Chapter 8
Countable Sets

While we were talking about sequences, we noted that care needs to be taken in distinguishing the sequence (a_n) from the set of its terms $\{a_n\}$. We will say that the sequence (a_n) **lists** the elements of H if $H = \{a_n\}$. (The elements of the set H and thus the terms of (a_n) can be arbitrary; we do not restrict ourselves to sequences of real numbers.) If there is a sequence that lists the elements of H, we say that H **can be listed** by that sequence. Clearly, every finite set can be listed, since if $H = \{a_1, \dots, a_k\}$, then the sequence $(a_1, \dots, a_k, a_k, a_k, \dots)$ satisfies $H = \{a_n\}$. Suppose now that H is infinite. We show that if there is a sequence (a_n) that lists the elements of H, then there is another sequence listing H whose terms are all distinct. Indeed, for each element x of H, we can choose a term a_n for which $a_n = x$. The terms chosen in this way (in the order of their indices) form (a_{n_k}), a subsequence of (a_n). If $c_k = a_{n_k}$ for all k, then the terms of the sequence (c_k) are distinct, and $H = \{c_k\}$.

Like every sequence, (c_k) is also a function defined on the set $\mathbb{N}^+$. Since its terms are distinct and $H = \{c_k\}$, this means that the map $k \mapsto c_k$ is a bijection between the sets $\mathbb{N}^+$ and H (see the footnote on p. 38).

We have shown that the elements of a set H can be listed if and only if H is finite or there is a bijection between H and $\mathbb{N}^+$.

Definition 8.1. We say the set H is *countably infinite*, if there exists a bijective map between $\mathbb{N}^+$ and H. The set H is *countable* if it is finite or countably infinite.

With our earlier notation, we can summarize our argument above by saying that a set can be listed if and only if it is countable.

It is clear that $\mathbb{N}$ is countable: consider the sequence $(0, 1, 2, \dots)$. It is also easy to see that $\mathbb{Z}$ is countable, since the sequence $(0, 1, -1, 2, -2, \dots)$ contains every integer. More surprising is the following theorem.

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Theorem 8.2. *The set of all rational numbers is countable.*

Proof. We have to convince ourselves that there is a sequence that lists the rational numbers. One example of such a sequence is

$$\begin{aligned}&\frac{0}{1},\ \frac{-1}{1},\ \frac{0}{2},\ \frac{1}{1},\ \frac{-2}{1},\ \frac{-1}{2},\ \frac{0}{3},\ \frac{1}{2},\ \frac{2}{1},\ \frac{-3}{1},\ \frac{-2}{2},\ \frac{-1}{3},\ \frac{0}{4},\ \frac{1}{3},\\ &\frac{2}{2},\ \frac{3}{1},\ \frac{-4}{1},\ \frac{-3}{2},\ \frac{-2}{3},\ \frac{-1}{4},\ \frac{0}{5},\ \frac{1}{4},\ \frac{2}{3},\ \frac{3}{2},\ \frac{4}{1},\ \frac{-5}{1},\ \frac{-4}{2},\ \ldots.\end{aligned} \tag{8.1}$$

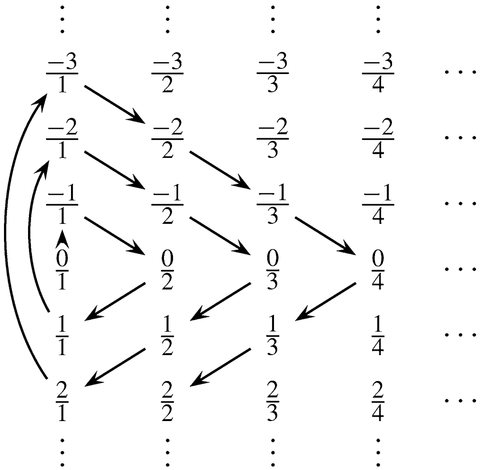

Fig. 8.1

Here fractions of the form p/q (where p,q are integers and $q>0$) are listed by the size of $|p|+q$. For all n, we list all fractions for which $|p|+q=n$ in some order, and then we attach the finite sequences we get for $n=1,2,\ldots$ to get an infinite sequence (Figure 8.1). It is clear that we have listed every rational number in this way. □

Using similar techniques, we can show that sets much "larger" than the set of rational numbers are also countable. We say that the complex number α is **algebraic** if it is the root of a not identically zero polynomial with integer coefficients. It is clear that every rational number is algebraic, since p/q is the root of the polynomial $qx-p$. The number $\sqrt{2}$ is also algebraic, since it is a root of x^2-2.

