

Chapter 17

Functions of Bounded Variation

We know that if f is integrable, then the lower and upper sums of every partition F approximate its integral from below and above, and so the difference between either sum and the integral is at most $S_F - s_F = \Omega_F$, the oscillatory sum corresponding to F .

Thus the oscillatory sum is an upper bound for the difference between the approximating sums and the integral.

We also know that if f is integrable, then the oscillating sum can become smaller than any fixed positive number for a sufficiently fine partition (see Theorem 14.23).

If the function f is monotone, we can say more: $\Omega_F(f) \leq |f(b) - f(a)| \cdot \delta(F)$ for all partitions F , where $\delta(F)$ denotes the mesh of the partition F (see Theorem 14.28 and inequality (14.19) in the proof of the theorem). A similar inequality holds for Lipschitz functions: if $|f(x) - f(y)| \leq K \cdot |x - y|$ for all $x, y \in [a, b]$, then $\Omega_F(f) \leq K \cdot (b - a) \cdot \delta(F)$ for every partition F (see Exercise 17.1). We can state this condition more concisely by saying that $\Omega_F(f) = O(\delta(F))$ holds for f if there exists a number C such that for an arbitrary partition F , $\Omega_F(f) \leq C \cdot \delta(F)$. (Here we used the big-oh notation seen on p. 141.) By the above, this condition holds for both monotone and Lipschitz functions.

Is it true that the condition $\Omega_F(f) = O(\delta(F))$ holds for every integrable function? The answer is no: one can show that the function

$$f(x) = \begin{cases} x \cdot \sin(1/x), & \text{if } 0 < x \leq 1; \\ 0, & \text{if } x = 0 \end{cases}$$

does not satisfy the condition (see Exercise 17.3). It is also true that for an arbitrary sequence ω_n that tends to zero, there exists a continuous function $f: [0, 1] \rightarrow \mathbb{R}$ such that $\Omega_{F_n}(f) \geq \omega_n$ for all n , where F_n denotes a partition of $[0, 1]$ into n equal subintervals (see Exercise 17.4). That is, monotone functions “are better behaved” than continuous functions in this aspect.

We characterize below the class of functions for which $\Omega_F(f) = O(\delta(F))$ holds. By what we stated above, every monotone and every Lipschitz function is included

in this class, but not every continuous function is. The elements of this class are the so-called functions of bounded variation, and they play an important role in analysis.

Definition 17.1. Let the function f be defined on the interval $[a, b]$. If we have a partition of the interval $[a, b]$ given by $F : a = x_0 < x_1 < \cdots < x_n = b$, let $V_F(f)$ denote the sum $\sum_{i=1}^n |f(x_i) - f(x_{i-1})|$. The *total variation* of f over $[a, b]$ is the supremum of the set of sums $V_F(f)$, where F ranges over all partitions of the interval $[a, b]$. We denote the total variation of f on $[a, b]$ by $V(f; [a, b])$ (which can be infinite). We say that the function $f : [a, b] \rightarrow \mathbb{R}$ is of *bounded variation* if $V(f; [a, b]) < \infty$.

Remarks 17.2. 1. Suppose that the graph of the function f consists of finitely many monotone segments. Let f be monotone on each of the intervals $[c_{i-1}, c_i]$ ($i = 1, \dots, k$), where $F_0 : a = c_0 < c_1 < \cdots < c_k = b$ is a suitable partition of $[a, b]$. It is easy to check that for an arbitrary partition F , we have

$$V_F(f) \leq \sum_{i=1}^k |f(c_i) - f(c_{i-1})| = V_{F_0}(f).$$

Thus the total variation of f is equal to $V_{F_0}(f)$, and so the supremum defining the total variation is actually a maximum. This statement can be turned around: if there is a largest value among $V_F(f)$, then the graph of f consists of finitely many monotone segments (see Exercise 17.5).

2. Suppose again that f is monotone on each of the intervals $[c_{i-1}, c_i]$ ($i = 1, \dots, k$), where $F_0 : a = c_0 < c_1 < \cdots < c_k = b$. Consider the graph of f to be the crest of a mountain along which a tourist is walking. Suppose that for this tourist, the effort required to change altitude is proportional to the change in altitude, independent of whether the tourist is ascending or descending (and thus the tourist floats effortlessly when the mountain crest is horizontal). Then the value $V_{F_0}(f)$ measures the required effort for the tourist to traverse the crest of the mountain.

