Chapter 4
Infinite Sequences I

In this chapter, we will be dealing with sequences of real numbers. For brevity, by a sequence we shall mean an infinite sequence whose terms are all real numbers.

We can present a sequence in various different ways. Here are a few examples (each one is defined for $n \in \mathbb{N}^+$):

Examples 4.1. 1. $a_n = \frac{1}{n}$: $(a_n) = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right)$;
2. $a_n = (-1)^{n+1} \cdot \frac{1}{n}$: $(a_n) = \left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\right)$;
3. $a_n = (-1)^n$: $(a_n) = (-1, 1, \ldots, -1, 1, \ldots)$;
4. $a_n = (n+1)^2$: $(a_n) = (4, 9, 16, \ldots)$;
5. $a_n = \sqrt{n+1} - \sqrt{n}$: $(a_n) = \left(\sqrt{2} - 1, \sqrt{3} - \sqrt{2}, \ldots\right)$;
6. $a_n = \frac{n+1}{n}$: $(a_n) = \left(2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\right)$;
7. $a_n = (-1)^n \cdot n^2$: $(a_n) = (-1, 4, -9, 16, \ldots)$;
8. $a_n = n + \frac{1}{n}$: $(a_n) = \left(2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \ldots\right)$;
9. $a_n = \sqrt{n+10}$: $(a_n) = \left(\sqrt{11}, \sqrt{12}, \ldots\right)$;
10. $a_n = \left(1 + \frac{1}{n}\right)^n$;
11. $a_n = \left(1 + \frac{1}{n}\right)^{n^2}$;
12. $a_n = \left(1 + \frac{1}{n^2}\right)^n$;
13. $a_1 = -1$, $a_2 = 2$, $a_n = (a_{n-1} + a_{n-2})/2$ $(n \geq 3)$: $(a_n) = \left(-1, 2, \frac{1}{2}, \frac{5}{4}, \frac{7}{8}, \frac{17}{16}, \ldots\right)$;
14. $a_1 = 1$, $a_2 = 3$, $a_{n+1} = \sqrt[n]{a_1 \cdot \cdots \cdot a_n}$ $(n \geq 2)$: $(a_n) = \left(1, 3, \sqrt{3}, \sqrt{3}, \sqrt{3}, \ldots\right)$;
15. $a_1 = 0$, $a_{n+1} = \sqrt{2 + a_n}$ $(n \geq 1)$: $(a_n) = \left(0, \sqrt{2}, \sqrt{2 + \sqrt{2}}, \ldots\right)$;
16. $a_n = \begin{cases} n, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd} \end{cases}$: $(a_n) = (1, 2, 1, 4, 1, 6, 1, 8, \ldots)$;
17. $a_n =$ the nth prime number: $(a_n) = (2, 3, 5, 7, 11, \ldots)$;
18. $a_n =$ the nth digit of the infinite decimal expansion of $\sqrt{2}$:
$(a_n) = (4, 1, 4, 2, 1, 3, 5, 6, \ldots)$.

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In the first 12 sequences, the value of a_n is given by an "explicit formula." The terms of (13)–(15) are given **recursively**. This means that we give the first few, say k, terms of the sequence, and if $n > k$, then the nth term is given by the terms with index less than n. In (16)–(18), the terms are not given with a specific "formula." There is no real difference, however, concerning the validity of the definitions; they are all valid sequences. As we will later see, whether a_n (or generally a function) can be expressed with some kind of formula depends only on how frequently such an expression occurs and how important it is. For if it defines an important map that we use frequently, then it is worthwhile to define some notation and nomenclature that converts the long definition (of a_n or the function) into a "formula." If it is not so important or frequently used, we usually leave it as a lengthy description. For example, in the above sequence (9), a_n is the positive number whose square is $n+10$. However, the map expressed here (the square root) occurs so frequently and with such importance that we create a new symbol for it, and the definition of a_n becomes a simple formula.

