# Chapter 10
# Continuity and Limits of Functions

If we want to compute the value of a specific function at some point $a$, it may happen that we can compute only the values of the function near $a$. Consider, for example, the distance a free-falling object covers. This is given by the equation $s(t) = g \cdot t^2/2$, where $t$ is the time elapsed, and $g$ is the gravitational constant. Knowing this equation, we can easily compute the value of $s(t)$. If, however, we want to calculate $s(t)$ at a particular time $t = a$ by measuring the time, then we will not be able to calculate the *precise* distance corresponding to this given time; we will obtain only a better or worse approximation—depending on the precision of our instruments. However, if we are careful, we will hope that if we use the value of $t$ that we get from the measurement to recover $s(t)$, the result will be close to the original $s(a)$. In essence, such difficulties always arise when we are trying to find some data with the help of another measured quantity. At those times, we assume that if our measured quantity differs from the real quantity by a very small amount, then the value computed from it will also be very close to its actual value.

In such cases, we have a function $f$, and we assume that $f(t)$ will be close to $f(a)$ as long as $t$ deviates little from $a$. We call this property continuity. The precise definition of this idea is the following.

**Definition 10.1.** Let $f$ be defined on an open interval containing $a$. The function $f$ is *continuous* at $a$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ (dependent on $\varepsilon$) such that

$$|f(x) - f(a)| < \varepsilon, \text{ if } |x - a| < \delta. \tag{10.1}$$

Continuity of the function $f$ at $a$ graphically means that the graph $G = \operatorname{graph} f$ has the following property: given an arbitrary (narrow) strip

$$\{(x,y) : f(a) - \varepsilon < y < f(a) + \varepsilon\},$$



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there exists an interval $(a-\delta, a+\delta)$ such that the part of $G$ over this interval lies inside the given strip (Figure 10.1).

It is clear that if (10.1) holds with some $\delta > 0$, then it holds for all $\delta' \in (0, \delta)$ as well. In other words, if for some $\varepsilon > 0$ a $\delta > 0$ is "good," then every positive $\delta' < \delta$ is also "good." On the other hand, if a $\delta > 0$ is "good" for some $\varepsilon$, then it is good for all $\varepsilon' > \varepsilon$ as well. If our goal is to deduce whether a function is continuous at a point, then generally there is no need to find the best (i.e., largest) $\delta$ for each $\varepsilon > 0$. It suffices to find just *one* such $\delta$. (The situation is analogous to determining the convergence of sequences: we do not have to find the smallest threshold for a given $\varepsilon$; it is enough to find one.)

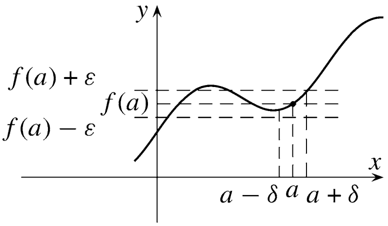


Fig. 10.1

*Examples 10.2.* **1.** The constant function $f(x) \equiv c$ is continuous at all $a$. For every $\varepsilon > 0$, every positive $\delta$ is good.
**2.** The function $f(x) = x$ is continuous at all $a$. For all $\varepsilon > 0$, the choice of $\delta = \varepsilon$ is good.
**3.** The function $f(x) = x^2$ is continuous at all $a$. If $0 < \delta \le 1$ and $|x-a| < \delta$, then

$$|x^2 - a^2| = |x-a| \cdot |x+a| = |x-a| \cdot |x-a+2a| < |x-a| \cdot (2|a|+1).$$

Thus, if

$$\delta = \min\left(1, \frac{\varepsilon}{2|a|+1}\right),$$

then $|x^2 - a^2| < \delta \cdot (2|a|+1) < \varepsilon$ whenever $|x-a| < \delta$.
**4.** The function $f(x) = 1/x$ is continuous at all $a \neq 0$. To see this, for a given $\varepsilon > 0$ we will choose the largest $\delta$, and in fact, we will determine all $x$ where $|f(x) - f(a)| < \varepsilon$. Let, for simplicity's sake, $0 < a < 1$ and $0 < \varepsilon < 1$. Then

$$\frac{1}{x} = \frac{1}{a} + \varepsilon, \text{ if } x = \frac{a}{1+\varepsilon a},$$

$$\frac{1}{x} = \frac{1}{a} - \varepsilon, \text{ if } x = \frac{a}{1-\varepsilon a}.$$

Then, because $1/x$ is strictly monotone on $(0, \infty)$,

$$|f(x) - f(a)| < \varepsilon, \text{ if } \frac{a}{1+\varepsilon a} < x < \frac{a}{1-\varepsilon a},$$

$$|f(x) - f(a)| \ge \varepsilon, \text{ if } x \notin \left(\frac{a}{1+\varepsilon a}, \frac{a}{1-\varepsilon a}\right).$$

It then follows that for a given $a$ and $\varepsilon$,

$$\delta = \min\left(\frac{a}{1-\varepsilon a} - a, a - \frac{a}{1+\varepsilon a}\right) = a - \frac{a}{1+\varepsilon a} = \frac{\varepsilon a^2}{1+\varepsilon a} \tag{10.2}$$

is the largest $\delta$ for which

$$|f(x) - f(a)| < \varepsilon, \quad \text{if} \quad x \in (a-\delta, a+\delta)$$

holds (Figure 10.2).

Later a more general theorem (10.44) will show that the continuity of $f(x) = x^2$ and $f(x) = 1/x$ follows directly from the continuity of $f(x) = x$ without extra work.

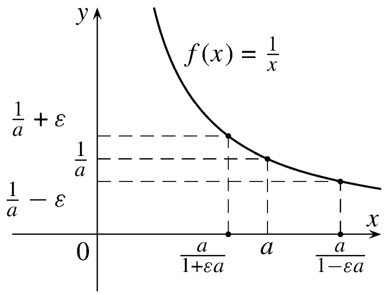


Fig. 10.2

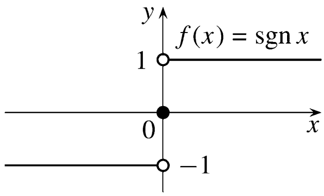


Fig. 10.3

**5.** The function

$$f(x) = \operatorname{sgn} x = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is continuous at all $a \neq 0$, but at 0, it is not continuous. Since $f \equiv 1$ on the half-line $(0, \infty)$, it follows that for all $a > 0$ and $\varepsilon > 0$, the choice $\delta = a$ is good. Similarly, if $a < 0$, then $\delta = |a|$ is a good $\delta$ for all $\varepsilon$. However, $|f(x) - f(0)| = 1$ if $x \neq 0$, so for $a = 0$, if $0 < \varepsilon < 1$, then we cannot choose a good delta (Figure 10.3).

**6.** The function

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous at 0, but not continuous at any $x \neq 0$. It is continuous at 0, since for all $x$, $|f(x) - f(0)| = |x|$, so

$$|f(x) - f(0)| < \varepsilon, \quad \text{if} \quad |x - 0| < \varepsilon,$$

that is, for all $\varepsilon > 0$, the choice $\delta = \varepsilon$ is good.

Now we show that the function is discontinuous at $a \neq 0$. Let $a$ be a rational number. Then at all irrational $x$ (sharing the same sign), $|f(x) - f(a)| > |a|$. This, in turn, means that for $0 < \varepsilon < |a|$, we cannot choose a good $\delta$ for that $\varepsilon$. The proof is similar if $a$ is irrational.

(We know that for all $(\delta > 0)$, there exist a rational number and an irrational number in the interval $(a - \delta, a + \delta)$. See Theorems 3.2 and 3.12.) This example is worth noting, for it shows that a function can be continuous at one point but not continuous anywhere else. This situation is perhaps less intuitive than a function that is continuous everywhere except at one point.

**7.** Let $f(x) = \{x\}$ be the fractional-part function (see Figure 9.1, graph (8)). We see that $f$ is continuous at $a$ if $a$ is not an integer, and it is discontinuous at $a$ if $a$ is an integer. Indeed, $f(a) = a - [a]$ and $f(x) = x - [a]$ if $[a] \leq x < [a+1]$, so $|f(x) - f(a)| = |x - a|$ if $[a] \leq x < [a+1]$. Clearly, if $a$ is not an integer, then

$$\delta = \min(\varepsilon, a - [a], [a] + 1 - a)$$

is a good $\delta$. If, however, $a$ is an integer, then

$$|f(x) - f(a)| = |x - (a-1)| > \tfrac{1}{2}, \quad \text{if} \quad a - \tfrac{1}{2} < x < a,$$

which implies that, for example, if $0 < \varepsilon < 1/2$, then there does not exist a good $\delta$ for $\varepsilon$.

We can see that if $a$ is an integer, then the continuity of the function $f(x) = \{x\}$ is prevented only by behavior on the left-hand side of $a$. If this is the case, we say the function is continuous from the right. We make this more precise below.

**Definition 10.3.** Let $f$ be defined on an interval $[a, b)$. We say that the function $f$ is *continuous from the right* at $a$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon, \quad \text{if} \quad 0 \leq x - a < \delta. \tag{10.3}$$

The function $f$ is *continuous from the left* at $a$ if it is defined on an interval $(c, a]$, and if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon, \quad \text{if} \quad 0 \leq a - x < \delta. \tag{10.4}$$

## *Exercises*

**10.1.** Show that the function $f$ is continuous at $a$ if and only if it is continuous both from the right and from the left there.

**10.2.** Show that the function $[x]$ is continuous at $a$ if $a$ is not an integer, and continuous from the right at $a$ if $a$ is an integer.

**10.3.** For a given $\varepsilon > 0$, find a good $\delta$ (according to Definition 10.1) in the following functions.

(a) $f(x) = (x+1)/(x-1)$, $a = 3$;
(b) $f(x) = x^3$, $a = 2$;
(c) $f(x) = \sqrt{x}$, $a = 2$.

**10.4.** Continuity of the function $f\colon \mathbb{R} \to \mathbb{R}$ at $a$ can be written using the following expression:
$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$.
Consider the following expressions:
$(\forall\varepsilon > 0)(\forall\delta > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$;
$(\exists\varepsilon > 0)(\forall\delta > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$;
$(\exists\varepsilon > 0)(\exists\delta > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$;
$(\forall\delta > 0)(\exists\varepsilon > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$;
$(\exists\delta > 0)(\forall\varepsilon > 0)(\forall x)(|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$.
What properties of $f$ do these express?

**10.5.** Prove that if $f$ is continuous at a point, then $|f|$ is also continuous there. Conversely, does the continuity of $f$ follow from the continuity of $|f|$?

**10.6.** Prove that if $f$ and $g$ are continuous at a point $a$, then $\max(f,g)$ and $\min(f,g)$ are also continuous at $a$.

**10.7.** Prove that if the function $f\colon \mathbb{R} \to \mathbb{R}$ is monotone increasing and assumes every rational number as a value, then it is continuous everywhere.

**10.8.** Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is not constant, continuous, and periodic, then it has a smallest positive period. (H)

## 10.1 Limits of Functions

Before explaining limits of functions, we discuss three problems that shed light on the need for defining limits and actually suggest a good definition for them. The first two problems raise questions that are of fundamental importance; one can almost say that the theory of analysis was developed to answer these questions. The third problem is a concrete exercise, but it also demonstrates the concept of a limit well.

1. The first problem is defining speed. For constant speed, the value of velocity is $v = s/t$, where $s$ is the length of the path traveled at a time $t$. Now consider movement with a variable speed, and let $s(t)$ denote the length of the path traveled at a time $t$. The problem is defining and computing *instantaneous* velocity at a given time $t_0$. Let $\omega(t)$ denote the average speed on the time interval $[t_0,t]$, that, is let

$$\omega(t) = \frac{s(t) - s(t_0)}{t - t_0}.$$

This is the velocity that in the case of constant speed a point would have moving a distance $s(t) - s(t_0)$ in time $t - t_0$. If, for example,

$$s(t) = t^3 \text{ and } t_0 = 2, \qquad \text{then} \qquad \omega(t) = \frac{t^3 - 8}{t - 2} = (t^2 + 2t + 4).$$

In this case, if $t$ is close to 2, then the average velocity in the interval $[2, t]$ is close to 12. It is clear that we should decide that 12 should be the instantaneous velocity at $t_0 = 2$.

Generally, if we know that there is a value $v$ to which $\omega(t)$ is "very" close for all $t$ close enough to $t$, then we will call this $v$ the instantaneous velocity at $t_0$.

2. The second problem concerns defining and finding a tangent line to the graph of a function. Consider the graph of $f$ and a fixed point $P = (a, f(a))$ on it. Let $h_a(x)$ denote the chord crossing $P$ and $(x, f(x))$ (Figure 10.4). We wish to define the tangent line to the curve, that is, the—yet to be precisely defined—line to which these lines tend. Since these lines cross $P$, their slopes uniquely determine them. The slope of the chord $h_a(x)$ is given by

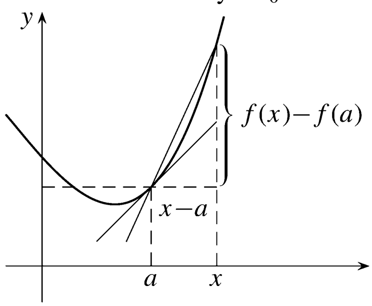


Fig. 10.4

$$m_a(x) = \frac{f(x) - f(a)}{x - a} \qquad (x \neq a).$$

For example, in the case $f(x) = 1/x$,

$$m_a(x) = \frac{\frac{1}{x} - \frac{1}{a}}{x - a} = -\frac{1}{xa}.$$

It can be seen that if $x$ tends to $a$, then $m_a(x)$ tends to $-1/a^2$, in a sense yet to be defined. Then it is only natural to say that the tangent at the point $P$ is the line whose slope is $-1/a^2$ that intersects the point $(a, 1/a)$. Thus the equation of the tangent is

$$y = -\frac{1}{a^2}(x - a) + \frac{1}{a}.$$

Generally, if the values

$$m_a(x) = \frac{f(x) - f(a)}{x - a}$$

tend to some value $m$ while $x$ tends to $a$ (again, in a yet to be defined way), then the line crossing $P$ with slope $m$ will be called the tangent of the graph of $f$ at the point $P$.

3. The third problem is finding the focal length of a spherical mirror. Consider a spherically curved concave mirror with radius $r$. Light rays traveling parallel to the principal axis at distance $x$ from it will reflect off the mirror and intersect the principal axis at a point $P_x$. The problem is to find the limit of $P_x$ as $x$ tends to 0 (Figure 10.5).

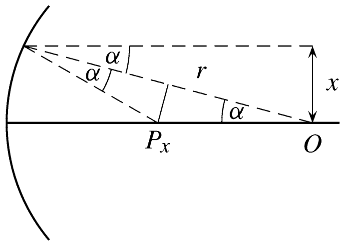


Fig. 10.5

Assuming knowledge of the law of reflection, we have

$$\frac{\overline{OP_x}}{(r/2)} = \frac{r}{\sqrt{r^2 - x^2}}, \quad \text{that is} \quad \overline{OP_x} = \frac{r^2}{2\sqrt{r^2 - x^2}}.$$

We see that if $x$ is close enough to 0, then $\overline{OP_x}$ gets arbitrarily close to the value $r/2$. Thus the focal length of the spherically curved mirror is $r/2$.

In all three cases, the essence of the problem is the following: how do we define what it means to say that "as $x$ tends to $a$, the values $f(x)$ tend to a number $b$" or that "the limit of the function $f$ at $a$ is $b$"? The above three problems indicate that we should define the limit of a function in a way that does not depend on the value of $f$ at $x = a$ or the fact that the function $f$ might not even be defined at $x = a$.

Continuity of $f$ is the precise definition that at places "close" to $a$, the values of $f$ are "close" to $f(a)$. Then by conveniently modifying the definition of continuity, we get the following definition for the limit of a function.

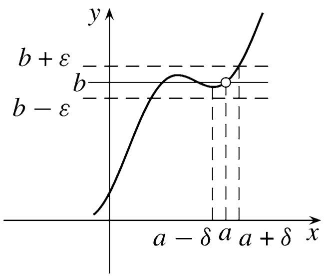


Fig. 10.6

**Definition 10.4.** Let $f$ be defined on an open interval containing $a$, excluding perhaps $a$ itself. The *limit* of $f$ at $a$ exists and has the value $b$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - b| < \varepsilon, \ \text{if} \ 0 < |x - a| < \delta. \quad (10.5)$$

(See Figure 10.6.) Noting the definition of continuity, we see that the following is clearly equivalent:

**Definition 10.5.** Let $f$ be defined on an open interval containing $a$, excluding perhaps $a$ itself. The limit of $f$ at $a$ exists and has the value $b$ if the function

$$f^*(x) = \begin{cases} f(x), & \text{if } x \neq a \\ b, & \text{if } x = a \end{cases}$$

is continuous at $a$.

We denote that $f$ has the limit $b$ at $a$ by

$$\lim_{x\to a} f(x) = b, \text{ or } f(x) \to b, \text{ if } x \to a;$$

$f(x) \not\to b$ denotes that $f$ does not tend to $b$.

If $f$ is continuous at $a$, then the conditions of Definition 10.4 are satisfied with $b = f(a)$. Thus, the connection between continuity and limits can also be expressed in the following way:

*Let $f$ be defined on an open interval containing $a$. The function $f$ is continuous at $a$ if and only if* $\lim_{x\to a} f(x)$ *exists and has the value $f(a)$.*

The statement $\lim_{x\to a} f(x) = b$ holds the following interpretation in the graph of $f$: given an arbitrarily "narrow" strip $\{(x,y) : b-\varepsilon < y < b+\varepsilon\}$, there exists a $\delta > 0$ such that the part of the graph over the set $(a-\delta, a+\delta)\setminus\{a\}$ lies inside the strip.

The following theorem states that limits are unique.