Generalizing this interpretation, we can say that the total variation of an arbitrary function is the effort required to “climb” the graph, and so a function is of bounded variation if the graph can be climbed with a finite amount of effort.

Theorem 17.3.

- (i) If f is monotone on $[a, b]$, then f is of bounded variation there, and $V(f; [a, b]) = |f(b) - f(a)|$.
- (ii) Let f be Lipschitz on $[a, b]$, and suppose that $|f(x) - f(y)| \leq K \cdot |x - y|$ for all $x, y \in [a, b]$. Then f is of bounded variation on $[a, b]$, and $V(f; [a, b]) \leq K \cdot (b - a)$.

Proof. (i) If f is monotone, then for an arbitrary partition $a = x_0 < x_1 < \cdots < x_n = b$,

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| = \left| \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \right| = |f(b) - f(a)|.$$

(ii) If $|f(x) - f(y)| \leq K \cdot |x - y|$ for all $x, y \in [a, b]$, then for an arbitrary partition $a = x_0 < x_1 < \cdots < x_n = b$,

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^n K \cdot (x_i - x_{i-1}) = K \cdot (b - a).$$

□

As the example mentioned in the introduction above demonstrates, not every continuous function is of bounded variation.

Example 17.4. Let $f(x) = x \cdot \sin(1/x)$ if $0 < x \leq 1$, and $f(0) = 0$. We show that f is not of bounded variation on $[0, 1]$. Let F_n be the partition that consists of the base points $0, 1$, and $x_i = 2/((2i - 1)\pi)$ ($i = 1, \dots, n$). Then $V_{F_n}(f) \geq \sum_{i=2}^n |f(x_i) - f(x_{i-1})|$. In Example 16.16, we saw that this sum can be arbitrarily large if we choose n to be sufficiently large, so f is not of bounded variation.

Theorem 17.5.

(i) For every $f: [a, b] \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$, we have

$$V(c \cdot f; [a, b]) = |c| \cdot V(f; [a, b]). \quad (17.1)$$

(ii) For arbitrary functions $f, g: [a, b] \rightarrow \mathbb{R}$,

$$V(f + g; [a, b]) \leq V(f; [a, b]) + V(g; [a, b]). \quad (17.2)$$

(iii) If both f and g are of bounded variation on $[a, b]$, then $a \cdot f + b \cdot g$ is also of bounded variation there for every $a, b \in \mathbb{R}$.

Proof. (i) For an arbitrary partition $a = x_0 < x_1 < \cdots < x_n = b$,

$$\sum_{i=1}^n |c \cdot f(x_i) - c \cdot f(x_{i-1})| = |c| \cdot \sum_{i=1}^n |f(x_i) - f(x_{i-1})|.$$

Taking the supremum of both sides over all partitions, we obtain (17.1).

(ii) Let $h = f + g$. Then for an arbitrary partition $a = x_0 < x_1 < \cdots < x_n = b$,

$$\begin{aligned} \sum_{i=1}^n |h(x_i) - h(x_{i-1})| &\leq \sum_{i=1}^n |f(x_i) - f(x_{i-1})| + \sum_{i=1}^n |g(x_i) - g(x_{i-1})| \\ &\leq V(f; [a, b]) + V(g; [a, b]). \end{aligned}$$

Since this is true for every partition, 17.2 holds. The third statement of the theorem follows from (i) and (ii). □

Theorem 17.6. If f is of bounded variation on $[a, b]$, then it also is of bounded variation in every subinterval.

Proof. If $[c, d] \subset [a, b]$, then extending an arbitrary partition $F: c = x_0 < x_1 < \cdots < x_n = d$ of $[c, d]$ to a partition F' of $[a, b]$, we obtain that $\sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq V_{F'}(f) \leq V(f; [a, b])$. Since this holds for every partition of $[c, d]$, $V(f; [c, d]) \leq V(f; [a, b]) < \infty$. □

Theorem 17.7. *Let $a < b < c$. If f is of bounded variation on $[a, b]$ and $[b, c]$, then it is of bounded variation in $[a, c]$ as well, and*

$$V(f; [a, c]) = V(f; [a, b]) + V(f; [b, c]). \quad (17.3)$$

Proof. Let $F : a = x_0 < x_1 < \cdots < x_n = c$ be an arbitrary partition of the interval $[a, c]$, and let $x_{k-1} \leq b \leq x_k$. It is clear that

$$\begin{aligned} V_F(f) &\leq \sum_{i=1}^{k-1} |f(x_i) - f(x_{i-1})| + |f(b) - f(x_{k-1})| + \\ &\quad + |f(x_k) - f(b)| + \sum_{i=k+1}^n |f(x_i) - f(x_{i-1})| \leq V(f; [a, b]) + V(f; [b, c]), \end{aligned}$$

and so $V(f; [a, c]) \leq V(f; [a, b]) + V(f; [b, c])$.

