also implies that $k \geq 2$.

If 0 occurs among the terms, then $G = 0$ and $A > 0$, so the statement is true. So we can suppose that the terms are positive. Replace a_n with A, and a_1 with $a_1 + a_n - A$. This new set of numbers will have the same arithmetic mean A, since the replacements did not affect the value of their sum. Also, the new geometric mean after the replacement must have increased, since $A(a_1 + a_n - A) > a_1 a_n$, which is equivalent to $(a_n - A)(A - a_1) > 0$, which is true.

In this new set of numbers, the number of terms different from A decreased by one or two. Then, by our induction hypothesis, $G' \leq A$, where G' denotes the geometric mean of the new set. Since $G < G'$, we have $G < A$, which concludes the proof. $\square$

Theorem 2.7. *If $a_1, \ldots, a_n$ are arbitrary positive numbers, then*

$$n \cdot \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n} \right)^{-1} \leq \sqrt[n]{a_1 \cdots a_n}.$$

Equality holds if and only if $a_1 = \cdots = a_n$.

Proof. Apply the previous theorem to the numbers $1/a_1, \ldots, 1/a_n$, and then take the reciprocal of both sides of the resulting inequality. $\square$

The more general form of induction (sometimes called *strong induction*, while the previous form is sometimes known as *weak induction*) is to show that A_1 holds, then in the induction step to show that if $A_1, \ldots, A_n$ are all true, then A_{n+1} is also true. This proves that every A_n is true just as our previous arguments did.

Exercises

2.5. Prove that the following identities hold for every positive integer n:

(a) $\dfrac{x^n - y^n}{x-y} = x^{n-1} + x^{n-2}\cdot y + \cdots + x\cdot y^{n-2} + y^{n-1}$;
(b) $1^2 + \cdots + n^2 = \dfrac{n\cdot(n+1)\cdot(2n+1)}{6}$;
(c) $1^3 + \cdots + n^3 = \left(\dfrac{n\cdot(n+1)}{2}\right)^2$;
(d) $1 - \dfrac{1}{2} + \dfrac{1}{3} - \cdots - \dfrac{1}{2n} = \dfrac{1}{n+1} + \cdots + \dfrac{1}{2n}$;
(e) $\dfrac{1}{1\cdot 2} + \cdots + \dfrac{1}{(n-1)\cdot n} = \dfrac{n-1}{n}$.

2.6. Express the following sums in simpler terms:

(a) $\dfrac{1}{1\cdot 2\cdot 3} + \cdots + \dfrac{1}{n\cdot(n+1)\cdot(n+2)}$;
(b) $1\cdot 2 + \cdots + n\cdot(n+1)$;
(c) $1\cdot 2\cdot 3 + \cdots + n\cdot(n+1)\cdot(n+2)$.

2.7. Prove that the following inequalities hold for every positive integer n:

(a) $\sqrt{n} \le 1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} < 2\sqrt{n}$;
(b) $1 + \dfrac{1}{2\cdot\sqrt{2}} + \cdots + \dfrac{1}{n\cdot\sqrt{n}} \le 3 - \dfrac{2}{\sqrt{n}}$.

2.8. Denote the nth Fibonacci number by u_n. Prove that $u_n > 1.6^n/3$ for every $n \ge 1$.

2.9. Prove the following equalities:

(a) $u_n^2 - u_{n-1}u_{n+1} = (-1)^{n+1}$;
(b) $u_1^2 + \cdots + u_n^2 = u_n u_{n+1}$.

2.10. Express the following sums in simpler terms:

(a) $u_0 + u_2 + \cdots + u_{2n}$;
(b) $u_1 + u_3 + \cdots + u_{2n+1}$;
(c) $u_0 + u_3 + \cdots + u_{3n}$;
(d) $u_1u_2 + \cdots + u_{2n-1}u_{2n}$.

2.11. Find the flaw in the following argument: We want to prove that no matter how we choose n lines in the plane such that no two are parallel, with $n > 1$, they will always intersect at a single point. The statement is clearly true for $n = 2$. Let $n \ge 2$, and suppose the statement is true for n lines. Let $l_1, \ldots, l_{n+1}$ be lines such that no two are parallel. By the induction hypothesis, the lines $l_1, \ldots, l_n$ intersect at the single point P, while the lines $l_2, \ldots, l_{n+1}$ intersect at the point Q. Since each of the lines $l_2, \ldots, l_n$ passes through both P and Q, necessarily $P = Q$. Thus we see that each line goes through the point P. (S)

2.12. If we have n lines such that no two are parallel and no three intersect at one point, into how many regions do they divide the plane? (H)

2.13. Prove that finitely many lines (or circles) partition the plane into regions that can be colored with two colors such that the borders of two regions with the same color do not share a common segment or circular arc.