**Theorem 10.6.** *If* $\lim_{x\to a} f(x) = b$ *and* $\lim_{x\to a} f(x) = b'$, *then* $b = b'$.

*Proof.* Suppose that $b' \neq b$. Let $0 < \varepsilon < |b'-b|/2$. Then the inequalities $|f(x)-b| < \varepsilon$ and $|f(x)-b'| < \varepsilon$ can never both hold at once (since if they did, then

$$|b-b'| \le |b-f(x)| + |f(x)-b'| < \varepsilon+\varepsilon = 2\varepsilon$$

would follow), which is impossible. □

*Examples 10.7.* **1.** The function $f(x) = \operatorname{sgn}^2 x$ is not continuous at 0, but its limit there exists and has the value 1. Clearly, the function

$$f^*(x) = \begin{cases} \operatorname{sgn}^2 x, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

has value 1 at all $x$, so $f^*$ is continuous at 0.

**2.** We show that

$$\lim_{x\to 2} \frac{x-2}{x^2-3x+2} = 1.$$

Since $x^2-3x+2 = (x-1)(x-2)$, then whenever $x \neq 2$,

$$\frac{x-2}{x^2-3x+2} = \frac{1}{x-1}.$$

From this, we have that

$$\left|\frac{x-2}{x^2-3x+2} - 1\right| = \left|\frac{1}{x-1} - 1\right| = \left|\frac{2-x}{x-1}\right|.$$

Since whenever $|x-2| < 1/2$, $|x-1| > 1/2$, it follows that $|(2-x)/(x-1)| < \varepsilon$ as long as $0 < |2-x| < \min(\varepsilon/2, 1/2)$.

**3.** Let

$$f(x) = \begin{cases} 0, & \text{if } x \text{ irrational} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q}, \text{ where } p,q \text{ integers, } q > 0 \text{ and } (p,q) = 1. \end{cases}$$

This function has the following strange properties from the point of view of continuity and limits:

*a) The function $f$ has a limit at every value $a$, which is* 0 (even though $f$ is not identically 0).
*b)* The function $f$ is continuous at every irrational point.
*c)* The function $f$ is not continuous at any rational point.

To prove statement *a)*, we need to show that if $a$ is an arbitrary value, then for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - 0| < \varepsilon, \qquad \text{if} \qquad 0 < |x - a| < \delta. \tag{10.6}$$

For simplicity, let us restrict ourselves to the interval $(-1,1)$. Let $\varepsilon > 0$ be given, and choose an integer $n > 1/\varepsilon$. By the definition of $f$, $|f(x)| < 1/n$ for all irrational numbers $x$ and for all rational numbers $x = p/q$ for which $(p,q) = 1$ and $q > n$. That is, $|f(x) - 0| \geq 1/n$ only on the following points of $(-1,1)$:

$$0,\ \pm 1,\ \pm\frac{1}{2},\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \ldots, \pm\frac{1}{n},\ \pm\frac{2}{n}, \ldots,\ \pm\frac{n-1}{n}. \tag{10.7}$$

Now if $a$ is an arbitrary point of the interval $(-1,1)$, then out of the finitely many numbers in (10.7), there is one that is different from $a$, and among those, the closest to $a$. Let this one be $p_1/q_1$, and let $\delta = |p_1/q_1 - a|$. Then inside the interval $(a - \delta, a + \delta)$, there is no number in (10.7) other than $a$, so $|f(x)| < 1/n < \varepsilon$ if $0 < |x - a| < \delta = |p_1/q_1 - a|$, that is, for $\varepsilon$, $\delta = |p_1/q_1 - a|$ is a good choice. Since $\varepsilon > 0$ was arbitrary, we have shown that $\lim_{x\to a} f(x) = 0$.

*b)* Since for an irrational number $a$ we have $f(a) = 0$, $\lim_{x\to a} f(x) = f(a)$ follows, so the function is continuous at every irrational point.
*c)* Since for a rational number $a$ we have $f(a) \neq 0$, $\lim_{x\to a} f(x) \neq f(a)$ follows, so the function is not continuous at the rational points.

*Remark 10.8.* The function defined in Example 3 is named—after its discoverer—the *Riemann function*.[1] This function is then continuous at all irrational points, and discontinuous at every rational point. It can be proven, however, that there *does not exist* a function which is continuous at each rational point but discontinuous at all irrational points (see Exercise 10.17).

Similar to the idea of one-sided continuity, there is also the notion of one-sided limits.

[1] Georg Friedrich Bernhard Riemann (1826–1866), German mathematician.

**Definition 10.9.** Let $f$ be defined on an open interval $(a,c)$. The right-hand limit of $f$ at $a$ is $b$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - b| < \varepsilon$ if $0 < x - a < \delta$.

**Notation:** $\lim_{x\to a+0} f(x) = b$, $\lim_{x\to a^+} f(x) = b$ or $f(x) \to b$, if $x \to a+0$, or even shorter, $f(a+0) = b$.

Left-hand limits can be defined and denoted in a similar way.

*Remark 10.10.* In the above notation, $a+0$ and $a-0$ are not numbers, simply symbols that allow a more abbreviated notation for expressing the property given in the definition.

The following theorem is clear from the definitions.

**Theorem 10.11.** $\lim_{x\to a} f(x) = b$ *if and only if both* $f(a+0)$ *and* $f(a-0)$ *exist, and* $f(a+0) = f(a-0) = b$.

Similarly to when we talked about limits of sequences, we will need to define the concept of a function tending to positive and negative infinity.

**Definition 10.12.** Let $f$ be defined on an open interval containing $a$, except possibly $a$ itself. The limit of the function $f$ at $a$ is $\infty$ if for all numbers $P$, there exists a $\delta > 0$ such that $f(x) > P$ whenever $0 < |x - a| < \delta$.

**Notation:** $\lim_{x\to a} f(x) = \infty$, or $f(x) \to \infty$ as $x \to a$.

The statement $\lim_{x\to a} f(x) = \infty$ expresses the following property of the graph of $f$: for arbitrary $P$, there exists a $\delta > 0$ such that the graph over the set $(a-\delta, a+\delta) \setminus \{a\}$ lies above the horizontal line $y = P$ (Figure 10.7).

We define the limit of $f$ as $-\infty$ at a point $a$ similarly. We will also need the one-sided equivalents for functions tending to $\infty$.

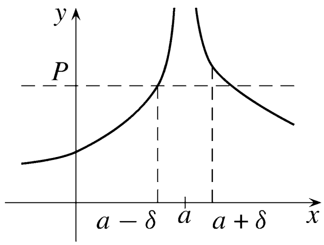


Fig. 10.7

**Definition 10.13.** Let $f$ be defined on an open interval $(a,c)$. The right-hand limit of $f$ at $a$ is $\infty$ if for every number $P$, there exists a $\delta > 0$ such that $f(x) > P$ if $0 < x - a < \delta$.

**Notation:** $\lim_{x\to a+0} f(x) = \infty$; $f(x) \to \infty$, if $x \to a+0$; or $f(a+0) = \infty$.
We define $\lim_{x\to a+0} f(x) = -\infty$ and $\lim_{x\to a-0} f(x) = \pm\infty$ similarly.

But we are not yet done with the different variations of definitions of limits.

**Definition 10.14.** Let $f$ be defined on the half-line $(a,\infty)$. We say that the limit of $f$ at $\infty$ is $b$ if for all $\varepsilon > 0$, there exists a $K$ such that $|f(x) - b| < \varepsilon$ if $x > K$ (Figure 10.8).

**Notation:** $\lim_{x\to\infty} f(x) = b$, or $f(x) \to b$, if $x \to \infty$.

We can similarly define the limit of $f$ at $-\infty$ being $b$.

Finally, we have one more type, in which both the "place" and the "value" are infinite.

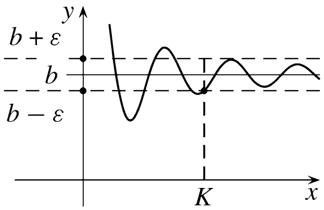


Fig. 10.8

**Definition 10.15.** Let $f$ be defined on the half-line $(a,\infty)$. We say that the limit of $f$ at $\infty$ is $\infty$ if for all $P$, there exists a $K$ such that $f(x) > P$ if $x > K$.

We get three more variations of the last definition if we define the limit at $\infty$ as $-\infty$, and the limit at $-\infty$ as $\infty$ or $-\infty$.

To summarize, we have defined the following variants of limits:

$$\lim_{x\to a} f(x) = \begin{cases} b \text{ is finite} \\ \infty \\ -\infty \end{cases} ; \quad \lim_{x\to a\pm 0} f(x) = \begin{cases} b \text{ is finite} \\ \infty \\ -\infty \end{cases} ; \quad \lim_{x\to \pm\infty} f(x) = \begin{cases} b \text{ is finite} \\ \infty \\ -\infty \end{cases} .$$

These are 15 variations of a concept that we can feel is based on a unifying idea. To make this unifying concept clear, we introduce the notion of a **neighborhood**.

**Definition 10.16.** We define a *neighborhood* of a real number $a$ as an interval of the form $(a-\delta, a+\delta)$, where $\delta$ is an arbitrary positive number. Sets of the form $[a, a+\delta)$ and $(a-\delta, a]$ are called the *right-hand*, and respectively the *left-hand neighborhoods* of the real number $a$.

A *punctured neighborhood* of the real number $a$ is a set of the form $(a-\delta, a+\delta)\setminus\{a\}$, where $\delta$ is an arbitrary positive number. Sets of the form $(a, a+\delta)$ and $(a-\delta, a)$ are called the *right-hand* and respectively the *left-hand punctured neighborhoods* of the real number $a$.

Finally, a *neighborhood* of $\infty$ is a half-line of the form $(K,\infty)$, where $K$ is an arbitrary real number, while a *neighborhood* of $-\infty$ is a half-line of the form $(-\infty, K)$, where $K$ is an arbitrary real number.

In the above definition, the prefix "punctured" indicates that we leave out the point itself from its neighborhood, or that we "puncture" the set at that point. Now with the help of the concept of neighborhoods, we can give a unified definition for the 15 different types of limits, which also demonstrates the meaning of a limit better.

**Definition 10.17.** Let $\alpha$ denote the real number $a$ or one of the symbols $a+0$, $a-0$, $\infty$, or $-\infty$. In each case, by the punctured neighborhood of $\alpha$ we mean the punctured neighborhood of $a$, the right-hand punctured neighborhood of $a$, the left-hand punctured neighborhood of $a$, the neighborhood of $\infty$, or the neighborhood of $-\infty$ respectively. Let $\beta$ denote the real number $b$, the symbol $\infty$, or $-\infty$.

Let $f$ be defined on a punctured neighborhood of $\alpha$. We say that $\lim_{x\to\alpha} f(x) = \beta$ if for each neighborhood $V$ of $\beta$, there exists a punctured neighborhood $\dot{U}$ of $\alpha$ such that $f(x)\in V$ if $x\in\dot{U}$.

We let the reader verify that each listed case of the definition of the limit can be obtained (as a special case) from this definition.

*Examples 10.18.* **1.** $\lim_{x\to 0+0} 1/x = \infty$, since $1/x > P$, if $0 < x < 1/P$.
$\lim_{x\to 0-0} 1/x = -\infty$, since $1/x < P$, if $1/P < x < 0$.
**2.** $\lim_{x\to 0} 1/x^2 = \infty$, since $1/x^2 > P$, if $0 < |x| < 1/\sqrt{P}$.
**3.** $\lim_{x\to\infty}(1-2x)/(1+x) = -2$, as for arbitrary $\varepsilon > 0$,

$$\left|\frac{1-2x}{1+x} - (-2)\right| = \frac{3}{|1+x|} < \varepsilon, \text{ if } x > \frac{3}{\varepsilon} - 1.$$

**4.** $\lim_{x\to-\infty} 10x/(2x^2+3) = 0$, since for arbitrary $\varepsilon > 0$,

$$\left|\frac{10x}{2x^2+3}\right| < \frac{5}{|x|} < \varepsilon, \text{ if } |x| > \frac{5}{\varepsilon},$$

and so the same holds if $x < -5/\varepsilon$.
**5.** $\lim_{x\to-\infty} x^2 = \infty$, since $x^2 > P$ if $x < -\sqrt{|P|}$.
**6.** We show that $\lim_{x\to\infty} a^x = \infty$ for all $a > 1$. Let $P$ be given. If $n = [P/(a-1)] + 1$ and $x > n$, then using the monotonicity of $a^x$ (Theorem 3.27) and Bernoulli's inequality, we get that

$$a^x > a^n = (1+(a-1))^n \geq 1 + n\cdot(a-1) > \frac{P}{a-1}\cdot(a-1) = P.$$

**7.** Let $f(x) = x\cdot[1/x]$. Then $\lim_{x\to 0} f(x) = 1$ (Figure 10.9).

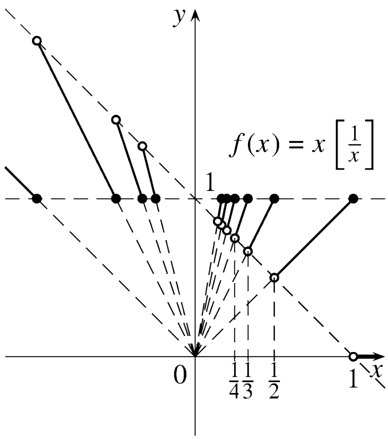


Fig. 10.9

Since

$$f(x) = \begin{cases} 0, & \text{if } x > 1, \\ nx, & \text{if } \frac{1}{n+1} < x \le \frac{1}{n}, \\ -(n+1)x, & \text{if } -\frac{1}{n} < x \le -\frac{1}{n+1}, \\ -x, & \text{if } x \le -1, \end{cases}$$

we can see that if $1/(n+1) < x \le 1/n$ holds, then $n/(n+1) < f(x) \le 1$, that is, $|f(x)-1| < 1/(n+1)$. Similarly, if $-1/n < x \le -1/(n+1)$, then $1 \le f(x) = -(n+1)x < (n+1)/n$, that is, $|f(x)-1| < 1/n$. It then follows that

$$|f(x)-1| < \frac{1}{n}, \text{ ha } \ 0 < |x| < \frac{1}{n}.$$

That is, for every $\varepsilon > 0$, the choice $\delta = 1/n$ is good if $n > 1/\varepsilon$.

**8.** Let $f(x) = [x]/x$ (Figure 10.10), that is

$$f(x) = \begin{cases} 0, & \text{if } 0 < x < 1, \\ n/x, & \text{if } n \le x < n+1, \\ -(n+1)/x, & \text{if } -(n+1) \le x < -n. \end{cases}$$

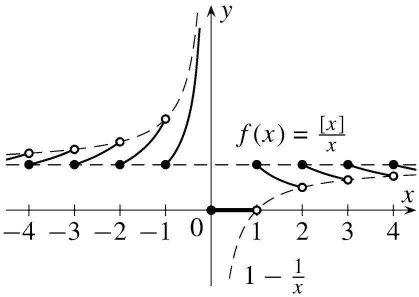


Fig. 10.10

Clearly $\lim_{x\to 0+0} f(x) = 0$; moreover, $\lim_{x\to 0-0} f(x) = \infty$, since $f(x) = -1/x$, if $-1 < x < 0$.

Finally, we show that

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 1.$$

Clearly, $n/(n+1) = 1 - 1/(n+1) < f(x) = n/x \le 1$ if $n \le x < n+1$, that is, $|f(x)-1| < 1/(n+1)$, if $x \ge n$, so

$$\lim_{x\to\infty} f(x) = 1.$$

Similarly,

$$1 = \frac{-(n+1)}{-(n+1)} \leq f(x) = \frac{-(n+1)}{x} < \frac{-(n+1)}{-n} = 1 + \frac{1}{n}$$

if $-(n+1) \leq x < -n$; therefore, $|f(x) - 1| < 1/n$ if $x < -n$. Thus we see that $\lim_{x\to-\infty} f(x) = 1$.

## Exercises

**10.9.** In the following functions, the limit $\beta$ at $\alpha$ exists. Find $\beta$, and for each neighborhood $V$ of $\beta$, give a neighborhood $\dot{U}$ of $\alpha$ such that $x \in \dot{U}$ implies $f(x) \in V$.

(a) $f(x) = [x]$, $\alpha = 2+0$;
(b) $f(x) = \{x\}$, $\alpha = 2+0$;
(c) $f(x) = \frac{x}{2x-1}$, $\alpha = \infty$;
(d) $f(x) = \frac{x}{2x-1}$, $\alpha = \frac{1}{2}+0$;
(e) $f(x) = \frac{x}{x^2-1}$, $\alpha = \infty$;
(f) $f(x) = \frac{x}{x^2-1}$, $\alpha = 1-0$.
(g) $f(x) = \sqrt{x+1} - \sqrt{x}$, $\alpha = \infty$;
(h) $\dfrac{\sqrt{x}+\sqrt[3]{x}}{x-\sqrt{x}}$, $\alpha = \infty$;
(i) $\dfrac{x^2+5x+6}{x^2+6x+5}$, $\alpha = \infty$;
(j) $2^{-[1/x]}$, $\alpha = \infty$;
(k) $\sqrt[3]{x^3+1} - x$, $\alpha = \infty$.

**10.10.** Can we define the function $(\sqrt{x}-1)/(x-1)$ at $x = 1$ such that it will be continuous there?

**10.11.** Let $n$ be a positive integer. Can we define the function $(\sqrt[n]{1+x}-1)/x$ at $x = 0$ such that it will be continuous there?

**10.12.** Prove that the value of the Riemann function at $x$ is

$$1 - \lim_{n\to\infty} \max(\{x\}, \{2x\}, \ldots, \{nx\}).$$ (H)

**10.13.** Suppose that the function $f\colon \mathbb{R} \to \mathbb{R}$ has a finite limit $b(x)$ at every point $x \in \mathbb{R}$. Show that the function $b$ is continuous everywhere.

**10.14.** Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is periodic and $\lim_{x\to\infty} f(x) = 0$, then $f$ is identically 0.