Now let $\varepsilon > 0$ be given, and let $a = x_0 < x_1 < \cdots < x_n = b$ and $b = y_0 < y_1 < \cdots < y_k = c$ be partitions of $[a, b]$ and $[b, c]$ respectively such that $\sum_{i=1}^n |f(x_i) - f(x_{i-1})| > V(f; [a, b]) - \varepsilon$ and $\sum_{i=1}^k |f(y_i) - f(y_{i-1})| > V(f; [b, c]) - \varepsilon$. Then

$$\begin{aligned} V(f; [a, c]) &\geq \sum_{i=1}^n |f(x_i) - f(x_{i-1})| + \sum_{i=1}^k |f(y_i) - f(y_{i-1})| > \\ &> V(f; [a, b]) + V(f; [b, c]) - 2\varepsilon. \end{aligned}$$

Since ε was arbitrary, $V(f; [a, c]) \geq V(f; [a, b]) + V(f; [b, c])$ follows, and so 17.3 holds. \square

The following theorem gives a simple characterization of functions of bounded variation.

Theorem 17.8. *The function $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation if and only if it can be expressed as the difference of two monotone increasing functions.*

Proof. Every monotone function is of bounded variation (Theorem 17.3), and the difference of two functions of bounded variation is also of bounded variation (Theorem 17.5), so the “if” part of the theorem is clearly true.

Now suppose that f is of bounded variation. Let $g(x) = V(f; [a, x])$ for all $x \in [a, b]$, and let $g(a) = 0$. If $a \leq x < y \leq b$, then by Theorem 17.7,

$$g(y) = g(x) + V(f; [x, y]) \geq g(x) + |f(y) - f(x)| \geq g(x) - f(y) + f(x).$$

Thus on the one hand, $g(y) \geq g(x)$, while on the other hand, $g(y) + f(y) \geq g(x) + f(x)$, so both g and $g + f$ are monotone increasing in $[a, b]$. Since $f = (g + f) - g$, this ends the proof of the theorem. \square

Corollary 17.9. *If f is of bounded variation on $[a, b]$, then it is integrable there.*

In Example 17.4, we saw that not every continuous function is of bounded variation. Thus the corollary above cannot be turned around.

The following theorem can often be applied to compute total variation. We leave its proof to the reader (in Exercise 17.12).

Theorem 17.10. *If f is differentiable and f' is integrable on $[a, b]$, then f is of bounded variation and $V(f; [a, b]) = \int_a^b |f'| dx$.*

The following theorem clarifies the condition for rectifiability.

Theorem 17.11.

- (i) *The graph of a function $f: [a, b] \rightarrow \mathbb{R}$ is rectifiable if and only if f is of bounded variation.*
(ii) *A curve $g: [a, b] \rightarrow \mathbb{R}^d$ is rectifiable if and only if its coordinate functions are of bounded variation.*

Proof. (i) Let the arc length of the graph f be denoted by $s(f; [a, b])$ (see Definition 10.78). Then for an arbitrary partition $F: a = x_0 < x_1 < \cdots < x_n = b$, we have

$$\begin{aligned} V_F(f) &= \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \leq \\ &\leq \sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \leq s(f; [a, b]) \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} &\leq \\ &\leq \sum_{i=1}^n (x_i - x_{i-1}) + \sum_{i=1}^n |f(x_i) - f(x_{i-1})| = \\ &= (b - a) + V_F(f) \leq (b - a) + V(f; [a, b]). \end{aligned}$$

Since these hold for every partition, we have $V(f; [a, b]) \leq s(f; [a, b])$ and $s(f; [a, b]) \leq (b - a) + V(f; [a, b])$. It is then clear that $s(f; [a, b])$ is finite if and only if $V(f; [a, b])$ is finite, that is, the graph of f is rectifiable if and only if f is of bounded variation.