Theorem 8.3. *The set of algebraic numbers is countable.*

Proof. We define the *weight* of the polynomial $a_kx^k+\cdots+a_0$ to be the number $k+|a_k|+|a_{k-1}|+\cdots+|a_0|$. It is clear that for every n, there are only finitely many integer-coefficient polynomials whose weight is n. It then follows that there is a sequence $f_1,f_2,\ldots$, that contains every nonconstant polynomial. We can get this by first listing all the integer-coefficient nonconstant polynomials whose weight is 2, followed by all those whose weight is 3, and so on. We have then listed every integer-coefficient nonconstant polynomial, since the weight of such a polynomial cannot be 0 or 1. Each polynomial f_i can have only finitely many roots (see Lemma 11.1). List the roots of f_1 in some order, followed by the roots of f_2, and so on. The sequence we get lists every algebraic number, which proves our theorem. □

According to the following theorem, set-theoretic operations (union, intersection, complement) do not lead us out of the class of countable sets.

Theorem 8.4.

(i) *Every subset of a countable set is countable.*
(ii) *The union of two countable sets is countable.*

Proof. (i) Let A be countable, and $B \subset A$. Suppose that the sequence (a_n) lists the elements of A. For each element x of B, choose a term a_n for which $a_n = x$. The terms chosen in this way (in the order of their indices) form a subsequence of (a_n) that lists the elements of B.
(ii) If the sequences (a_n) and (b_n) list the elements of the sets A and B, then the sequence $(a_1, b_1, a_2, b_2, \ldots)$ lists the elements of $A \cup B$. □

An immediate consequence of statement (ii) above (by induction) is that the union of finitely many countable sets is also countable. By the following theorem, more is true.

Theorem 8.5. *The union of a countable number of countable sets is countable.*

Proof. Let $A_1, A_2, \ldots$ be countable sets, and for each k, let $\left(a_n^k\right)$ be a sequence that lists the elements of A_k. Then the sequence

$$\left(a_1^1, a_2^1, a_1^2, a_3^1, a_2^2, a_1^3, a_4^1, a_3^2, a_2^3, a_1^4, \ldots\right)$$

lists the elements of $\bigcup_{k=1}^{\infty} A_k$. We get the above sequence by writing all the finite sequences $\left(a_n^1, a_{n-1}^2, \ldots, a_2^{n-1}, a_1^n\right)$ one after another for each n. □

Based on the previous theorems, the question whether there are uncountable sets at all arises naturally. The following theorem gives an answer to this.

Theorem 8.6. *The set of real numbers is uncountable.*

Proof. Suppose that $\mathbb{R}$ is countable, and let (x_n) be a sequence of real numbers that contains every real number. We will work toward a contradiction by constructing a real number x that is not in the sequence. We outline two constructions.

I. The first construction is based on the following simple observation: if I is a closed interval and c is a given number, then I has a closed subinterval that does not contain c. This is clear: if we choose two disjoint closed subintervals, at least one will not contain c.

Let I_1 be a closed interval that does not contain x_1. Let I_2 be a closed subinterval of I_1 such that $x_2 \notin I_2$. Following this procedure, let I_n be a closed subinterval of

I_{n-1} such that $x_n \notin I_n$. According to Cantor's axiom, the intervals I_n have a shared point. If $x \in \bigcap_{n=1}^{\infty} I_n$, then $x \neq x_n$ for all n, since $x \in I_n$, but $x_n \notin I_n$. Thus x cannot be a term in the sequence, which is what we were trying to show.

II. A second construction for a similar x is the following: Consider the decimal expansion of $x_1, x_2, \ldots$:

$$x_1 = \pm n_1.a_1^1 a_2^1 \ldots$$
$$x_2 = \pm n_2.a_1^2 a_2^2 \ldots$$
$$\vdots$$

Let $x = 0.b_1 b_2 \ldots$, where $b_i = 5$ if $a_i^i \neq 5$, and $b_i = 4$ if $a_i^i = 5$. Clearly, x is different from each x_n. □

Theorems 8.4 and 8.6 imply that the set of irrational numbers is uncountable. If it were countable, then—since $\mathbb{Q}$ is also countable—$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ would also be countable, whereas it is not. We call a number **transcendental** if it is not algebraic. Repeating the above argument—using the fact that the set of algebraic numbers is countable—we get that *the set of transcendental numbers is uncountable.*

Definition 8.7. If there exists a bijection between two sets A and B, then we say that the two sets are *equivalent*, or that A and B have the same *cardinality*, and we denote this by $A \sim B$.