**10.15.** Prove that there is no function $f\colon \mathbb{R} \to \mathbb{R}$ whose limit at every point $x$ is infinite. (H)

**10.16.** Prove that if at each point $x$ the limit of the function $f\colon \mathbb{R} \to \mathbb{R}$ equals zero, then there exists a point $x$ at which $f(x) = 0$. (H)

**10.17.** Prove that if the function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at every rational point, then there is an irrational point at which it is also continuous. ($*$ S)

## 10.2 The Transference Principle

The concept of a limit of a function is closely related to the concept of a limit of a sequence. This is expressed by the following theorem. (In the theorem, the meanings of $\alpha$ and $\beta$ are the same as in Definition 10.17.)

**Theorem 10.19.** *Let $f$ be defined on a punctured neighborhood $\dot{U}$ of $\alpha$. We have $\lim_{x\to\alpha} f(x) = \beta$ if and only if whenever a sequence $(x_n)$ satisfies*

$$\{x_n\} \subset \dot{U} \ \text{ and } \ x_n \to \alpha, \tag{10.8}$$

*then* $\lim_{n\to\infty} f(x_n) = \beta$.

*Proof.* Suppose that $\alpha = a$ and $\beta = b$ are finite real numbers. We first prove that if $\lim_{x\to a} f(x) = b$, then $f(x_n) \to b$ whenever $\{x_n\} \subset \dot{U}$ and $x_n \to a$.

Let $\varepsilon > 0$ be given. We know that there exists a $\delta > 0$ such that $|f(x) - b| < \varepsilon$ if $0 < |x-a| < \delta$. If $x_n \to a$, then there exists an $n_0$ for $\delta > 0$ such that $|x_n - a| < \delta$ for all $n > n_0$. Since by (10.8), $x_n$ is in a punctured neighborhood of $a$, $x_n \neq a$ for all $n$. Thus if $n > n_0$, then $0 < |x_n - a| < \delta$, and so $|f(x_n) - b| < \varepsilon$. This shows that $f(x_n) \to b$.

Now we show that if for all sequences satisfying (10.8) we have $f(x_n) \to b$, then $\lim_{x\to a} f(x) = b$. We prove this by contradiction.

Suppose that $\lim_{x\to a} f(x) = b$ does not hold. This means that there exists an $\varepsilon > 0$ for which there does not exist a good $\delta > 0$, that is, in all $(a-\delta, a+\delta) \cap \dot{U}$, there exists an $x$ for which $|f(x) - b| \geq \varepsilon$. This is true for all $\delta = 1/n$ specifically, so for all $n \in \mathbb{N}^+$, there is an $x_n \in \dot{U}$ for which $0 < |x_n - a| < 1/n$ and $|f(x_n) - b| \geq \varepsilon$. The sequence $(x_n)$ we get in this way has the properties that $x_n \to a$ and $x_n \in \dot{U}$, but $f(x_n) \not\to b$. This contradicts our assumptions.

Let us now consider the case that $\alpha = a + 0$ and also $\beta = \infty$. Suppose that $\lim_{x\to a+0} f(x) = \infty$, and let $(x_n)$ be a sequence approaching $a$ from the right.[2] We want to show that $f(x_n) \to \infty$. Let $K$ be fixed. Then there exists a $\delta > 0$ such that $f(x) > K$ for all $a < x < a + \delta$. Since $x_n \to a$ and $x_n > a$, there exists an $n_0$ such that $a < x_n < a + \delta$ holds for all $n > n_0$. Then $f(x_n) > K$ if $n > n_0$, which shows that $f(x_n) \to \infty$.

Now suppose that $f(x_n) \to \infty$ for all sequences for which $x_n \to a$ and $x_n > a$. We want to show that $\lim_{x\to a+0} f(x) = \infty$. We prove this by contradiction. If the statement is false, then there exists a $K$ for which there is no good $\delta$, that is, for all $\delta > 0$, there exists an $x \in (a, a+\delta)$ such that $f(x) \leq K$. This is also true for each $\delta = 1/n$, so for each $n \in \mathbb{N}^+$, there is an $a < x_n < a + 1/n$ such that $f(x_n) \leq K$. The sequence $(x_n)$ we then get tends to $a$ from the right, and $f(x_n) \not\to \infty$, which contradicts our initial assumption.

The statements for the rest of the cases can be shown similarly. □

*Remark 10.20.* We see that a necessary and sufficient condition for the existence of a limit is that for each sequence $x_n \to \alpha$, $\{x_n\} \subset \dot{U}$ (i) $(f(x_n))$ has a limit, and (ii) the value of $\lim_{n\to\infty} f(x_n)$ is independent of the choice of the sequence $(x_n)$.

[2] This naturally means that $x_n > a$ for all $n$, and $x_n \to a$.

Here condition (ii) can actually be dropped, since it follows from (i). We can show this using contradiction: suppose that (i) holds, but (ii) is false. This would mean that there exist a sequence

$$x'_n \to \alpha, \ \{x'_n\} \subset \dot{U}$$

and a sequence

$$x''_n \to \alpha, \ \{x''_n\} \subset \dot{U}$$

such that

$$\lim_{n\to\infty} f(x'_n) \neq \lim_{n\to\infty} f(x''_n).$$

But then the sequence $(x'_1, x''_1, x'_2, x''_2, \dots, x'_n, x''_n, \dots)$, which also tends to $\alpha$, provides us with a sequence

$$(f(x'_1), f(x''_1), f(x'_2), f(x''_2), \dots, f(x'_n), f(x''_n), \dots),$$

which would oscillate at infinity, since it would have two subsequences that tend to two different limits. This, however, is impossible by (i).

We call Theorem 10.19 the **transference principle**, since it "transfers" the concept (and value) of limits of functions to limits of sequences.

The significance of the theorem lies in its ability to convert results pertaining to limits of sequences into results about limits of functions. We will also want to use a theorem connecting continuity to limits, but stating this will be much simpler than Theorem 10.19, and we can actually reduce our new theorem to it.

**Theorem 10.21.** *The function $f$ is continuous at a point $a$ if and only if $f$ is defined in a neighborhood of $a$ and for each sequence $(x_n)$, $x_n \to a$ implies $f(x_n) \to f(a)$.*

*Proof.* Suppose that $f$ is continuous at $a$, and let $(x_n)$ be a sequence that tends to $a$. For a fixed $\varepsilon$, there exists a $\delta$ such that $|f(x) - f(a)| < \varepsilon$ for all $x \in (a-\delta, a+\delta)$. Since $x_n \to a$, $x_n \in (a-\delta, a+\delta)$ for all sufficiently large $n$. Thus $|f(x_n) - f(a)| < \varepsilon$ for all sufficiently large $n$, which shows that $f(x_n) \to f(a)$.

Now suppose that $f(x_n) \to f(a)$ if $x_n \to a$. By Theorem 10.19, it then follows that $\lim_{x\to a} f(x) = f(a)$, that is, $f$ is continuous at $a$. $\square$

For applications in the future, it is worth stating the following theorem.

**Theorem 10.22.** *The finite limit $\lim_{x\to a-0} f(x)$ exists if and only if for for each sequence $x_n \nearrow a$, $(f(x_n))$ is convergent. That is, in the case of a left-hand limit, it suffices to consider monotone increasing sequences $(x_n)$. A similar statement holds for right-hand limits.*

*Proof.* To prove the theorem, it is enough to show that if for every sequence $x_n \nearrow a$, $(f(x_n))$ is convergent, then it follows that for all sequences $x_n \to a$ such that $x_n < a$, the sequence $(f(x_n))$ is convergent.

But this is a simple corollary of the fact that every sequence $x_n < a$, $x_n \to a$, can be rearranged into a monotone increasing sequence $(x_{k_n})$ (See Theorem 6.7 and the remark following it), and if the rearranged sequence $(f(x_{k_n}))$ is convergent, then the original sequence $(f(x_n))$ is also convergent (see Theorem 5.5). □

Another—albeit not as deep—connection between limits of functions and limits of sequences is as follows. An infinite sequence is actually just a function defined on the positive integers. Then the limit of the sequence $a_1 = f(1)$, $a_2 = f(2)$, $\ldots$, $a_n = f(n)$, $\ldots$ is the limit of $f$ at $\infty$, at least restricted to the set of positive integers. To make this precise, we define the notion of a limit restricted to a set.

**Definition 10.23.** Let $\alpha$ denote a real number, the symbol $\infty$, or $-\infty$. We say that $\alpha$ is a *limit point* or an *accumulation point* of the set $A$ if every neighborhood of $\alpha$ contains infinitely many points of $A$.

**Definition 10.24.** Let $\alpha$ be a limit point of $A = b \in \mathbb{R}$. The *limit of $f$ at $\alpha$ restricted to $A$* is $\beta$ if for each neighborhood $V$ of $\beta$ there exists a punctured neighborhood $\dot{U}$ of $\alpha$ such that

$$f(x) \in V, \text{ if } x \in \dot{U} \cap A. \tag{10.9}$$

**Notation:** $\lim_{\substack{x\to\alpha \\ x\in A}} f(x) = \beta$.

*Example 10.25.* Let $f$ be the Dirichlet function (as in (9) of Examples 9.7). Then for an arbitrary real number $c$, we have $\lim_{x\to c, x\in\mathbb{Q}} f(x) = 1$ and $\lim_{x\to c, x\in\mathbb{R}\setminus\mathbb{Q}} f(x) = 0$, since $f(x) = 1$ if $x$ is rational, and $f(x) = 0$ if $x$ is irrational. It is clear that every real number $c$ is a limit point of the set of rational numbers as well as the set of irrational numbers, so the above limits have meaning.

*Remarks 10.26.* **1.** By this definition, the limit $\lim_{x\to a+0} f(x)$ is simply the limit of $f$ at $a$ restricted to the set $(a,\infty)$.
**2.** The limit of the sequence $a_n = f(n)$ (in the case $n \to \infty$) is simply the limit of $f$ as $x \to \infty$ restricted to the set $\mathbb{N}^+$.
**3.** If $\alpha$ is not a limit point of the set $A$, then $\alpha$ has a punctured neighborhood $\dot{U}$ such that $\dot{U} \cap A = \emptyset$. In this case, the conditions of the definition automatically hold (since the statement $x \in \dot{U} \cap A$ is then vacuous). Thus (10.9) is true for *all* neighborhoods $V$ of $\beta$. This means that we get a meaningful definition only if $\alpha$ is a limit point of $A$.

With regard to the above, the following is the most natural definition.

**Definition 10.27.** Let $a \in A \subset D(f)$. The function $f$ *is continuous at the point $a$ restricted to the set $A$* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ if $x \in (a-\delta, a+\delta) \cap A$. If $A = D(f)$, then instead of saying that $f$ is continuous at the point $a$ restricted to $D(f)$, we say that $f$ is continuous at $a$ for short.

*Remarks 10.28.* **1.** By this definition, $f$ being continuous from the right at $a$ can also be defined as $f$ being continuous at $a$ restricted to the interval $[a,\infty)$.
**2.** In contrast to the definition of defining limits, we have to assume that $f$ is defined at $a$ when we define continuity. However, we *do not* need to assume that $a$ is a limit point of the set $A$. If $a \in A$, but $a$ is not a limit point of $A$, then we say that $a$ is an **isolated** point of $A$. It is easy to see that $a$ is an isolated point of $A$ if and only if there exists a $\delta > 0$ such that $(a-\delta, a+\delta) \cap A = \{a\}$. It then follows that *if $a$ is an isolated point of $A$, then every function $f\colon A \to \mathbb{R}$ is continuous at $a$ restricted to $A$.* Clearly, for every $\varepsilon > 0$, the $\delta$ above has the property that $|f(x) - f(a)| < \varepsilon$ if $x \in (a-\delta, a+\delta) \cap A$, since the last condition is satisfied only by $x = a$, and $|f(a) - f(a)| = 0 < \varepsilon$.

The following statements are often used; they are the equivalents to Theorems 5.7, 5.8, and 5.10.

**Theorem 10.29 (Squeeze Theorem).** *If the inequality $f(x) \le g(x) \le h(x)$ is satisfied in a punctured neighborhood of $\alpha$ and $\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} h(x) = \beta$, then $\lim_{x\to\alpha} g(x) = \beta$.*

*Proof.* The statement follows from Theorems 5.7 and 10.19. □

**Theorem 10.30.** *If*

$$\lim_{x\to\alpha,\ x\in A} f(x) = b < c = \lim_{x\to\alpha,\ x\in A} g(x),$$

*then there exists a punctured neighborhood $\dot{U}$ of $\alpha$ such that $f(x) < g(x)$ for all $x \in \dot{U} \cap A$.*

*Proof.* By the definition of the limit, for $\varepsilon = (c-b)/2$, there exists a punctured neighborhood $\dot{U}_1$ of $\alpha$ such that $|f(x) - b| < (c-b)/2$ whenever $x \in A \cap \dot{U}_1$. Similarly, there is a $\dot{U}_2$ such that $|g(x) - c| < (c-b)/2$ if $x \in A \cap \dot{U}_2$. Let $\dot{U} = \dot{U}_1 \cap \dot{U}_2$. Then $\dot{U}$ is also a punctured neighborhood of $\alpha$, and if $x \in A \cap \dot{U}$, then $f(x) < b + (c-b)/2 = c - (c-b)/2 < g(x)$. □

**Theorem 10.31.** *If the limits $\lim_{x\to\alpha} f(x) = b$ and $\lim_{x\to\alpha} g(x) = c$ exist, and if $f(x) \le g(x)$ holds on a punctured neighborhood of $\alpha$, then $b \le c$.*

*Proof.* Let $\dot{U}$ be a punctured neighborhood of $\alpha$ in which $f(x) \le g(x)$. Suppose that $b > c$. Then by the previous theorem, there exists a punctured neighborhood $\dot{V}$ of $\alpha$ such that $f(x) > g(x)$ if $x \in \dot{V}$. This, however, is impossible, since the set $\dot{U} \cap \dot{V}$ is nonempty, and for each of its elements $x$, $f(x) \le g(x)$. □

**Corollary 10.32.** *If $f$ is continuous in $a$ and $f(a) > 0$, then there exists a $\delta > 0$ such that $f(x) > 0$ for all $x \in (a-\delta, a+\delta)$. If $f \ge 0$ holds in a punctured neighborhood of $a$ and $f$ is continuous at $a$, then $f(a) \ge 0$.* □

*Remark 10.33.* The converse of Theorem 10.30 is not true: if $f(x) < g(x)$ holds on a punctured neighborhood of $\alpha$, then we cannot conclude that $\lim_{x\to\alpha} f(x) < \lim_{x\to\alpha} g(x)$. If, for example, $f(x) = 0$ and $g(x) = |x|$, then $f(x) < g(x)$ for all $x \neq 0$, but $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$.

The converse of Theorem 10.31 is also not true: if $\lim_{x\to\alpha} f(x) \le \lim_{x\to\alpha} g(x)$, then we cannot conclude that $f(x) \le g(x)$ holds in a punctured neighborhood of $\alpha$. If, for example, $f(x) = |x|$ and $g(x) = 0$, then $\lim_{x\to 0} f(x) \le \lim_{x\to 0} g(x) = 0$, but $f(x) > g(x)$ for all $x \neq 0$.

The following theorem for function limits is the analogue to Cauchy's criterion.

**Theorem 10.34.** *Let $f$ be defined on a punctured neighborhood of $\alpha$. The limit $\lim_{x\to\alpha} f(x)$ exists and is finite if and only if for all $\varepsilon > 0$, there exists a punctured neighborhood $\dot{U}$ of $\alpha$ such that*

$$|f(x_1) - f(x_2)| < \varepsilon \tag{10.10}$$

*if $x_1, x_2 \in \dot{U}$.*

*Proof.* Suppose that $\lim_{x\to\alpha} f(x) = b \in \mathbb{R}$, and let $\varepsilon > 0$ be fixed. Then there exists a punctured neighborhood $\dot{U}$ of $\alpha$ such that $|f(x) - b| < \varepsilon/2$ if $x \in \dot{U}$. It is clear that (10.10) holds for all $x_1, x_2 \in \dot{U}$.

Now suppose that the condition formulated in the theorem holds. If $x_n \to \alpha$ and $x_n \neq \alpha$ for all $n$, then the sequence $f(x_n)$ satisfies the Cauchy criterion. Indeed, for a given $\varepsilon$, choose a punctured neighborhood $\dot{U}$ such that (10.10) holds for all $x_1, x_2 \in \dot{U}$. Since $x_n \to \alpha$ and $x_n \neq \alpha$ for all $n$, there exists an $N$ such that $x_n \in \dot{U}$ for all $n \ge N$. If $n, m \ge N$, then by (10.10), we have $|f(x_n) - f(x_m)| < \varepsilon$. Then by Theorem 6.13, the sequence $(f(x_n))$ is convergent.

Fix a sequence $x_n \to \alpha$ that satisfies $x_n \neq \alpha$ for all $n$, and let $\lim_{n\to\infty} f(x_n) = b$. If $y_n \to \alpha$ is another sequence satisfying $y_n \neq \alpha$ for all $n$, then the combined sequence $(x_1, y_1, x_2, y_2, \ldots)$ also satisfies this assumption, and so the sequence of function values $s = (f(x_1), f(y_1), f(x_2), f(y_2), \ldots)$ is also convergent. Since $(f(x_n))$ is a subsequence of this, the limit of $s$ can only be $b$. On the other hand, $(f(y_n))$ is also a subsequence of $s$, so $f(y_n) \to b$. This holds for all sequences $y_n \to \alpha$ for which $y_n \neq \alpha$ for all $n$, so by the transference principle, $\lim_{x\to a} f(x) = b$. □

## *Exercises*

**10.18.** Show that for all functions $f\colon \mathbb{R} \to \mathbb{R}$, there exists a sequence $x_n \to \infty$ such that $(f(x_n))$ has a limit.