Exercises

4.1. Give a closed formula for the nth term of sequence (13) in Example 4.1. (S)

4.2. Let $p(x) = x^k - c_1x^{k-1} - c_2x^{k-2} - \ldots - c_k$, let $\alpha_1, \ldots, \alpha_m$ be roots of the polynomial p (not necessarily all of the roots), and let $\beta_1, \ldots, \beta_m$ be arbitrary real numbers. Show that the sequence

$$a_n = \beta_1 \cdot \alpha_1^n + \cdots + \beta_m \cdot \alpha_m^n \quad (n = 1, 2, \ldots)$$

satisfies the recurrence relation (recursion) $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ for all $n > k$. (H)

4.3. Give a closed formula for the nth term of the following sequences, given by recursion.

(a) $u_0 = 0,\ u_1 = 1,\ u_n = u_{n-1} + u_{n-2}\ \ (n \geq 2)$ (HS);
(b) $a_0 = 0,\ a_1 = 1,\ a_n = a_{n-1} + 2 \cdot a_{n-2}\ \ (n \geq 2)$;
(c) $a_0 = 0,\ a_1 = 1,\ a_n = 2 \cdot a_{n-1} + a_{n-2}\ \ (n \geq 2)$.

4.4. Let $p(x) = x^k - c_1x^{k-1} - c_2x^{k-2} - \ldots - c_k$, and let α be a double root of the polynomial p (this means that $(x-\alpha)^2$ can be factored from p). Show that the sequence $a_n = n \cdot \alpha^n$ $(n = 1, 2, \ldots)$ satisfies the following recurrence for all $n > k$: $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$.

4.5. Give a closed formula for the nth term of the sequence

$$a_0 = 0,\ a_1 = 0,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2} - a_{n-3}\ (n \geq 3)$$

given by a recurrence relation.

4.1 Convergent and Divergent Sequences

When we make measurements, we are often faced with a value that we can only express through approximation—although arbitrarily precisely. For example, we define the circumference (or area) of a circle in terms of the perimeter (or area) of the inscribed or circumscribed regular n-gons. According to this definition, the circumference (or area) of the circle is the number that the perimeter (or area) of the inscribed regular n-gon "approximates arbitrarily well as n increases past every bound."

Some of the above sequences also have this property that the terms "tend" to some "limit value." The terms of the sequence (1) "tend" to 0 in the sense that if n is large, the value of $1/n$ is "very small," that is it is very close to 0. More precisely, no matter how small an interval we take around 0, if n is large enough, then $1/n$ is inside this interval (there are only finitely many n such that $1/n$ is outside the interval).

The terms of the sequence (2) also "tend" to 0, while the terms of (6) "tend" to 1 by the above notion.

For the sequences (3), (4), (7), (8), (9), (16), (17), and (18), no number can be found that the terms of any of these sequences "tend" to, by the above notion.

The powers defining the a_n in the sequences (10), (11), and (12) all behave differently as n increases: the base approaches 1, but the exponent gets very big. Without detailed examination, we cannot "see" whether they tend to some value, and if they do, to what. We will see later that these three sequences each behave differently from the point of view of limits.

To define limits precisely, we use the notion outlined in the examination of (1). We give two definitions, which we promptly show to be equivalent.

Definition 4.2. The sequence (a_n) *tends to* b (or b *is the limit of the sequence*) if for every $\varepsilon > 0$, there are only finitely many terms falling outside the interval $(b-\varepsilon, b+\varepsilon)$. In other words, the limit of (a_n) is b if for every $\varepsilon > 0$, the terms of the sequence satisfy, with finitely many exceptions, the inequality $b-\varepsilon < a_n < b+\varepsilon$.

Definition 4.3. The sequence (a_n) *tends to* b (or b *is the limit of the sequence*), if for every $\varepsilon > 0$ there exists a number n_0 (depending on ε) such that

$$|a_n - b| < \varepsilon \text{ for all indices } n > n_0. \tag{4.1}$$

Let us show that the *two definitions are equivalent*. Suppose first that the sequence (a_n) tends to b by Definition 4.2. Consider an arbitrary $\varepsilon > 0$. Then only finitely many terms of the sequence fall outside $(b-\varepsilon, b+\varepsilon)$. If there are no terms of the sequence outside the interval, then (4.1) holds for any choice of n_0. If terms of the sequence fall outside $(b-\varepsilon, b+\varepsilon)$, then out of those finitely many terms, there is one with maximal index. Denote this index by n_0. Then for each $n > n_0$, the a_n are in the interval $(b-\varepsilon, b+\varepsilon)$, that is, $|a_n - b| < \varepsilon$ if $n > n_0$. Thus we see that (a_n) satisfies Definition 4.3.