By the above definition, a set A is countably infinite if and only if $A \sim \mathbb{N}^+$. It can be seen immediately that if $A \sim B$ and $B \sim C$, then $A \sim C$; if f is a bijection from A to B and g is a bijection from B to C, then the map $x \mapsto g(f(x))$ $(x \in A)$ is a bijection from A to C.

Definition 8.8. We say that a set H has the *cardinality of the continuum* if it is equivalent to $\mathbb{R}$.

We show that both the set of irrational numbers and the set of transcendental numbers have the cardinality of the continuum. For this, we need the following simple lemma.

Lemma 8.9. *If A is infinite and B is countable, then $A \cup B \sim A$.*

Proof. First of all, we show that A contains a countably infinite subsequence. Since A is infinite, it is nonempty, and we can choose an $x_1 \in A$. If we have already chosen $x_1, \ldots, x_n \in A$, then $A \neq \{x_1, \ldots, x_n\}$ (since then A would be finite), so we can choose an element $x_{n+1} \in A \setminus \{x_1, \ldots, x_n\}$. Thus by induction, we have chosen distinct x_n for each n. Then $X = \{x_n \colon n = 1, 2, \ldots\}$ is a countably infinite subset of A.

To prove the lemma, we can suppose that $A \cap B = \emptyset$, since we can substitute B with $B \setminus A$ (which is also countable). By Theorem 8.4, $X \cup B$ is countable. Since it is also infinite, $X \cup B \sim \mathbb{N}^+$, and so $X \cup B \sim X$, since $\mathbb{N}^+ \sim X$. Let f be a bijection from X to $X \cup B$. Then

$$g(x) = \begin{cases} x, & \text{if } x \in A \setminus X, \\ f(x), & \text{if } x \in X \end{cases}$$

is a bijection from A to $A \cup B$. □

Theorem 8.10. *Both the set of irrational numbers and the set of transcendental numbers have the cardinality of the continuum.*

Proof. By the previous theorem, $\mathbb{R} \setminus \mathbb{Q} \sim (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$, so $\mathbb{R} \setminus \mathbb{Q}$ has the cardinality of the continuum. By a straightforward modification of the argument, we find that the set of transcendental numbers also has the cardinality of the continuum. □

Theorem 8.11. *Every nondegenerate interval has the cardinality of the continuum.*

Proof. The interval $(-1,1)$ has the cardinality of the continuum, since the map $f(x) = x/(1+|x|)$ is a bijection from $\mathbb{R}$ to $(-1,1)$. (The inverse of f is $f^{-1}(x) = x/(1-|x|)$ $(x \in (-1,1))$.) Since every open interval is equivalent to $(0,1)$ (the function $(b-a)x+a$ maps $(0,1)$ to (a,b)), every open interval has the cardinality of the continuum.

Moreover, Lemma 8.9 gives $[a,b] \sim (a,b)$, $(a,b] \sim (a,b)$ and $[a,b) \sim (a,b)$, so we get that every bounded nondegenerate interval has the cardinality of the continuum.

The proof for rays (or half-lines) having the cardinality of the continuum is left as an exercise for the reader. □

Exercises

8.1. Let a_n denote the nth term of sequence (8.1). What is the smallest n for which $a_n = -17/39$?

8.2. Prove that the set of finite sequences with integer terms is countable. (H)

8.3. Show that the set of finite-length English texts is countable.

8.4. Prove that every set of disjoint intervals is countable. (H)

8.5. Prove that a set is infinite if and only if it is equivalent to a proper subset of itself.

8.6. Prove that every ray (half-line) has the cardinality of the continuum.

8.7. Prove that if both A and B have the cardinality of the continuum, then so does $A \cup B$.

8.8. Prove that every circle has the cardinality of the continuum.

8.9. Give a function that maps $(0,1]$ bijectively to the set of infinite sequences made up of positive integers. (H)

8.10. Prove that the set of all subsets of $\mathbb{N}$ has the cardinality of the continuum.

8.11. Prove that the plane (that is, the set $\{(x,y)\colon x,y \in \mathbb{R}\}$) has the cardinality of the continuum. (H)