**10.19.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be arbitrary. Prove that the limit $\lim_{x\to\infty} f(x)$ exists if and only if whenever the sequences $(x_n)$ and $(y_n)$ tend to $\infty$, and the limits $\lim_{n\to\infty} f(x_n)$, $\lim_{n\to\infty} f(y_n)$ exist, then they are equal.

**10.20.** Construct a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(a \cdot n) \to 0$ $(n \to \infty)$ for all $a > 0$, but the limit $\lim_{x\to\infty} f(x)$ does not exist. (H)

**10.21.** Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $f(a \cdot n) \to 0$ $(n \to \infty)$ for all $a > 0$, then $\lim_{x\to\infty} f(x) = 0$. $(*\mathrm{S})$

**10.22.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that the sequence $(f(x_n))$ has a limit for all sequences $x_n \to \infty$ for which $x_{n+1}/x_n \to \infty$. Show that the limit $\lim_{x\to\infty} f(x)$ exists.

**10.23.** Prove that if the points $1/n$ for all $n \in \mathbb{N}^+$ are limit points of $H$, then 0 is also a limit point of $H$.

**10.24.** Show that every point $x \in \mathbb{R}$ is a limit point of each of the sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$.

**10.25.** Prove that (a) every bounded infinite set has a finite limit point; and (b) every infinite set has a limit point.

**10.26.** Prove that if the set $H$ has only one limit point, then $H$ is countable, and it can be listed as a sequence $(x_n)$ such that $\lim_{n\to\infty} x_n$ exists and is equal to the limit point of $H$.

**10.27.** What are the sets that have exactly two limit points?

**10.28.** Let $f(x) = x$ if $x$ is rational, and $f(x) = -x$ if $x$ is irrational. What can we say about the limits $\lim_{x\to c, x\in\mathbb{Q}} f(x)$, and $\lim_{x\to c, x\in\mathbb{R}\setminus\mathbb{Q}} f(x)$?

## 10.3 Limits and Operations

In examples up until now, we deduced continuity and values of limits directly from the definitions. The following theorems—which follow from the transference principle and theorems analogous to theorems on limits of sequences—let us deduce the continuity or limits of more complex functions using what we already know about continuity or limits of simple functions.

**Theorem 10.35.** *Let $\alpha$ denote a number $a$ or one of the symbols $a-0$, $a+0$, $\infty$, $-\infty$. If the finite limits $\lim_{x\to\alpha} f(x) = b$ and $\lim_{x\to\alpha} g(x) = c$ exist, then*

(i) $\lim_{x\to\alpha}(f(x)+g(x))$ *exists and is equal to $b+c$;*
(ii) $\lim_{x\to\alpha}(f(x)\cdot g(x))$ *exists and is equal to $b\cdot c$;*
(iii) *If $c \neq 0$, then* $\lim_{x\to\alpha}(f(x)/g(x))$ *exists and is equal to $b/c$.*

*Proof.* We will go into detail only with (i). Let $f$ and $g$ be defined on a punctured neighborhood $\dot{U}$ of $\alpha$. By the transference principle, we know that for every sequence $x_n \to \alpha$, $x_n \in \dot{U}$, $\lim_{n\to\infty} f(x_n) = b$ and $\lim_{n\to\infty} g(x_n) = c$. Then by Theorem 5.12,

$$\lim_{n\to\infty}(f(x_n)+g(x_n)) = b+c,$$

which, by the transference principle again, gives us (i). Statements (ii) and (iii) can be shown similarly. □

*Remarks 10.36.* **1.** In the first half of the proof, we used that the condition for sequences is necessary, while in the second half, we used that it is sufficient for the limit to exist.
**2.** In statement (iii), we did not suppose that $g(x) \neq 0$. To ensure that the limit $\lim_{x\to\alpha}(f(x)/g(x))$ has meaning, we see that if $c \neq 0$, then there exists a punctured neighborhood $\dot{U}$ of $\alpha$ in which $g(x) \neq 0$. Theorem 10.30 states that if $c < 0$, then in a suitable punctured neighborhood, $g(x) < 0$, and if $c > 0$, then in a suitable $\dot{U}$, $g(x) > 0$.

*Examples 10.37.* **1.** A simple application of Theorem 10.35 gives

$$\lim_{x\to 1} \frac{x^n - 1}{x^m - 1} = \frac{n}{m}$$

for all $n, m \in \mathbb{N}^+$, since if $x \neq 1$, then

$$\frac{x^n - 1}{x^m - 1} = \frac{x^{n-1} + x^{n-2} + \cdots + 1}{x^{m-1} + x^{m-2} + \cdots + 1}.$$

Here the numerator has $n$ summands, while the denominator has $m$, each of which tends to 1 if $x \to 1$.
**2.** Consider the following problem. Find the values of $a$ and $b$ such that

$$\lim_{x\to\infty} \left( \sqrt{x^2 - x + 1} \; - \; (ax + b) \right) = 0 \tag{10.11}$$

holds. It is clear that only positive values of $a$ can come into play, and so we have to find limits of type "$\infty - \infty$." The following argument will lead us to an answer:

$$\begin{aligned}
&\sqrt{x^2 - x + 1} - (ax + b) = \\
&= \frac{(\sqrt{x^2 - x + 1} - (ax + b)) \cdot (\sqrt{x^2 - x + 1} + (ax + b))}{\sqrt{x^2 - x + 1} + (ax + b)} = \\
&= \frac{x^2 - x + 1 - (ax + b)^2}{\sqrt{x^2 - x + 1} + (ax + b)} = \frac{(1 - a^2)x^2 - (2ab + 1)x + (1 - b^2)}{\sqrt{x^2 - x + 1} + (ax + b)} = \\
&= \frac{(1 - a^2)x - (2ab + 1) + \frac{1 - b^2}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + a + \frac{b}{x}}.
\end{aligned}$$

Since

$$\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + a + \frac{b}{x} \to a + 1 \quad \text{and} \quad 2ab + 1 + \frac{b^2 - 1}{x} \to 2ab + 1,$$

if $x \to \infty$, the quotient can tend to 0 only if $1 - a^2 = 0$, that is, considering $a > 0$, only if $a = 1$. In this case,

$$\lim_{x\to\infty} \left( \sqrt{x^2 - x + 1} - (x + b) \right) = -\frac{2b + 1}{2}.$$

This is 0 if $b = -1/2$. Thus (10.11) holds if and only if $a = 1$ and $b = -1/2$.

**Definition 10.38.** If

$$\lim_{x\to\infty}(f(x)-(ax+b))=0,$$

then we say that the *asymptote* of $f(x)$ at $\infty$ is the linear function $ax+b$. (Or from a geometric viewpoint: the asymptote of the curve $y=f(x)$ at $\infty$ is the line $y=ax+b$.) The asymptote of $f(x)$ at $-\infty$ is defined similarly.

*Remark 10.39.* In Theorems 5.12, 5.14 and 5.16, we saw that we can interchange the order of taking limits of sequences and performing basic operations on them. We also proved this in numerous cases in which the limit of one (or both) of the sequences considered is infinity. These cases apply to the corresponding statements for functions (with identical proofs), just as with the cases with finite limits. So for example:

If $\lim_{x\to\alpha} f(x)=b$ is finite and $\lim_{x\to\alpha} g(x)=\infty$, then $\lim_{x\to\alpha}(f(x)+g(x))=\infty$.

Or if $\lim_{x\to\alpha} f(x)=a\neq 0$, $\lim_{x\to\alpha} g(x)=0$ and $g\neq 0$ in a punctured neighborhood of $a$, then $\lim_{x\to\alpha}|f(x)/g(x)|=\infty$. If we also know that $f/g$ has unchanging sign, then it follows that $\lim_{x\to\alpha} f(x)/g(x)=\infty$ or $\lim_{x\to\alpha} f(x)/g(x)=-\infty$ depending on the sign.

*Example 10.40.* Let $x_1$ and $x_2$ be the roots of the equation $ax^2+bx+c=0$. Find the limits of $x_1$ and $x_2$ if $b$ and $c$ are fixed, $b\neq 0$, and $a\to 0$.

Let

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \qquad \text{and} \qquad x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

We can suppose $b>0$ without sacrificing generality. We see that then $\lim_{a\to 0} x_1$ is a limit of type $0/0$. A simple rearrangement yields

$$x_1=\frac{\sqrt{b^2-4ac}-b}{2a}=\frac{-4ac}{2a(\sqrt{b^2-4ac}+b)}=\frac{-4c}{2(\sqrt{b^2-4ac}+b)}\to -\frac{c}{b}.$$

For the other root, $\lim_{a\to 0}(-b-\sqrt{b^2-4ac})=-2b<0$ implies

$$\lim_{a\to 0+0} x_2=-\infty, \qquad \text{and} \qquad \lim_{a\to 0-0} x_2=\infty.$$

We note that critical limits can arise in the case of functions as well—in fact, exactly in the same situations as for sequences. Critical limits occur when the limits of $f$ and $g$ on their own do not determine the limits of $f+g$, $f\cdot g$, or $f/g$. So for example, if $\lim_{x\to 0} f(x)=\lim_{x\to 0} g(x)=0$, then the limit $\lim_{x\to 0} f(x)/g(x)$ can be finite or infinite, or it might not exist. Clearly, $\lim_{x\to 0} x/x=1$, $\lim_{x\to 0} x/x^3=\infty$, $\lim_{x\to 0} -x/x^3=-\infty$, and if $f$ is the Riemann function, then the limit $\lim_{x\to 0} f(x)/x$ does not exist (show this). The examples that illustrate critical limits for sequences can usually be transformed into examples involving functions without much difficulty.

The following theorem addresses limits of compositions of functions.

**Theorem 10.41.** *Suppose that* $\lim_{x\to\alpha} g(x) = \gamma$ *and* $\lim_{t\to\gamma} f(t) = \beta$. *If* $g(x) \neq \gamma$ *on a punctured neighborhood of* $\alpha$, *or* $\gamma$ *is finite and* $f$ *continuous at* $\gamma$, *then* $\lim_{x\to\alpha} f(g(x)) = \beta$.

*Proof.* For brevity, we denote the punctured neighborhoods of $\alpha$ by $\dot{U}(\alpha)$.

We have to show that for every neighborhood $V$ of $\beta$, we can find a $\dot{U}(\alpha)$ such that if $x \in \dot{U}(\alpha)$, then $f(g(x)) \in V$. Since $\lim_{t\to\gamma} f(t) = \beta$, there exists a $\dot{W}(\gamma)$ such that $f(t) \in V$ whenever $t \in \dot{W}(\gamma)$. Let $W(\gamma) = \dot{W}(\gamma)$ if $\gamma = \infty$ or $-\infty$, and let $W(\gamma) = \dot{W}(\gamma) \cup \{\gamma\}$ if $\gamma$ is finite. Then $W(\gamma)$ is a neighborhood of $\gamma$, so by $\lim_{x\to\alpha} g(x) = \gamma$, there exists a $\dot{U}_1(\alpha)$ such that if $x \in \dot{U}_1(\alpha)$, then $g(x) \in W(\gamma)$.

If we know that $g(x) \neq \gamma$ in a punctured neighborhood $\dot{U}_2(\alpha)$ of $\alpha$, then $\dot{U}(\alpha) = \dot{U}_1(\alpha) \cap \dot{U}_2(\alpha)$ is a punctured neighborhood of $\alpha$ such that whenever $x \in \dot{U}(\alpha)$, $g(x) \in W(\gamma)$ and $g(x) \neq \gamma$, that is, $g(x) \in \dot{W}(\gamma)$, and thus $f(g(x)) \in V$.

Now let $f$ be continuous at $\gamma$. Then $f(t) \in V$ for all $t \in W(\gamma)$, since $f(\gamma) = \beta \in V$. Thus if $x \in \dot{U}_1(\alpha)$, then $g(x) \in W(\gamma)$ and $f(g(x)) \in V$. □

*Remark 10.42.* The crucial condition in the theorem is that either $g(x) \neq \gamma$ in $\dot{U}(\alpha)$, or that $f$ is continuous at $\gamma$. If neither of these holds, then the statement of the theorem is not always true, which is illustrated by the following example. Let $g$ be the Riemann function (that is, function 3 in Example 10.7), and let

$$f(t) = \begin{cases} 1, & \text{if } t \neq 0, \\ 0, & \text{if } t = 0. \end{cases}$$

It is easy to see that then $(f \circ g)(x) = f(g(x))$ is exactly the Dirichlet function. In Example 10.7, we saw that $\lim_{x\to 0} g(x) = 0$. On the other hand, it is clear that $\lim_{t\to 0} f(t) = 1$, while the limit $\lim_{x\to 0} f(g(x))$ does not exist.

If $\gamma = \infty$, then the condition $g(x) \neq \gamma$ $(x \in \dot{U}(\alpha))$ automatically holds. This means that if $\lim_{x\to\alpha} g(x) = \infty$ and $\lim_{t\to\infty} f(t) = \beta$, then $\lim_{x\to\alpha} f(g(x)) = \beta$ holds without further assumptions. A special case of the converse of this is also true.

**Theorem 10.43.** *Let* $f$ *be defined in a neighborhood of* $\infty$. *Then*

$$\lim_{x\to 0+0} f\left(\frac{1}{x}\right) = \lim_{x\to\infty} f(x) \tag{10.12}$$

*in the sense that if one limit exists, then so does the other, and they are equal.*

*Proof.* Since $\lim_{x\to 0+0} 1/x = \infty$, as we saw before, the statement is true whenever the right-hand side exists. If the left-hand side exists, then we apply Theorem 10.41 with the choices $\alpha = \infty$, $g(x) = 1/x$, and $\gamma = 0$. Since $g(x)$ is nowhere zero,

$$\lim_{x\to\infty} h(1/x) = \lim_{x\to 0+0} h(x)$$

whenever the right-hand side exists. If we apply this to the function $h(x) = f(1/x)$, then we get (10.12). □

As an application of Theorems 10.35 and 10.41, we immediately get the following theorem.

**Theorem 10.44.**

(i) *If $f$ and $g$ are continuous at $a$, then $f+g$ and $f \cdot g$ are also continuous at $a$, and if $g(a) \neq 0$, then $f/g$ is continuous at $a$ as well.*
(ii) *If $g(x)$ is continuous at $a$ and $f(t)$ is continuous at $g(a)$, then $f \circ g$ is also continuous at $a$.*

We prove the following regarding continuity of the inverse of a function. (We would like to draw the reader's attention to the fact that in the theorem, we do not suppose that the function itself is continuous.)

**Theorem 10.45.** *Let $f$ be strictly monotone increasing (decreasing) on the interval $I$. Then*

(i) *The inverse of $f$, $f^{-1}$, is strictly monotone increasing (decreasing) on the set $f(I)$, and moreover,*
(ii) *$f^{-1}$ is continuous at every point of $f(I)$ restricted to the set $f(I)$.*

*Proof.* We can suppose that the interval $I$ is nondegenerate, since otherwise, the statement is trivial: on a set of one element, every function is strictly increasing, decreasing, and continuous (restricted to the set). We can also assume that $f$ is strictly increasing, since the decreasing case can be proved the same way.

Since if $u_1, u_2 \in I$, $u_1 < u_2$, then $f(u_1) < f(u_2)$, $f$ is one-to-one, so it has an inverse. Let $x_1, x_2 \in f(I)$, and suppose that $x_1 < x_2$, but $f^{-1}(x_1) \geq f^{-1}(x_2)$. Since $f$ is monotone increasing, we would have

$$x_1 = f\left(f^{-1}(x_1)\right) \geq f\left(f^{-1}(x_2)\right) = x_2,$$

which is impossible. Thus we see that $f^{-1}(x_1) < f^{-1}(x_2)$ if $x_1 < x_2$, that is, $f^{-1}$ is monotone increasing.

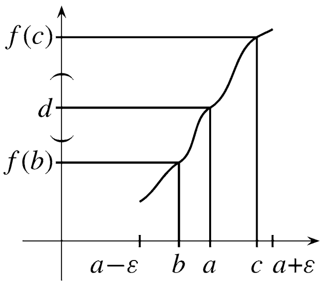


Fig. 10.11

Let $d \in f(I)$ be arbitrary. Then $d = f(a)$ for a suitable $a \in I$. We show that $f^{-1}$ is continuous at $d$ restricted to $f(I)$. Let $\varepsilon > 0$ be fixed. Since $f^{-1}(d) = a$, we have to show that there exists a $\delta > 0$ such that

$$f^{-1}(x) \in (a-\varepsilon, a+\varepsilon) \text{ if } x \in (d-\delta, d+\delta) \cap f(I). \tag{10.13}$$

Consider first the case that $a$ is an interior point of $I$ (that is, not an endpoint). Then we can choose points $b,c \in I$ such that $a-\varepsilon < b < a < c < a+\varepsilon$ (see Figure 10.11). Since $f$ is strictly monotone increasing, $f(b) < f(a) < f(c)$, that is, $f(b) < d < f(c)$. Choose a positive $\delta$ such that $f(b) < d-\delta < d+\delta < f(c)$. If $x \in (d-\delta, d+\delta) \cap f(I)$, then by the strict monotonicity of $f^{-1}$,

$$b = f^{-1}(f(b)) < f^{-1}(x) < f^{-1}(f(c)) = c.$$

Thus $f^{-1}(x) \in (b,c) \subset (a-\varepsilon, a+\varepsilon)$, which proves (10.13).

If $a$ is the left endpoint of $a$, then choose a point $c \in I$ such that $a < c < a+\varepsilon$. Then $d = f(a) < f(c)$, so for a suitable $\delta > 0$, $d+\delta < f(c)$. If $x \in (d-\delta, d+\delta) \cap f(I)$, then

$$a \le f^{-1}(x) < f^{-1}(f(c)) = c.$$

Thus $f^{-1}(x) \in [a,c) \subset (a-\varepsilon, a+\varepsilon)$, so (10.13) is again true. The argument for $a$ being the right endpoint of $I$ is the same. □

Just as with sequences, we define order of magnitude and asymptotic equivalence for functions as well.