(ii) Let the coordinate functions of g be g_1, \dots, g_d , and let $F: a = t_0 < t_1 < \cdots < t_n = b$ be a partition of the interval $[a, b]$. If $p_i = g(t_i)$ ($i = 0, \dots, n$), then for every $i = 1, \dots, n$ and $j = 1, \dots, d$,

$$\begin{aligned} |g_j(x_i) - g_j(x_{i-1})| &\leq \\ &\leq |p_i - p_{i-1}| = \sqrt{(g_1(x_i) - g_1(x_{i-1}))^2 + \cdots + (g_d(x_i) - g_d(x_{i-1}))^2} \leq \\ &\leq |g_1(x_i) - g_1(x_{i-1})| + \cdots + |g_d(x_i) - g_d(x_{i-1})|. \end{aligned}$$

If we sum these equations for $i = 1, \dots, n$, then we get that

$$V_F(g_j) \leq \ell_F \leq V_F(g_1) + \dots + V_F(g_d) \leq V(g_1; [a, b]) + \dots + V(g_d; [a, b]),$$

where ℓ_F denotes the length of the inscribed polygon corresponding to the partition F . Since this holds for every partition, we have

$$V(g_j; [a, b]) \leq s(g) \leq V(g_1; [a, b]) + \dots + V(g_d; [a, b])$$

for all $j = 1, \dots, d$, where $s(g)$ is the arc length of the curve. It is then clear that $s(g)$ is finite if and only if $V(g_j; [a, b])$ is finite for all $j = 1, \dots, d$, that is, g is rectifiable if and only if its coordinate functions are of bounded variation. \square

Now we prove the statement from the introduction.

Theorem 17.12. *A function $f: [a, b] \rightarrow \mathbb{R}$ satisfies $\Omega_F(f) = O(\delta(F))$ for every partition F if and only if f is of bounded variation.*

Proof. We first prove that if f is of bounded variation, then $\Omega_F(f) = O(\delta(F))$ holds. At the beginning of the chapter, we saw that if g is monotone in $[a, b]$, then $\Omega_F(g) \leq |g(b) - g(a)| \cdot \delta(F)$ for every partition F . We also know that if f is of bounded variation in $[a, b]$, then it can be expressed as $f = g - h$, where g and h are monotone increasing functions (Theorem 17.8). Thus for an arbitrary partition F ,

$$\Omega_F(f) \leq \Omega_F(g) + \Omega_F(h) \leq |g(b) - g(a)| \cdot \delta(F) + |h(b) - h(a)| \cdot \delta(F) = C \cdot \delta(F),$$

where $C = |g(b) - g(a)| + |h(b) - h(a)|$. Thus the condition $\Omega_F(f) = O(\delta(F))$ indeed holds.

Now we show that if f is not of bounded variation, then the property $\Omega_F(f) = O(\delta(F))$ does not hold, that is, for every real number A , there exists a partition F such that $\Omega_F(f) > A \cdot \delta(F)$. The proof relies on the observation that for an arbitrary bounded function $f: [a, b] \rightarrow \mathbb{R}$ and partition $F: a = x_0 < x_1 < \dots < x_n = b$,

$$\begin{aligned} \Omega_F(f) &= \sum_{i=1}^n \omega(f; [x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) \geq \\ &\geq \left[\sum_{i=1}^n |f(x_i) - f(x_{i-1})| \right] \cdot \min_{1 \leq i \leq n} (x_i - x_{i-1}) = \\ &= V_F(f) \cdot \rho(F), \end{aligned}$$

where $\rho(F) = \min_{1 \leq i \leq n} (x_i - x_{i-1})$.

If f is not of bounded variation, then for every real number A , there exists a partition F_0 such that $V_{F_0}(f) > A$. However, we know only that this partition F_0 satisfies $\Omega_{F_0}(f) \geq V_{F_0}(f) \cdot \rho(F_0) > A \cdot \rho(F_0)$, while we want $\Omega_F(f) > A \cdot \delta(F)$ to hold for some F .

So consider a refinement F of F_0 such that $\rho(F) \geq \delta(F)/2$. We can get such a refinement by further subdividing the intervals in F_0 into pieces whose lengths are

between $\rho(F_0)/2$ and $\rho(F_0)$. In this case, $\delta(F) \leq \rho(F_0)$ and $\rho(F) \geq \rho(F_0)/2$, so $\rho(F) \geq \delta(F)/2$ holds. Since F is a refinement of F_0 , we easily see that $V_F(f) \geq V_{F_0}(f) > A$, so $\Omega_F(f) \geq V_F(f) \cdot \rho(F) > A \cdot \delta(F)/2$. Since A was arbitrary, this concludes the proof. \square

With the theorems above in hand, we might ask whether there exist functions for which we can say more than $\Omega_F(f) = O(\delta(F))$? Could it be possible for $\Omega_F(f) \leq C \cdot \delta(F)^2$ to hold for every partition with some constant C ? The answer to this question is no. If f is constant, then of course $\Omega_F(f) = 0$ for every partition. If, however, f is not constant, then there exists a $c > 0$ such that $\Omega_{F_n}(f) \geq c \cdot \delta(F_n)$ for all n , where F_n denotes the uniform partition of $[a, b]$ into n equal subintervals (see Exercise 17.2).