2.14. Deduce Bernoulli's inequality from the inequality of arithmetic and geometric means. (S)

2.15. Prove that if $a_1, \dots, a_n \ge 0$, then

$$\frac{a_1 + \cdots + a_n}{n} \le \sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}}.$$

(Use the arguments of the proof of Theorem 2.7.)

2.16. Find the largest value of $x^2 \cdot (1-x)$ for $x \in [0,1]$. (H)

2.17. Find the cylinder of maximal volume contained in a given right circular cone.

2.18. Find the cylinder of maximal volume contained in a given sphere.

2.4 Sets, Functions, Sequences

Sets

Every branch of mathematics involves inspecting some set of objects or elements defined in various ways. In geometry, those elements are points, lines, and planes; in analysis, they are numbers, sequences, and functions; and so on. Therefore, we need to clarify some basic notions regarding sets.

What is a set? Intuitively, a set is a collection, family, system, assemblage, or aggregation of specific things. It is important to note that these terms are redundant and are rather synonyms than definitions. In fact, we could not even accept them as definitions, since we would first have to define a collection, family, system, assemblage, or aggregation, putting us back right where we started. Sometimes, a set is defined as a collection of things sharing some property. Disregarding the fact that we again used the term "collection," we can raise another objection: what do we mean by a shared property? That is a very subjective notion, which we cannot allow in a formal definition. (Take, for example, the set that consists of all natural numbers and all circles in the plane. Whether there is a common property here could be a matter of debate.) So we cannot accept this definition either.

It seems that we are unable to define sets using simpler language. In fact, we have to accept that there are concepts that we cannot express in simpler terms (since otherwise, the process of defining things would never end), so we need to have some fundamental concepts that we do not define. *A set is one of those fundamental concepts. All that we suppose is that every item in our universe of discourse is either an element of a given set or not an element of that set.* (Here we are using the word "or" to mean the complementary or.)

If x is an element of the set H (we also say that x belongs to H, or is in H), we denote this relationship by $x \in H$. If x is not an element of the set (we also say that x does not belong to H, or is not in H), we denote this fact by $x \notin H$.

We can represent sets themselves in two different ways. The ostensibly simpler way is to list the set's elements inside a pair of curly braces: $A = \{1,2,3,4,6,8,12,24\}$. If a set H contains only one element, x, then we denote it by $H = \{x\}$. We can even list an infinite set of elements if the relationship among its elements is unambiguous, for example $\mathbb{N} = \{0,1,2,3,\dots\}$. The other common way of denoting sets is that after a letter or symbol denoting a generic element we put a colon, and then provide a rule that defines such elements:

$$\begin{aligned}
\mathbb{N} &= \{n : n \text{ is a nonnegative integer}\} = \{0,1,2,\dots\},\\
\mathbb{N}^+ &= \{n : n \text{ is a positive integer}\} = \{1,2,\dots\},\\
B &= \{n : n \text{ is an odd natural number}\} = \{2n-1 :\ n = 1,2,\dots\} = \{1,3,5,\dots\},\\
C &= \left\{\frac{1}{n} : n = 1,2,\dots\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\},\\
D &= \{n : n \mid 24 \text{ and } n > 0\} = \{1,2,3,4,6,8,12,24\}.
\end{aligned}$$

What do we mean by the equalities here? By agreement, we consider two sets A and B to be equal if they have the same elements, that is, for every object x, $x \in A$ if and only if $x \in B$. Using notation:

$$A = B \iff (\forall x)(x \in A \iff x \in B).$$

In other words, $A = B$ if every element of A is an element of B and vice versa. Let us note that if we list an element multiple times, it does not affect the set. For example, $\{1,1,2,2,3,3,3\} = \{1,2,3\}$, and $\{n^2 : n \text{ is an integer}\} = \{n^2 : n \geq 0 \text{ is an integer}\} = \{0,1,4,9,16,\dots\}$.

As we have seen, a set can consist of a single element, for example $\{5\}$. Such a set is called a *singleton*; this set, of course, is not equal to the number 5 (which is not a set), nor to $\{\{5\}\}$ which is indeed a set (and a singleton), but its only element is not 5 but $\{5\}$. We see that an element of a set can be a set itself.

Is it possible for a set to have no elements? Consider the following examples:

$$\begin{aligned}
G &= \{p : p \text{ is prime, and } 888 \leq p \leq 906\},\\
H &= \left\{n^2 : n \in \mathbb{N} \text{ and the sum of the decimal digits of } n^2 \text{ is } 300\right\}.
\end{aligned}$$

At first glance, these sets seem just as legitimate as all the previously listed ones. But if we inspect them more carefully, we find out that they are empty, that is, they have no elements. Should we exclude them from being sets? Were we to do so, then before specifying any set, we would have to make sure that it had an element. Other than making life more difficult, there would be another drawback: we are not always able to check this. *It is a longstanding open problem whether or not there is an odd prefect number* (a number is perfect if it is equal to the sum of its positive proper divisors). If we exclude sets with no elements from being sets, we would not be able to determine whether $\{n : n \text{ is an odd perfect number}\}$ is a well-defined set.