**Definition 10.46.** Suppose that $\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} g(x) = \infty$. If

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \infty,$$

then we say that $f$ *tends to infinity faster than* $g$ (or $g$ *tends to infinity more slowly than* $f$). We can also say that the *order of magnitude of* $f$ *is greater than the order of magnitude of* $g$.

Similarly, if $\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} g(x) = 0$ and $\lim_{x\to\alpha} f(x)/g(x) = 0$, then we say that $f$ *tends to zero faster than* $g$ (or $g$ *tends to zero more slowly than* $f$).

The statement $\lim_{x\to\alpha} f(x)/g(x) = 0$ is sometimes denoted by $f(x) = o(g(x))$ (read: $f(x)$ is little-o of $g(x)$).

If we know only that $f(x)/g(x)$ is bounded in $\dot{U}(\alpha)$, then we denote this by $f(x) = O(g(x))$ (read: $f(x)$ is big-O of $g(x)$).

*Example 10.47.* We show that if $x \to \infty$, then the function $a^x$ tends to infinity faster than $x^k$ for every $a > 1$ and $k > 0$. To see this, we have to show that $\lim_{x\to\infty} a^x/x^k = \infty$. Let $P$ be a fixed real number. Since the sequence $(a^n/n^k)$ tends to infinity, there exists an $n_0$ such that $a^n/n^k > 2^k \cdot P$ if $n \ge n_0$. Now if $x \ge 1$, then $a^x \ge a^{[x]}$ and $x^k \le (2\cdot[x])^k = 2^k \cdot [x]^k$. Thus if $x \ge n_0$, then $[x] \ge n_0$, and so

$$\frac{a^x}{x^k} \ge \frac{a^{[x]}}{2^k \cdot [x]^k} > P,$$

which concludes the argument.

Consider the following functions:

$$\ldots, \sqrt[n]{x}, \ldots, \sqrt[3]{x}, \sqrt{x}, x, x^2, x^3, \ldots, x^n, \ldots, 2^x, 3^x, \ldots, n^x, \ldots, x^x. \tag{10.14}$$

It is easy to see that if $x \to \infty$, then in the above arrangement, each function tends to infinity faster than the functions to the left of it. (For the function $2^x$, this follows from the previous example.)

**Definition 10.48.** Suppose that

$$\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} g(x) = 0 \qquad \text{or} \qquad \lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} g(x) = \pm\infty.$$

If

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = 1,$$

then we say that $f$ and $g$ are *asymptotically equal*. We denote this by $f \sim g$ if $x \to \alpha$.

*Example 10.49.* We show that if $x \to 0$, then $\sqrt{1+x} - 1 \sim x/2$. Indeed,

$$\frac{\sqrt{1+x}-1}{(x/2)} = \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{(x/2)\cdot(\sqrt{1+x}+1)} = \frac{2}{\sqrt{1+x}+1} \to 1$$

if $x \to 0$.

## *Exercises*

**10.29.** $\lim_{x\to 7} \frac{\sqrt{x+2} - \sqrt[3]{x+20}}{\sqrt[4]{x+9}-2} = ?$

**10.30.** $\lim_{x\to 1} \frac{\sqrt[359]{x}-1}{\sqrt[5]{x}-1} = ?$

**10.31.** $\lim_{x\to\infty} x \cdot \left[\sqrt{x^2+2x} - 2\sqrt{x^2+x} + x\right] = ?$

**10.32.** $\lim_{x\to\infty} x^{3/2} \cdot \left[\sqrt{x+2} + \sqrt{x} - 2\sqrt{x+1}\right] = ?$

**10.33.** $\lim_{x\to 1} \frac{(1-x)(1-\sqrt{x})(1-\sqrt[3]{x})\cdot\ldots\cdot(1-\sqrt[n]{x})}{(1-x)^n} = ?$

**10.34.** Prove that

$$\lim_{x\to -d/c+0} \frac{ax+b}{cx+d} = \begin{cases} \infty, & \text{if } bc-ad>0 \\ -\infty, & \text{if } bc-ad<0, \end{cases}$$

$$\lim_{x\to -d/c-0} \frac{ax+b}{cx+d} = \begin{cases} -\infty, & \text{if } bc-ad>0 \\ \infty, & \text{if } bc-ad<0, \end{cases}$$

and

$$\lim_{x\to\pm\infty} \frac{ax+b}{cx+d} = \frac{a}{c} \qquad (c \neq 0).$$

**10.35.** Let $p(x)$ be a polynomial of degree at most $n$; that is, let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Prove that if

$$\lim_{x\to 0} \frac{p(x)}{x^n} = 0,$$

then $p(x) = 0$ for all $x$.

**10.36.** Construct functions $f$ and $g$ such that $\lim_{x\to 0} f(x) = \infty$, $\lim_{x\to 0} g(x) = -\infty$, and moreover,

(a) $\lim_{x\to 0}(f(x)+g(x))$ exists and is finite;
(b) $\lim_{x\to 0}(f(x)+g(x)) = \infty$;
(c) $\lim_{x\to 0}(f(x)+g(x)) = -\infty$;
(d) $\lim_{x\to 0}(f(x)+g(x))$ does not exist.

**10.37.** (a) If at $x = a$, $f$ is continuous while $g$ is not, then can $f+g$ be continuous here?
(b) If at $x = a$, neither $f$ nor $g$ is continuous, then can $f+g$ be continuous here?

**10.38.** Suppose that $\varphi\colon \mathbb{R} \to \mathbb{R}$ is strictly monotone, and let $R(\varphi) = \mathbb{R}$. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ and if $f \circ \varphi$ is continuous, then $f$ is continuous.

**10.39.** What can we say about the continuity of $f \circ g$ if in $\mathbb{R}$,

(i) both $f$ and $g$ are continuous,
(ii) $f$ is continuous, $g$ is not continuous,
(iii) neither $f$ nor $g$ is continuous.

**10.40.** Give an example of functions $f, g\colon \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0, \qquad \text{but} \qquad \lim_{x\to 0} f(g(x)) = 1. \tag{S}$$

**10.41.** Prove that if $f$ is the Riemann function, then the limit $\lim_{x\to 0} f(x)/x$ does not exist.

**10.42.** Is the following statement true? If $f_1, f_2, \ldots$ is an infinite sequence of continuous functions and $F(x) = \inf_k \{f_k(x)\}$, then $F(x)$ is also continuous.

**10.43.** Is it true that if $f$ is strictly monotone on the set $A \subset \mathbb{R}$, then its inverse is continuous on the set $f(A)$?

**10.44.** Let $a$ and $b$ be positive numbers. Prove that (a) if $x \to \infty$, then the order of magnitude of $x^a$ is greater than the order of magnitude of $x^b$ if and only if $a > b$; and (b) if $x \to 0+0$, then the order of magnitude of $x^{-a}$ is greater than the order of magnitude of $x^{-b}$ if and only if $a > b$.

**10.45.** Prove that in the ordering (10.14), each function tends to infinity faster than the functions to the left of it.

**10.46.** Let $a > 1$ and $k > 0$. Prove that if $x \to \infty$, then the order of magnitude of $a^{\sqrt{x}}$ is larger than the order of magnitude of $x^k$.

**10.47.** Suppose that all of the functions $f_1, f_2, \dots$ tend to infinity as $x \to \infty$. Prove that there is a function $f$ whose order of magnitude is greater than the order of magnitude of each $f_n$.

## 10.4 Continuous Functions in Closed and Bounded Intervals

The following theorems show that if $f$ is continuous on a closed and bounded interval, then it automatically follows that $f$ possesses numerous other important properties.

**Definition 10.50.** Let $a < b$. The function $f$ *is continuous in the interval* $[a,b]$ if it is continuous at all $x \in (a,b)$, is continuous from the right at $a$, and continuous from the left at $b$.

More generally:

**Definition 10.51.** Let $A \subset D(f)$. The function $f$ is *continuous on the set* $A$ if at each $x \in A$, it is continuous when restricted to the set $A$.

From now on, we denote the set of all continuous functions on the closed and bounded interval $[a,b]$ by $C[a,b]$.

**Theorem 10.52.** *If $f \in C[a,b]$, then $f$ is bounded on $[a,b]$.*

*Proof.* We prove the statement by contradiction. Suppose that $f$ is not bounded on $[a,b]$. Then for no number $K$ does $|f(x)| \le K$ for all $x \in [a,b]$ hold. Namely, for every $n$, there exists an $x_n \in [a,b]$ such that $|f(x_n)| > n$.

Consider the sequence $(x_n)$. This is bounded, since each of its terms falls inside $[a,b]$, so it has a convergent subsequence $(x_{n_k})$. Let $\lim_{k\to\infty} x_{n_k} = \alpha$. Since $\{x_n\} \subset [a,b]$, $\alpha$ is also in $[a,b]$. However, $f$ is continuous at $\alpha$, so by the transference principle, the sequence $(f(x_{n_k}))$ is convergent (and tends to $f(\alpha)$). It follows that the sequence $(f(x_{n_k}))$ is bounded. This, however, contradicts that $|f(x_{n_k})| > n_k$ of all $k$. □

*Remark 10.53.* In the theorem, it is necessary to suppose that the function $f$ is continuous on a *closed and bounded* interval. Dropping either assumption leads to the statement of the theorem not being true. So for example, the function $f(x) = 1/x$ is continuous on the bounded interval $(0,1]$, but $f$ is not bounded there. The function $f(x) = x^2$ is continuous on $[0,\infty)$, but is also not bounded there.

**Definition 10.54.** Let $f$ be defined on the set $A$. If the image of $A$, $f(A)$, has a greatest element, then we call it the (global) *maximum* of the function $f$ over $A$, and denote it by $\max f(A)$ or $\max_{x\in A} f(x)$. If $a \in A$ and $f(a) = \max f(A)$, then we say that $a$ is an *global maximum point* of $f$ over $A$.

If $f(A)$ has a smallest element, then we call this the (global) *minimum* of the function $f$ over $A$, and denote it by $\min f(A)$ or $\min_{x\in A} f(x)$. If $b \in A$ and $f(b) = \min f(A)$, then we say that $b$ is a *global minimum point* of $f$ over $A$.

We collectively call the global maximum and minimum points *global extrema*.

Naturally, on a set $A$, a function can have numerous maximum (or minimum) points.

A set of numbers can have a maximum (or minimum) only if it is bounded from above (below). However, as we have already seen, not every bounded set has a maximum or minimum element. If the image of a set $A$ under a function $f$ is bounded, that alone does not guarantee that there will be a largest or smallest value of $f$.

For example, the fractional part $f(x) = \{x\}$ is bounded on $[0,2]$. The least upper bound of the image of this set is 1, but the function is never equal to 1. Thus this function does not have a maximum point in $[0,2]$.

The following theorem shows that this cannot happen with a continuous function on a closed and bounded interval.

**Theorem 10.55 (Weierstrass's Theorem).** *If $f \in C[a,b]$, then there exist $\alpha \in [a,b]$ and $\beta \in [a,b]$, such that $f(\alpha) \le f(x) \le f(\beta)$ if $x \in [a,b]$. In other words, a continuous function always has an absolute maximum and minimum point over a closed and bounded interval.*

We give two proofs of the theorem.

*Proof I.* By Theorem 10.52, $f([a,b])$ is bounded. Let the least upper bound of $f([a,b])$ be $M$. If $M \in f([a,b])$, then this means that $M = \max_{x\in[a,b]} f(x)$. Thus we have only to show that $M \notin f([a,b])$ is impossible. We prove this by contradiction. If $M \notin f([a,b])$, then the values of the function $F(x) = M - f(x)$ are positive for all $x \in [a,b]$. Thus the function $1/F$ is also continuous on $[a,b]$ (see Theorem 10.44), so it is bounded there (by Theorem 10.52 again). Then there exists a $K > 0$ such that

$$\frac{1}{M - f(x)} \le K$$

for all $x \in [a,b]$. Taking reciprocals of both sides and rearranging (and using that $M - f(x) > 0$ everywhere) yields the inequality

$$f(x) \le M - \frac{1}{K}$$

if $x \in [a,b]$. However, this contradicts $M$ being the least upper bound of $f([a,b])$.

The existence of $\min f[a,b]$ can be proven similarly. (Or we can reduce it to the statement regarding the maximum if we apply that to $-f$ instead of $f$.) $\square$

*Proof II.* Once again, let $M = \sup f([a,b])$; we will show that $M \in f([a,b])$. If $n$ is a positive integer, then $M-(1/n)$ is not an upper bound of $f([a,b])$, since $M$ was the least upper bound. Thus there exists a point $x_n \in [a,b]$ such that $f(x_n) > M-(1/n)$. The sequence $(x_n)$ is bounded (since each of its terms falls in $[a,b]$), so it has a convergent subsequence $(x_{n_k})$. Let $\lim_{k\to\infty} x_{n_k} = \alpha$. Since $\{x_n\} \subset [a,b]$, we have $\alpha \in [a,b]$. Now $f$ is continuous at $\alpha$, so by the transference principle, $f(x_{n_k}) \to f(\alpha)$. Since

$$M - \frac{1}{n_k} < f(x_{n_k}) \le M$$

for all $k$, by the squeeze theorem, $M \le f(\alpha) \le M$, that is, $f(\alpha) = M$. This shows that $M \in f([a,b])$.

The existence of $\min f[a,b]$ can be proven similarly. □

*Remark 10.56.* Looking at the conditions of the theorem, it is again essential that we are talking about continuous functions in *closed, bounded* intervals. As we saw in Remark 10.53, if $f$ is continuous on an open interval $(a,b)$, then it might happen that $f((a,b))$ is not bounded from above, and then $\max f((a,b))$ does not exist. But this can occur even if $f$ is bounded. For example, the function $f(x) = x$ is continuous and bounded on the open interval $(0,1)$, but it does not have a greatest value there.

It is equally important for the interval to be bounded. This is illustrated by the function $f(x) = -1/(1+x^2)$, which is bounded in $[0,\infty)$, but does not have a greatest value.

Another important property of continuous functions over closed and bounded intervals is given by the following theorem.

**Theorem 10.57 (Bolzano–Darboux[3] Theorem).** *If $f \in C[a,b]$, then $f$ takes on every value between $f(a)$ and $f(b)$ on the interval $[a,b]$.*

We give two proofs of this theorem again, since both proofs embody ideas that are frequently used in analysis.

*Proof I.* Without loss of generality, we may assume that $f(a) < c < f(b)$. We will prove the existence of a point $\alpha \in [a,b]$ such that the function takes on values not larger than $c$ and not smaller than $c$ in every neighborhood of the point. Then by the continuity of $f$ at $\alpha$, it follows that $f(\alpha) = c$.

We will define $\alpha$ as the intersection of a sequence of nested closed intervals $I_0 \supset I_1 \supset \ldots$. Let $I_0 = [a,b]$.

If $\quad f\left(\dfrac{a+b}{2}\right) \le c$, then let $I_1 = [a_1,b_1] = \left[\dfrac{a+b}{2}, b\right]$,

but if $f\left(\dfrac{a+b}{2}\right) > c$, then let $I_1 = [a_1,b_1] = \left[a, \dfrac{a+b}{2}\right]$.

We continue this process. If $I_n = [a_n,b_n]$ is already defined

[3] Jean Gaston Darboux (1842–1917), French mathematician.

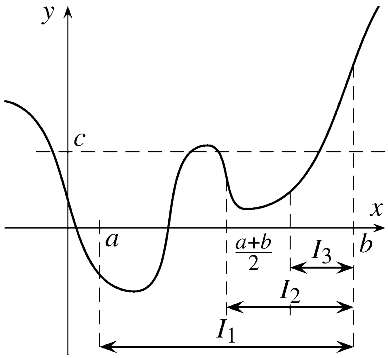


Fig. 10.12

and $f\left(\frac{a_n+b_n}{2}\right) \le c$, then let $I_{n+1} = [a_{n+1}, b_{n+1}] = \left[\frac{a_n+b_n}{2}, b_n\right]$,

but if $f\left(\frac{a_n+b_n}{2}\right) > c$, then let $I_{n+1} = [a_{n+1}, b_{n+1}] = \left[a_n, \frac{a_n+b_n}{2}\right]$.

The interval sequence $I_0 \supset I_1 \supset \ldots$ is defined such that

$$f(a_n) \le c < f(b_n) \tag{10.15}$$

holds for all $n$ (Figure 10.12). Since $|I_n| = (b-a)/2^n \to 0$, the interval sequence $(I_n)$ has exactly one shared point. Let this be $\alpha$. Clearly,

$$\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

and since $f$ is continuous in $\alpha$,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(\alpha). \tag{10.16}$$

But by (10.15),

$$\lim_{n\to\infty} f(a_n) \le c \le \lim_{n\to\infty} f(b_n),$$

that is, (10.16) can hold only if $f(\alpha) = c$. □

*Proof II.* Let us suppose once again that $f(a) < c < f(b)$, and let

$$A = \{x \in [a,b] : f(x) < c\}.$$

The set $A$ is then bounded and nonempty, since $a \in A$. Thus $\alpha = \sup A$ exists, and since $A \subset [a,b]$, we have $\alpha \in [a,b]$. Since $f$ is continuous at $a$ and $f(a) < c$, $f(x) < c$ holds over a suitable interval $[a, a+\delta)$, and so $a < \alpha$. Moreover, since $f$ is continuous at $b$ and $f(b) > c$, $f(x) > c$ holds on a suitable interval $(b-\delta, b]$, and so $\alpha < b$. We will show that $f(\alpha) = c$.

If $f(\alpha)$ were larger than $c$, then there would exist an interval $(\alpha-\delta,\alpha+\delta)$ in which $f(x)>c$ would hold. But then $\alpha$ could not be the upper limit of the set $A$, that is, its *least* upper bound, since the smaller $\alpha-\delta$ would also be an upper bound of $A$.