Exercises

17.1. Prove that if $|f(x) - f(y)| \leq K \cdot |x - y|$ for all $x, y \in [a, b]$, then $\Omega_F(f) \leq K \cdot (b - a) \cdot \delta(F)$ for every partition F of $[a, b]$. (S)

17.2. Prove that if f is bounded in $[a, b]$ and F_n denotes the uniform partition of $[a, b]$ into n equal subintervals, then

$$\Omega_{F_n}(f) \geq \omega(f; [a, b]) \cdot (b - a)/n,$$

where $\omega(f; [a, b])$ is the oscillation of the function f on the interval $[a, b]$. (H)

17.3. Let $f(x) = x \cdot \sin(1/x)$ if $0 < x \leq 1$, and $f(0) = 0$. Prove that there exists a constant $c > 0$ such that $\Omega_{F_n}(f) \geq c \cdot (\log n)/n = c \cdot (\log n) \cdot \delta(F_n)$ for all n , where F_n denotes the uniform partition of $[0, 1]$ into n equal subintervals. (*H)

17.4. Show that if an arbitrary sequence ω_n tends to zero, then there exists a continuous function $f: [0, 1] \rightarrow \mathbb{R}$ such that $\Omega_{F_n}(f) \geq \omega_n$ for all n , where F_n denotes the uniform partition of $[0, 1]$ into n equal subintervals. (*)

17.5. Let $f: [a, b] \rightarrow \mathbb{R}$ and suppose that there is a largest value among $V_F(f)$ (where F runs over the partitions of $[a, b]$). Show that in this case, the graph of f is made up of finitely many monotone segments. (H)

17.6. Show that if f is of bounded variation in $[a, b]$, then so is f^2 .

17.7. Show that if f and g are of bounded variation in $[a, b]$, then so is $f \cdot g$. Moreover, if $\inf |g| > 0$, then so is f/g .

17.8. Let $f(x) = x^\alpha \cdot \sin(1/x)$ if $0 < x \leq 1$ and $f(0) = 0$. For what α will f be of bounded variation in $[0, 1]$?

17.9. Give an example for a function f that is differentiable in $[0, 1]$ but is not of bounded variation there.

17.10. For what $c > 0$ will the c th power of the Riemann function be of bounded variation in $[0, 1]$? (H)

17.11. Prove that if f is differentiable on $[a, b]$ and f' is bounded there, then f is of bounded variation on $[a, b]$.

17.12. Prove Theorem 17.10. (S)

17.13. Let $\alpha > 0$ be given. We say that f is **Hölder α** in the interval $[a, b]$ if there exists a number C such that $|f(x) - f(y)| \leq C \cdot |x - y|^\alpha$ for all $x, y \in [a, b]$. Show that if $\alpha > 0$, then the function x^α is Hölder β in the interval $[0, 1]$, where $\beta = \min(\alpha, 1)$. (S)

17.14. Show that if f is Hölder α in the interval $[a, b]$, where $\alpha > 1$, then f is constant. (H)

17.15. Let $f(x) = x^\alpha \cdot \sin x^{-\beta}$ if $0 < x \leq 1$ and $f(0) = 0$, where α and β are positive constants. Show that f is Hölder γ in the interval $[0, 1]$, where

$$\gamma = \min\left(\frac{\alpha}{\beta + 1}, 1\right). \quad (* H S)$$

17.16. For what α can we say that if f is Hölder α , then f is of bounded variation in $[a, b]$? (H)

17.17. Prove that a function of bounded variation in $[a, b]$ has at most countably many points of discontinuity.

17.18. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Prove that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that every partition F with mesh smaller than δ satisfies $V_F(f) > V(f; [a, b]) - \varepsilon$.

17.19. Prove that a function defined on $[a, b]$ is not of bounded variation in $[a, b]$ if and only if there exists a strictly monotone sequence (x_n) in $[a, b]$ such that $\sum_{i=1}^{\infty} |f(x_{i+1}) - f(x_i)| = \infty$. (* H)