Thus it seems practical to agree that the above definitions form sets, and so (certainly in the case of G or H above) we accept sets that do not have any elements. How many such sets are there? By our previous notion of equality, there is just one, since if neither A nor B has any elements, then it is vacuously true that every element of A is an element of B, and vice versa, since there are no such elements. We call this single set the **empty set** and denote it by $\emptyset$.

If every element of a set B is an element of a set A, then we say that B is a **subset** of A, or that B is contained in A. Notation: $B \subset A$ or $A \supset B$; we can say equivalently that B is a subset of A and that A contains B.[3]

It is clear that $A = B$ if and only if $A \subset B$ and $B \subset A$. If $B \subset A$, but $B \neq A$, then we say that B is a **proper subset** of A. We denote this by $B \subsetneq A$.

Just as we define operations between numbers (such as addition and multiplication), we can define operations between sets. For two sets A and B, their **union** is the set of elements that belong to at least one of A and B. We denote the union of A and B by $A \cup B$, so

$$A \cup B = \{x : x \in A \vee x \in B\}.$$

We can define the union of any number of sets, even infinitely many: $A_1 \cup A_2 \cup \cdots \cup A_n$ denotes the set of elements that belong to at least one of $A_1, \ldots, A_n$. This same set can also be denoted more concisely by $\bigcup_{i=1}^{n} A_i$. Similarly, $A_1 \cup A_2 \cup \cdots$ or $\bigcup_{i=1}^{\infty} A_i$ denotes the set of all elements that belong to at least one of $A_1, A_2, \ldots$.

For two sets A and B, their **intersection** is the set of elements that belong to both A and B. We denote the intersection of A and B by $A \cap B$, so

$$A \cap B = \{x : x \in A \wedge x \in B\}.$$

The intersection of any finite number of sets or infinitely many sets is defined similarly as for union.

We say that two sets A and B are **disjoint** if $A \cap B = \emptyset$.

For two sets A and B, their **set-theoretic difference** is the set of elements that are in A but not in B. We denote the difference of sets A and B by $A \setminus B$, so

$$A \setminus B = \{x : x \in A \wedge x \notin B\}.$$

Let H be a fixed set, and let $X \subset H$. We call $H \setminus X$ the **complement** (with respect to H) of X, and we denote it by $\overline{X}$. It is easy to see that

$$\overline{\overline{A}} = A,$$

[3] Sometimes, containment is denoted by $\subseteq$.

and the so-called De Morgan identities are also straightforward:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}, \quad \overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Here are some further identities:

$$\begin{aligned}
A \setminus A &= \emptyset, \\
A \setminus \emptyset &= A, \\
A \cup A &= A, \\
A \cap A &= A, \\
A \cup B &= B \cup A, \\
A \cap B &= B \cap A, \\
A \cup (B \cup C) &= (A \cup B) \cup C = A \cup B \cup C, \\
A \cap (B \cap C) &= (A \cap B) \cap C = A \cap B \cap C, \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C), \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\
A &\subset A, \\
A \subset B,\ B \subset C &\Rightarrow A \subset C, \\
\emptyset &\subset A.
\end{aligned} \tag{2.1}$$

Functions

Consider a mathematical expression that contains the variable x, such as

$$x^2 + 1 \quad \text{or} \quad \frac{x+3}{x-2}.$$

To evaluate such an expression for a particular numeric value of x, we have to compute the value of the expression when the given number is substituted for x (this is possible, of course, only if the expression makes sense; in the second example, $x = 2$ is impermissible). These formulas thereby associate certain numbers with other numbers. There are many ways of creating such correspondences between numbers. For example, we could consider the number of positive divisors of an integer n, the sum of the digits of n, etc. We can even associate numbers with other things, for example to each person their weight or how many strands of hair they have. Even more generally, we could associate to each person their name. These examples define functions, or mappings. These two terms are synonymous, and they mean the following.

Consider two sets A and B. Suppose that for each $a \in A$, we somehow associate with a an element $b \in B$. Then this association is called a **function** or **mapping**.

If this function is denoted by f, then we say that f is a function from A to B, and we write $f : A \to B$. If the function f associates $a \in A$ to $b \in B$, we say that f maps a to b (or a is mapped to b under f), or that the value of f at a is b, and we denote this by $b = f(a)$. We call A the **domain** of f.