If, however, $f(\alpha)$ were smaller than $c$, then there would exist an interval $(\alpha-\delta,\alpha+\delta)$ in which $f(x)<c$ held. But then $\alpha$ cannot be the upper limit of the set $A$ once again, since there would be values $x$ in $A$ that were larger than $\alpha$. Thus neither $f(\alpha)>c$ nor $f(\alpha)<c$ can hold, so $f(\alpha)=c$. □

**Corollary 10.58.** *If $f\in C[a,b]$, then the image of $f$ (the set $f([a,b])$) is a closed and bounded interval; in fact,*

$$f\big([a,b]\big)=\left[\min_{x\in[a,b]}f(x),\ \max_{x\in[a,b]}f(x)\right].$$

*Proof.* It follows from Weierstrass's theorem that $M=\max f([a,b])$ and $m=\min f([a,b])$ exist. It is clear that $f([a,b])\subset[m,M]$. By Theorem 10.57, we also know that the function takes on every value of $[m,M]$ in $[a,b]$, so $f([a,b])=[m,M]$. □

It is easy to see from the above theorems that if $I$ is any kind of interval and $f$ is continuous on $I$, then $f(I)$ is also an interval (see Exercise 10.61).

Using the Bolzano–Darboux theorem, we can give a simple proof of the existence of the $k$th roots of nonnegative numbers (Theorem (3.6)).

**Corollary 10.59.** *If $a\geq 0$ and $k$ is a positive integer, then there exists a nonnegative real number $b$ such that $b^k=a$.*

*Proof.* The function $x^k$ is continuous on the interval $[0,a+1]$ (why?). Since we have $f(0)=0\leq a$ and $f(a+1)=(a+1)^k\geq a+1>a$, by the Bolzano–Darboux theorem, there exists a $b\in[0,a+1]$ such that $b^k=f(b)=a$. □

## *Exercises*

**10.48.** Give an example of a function $f\colon [a,b]\to\mathbb{R}$ that is continuous on all of $[a,b]$ except for one point, and that is (a) not bounded, (b) bounded, but does not have a greatest value.

**10.49.** Show that if $f\colon\mathbb{R}\to\mathbb{R}$ is continuous and $\lim_{x\to\infty}f(x)=\lim_{x\to-\infty}f(x)$, then $f$ has either a largest or smallest value (not necessarily both).

**10.50.** Which are the functions $f\colon [a,b] \to \mathbb{R}$ which have smallest and largest values on every nonempty set $A \subset [a,b]$?

**10.51.** Suppose that the function $f\colon [a,b] \to \mathbb{R}$ satisfies the following properties: (a) If $a \le c < d \le b$, then $f$ takes on every value between $f(c)$ and $f(d)$ on $[c,d]$; moreover, (b) whenever $x_n \in [a,b]$ for all $n$ and $x_n \to c$, then on the set $\{x_n : n = 1,2,\dots\} \cup \{c\}$, the function $f$ has largest and smallest values. Prove that $f$ is continuous.

**10.52.** Let $f\colon [a,b] \to (0,\infty)$ be continuous. Prove that for suitable $\delta > 0$, $f(x) > \delta$ for all $x \in [a,b]$. Give a counterexample if we write $(a,b)$ instead of $[a,b]$.

**10.53.** Let $f$, $g\colon [a,b] \to \mathbb{R}$ be continuous and suppose that $f(x) < g(x)$ for all $x \in [a,b]$. Prove that for suitable $\delta > 0$, $f(x)+\delta < g(x)$ for all $x \in [a,b]$. Give a counterexample if we write $(a,b)$ instead of $[a,b]$.

**10.54.** Prove that if $f\colon [a,b] \to \mathbb{R}$ is continuous and one-to-one, then it is strictly monotone.

**10.55.** Show that if $f\colon [a,b] \to \mathbb{R}$ is monotone increasing and the image contains $[f(a), f(b)] \cap \mathbb{Q}$, then $f$ is continuous.

**10.56.** Prove that if $f\colon [a,b] \to \mathbb{R}$ is continuous, then for every $x_1,\dots,x_n \in [a,b]$, there exists a $c \in [a,b]$ such that $(f(x_1)+\cdots+f(x_n))/n = f(c)$.

**10.57.** Prove that every cubic polynomial has a real root. Is it true that every fourth-degree polynomial has a real root? (H)

**10.58.** Prove that if $f\colon [a,b] \to [a,b]$ is continuous, then there exists $c \in [a,b]$ for which $f(c) = c$. Give a counterexample if we take any other type of interval than $[a,b]$. (H)

**10.59.** Prove that if $f\colon [0,1] \to \mathbb{R}$ is continuous and $f(0) = f(1)$, then there exists an $x \in [0,1/2]$ such that $f(x) = f(x+(1/2))$. In fact, for every $n \in \mathbb{N}^+$, there exists a $0 \le x \le 1-(1/n)$ such that $f(x) = f(x+(1/n))$.

**10.60.** Does there exist a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ for which $f(f(x)) = -x$ for all $x$? (H)

**10.61.** Prove that if $I$ is an interval (closed or not, bounded or not, degenerate or not) and $f\colon I \to \mathbb{R}$ is continuous, then $f(I)$ is also an interval. (S)

## 10.5 Uniform Continuity

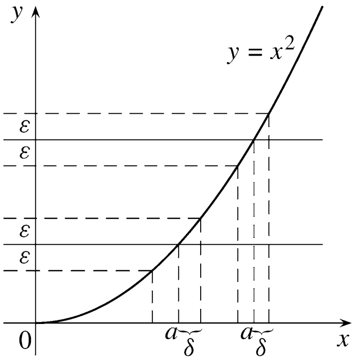


Fig. 10.13

Let the function $f$ be continuous on the interval $I$. This means that for all $a \in I$ and arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon, \text{ if } x \in (a-\delta, a+\delta) \cap I. \tag{10.17}$$

In many cases, we can determine the largest possible $\delta$ for a given $a$ such that (10.17) holds. Let us denote this by $\delta(a)$. If $\varepsilon > 0$ is fixed, then for different $a$, usually different $\delta(a)$ correspond. It is easy to see, for example, that for the function $f(x) = x^2$, the larger $|a|$ is, the smaller the corresponding $\delta(a)$ is (Figure 10.13). Thus in the interval $[0,1]$, the $\delta(a)$ corresponding to $a = 1$ is the smallest, so for each $a \in [0,1]$, we can choose $\delta$ to be $\delta(1)$. In other words, this means that for all $a \in [0,1]$

$$|f(x) - f(a)| < \varepsilon, \quad \text{if} \quad |x - a| < \delta(1).$$

This argument, of course, does not usually work. Since there is not always a smallest number out of infinitely many, we cannot always—at least using the above method—find a $\delta$ that is good for all $a \in I$ if we have a continuous function $f\colon I \to \mathbb{R}$. But such a $\delta$ does not always exist. In the case of $f(x) = 1/x$, we found the value of $\delta(a)$ in Example 10.2.4 (see equation (10.2)). We can see that in this case, $\delta(a) \to 0$ if $a \to 0$, that is, there does not exist a $\delta$ that would be good at every point of the interval $(0,1)$. As we will soon show, this phenomenon cannot occur in functions that are continuous on a closed and bounded interval: in this case, there must exist a shared, good $\delta$ for the whole interval. We call this property uniform continuity. The following is the precise definition:

**Definition 10.60.** The function $f$ is *uniformly continuous* on the interval $I$ if for every $\varepsilon > 0$, there exists a (shared, independent of the position) $\delta > 0$ such that

$$\text{if} \quad x_0, x_1 \in I \quad \text{and} \quad |x_1 - x_0| < \delta, \quad \text{then} \quad |f(x_1) - f(x_0)| < \varepsilon. \tag{10.18}$$

We can define uniform continuity on an arbitrary set $A \subset \mathbb{R}$ similarly: in the above definition, write $A$ in place of $I$ everywhere.

**Theorem 10.61 (Heine's Theorem[4]).** *If $f \in C[a,b]$, then $f$ is uniformly continuous in $[a,b]$.*

*Proof.* We prove the statement by contradiction. Suppose that $f$ is not uniformly continuous in $[a,b]$. This means that there exists an $\varepsilon_0 > 0$ for which there does not exist a $\delta > 0$ such that (10.18) holds. Then (10.18) does not hold with the choice

[4] Heinrich Eduard Heine (1821–1881), German mathematician.

$\delta = 1/n$ either, that is, for every $n$, there exist $\alpha_n \in [a,b]$ and $\beta_n \in [a,b]$ for which

$$|\alpha_n - \beta_n| < \frac{1}{n}, \tag{10.19}$$

but at the same time,

$$|f(\alpha_n) - f(\beta_n)| \geq \varepsilon_0. \tag{10.20}$$

Since $\alpha_n \in [a,b]$, there exists a convergent subsequence $(\alpha_{n_k})$ whose limit, $\alpha$, is also in $[a,b]$. Now by (10.19),

$$\beta_{n_k} = \left(\beta_{n_k} - \alpha_{n_k}\right) + \alpha_{n_k} \to 0 + \alpha = \alpha.$$

Since $f$ is continuous on $[a,b]$, it is continuous at $\alpha$ (restricted to $[a,b]$). Thus by the transference principle, $f\left(\alpha_{n_k}\right) \to f(\alpha)$ and $f\left(\beta_{n_k}\right) \to f(\alpha)$, so

$$\lim_{k\to\infty} \left(f\left(\alpha_{n_k}\right) - f\left(\beta_{n_k}\right)\right) = 0.$$

This, however, contradicts (10.20). □

*Remark 10.62.* In Theorem 10.61, both the boundedness and the closedness of the interval $[a,b]$ are necessary. For example, the function $f(x) = 1/x$ is continuous on $(0,1)$, but it is not uniformly continuous there. This shows that the closedness assumption cannot be dropped. The function $f(x) = x^2$ is continuous on $(-\infty,\infty)$, but it is not uniformly continuous there. This shows that the boundedness assumption cannot be dropped either.

Later, we will see that uniform continuity is a very useful property, and we often need to determine whether a function is uniformly continuous on a set $A$. If $A$ is a closed and bounded interval, then our job is easy: by Heine's theorem, the function is uniformly continuous on $A$ if and only if it is continuous at every point of $A$. If, however, $A$ is an interval that is neither bounded nor closed (perhaps $A$ is not even an interval), then Heine's theorem does not help. This is why it is important for us to know that there is a simple property that is easy to check that implies uniform continuity.

**Definition 10.63.** The function $f$ is said to have the *Lipschitz*[5] *property* (is Lipschitz, for short) on the set $A$ if there exists a constant $K \geq 0$ such that

$$|f(x_1) - f(x_0)| \leq K \cdot |x_1 - x_0| \tag{10.21}$$

for all $x_0, x_1 \in A$.

**Theorem 10.64.** *If $f$ is Lipschitz on the set $A$, then $f$ is uniformly continuous on $A$.*

[5] Rudolph Otto Sigismund Lipschitz (1832–1903), German mathematician.

*Proof.* If (10.21) holds for all $x_0, x_1 \in A$, then $x_0, x_1 \in A$ and $|x_1 - x_0| < \varepsilon/K$ imply

$$|f(x_1) - f(x_0)| \le K \cdot |x_1 - x_0| < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

□

*Remark 10.65.* The converse is generally not true: uniform continuity does not generally imply the Lipschitz property. (That is, the Lipschitz property is stronger than uniform continuity.) So for example, the function $\sqrt{x}$ is *not* Lipschitz on the interval $[0,1]$. Indeed, for every constant $K > 0$, if $x_0 = 0$ and $0 < x_1 < \min(1, 1/K^2)$, then $x_1 > K^2 \cdot x_1^2$, and so

$$|\sqrt{x_1} - \sqrt{x_0}| = \sqrt{x_1} > K \cdot x_1 = K \cdot |x_1 - x_0|.$$

On the other hand, $\sqrt{x}$ is uniformly continuous on $[0,1]$, since it is continuous there.

## *Exercises*

**10.62.** The functions given below are uniformly continuous on the given intervals by Heine's theorem. For all $\varepsilon > 0$, give a $\delta$ that satisfies the definition of uniform continuity.

(a) $x^2$ on $[0,1]$; (b) $x^3$ on $[-2,2]$; (c) $\sqrt{x}$ on $[0,1]$.

**10.63.** Prove that (a) $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$; and (b) $f(x) = 1/x^2$ is not uniformly continuous on $(0,1)$, but is uniformly continuous on $[1,+\infty)$.

**10.64.** Prove that if $f$ is continuous on $\mathbb{R}$ and

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0,$$

then $f$ is uniformly continuous on $\mathbb{R}$.

**10.65.** Prove that if $f$ is uniformly continuous on a bounded set $A$, then $f$ is bounded on $A$. Does this statement still hold if we do not assume that $A$ is bounded?

**10.66.** Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are uniformly continuous on $\mathbb{R}$, then $f + g$ is also uniformly continuous on $\mathbb{R}$.

**10.67.** Is it true that if $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are uniformly continuous on $\mathbb{R}$, then $f \cdot g$ is also uniformly continuous on $\mathbb{R}$?

**10.68.** Prove that if $f$ is continuous on $[a,b]$, then for every $\varepsilon > 0$, we can find a piecewise linear function $\ell(x)$ in $[a,b]$ such that $|f(x) - \ell(x)| < \varepsilon$ for all $x \in [a,b]$ (that is, the graph of $f$ can be approximated to within less than $\varepsilon$ by a piecewise linear function).

(The function $\ell(x)$ is a piecewise linear function on $[a,b]$ if the interval $[a,b]$ can be subdivided with points $a_0 = a < a_1 < \cdots < a_{n-1} < a_n = b$ into subintervals $[a_{k-1}, a_k]$ on which $\ell(x)$ is linear, that is, $\ell(x) = c_k x + d_k$ if $x \in [a_{k-1}, a_k]$ and $k = 1, \ldots, n$.)

**10.69.** Prove that the function $x^k$ is Lipschitz on every bounded set (where $k$ is an arbitrary positive integer).

**10.70.** Prove that the function $\sqrt{x}$ is Lipschitz on the interval $[a,b]$ for all $0 < a < b$.

**10.71.** Suppose that $f$ and $g$ are Lipschitz on $A$. Prove that then

(i) $f+g$ and $c \cdot f$ are Lipschitz on the set $A$ for all $c \in \mathbb{R}$; and
(ii) if the set $A$ is bounded, then $f \cdot g$ is also Lipschitz on $A$. (H)

**10.72.** Give an example for Lipschitz functions $f$, $g\colon \mathbb{R} \to \mathbb{R}$ for which $f \cdot g$ is not Lipschitz.

**10.73.** Suppose that $f$ is Lipschitz on the closed and bounded interval $[a,b]$. Prove that if $f$ is nowhere zero, then $1/f$ is also Lipschitz on $[a,b]$.

**10.74.** Suppose that the function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $|f(x_1) - f(x_2)| \le |x_1 - x_2|^2$ for all $x_1, x_2 \in \mathbb{R}$. Prove that then $f$ is constant.

## 10.6 Monotonicity and Continuity

Let $f$ be defined on a punctured neighborhood of $a$. The function $f$ is continuous at $a$ if and only if all of the following conditions hold:

(i) $\lim_{x\to a} f(x)$ exists,
(ii) $a \in D(f)$,
(iii) $\lim_{x\to a} f(x) = f(a)$.

If any one of these three conditions does not hold, then the function is not continuous at $a$; we then say that $f$ has a **point of discontinuity** at $a$. We classify points of discontinuity as follows.

**Definition 10.66.** Let $f$ be defined on a punctured neighborhood of $a$, and suppose that $f$ is not continuous at $a$. If $\lim_{x\to a} f(x)$ exists and is finite, but $a \notin D(f)$ or $f(a) \neq \lim_{x\to a} f(x)$, then we say that $a$ is a *removable discontinuity* of $f$.[6]

If $\lim_{x\to a} f(x)$ does not exist, but the finite limits

$$\lim_{x\to a+0} f(x) = f(a+0) \quad \text{and} \quad \lim_{x\to a-0} f(x) = f(a-0)$$

both exist (and then are necessarily different), then we say that $f$ has a *jump discontinuity* at $a$. We call removable discontinuities and jump discontinuities *discontinuities of the first type* collectively.

In all other cases, we say that $f$ *has a discontinuity of the second type at a.*

[6] Since then, by setting $f(a) = \lim_{x\to a} f(x)$, $f$ can be made continuous at $a$.

*Examples 10.67.* **1.** The functions $\{x\}$ and $[x]$ have jump discontinuities at every positive integer value. Similarly, the function $\operatorname{sgn} x$ has a jump discontinuity at 0.
**2.** The Riemann function (function 3. in Example 10.7) has a removable discontinuity at every rational point.
**3.** The Dirichlet function (function (9) in Example 9.7) has discontinuities of the second type at every point.

Below, we will show that the points of discontinuity of a monotone function are of the first type, and these can only be jump discontinuities. This is equivalent to saying that a monotone function possesses both one-sided limits at every point.

**Theorem 10.68.** *Let $f$ be monotone increasing on the finite or infinite open interval $(a,b)$. Then*

(i) *for every $x_0 \in (a,b)$, the finite limits $f(x_0-0)$ and $f(x_0+0)$ exist, and*

$$f(x_0-0) \le f(x_0) \le f(x_0+0).$$

(ii) *If $f$ is bounded from above on $(a,b)$, then the finite limit $f(b-0)$ exists. If $f$ is bounded from below on $(a,b)$, then the finite limit $f(a+0)$ exists.*
(iii) *If $f$ is not bounded from above on $(a,b)$, then $f(b-0)=\infty$; if $f$ is not bounded from below on $(a,b)$, then $f(a+0)=-\infty$.*

A similar statement can be formulated for monotone decreasing functions, as well as for intervals that are unbounded. We give two proofs of the theorem.