When we write down a formula, for example $n^2 + 1$, and we want to emphasize that we are not talking about the *number* $n^2 + 1$ but the *mapping* that maps n to $n^2 + 1$, then we can denote this by

$$n \mapsto n^2 + 1 \quad (n \in \mathbb{N}).$$

Sequences

Writing arbitrary elements sequentially, that is, in a particular order, gives us a **sequence**. If the number of elements in the list is n, then we say that we have a sequence of length n. To define a sequence, we need to specify what the first, second, and generally the kth element is, for each $k = 1, \ldots, n$. A sequence of length n is usually denoted by $(a_1, a_2, \ldots, a_n)$, where, of course, instead of a we can use any other letter or symbol. We often call the elements $a_1, \ldots, a_n$ the **terms** of the sequence; the number representing each term's position is called its **index**. We consider two sequences of length n to be equal if their kth terms agree for each $k = 1, \ldots, n$, so the order of the terms matters. That is, $(a_1, a_2, a_3) = (\text{Chicago}, \sqrt{2}, \text{Shakespeare})$ if and only if $a_1 = \text{Chicago}$, $a_2 = \sqrt{2}$, and $a_3 = \text{Shakespeare}$. The terms of the sequence need not be distinct, so $(2, 3, 3, 3)$ is a sequence of length 4.

We sometimes also call sequences of length n **ordered n-tuples**. For $n = 2$, instead of ordered two-tuples we say **ordered pairs**, and for $n = 3$, we usually say **ordered triples**.

If we want to make the ambiguous "sequentially" above more precise, we may say that a sequence of length n is a mapping with domain $\{1, 2, \ldots, n\}$ that maps k to the kth term of the sequence. So the sequence defined by the map $a : \{1, 2, \ldots, n\} \to B$ is $(a(1), a(2), \ldots, a(n))$, or as we previously denoted it, $(a_1, a_2, \ldots, a_n)$.

We will often work with infinite sequences. We obtain these by writing infinitely many elements sequentially. More precisely, we define an **infinite sequence** to be a function with domain $\mathbb{N}^+ = \{1, 2, \ldots\}$. That is, the function $a : \mathbb{N}^+ \to B$ defines a sequence $(a(1), a(2), \ldots)$, which we can also denote by $(a_1, a_2, \ldots)$ or by $(a_i)_{i=1}^{\infty}$.

We also consider functions with domain $\mathbb{N}$ as infinite sequences. These can be denoted by $(a(0), a(1), \ldots)$, or $(a_0, a_1, \ldots)$, or even $(a_i)_{i=0}^{\infty}$. More generally, every function from a set of the form $\{k, k+1, \ldots\}$ is called an infinite sequence; the notation is straightforward.

Exercises

2.19. Prove the identities in (2.1).

2.20. Prove that $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

2.21. Let $A\triangle B$ denote the set $(A \setminus B)\cup(B\setminus A)$. (We call $A\triangle B$ the **symmetric difference** of A and B.) Show that for arbitrary sets A, B, C,

(a) $A\triangle\emptyset = A$,
(b) $A\triangle A = \emptyset$,
(c) $(A\triangle B)\triangle C = A\triangle(B\triangle C)$.

2.22. Prove that $x \in A_1\triangle A_2\triangle\cdots\triangle A_n$ holds if and only if x is an element of an odd number of $A_1,\ldots,A_n$.[4]

2.23. Determine which of the following statements are true and which are false:[5]

(a) $(A\cup B)\setminus A = B$,
(b) $(A\cup B)\setminus C = A\cup(B\setminus C)$,
(c) $(A\setminus B)\cap C = (A\cap C)\setminus B = (A\cap C)\setminus(B\cap C)$,
(d) $A\setminus B = A\setminus(A\cap B)$.

2.24. Let $U(A_1,\ldots,A_n)$ and $V(A_1,\ldots,A_n)$ be expressions that are formed from the operations $\cup$, $\cap$, and $\setminus$ on the sets $A_1,\ldots,A_n$. (For example, $U = A_1\cap(A_2\cup A_3)$ and $V = (A_1\cap A_2)\cup(A_1\cap A_3)$.) Prove that $U(A_1,\ldots,A_n) = V(A_1,\ldots,A_n)$ holds for every $A_1,\ldots,A_n$ if and only if it holds for all sets $A_1,\ldots,A_n$ such that the nonempty ones are all equal. ($*$ H)

[4] We can write $A_1\triangle A_2\triangle\cdots\triangle A_n$, since we showed in part (c) of Exercise 2.21 that different placements of parentheses define the same set (so we may simply omit them).

[5] That is, either prove that the identity holds for every A, B, C, or give some A, B, C for which it does not hold.