*Proof I.* (i) Since $f(x) \le f(x_0)$ for all $x \in (a,x_0)$, the set $f((a,x_0))$ is bounded from above, and $f(x_0)$ is an upper bound. Let $\alpha = \sup f((a,x_0))$; then $\alpha \le f(x_0)$.

Let $\varepsilon > 0$ be fixed. Since $\alpha$ is the least upper bound of the set $f((a,x_0))$, $\alpha-\varepsilon$ cannot be an upper bound. Thus there exists $x_\varepsilon \in (a,x_0)$ for which $\alpha-\varepsilon < f(x_\varepsilon)$. Now by the monotonicity of $f$ and the definition of $\alpha$,

$$\alpha-\varepsilon < f(x_\varepsilon) \le f(x) \le \alpha$$

if $a < x_\varepsilon < x < x_0$, which clearly shows that $\lim_{x\to x_0-0} f(x) = \alpha$. Thus we saw that $f(x_0-0)$ exists and is finite, as well as that $f(x_0-0) \le f(x_0)$. The argument is similar for $f(x_0+0) \ge f(x_0)$.

Statements (ii) and (iii) can be proven similarly; in the first statement of (ii), $\sup f((a,b))$ takes on the role of $f(x_0)$. $\square$

*Proof II.* We will go into detail only in proving (i). By Theorem 10.22, it suffices to show that for every sequence $x_n \nearrow x_0$, $(f(x_n))$ is convergent, and its limit is at most $f(x_0)$. By the monotonicity of $f$, if $x_n \nearrow x_0$, then $(f(x_n))$ is monotone increasing; it has a (finite or infinite) limit. Since also $f(x_n) \le f(x_0)$ for all $n$, $\lim_{n\to\infty} f(x_n) \le f(x_0)$. $\square$

**Corollary 10.69.** *If $f$ is monotone on $(a,b)$, then at every point $x_0 \in (a,b)$, it either is continuous or has a jump discontinuity: a monotone function on $(a,b)$ can have only jump discontinuities.*

We now show that discontinuities of a monotone function are limited not only by type, but by quantity.

**Theorem 10.70.** *If $f$ is monotone on the open interval $I$, then it can have at most countably many discontinuities on $I$.*

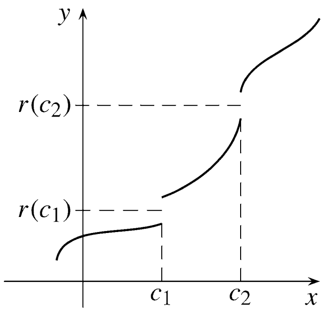


Fig. 10.14

*Proof.* Without loss of generality, we may assume that $f$ is monotone increasing on $I$. If $f$ is not continuous at a $c \in I$, then $f(c-0) < f(c+0)$. Let $r(c)$ be a rational number for which $f(c-0) < r(c) < f(c+0)$. If $c_1 < c_2$, then by the monotonicity of $f$, $f(c_1+0) \le f(c_2-0)$. Thus if $f$ has both $c_1$ and $c_2$ as points of discontinuity, then $r(c_1) < r(c_2)$ (Figure 10.14).

This means that we have created a one-to-one correspondence between the points of discontinuity and a subset of the rational numbers. Since the set of rational numbers is countable, $f$ can have only a countable number of discontinuities. □

*Remark 10.71.* Given an arbitrary countable set of numbers $A$, we can construct a function $f$ that is monotone on $(-\infty,\infty)$ and whose set of points of discontinuity is exactly $A$ (see Exercise 10.76). So for example, we can construct a monotone increasing function on $(-\infty,\infty)$ that is continuous at every irrational point and discontinuous at every rational point.

In Theorem 10.45, we saw that if $f$ is strictly monotone in the interval $I$, then its inverse is continuous on the interval $f(I)$. If the function $f$ is also continuous, then we can expand on this in the following way.

**Theorem 10.72.** *Let $f$ be strictly increasing and continuous on the interval $I$. Then*

(i) *$f(I)$ is also an interval; namely,*
*if $I=[a,b]$, then $f(I)=[f(a),f(b)]$;*
*if $I=[a,b)$, where $b$ is finite or infinite, then $f(I)=[f(a),\sup f(I))$;*
*if $I=(a,b]$, where $a$ is finite or infinite, then $f(I)=(\inf f(I),f(b)]$;*
*if $I=(a,b)$, where each of $a$ and $b$ is either finite or infinite, then $f(I)=(\inf f(I),\sup f(I))$.*

(ii) *The inverse of $f$, $f^{-1}$, is strictly monotone increasing and continuous on the interval $f(I)$ restricted to $f(I)$.*

*A similar statement can be made for strictly monotone decreasing and continuous functions.*

*Proof.* We need only prove (i). If $I=[a,b]$, then $f(I)=[f(a),f(b)]$ is clear from the Bolzano–Darboux theorem (Figure 10.15).

Next suppose that $I=[a,b)$. It is clear that then, $f(I)\subset[f(a),\sup f(I)]$. If $f(a)\le c<\sup f(I)$, then let us choose a point $u\in I$ for which $c<f(u)$. By the

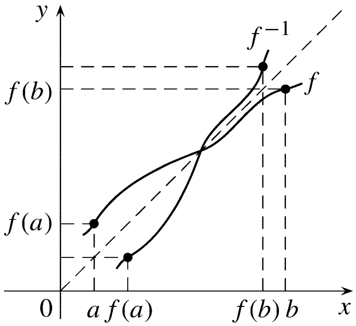


Fig. 10.15

Bolzano–Darboux theorem, $f$ takes on every value between $f(a)$ and $f(u)$ on the interval $[a,u]$, so $c \in f([a,u]) \subset f(I)$.

This shows that $[f(a), \sup f(I)) \subset f(I)$. To prove that $f(I) = [f(a), \sup f(I))$, we just need to show that $\sup f(I) \notin f(I)$. Indeed, if $c \in f(I)$ and $c = f(u)$, where $u \in I$, then $u < v \in I$ implies $c = f(u) < f(v) \leq \sup f(I)$, so $c \neq \sup f(I)$.

The rest of the statements can be proved similarly. □

*Remark 10.73.* By the previous theorem, the inverse of a function $f$ defined on an interval $[a,b]$ exists and is also defined on a closed and bounded interval if the function $f$ is strictly monotone and continuous. This condition is far from necessary, as the following example illustrates (Figure 10.16).

Let $f$ be defined for $x \in [0,1]$ as

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 1-x, & \text{if } x \text{ is irrational.} \end{cases}$$

It is easy to see that in $[0,1]$, $f$

*a)* is not monotone on any subinterval,
*b)* is nowhere continuous except for the point $x = 1/2$; yet
*c)* the inverse of $f$ exists.

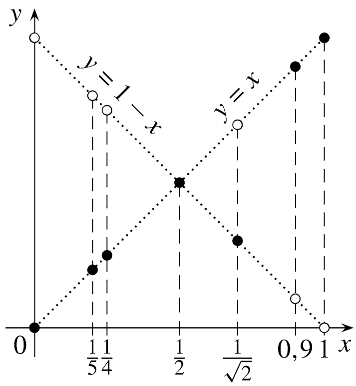


Fig. 10.16

Moreover, $f([0,1]) = [0,1]$, so $f$ is a one-to-one correspondence from $[0,1]$ to itself that is nowhere monotone and is continuous nowhere except at one point.

We see, however, that if $f$ is continuous on an interval, then strict monotonicity of $f$ is necessary and sufficient for the inverse function to exist (see Exercise 10.54).

## Exercises

**10.75.** Give a function $f\colon [0,1] \to [0,1]$ that is monotone and has infinitely many jump discontinuities.

**10.76.** Prove that for every countable set $A \subset \mathbb{R}$, there exists a monotone increasing function $f : \mathbb{R} \to \mathbb{R}$ that jumps at every point of $A$ but is continuous at every point of $\mathbb{R} \setminus A$. (H)

**10.77.** Let $f$ be defined on a neighborhood of $a$, and let

$$m(h) = \inf\{f(x) : x \in [a-h, a+h]\}, \quad M(h) = \sup\{f(x) : x \in [a-h, a+h]\}$$

for all $h > 0$. Prove that the limits $\lim_{h\to 0+0} M(h) = M$ and $\lim_{h\to 0+0} m(h) = m$ exist, and moreover, that $f$ is continuous at $a$ if and only if $m = M$.

**10.78.** Can $f$ have an inverse function in $[-1,1]$ if $f([-1,1]) = [-1,1]$ and $f$ has exactly two points of discontinuity in $[-1,1]$?

**10.79.** Construct a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at every point different from zero and has a discontinuity of the second type at zero.

**10.80.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $f(x-0) \le f(x) \le f(x+0)$ for all $x$. Is it true that $f$ is monotone increasing? (H)

**10.81.** Prove that the set of discontinuities of the first type of every function $f\colon \mathbb{R} \to \mathbb{R}$ is countable. (H)

**10.82.** Prove that if there is a discontinuity of the first type at every rational point of the function $f\colon \mathbb{R} \to \mathbb{R}$, then there is an irrational point where it is continuous. (H)

## 10.7 Convexity and Continuity

Our first goal is to prove that a convex function in an open interval is necessarily continuous. As we will see, this follows from the fact that if $f$ is convex, then every point $c$ has a neighborhood in which $f$ can be surrounded by two continuous (linear) functions that share the value $f(c)$ at $c$. To see this, we first prove a helping theorem. We recall that (for a given $f$) the linear function that agrees with $f$ at $a$ and $b$ is $h_{a,b}$, that is,

$$h_{a,b}(x) = \frac{f(b) - f(a)}{b-a} \cdot (x-a) + f(a).$$

**Lemma 10.74.** *Let $f$ be convex on the interval $I$. If $a$, $b \in I$, $a < b$, and $x \in I \setminus [a,b]$, then*

$$f(x) \ge h_{a,b}(x). \tag{10.22}$$

*If $f$ is strictly convex on $I$, then strict inequality holds in* (10.22). *(That is, outside the interval $[a,b]$, the points of the graph of $f$ lie above the line connecting the points $(a, f(a))$ and $(b, f(b))$; see Figure 9.2.)*

*Proof.* Suppose that $a < b < x$. By the definition of convexity, $f(b) \le h_{a,x}(b)$, that is,

$$f(b) \le \frac{f(x) - f(a)}{x - a} \cdot (b - a) + f(a),$$

which yields (10.22) after a simple rearrangement. If instead $x < a < b$, then $f(a) \le h_{x,b}(a)$, that is,

$$f(a) \le \frac{f(b) - f(x)}{b - x} \cdot (a - x) + f(x),$$

which yields (10.22) after a simple rearrangement.

If $f$ is strictly convex, then we can repeat the above argument using strict inequalities. □

Now we can easily prove the continuity of convex functions.

**Theorem 10.75.** *If $f$ is convex on the open interval $I$, then $f$ is continuous on $I$.*

*Proof.* Let $c \in I$ be fixed, and choose points $a, b \in I$ such that $a < c < b$. If $x \in (c, b)$, then by the above lemma and the convexity of $f$,

$$h_{a,c}(x) \le f(x) \le h_{c,b}(x).$$

Since $\lim_{x \to c} h_{a,c}(x) = \lim_{x \to c} h_{c,b}(x) = f(c)$, by the squeeze theorem we have $\lim_{x \to c+0} f(x) = f(c)$. We can similarly get that $\lim_{x \to c-0} f(x) = f(c)$. □

If $f$ is convex on the interval $I$, then for arbitrary $a, b \in I$,

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a) + f(b)}{2}. \tag{10.23}$$

Indeed, if $a = b$, then (10.23) is clear, while if $a < b$, (10.23) follows from the inequality $f(x) \le h_{a,b}(x)$ applied to $x = (a+b)/2$. The functions that satisfy (10.23) for all $a, b \in I$ are called **weakly convex** functions.[7] The function $f$ is **weakly concave** if $f((a+b)/2) \ge (f(a) + f(b))/2$ for all $a,b \in I$.

The condition for weak convexity—true to its name—is a weaker condition than convexity, that is, there exist functions that are weakly convex but not convex. One can show that there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ that is **additive** in the sense that $f(x+y) = f(x) + f(y)$ holds for all $x, y \in \mathbb{R}$, but is not continuous. (The proof of this fact, however, is beyond the scope of this book.) Now it is easy to see that such a function is weakly convex, and it actually satisfies the stronger condition $f((a+b)/2) = (f(a) + f(b))/2$ as well for all $a, b$. On the other hand, $f$ is not convex, since it is not continuous.

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[7] Weakly convex functions are often called **Jensen-convex** functions as well.

In the following theorem, we prove that if $f$ is continuous, then the weak convexity of $f$ is equivalent to the convexity of $f$. This means that in talking about continuous functions, to determine convexity it is enough to check the conditions for weak convexity, which is usually easier to do.

**Theorem 10.76.** *Suppose that $f$ is continuous and is weakly convex on the interval $I$. Then $f$ is convex on $I$.*

*Proof.* We have to show that if $a, x_0, b \in I$ and $a < x_0 < b$, then $f(x_0) \le h_{a,b}(x_0)$. Suppose that this is not true, that is, $f(x_0) > h_{a,b}(x_0)$. This means that the function $g(x) = f(x) - h_{a,b}(x)$ is positive at $x_0$. Since $g(a) = 0$, there exists a last point before $x_0$ where $g$ vanishes. To see this, let $A = \{x \in [a, x_0] : g(x) = 0\}$, and let $\alpha = \sup A$. Then $a \le \alpha \le x_0$. We show that $g(\alpha) = 0$. We can choose a sequence $x_n \in A$ that tends to $\alpha$, and so by the continuity of $g$, we have $g(x_n) \to g(\alpha)$. Since $g(x_n) = 0$ for all $n$, we must have $g(\alpha) = 0$. It follows that $\alpha < x_0$, and the function $g$ is positive on the interval $(\alpha, x_0]$: if there were a point $\alpha < x_1 \le x_0$ such that $g(x_1) \le 0$, then by the Bolzano–Darboux theorem, $g$ would have a root in $[x_1, x_0]$, which contradicts the fact that $\alpha$ is the supremum of the set $A$.

By the exact same argument, there is a first point $\beta$ after $x_0$ where $g$ vanishes, and so the function $g$ is positive in the interval $[x_0, \beta)$. Then $g(\alpha) = g(\beta) = 0$, and $g(x) > 0$ for all $x \in (\alpha, \beta)$. Now we got $g$ by subtracting a linear function $\ell$ from $f$. It follows that $g$ is also weakly convex; since $\ell$ is linear, $\ell((a+b)/2) = (\ell(a) + \ell(b))/2$ for all $a, b$, so if $f$ satisfies inequality (10.23), then subtracting $\ell$ does not change this. However, $g(\alpha) = g(\beta) = 0$ and $g((\alpha+\beta)/2) > 0$, so (10.23) is not satisfied with the choices $a = \alpha$ and $b = \beta$. This is a contradiction, which shows that $f$ is convex. □

*Remark 10.77.* If the function $f : I \to \mathbb{R}$ satisfies the condition

$$f\left(\frac{a+b}{2}\right) < \frac{f(a)+f(b)}{2} \tag{10.24}$$

for all $a, b \in I$, $a \neq b$, then we call $f$ **strictly weakly convex**. We can similarly define **strictly weakly concave** functions. By the previous theorem, it follows that *if $f$ is continuous and strictly weakly convex on the interval $I$, then $f$ is strictly convex on $I$.* Indeed, it is easy to see that if $f$ is convex but not strictly convex on the interval $I$, then there is a subinterval $J$ on which $f$ is linear (see Exercise 10.83). Then, however, (10.24) does not hold for the points of $J$, since if $a, b \in J$, then equality holds in (10.24).

We can similarly see that every continuous and strictly weakly concave function is strictly concave.

We mention that the conditions of Theorem 10.76 can be greatly weakened: instead of assuming the continuity of $f$, it suffices to assume that $I$ has a subinterval on which $f$ is bounded from above (see Exercises 10.99–10.102).

## *Exercises*

**10.83.** Prove that if $f$ is convex but not strictly convex on the interval $I$, then $I$ has a subinterval on which $f$ is linear.

**10.84.** Let us call a function $f\colon I \to \mathbb{R}$ **barely convex** if whenever $a, b, c \in I$ and $a < b < c$, then $f(b) \le \max(f(a), f(c))$. Prove that if $f\colon I \to \mathbb{R}$ is convex on the interval $I$, then $f$ is barely convex.

**10.85.** Let $f$ be barely convex on the interval $(a,b)$, and suppose that $a < c < d < b$ and $f(c) > f(d)$. Show that $f$ is monotone decreasing on $(a,c]$. Similarly, show that if $a < c < d < b$ and $f(c) < f(d)$, then $f$ is monotone increasing on $[d,b)$.

**10.86.** Prove that the function $f\colon I \to \mathbb{R}$ is barely convex on the interval $(a,b)$ if and only if one of the following cases applies:

(a) $f$ is monotone decreasing on $(a,b)$.
(b) $f$ is monotone increasing on $(a,b)$.
(c) There exists a point $c \in (a,b)$ such that $f$ is monotone decreasing on $(a,c)$, monotone increasing on $(c,b)$, and $f(c) \le \max(f(c-0), f(c+0))$.

**10.87.** Prove that if $f\colon I \to \mathbb{R}$ is convex on the interval $(a,b)$, then one of the following cases applies:

(a) $f$ is monotone decreasing on $(a,b)$.
(b) $f$ is monotone increasing on $(a,b)$.
(c) There exists a point $c \in (a,b)$ such that $f$ is monotone decreasing on $(a,c]$ and monotone increasing on $[c,b)$.

**10.88.** Let $f$ be convex on $(-\infty,\infty)$, and suppose that $\lim_{x\to-\infty} f(x) = \infty$. Is it possible that $\lim_{x\to\infty} f(x) = -\infty$? (S)

**10.89.** Let $f$ be convex on $(-\infty,\infty)$, and suppose that $\lim_{x\to-\infty} f(x) = 0$. Is it possible that $\lim_{x\to\infty} f(x) = -\infty$? (H)

**10.90.** Let $f$ be convex on $(0,1)$. Is it possible that $\lim_{x\to 1-0} f(x) = -\infty$? (H)

**10.91.** Let $f$ be weakly convex on the interval $I$. Prove that

$$f\left(\frac{x_1+\cdots+x_n}{n}\right) \le \frac{f(x_1)+\cdots+f(x_n)}{n}$$

for all $x_1,\ldots,x_n \in I$. (S)

**10.92.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be an additive function (that is, suppose that for all $x, y$, $f(x+y) = f(x) + f(y)$). Prove that $f(rx) = r \cdot f(x)$ for every real number $x$ and rational number $r$.

**10.93.** Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is additive, then the function $g(x) = f(x) - f(1)\cdot x$ is also additive and periodic, namely that every rational number is a period.

**10.94.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be an additive function. Prove that if $f$ is bounded from above on an interval, then $f(x) = f(1) \cdot x$ for all $x$. (H)

**10.95.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be an additive function. Prove that $f^2$ is weakly convex. (If $f$ is not a linear function, then $f^2$ is a weakly convex function that is bounded from below, but is not convex.)

**10.96.** Let $f$ be continuous on the interval $I$, and suppose that for all $a$, $b \in I$, $a < b$, there exists a point $a < x < b$ such that $f(x) \le h_{a,b}(x)$. Prove that $f$ is convex. (H)

**10.97.** Let $f$ be bounded on the interval $I$, and suppose that for all $a$, $b \in I$, $a < b$, there exists a point $a < x < b$ such that $f(x) \le h_{a,b}(x)$. Does it then follow that $f$ is convex?

**10.98.** Let $f$ be convex on the open interval $I$. Prove that $f$ is Lipschitz on every closed and bounded subinterval of $I$.

The following four questions will take us through the proof that if $f$ is weakly convex on an open interval $I$, and $I$ has a subinterval in which $f$ is bounded from above, then $f$ is convex.

**10.99.** Let $f$ be weakly convex on the open interval $I$, and let $x_0 \in I$. Prove that if $f$ is bounded from above on $(x_0 - \delta, x_0 + \delta)$, then $f$ is bounded on $(x_0 - \delta, x_0 + \delta)$. (S)

**10.100.** Let $f$ be weakly convex on the open interval $I$. Let $n \ge 1$ be an integer, and let $x$ and $h$ be numbers such that $x \in I$ and $x + 2^n h \in I$. Prove that

$$f(x+h) - f(x) \le \frac{1}{2^n} \cdot [f(x+2^n h) - f(x)]. \qquad \text{(S)}$$

**10.101.** Let $f$ be weakly convex on the open interval $I$, and let $x_0 \in I$. Prove that if $f$ is bounded from above on $(x_0 - \delta, x_0 + \delta)$, then $f$ is continuous at $x_0$. (S)

**10.102.** Let $f$ be weakly convex on the interval $I$, and suppose that $I$ contains a nondegenerate subinterval on which $f$ is bounded from above. Prove that $f$ is continuous (and so by Theorem 10.76, convex) on $I$. (H)

## 10.8 Arc Lengths of Graphs of Functions

One of the key objectives of analysis is the measurement of lengths, areas, and volumes. Our next goal is to deal with a special case: the notion of the arc length of the graph of a function[8].

We denote the line segment connecting the points $p, q \in \mathbb{R}^2$ by $[p,q]$, that is, $[p,q] = \{p + t(q-p) : t \in [0,1]\}$. The length of the line segment $[p,q]$ (by definition) is the distance between its endpoints, which is $|q-p|$. We call a set that is

[8] We will have need of this in defining trigonometric functions. We return to dealing with arc lengths of more general curves in Chapter 16.

a union of connected line segments a **broken line** (or a polygonal path). Thus a broken line is of the form $[p_0, p_1] \cup [p_1, p_2] \cup \ldots \cup [p_{n-1}, p_n]$, where $p_0, \ldots, p_n$ are points of the plane. The length of the broken line is the sum of the lengths of the lines that it comprises, that is, $|p_1 - p_0| + |p_2 - p_1| + \cdots + |p_n - p_{n-1}|$.

Since "the shortest distance between two points is a straight line", the length of a curve (no matter how we define it) should not be smaller than the distance between its endpoints. If we inscribe a broken line $[p_0, p_1] \cup [p_1, p_2] \cup \ldots \cup [p_{n-1}, p_n]$ "on top of" a curve, then the part of the arc connecting $p_{i-1}$ and $p_i$ has length at least $|p_i - p_{i-1}|$, and so the length of the whole curve needs to be at least $|p_1 - p_0| + |p_2 - p_1| + \cdots + |p_n - p_{n-1}|$. On the other hand—again just using intuition—we can expect a "very fine" broken line inscribed on the curve to "approximate" it well enough so that the two lengths will be close. What we can take away from this is that the arc length should be equal to the supremum of the lengths of the broken lines on the curve. This finding is what we will accept as the definition. We remind ourselves that we denote the graph of the function $f\colon [a,b] \to \mathbb{R}$ by graph $f$.

**Definition 10.78.** Let $f\colon [a,b] \to \mathbb{R}$ be an arbitrary function and let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition $F$ of the interval $[a,b]$. The *inscribed polygonal path* over the partition $F$ of $f$ is the broken line over the points

$$(x_0, f(x_0)), \ldots, (x_n, f(x_n)).$$

The *arc length* of graph $f$ is the least upper bound of the set of lengths of all inscribed polygonal paths on $f$. (The arc length can be infinite). We denote the arc length of the graph of $f$ by $s(f;[a,b])$. Thus

$$s(f;[a,b]) = \sup \Bigg\{ \sum_{i=1}^{n} |p_i - p_{i-1}| : a = x_0 < x_1 < \cdots < x_n = b,$$
$$p_i = (x_i, f(x_i))\ (i = 0, \ldots, n) \Bigg\}.$$

We say that graph $f$ is *rectifiable* if $s(f;[a,b])$ is finite.

Let us note that if $a = b$, then $s(f;[a,b]) = 0$ for all functions $f$.

**Theorem 10.79.**

(i) *For an arbitrary function $f\colon [a,b] \to \mathbb{R}$,*

$$\sqrt{(b-a)^2 + (f(b) - f(a))^2} \le s(f;[a,b]), \tag{10.25}$$

*and so if $a < b$, then $s(f;[a,b]) > 0$.*

(ii) *If $f\colon [a,b] \to \mathbb{R}$ is monotone, then* graph $f$ *is rectifiable, and*

$$s(f;[a,b]) \le (b-a) + |f(b) - f(a)|. \tag{10.26}$$

*Proof.* It is clear that $s(f;[a,b])$ is not smaller than any of its inscribed polygonal paths. Now the segment connecting $(a, f(a))$ and $(b, f(b))$ is such a path that corresponds to the partition $a = x_0 < x_1 = b$. Since the length of this is $\sqrt{(b-a)^2 + (f(a) - f(b))^2}$, (10.25) holds.

Now suppose that $f$ is monotone increasing, let $F: a = x_0 < x_1 < \cdots < x_n = b$ be a partition of the interval $[a,b]$, and denote the point $(x_i, f(x_i))$ by $p_i$ for all $i = 0, \ldots, n$. Then, using the monotonicity of $f$,

$$\begin{aligned}|p_i - p_{i-1}| = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \le \\ \le (x_i - x_{i-1}) + (f(x_i) - f(x_{i-1}))\end{aligned}$$

for all $i$, so

$$\begin{aligned}\sum_{i=1}^{n} |p_i - p_{i-1}| \le \left[\sum_{i=1}^{n} (x_i - x_{i-1})\right] + \left[\sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\right] = \\ = (x_n - x_0) + (f(x_n) - f(x_0)) = (b-a) + (f(b) - f(a)).\end{aligned}$$

Since the partition was arbitrary, we have established (10.26). If $f$ is monotone decreasing, then we can argue similarly, or we can reduce the statement to one about monotone increasing functions by considering the function $-f$. □

*Remark 10.80.* Since not every monotone function is continuous, by (ii) of the previous theorem, there exist functions that are not everywhere continuous but whose graphs are rectifiable. Thus rectifiability is a more general concept then what the words “arc length” intuitively suggest.

The statement of the following theorem can also be expressed by saying that *arc lengths are additive*.

**Theorem 10.81.** *Let $a < b < c$ and $f: [a,c] \to \mathbb{R}$. If* graph $f$ *is rectifiable, then*

$$s(f;[a,c]) = s(f;[a,b]) + s(f;[b,c]). \tag{10.27}$$

We give a proof for this theorem in the appendix of the chapter.

We will need the following simple geometric fact soon.

**Lemma 10.82.** *If $A, B$ are convex polygons, and $A \subset B$, then the perimeter of $A$ is not larger than the perimeter of $B$.*

*Proof.* If we cut off part of the polygon $B$ at a line given by an edge of the polygon $A$, then we get a polygon $B_1$ with perimeter not larger than $B$ but still containing $A$. Repeating the process, we get the sequence $B, B_1, \ldots, B_n = A$, in which the perimeter of each polygon is at most as big as the one before it. □

**Arc length of a circle.** Let $C$ denote the unit circle centered at the origin. Let the part of $C$ falling into the upper half of the plane $\{(x,y) : y \ge 0\}$ be denoted by $C^+$. It is clear that $C^+$ agrees with the graph of the function $c(x) = \sqrt{1-x^2}$ defined on the interval $[-1,1]$. Since $c$ is monotone on both the intervals $[-1,0]$ and $[0,1]$, by the above theorems, the graph of $c$ is rectifiable.

*We denote the arc length of* graph $c$ *(that is, of the half-circle* $C^+$*) by* $\pi$.

By the previous two theorems, we can extract the approximation $2\sqrt{2} \le \pi \le 4$, where the value $2\sqrt{2}$ is the length of the inscribed broken path corresponding to the partition $-1 = x_0 < 0 = x_1 < 1 = x_2$. Inscribing different broken lines into $C^+$ gives us different lower bounds for $\pi$, and with the help of these, we can approximate $\pi$ with arbitrary precision (at least in theory).

If we inscribe $C$ into an arbitrary convex polygon $P$, then by Lemma 10.82, every polygon inscribed in $C$ will have smaller or equal perimeter than the perimeter of $P$. Thus the supremum of the perimeters of the inscribed polygons, $2\pi$, cannot be larger than the perimeter of $P$.

The lower and upper approximations that we get this way can help us show that $\pi = 3.14159265\ldots$. The number $\pi$, like $e$, is irrational. One can also show that $\pi$, again like $e$, is transcendental, but the proof of that is beyond the scope of this book.

*Remark 10.83.* To define trigonometric functions, we will need the (seemingly trivial) fact that starting from the point $(0,1)$, we can measure arcs of any length on $C$. Consider the case $x \in [0,\pi]$. We have to show that there is a number $u \in [-1,1]$ such that $s(c;[u,1]) = x$. With the notation $S(u) = s(c;[u,1])$, this means that the function $S(u)$ takes on every value between 0 and $\pi$ on the interval $[-1,1]$ (Figure 10.17).

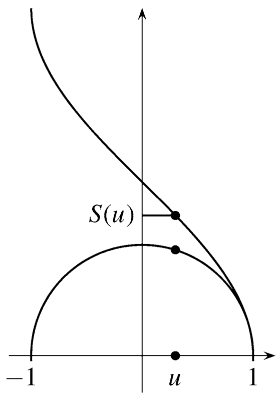


Fig. 10.17

**Theorem 10.84.** *The function* $S$ *is strictly monotone decreasing and continuous on* $[-1,1]$.

*Proof.* If $-1 \le u < v \le 1$, then by Theorem 10.81,

$$S(u) = s(c;[u,1]) = s(c;[u,v]) + s(c;[v,1]) = S(v) + s(c;[u,v]).$$

Since $s(c;[u,v]) > 0$, we know that $S$ is strictly monotone decreasing on $[-1,1]$.

Since the function $c$ is monotone both on $[-1,0]$ and on $[0,1]$, by (10.26),

$$s(c;[u,v]) \le (v-u) + |c(v) - c(u)|$$

if $-1 \le u < v \le 0$ or $0 \le u < v \le 1$.

Thus

$$|S(u) - S(v)| \le |v-u| + |c(v) - c(u)| \tag{10.28}$$

whenever $u, v \in [-1, 0]$ or $u, v \in [0, 1]$. Since the function $c(u) = \sqrt{1-u^2}$ is continuous on $[-1, 1]$, we have that

$$\lim_{v \to u}(|v-u| + |c(v) - c(u)|) = 0$$

for all $u \in [-1, 1]$, so by (10.28) we immediately have that $S$ is continuous on $[-1, 1]$. $\square$

By the previous theorem and by the Bolzano–Darboux theorem, the function $S(u)$ takes on every value between $S(-1)$ and $S(1)$, moreover exactly once. Since $S(-1) = \pi$ (since this was the definition of $\pi$) and $S(1) = 0$, we have seen that if $0 \leq x \leq \pi$, then we can measure out an arc of length $x$ onto the circle $C$. What about other lengths? Since the arc length of the semicircle is $\pi$, if we can measure one of length $x$, then we can measure one of length $x + \pi$ (or $x - \pi$) as well, in which case we just jump to the antipodal point.

## *Exercises*

**10.103.** Let $f\colon [a,b] \to \mathbb{R}$ be a function for which $s(f; [a,b]) = b - a$. Prove that $f$ is constant.

**10.104.** Prove that the function $f\colon [a,b] \to \mathbb{R}$ is linear (that is, it is of the form $mx+b$ with suitable constants $m$ and $b$) if and only if

$$s(f; [a,b]) = \sqrt{(b-a)^2 + (f(b) - f(a))^2}.$$

**10.105.** Prove that if the graph of $f\colon [a,b] \to \mathbb{R}$ is rectifiable, then $f$ is bounded on $[a,b]$.

**10.106.** Prove that if the graph of $f\colon [a,b] \to \mathbb{R}$ is rectifiable, then at every point $x \in [a,b)$, the right-hand limit of $f$ exists, and at every point $x \in (a,b]$, the left-hand limit exists.

**10.107.** Prove that neither the graph of the Dirichlet function nor the graph of the Riemann function over the interval $[0,1]$ is rectifiable.

**10.108.** Let the function $f\colon [0,1] \to \mathbb{R}$ be defined as follows: $f(x) = x$ if $x = 1/2^n$ $(n = 1, 2, \ldots)$, and $f(x) = 0$ otherwise. Prove that the graph of $f$ is rectifiable. What is its arc length?

**10.109.** Prove that if $f\colon [a,b] \to \mathbb{R}$ is Lipschitz, then its graph is rectifiable.

## 10.9 Appendix: Proof of Theorem 10.81

*Proof.* Let us denote by $S_1$, $S_2$, and $S_3$ the sets of the lengths of the inscribed polygonal paths of the intervals $[a,b]$, $[b,c]$, and $[a,c]$, respectively. Then $s(f;[a,b]) = \sup S_1$, $s(f;[b,c]) = \sup S_2$, and $s(f;[a,c]) = \sup S$ by the definition of arc length.

Since one partition of $[a,b]$ and one of $[b,c]$ together yield a partition of the interval $[a,c]$, the sum of any number in $S_1$ with any number in $S_2$ is in $S$. This means that $S \supset S_1 + S_2$. By Theorem 3.20, $\sup(S_1+S_2) = \sup S_1 + \sup S_2$, which implies

$$s(f;[a,c]) = \sup S \geq \sup(S_1+S_2) = \sup S_1 + \sup S_2 = s(f;[a,b]) + s(f;[b,c]).$$

Now we show that

$$s(f;[a,c]) \leq s(f;[a,b]) + s(f;[b,c]). \tag{10.29}$$

Let $F\colon a = x_0 < x_1 < \cdots < x_n = c$ be a partition of the interval $[a,c]$, and denote the point $(x_i, f(x_i))$ by $p_i$. Then the length of the inscribed polygonal path on $F$ is $h_F = \sum_{i=1}^{n} |p_i - p_{i-1}|$. If the point $b$ is equal to one of the points $x_i$, say to $x_k$, then $F_1 : a = x_0 < x_1 < \cdots < x_k = b$, and $F_2 : b = x_k < x_{k+1} < \cdots < x_n = c$ are partitions of the intervals $[a,b]$ and $[b,c]$ respectively, so

$$h_{F_1} = \sum_{i=1}^{k} |p_i - p_{i-1}| \leq s(f;[a,b]) \quad \text{and} \quad h_{F_2} = \sum_{i=k+1}^{n} |p_i - p_{i-1}| \leq s(f;[b,c]).$$

Since $h_F = h_{F_1} + h_{F_2}$, then $h_F \leq s(f;[a,b]) + s(f;[b,c])$. If the point $b$ is not equal to any of the points $x_i$ and $x_{k-1} < b < x_k$, then let $F_1 : a = x_0 < x_1 < \cdots < x_{k-1} < b$ and $F_2 : b < x_k < x_{k+1} < \cdots < x_n = c$. Let $q$ denote the point $(b, f(b))$. The lengths of the inscribed polygonal paths corresponding to partitions $F_1$ and $F_2$ are

$$h_{F_1} = \sum_{i=1}^{k-1} |p_i - p_{i-1}| + |q - p_{k-1}| \leq s(f;[a,b])$$

and

$$h_{F_2} = |p_k - q| + \sum_{i=k+1}^{n} |p_i - p_{i-1}| \leq s(f;[b,c]).$$

Now by the triangle inequality,

$$|p_k - p_{k-1}| \leq |q - p_{k-1}| + |p_k - q|,$$

so it follows that $h_F \leq h_{F_1} + h_{F_2} \leq s(f;[a,b]) + s(f;[b,c])$. Thus $h_F \leq s(f;[a,b]) + s(f;[b,c])$ for all partitions $F$, which makes (10.29) clear. □