Secondly, suppose that (a_n) tends to b by Definition 4.3. Let $\varepsilon > 0$ be given. Then there exists an n_0 such that if $n > n_0$, then a_n is in the interval $I = (b-\varepsilon, b+\varepsilon)$. Thus only among the terms a_i $(i \le n_0)$ can there be terms that do not fall in the interval I. The number of these is at most n_0, thus finite. It follows that the sequence satisfies Definition 4.2.

If the sequence (a_n) tends to b, then we denote this by

$$\lim_{n\to\infty} a_n = b \qquad \text{or} \qquad a_n \to b, \text{ as } n \to \infty \text{ (or just } a_n \to b\text{)}.$$

If there is a real number b such that $\lim_{n\to\infty} a_n = b$, then we say that the sequence (a_n) is **convergent**. If there is no such number, then the sequence (a_n) is **divergent**.

Examples 4.4. **1.** By Definition 4.3, it is easy to see that the sequence (1) in 4.1 truly tends to 0, that is, $\lim_{n\to\infty} 1/n = 0$. For if ε is an arbitrary positive number, then $n > 1/\varepsilon$ implies $1/n < \varepsilon$, and thus $|(1/n) - 0| = 1/n < \varepsilon$. That is, we can pick the n_0 in the definition to be $1/\varepsilon$. (The definition did not require that n_0 be an integer. But it is clear that if some n_0 satisfies the conditions of the definition, then every number larger than n_0 does as well, and among these—by the axiom of Archimedes—is an integer.)

2. In the same way, we can see that the sequence (2) in 4.1 tends to 0. Similarly,

$$\lim_{n\to\infty} \frac{n+1}{n} = 1,$$

or (6) is also convergent, and it has limit 1. Clearly, $(n+1)/n \in (1-\varepsilon,\ 1+\varepsilon)$ if $1/n < \varepsilon$, that is, the only a_n outside the interval $(1-\varepsilon, 1+\varepsilon)$ are those for which $1/n \ge \varepsilon$, that is, $n \le 1/\varepsilon$.

3. Now we show that the sequence (5) tends to 0, that is,

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0. \tag{4.2}$$

Since

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}},$$

if $n > 1/(4\varepsilon^2)$, then $1/(2\sqrt{n}) < \varepsilon$ and $a_n \in (-\varepsilon, \varepsilon)$.

Remarks 4.5. **1.** It is clear that if a threshold n_0 is good, that is it satisfies the conditions of (4.1), then every number larger than n_0 is also a good index. Generally when finding a threshold n_0 we do not strive to find the smallest one.

2. It is important to note the following regarding Definition 4.2. If the infinite sequence (a_n) has only finitely many terms *outside* the interval $(a-\varepsilon, a+\varepsilon)$, then naturally there are infinitely many terms *inside* the interval $(a-\varepsilon, a+\varepsilon)$. It is clear, however, that for the sequence (a_n), if for every $\varepsilon > 0$ there are infinitely many terms *inside* the interval $(a-\varepsilon, a+\varepsilon)$ it does not necessarily mean that there are only finitely many terms *outside* the interval; that is, it does not follow that $\lim_{n\to\infty} a_n = a$. For example, in Example 4.1, for the sequence (3) there are infinitely many terms inside $(1-\varepsilon, 1+\varepsilon)$ for every $\varepsilon > 0$ (and the same holds for $(-1-\varepsilon, -1+\varepsilon)$), but if $\varepsilon < 2$, then it is not true that there are only finitely many terms outside $(1-\varepsilon, 1+\varepsilon)$. That is, 1 is not a limit of the sequence, and it is easy to see that the sequence is divergent.

3. Denote the set of numbers occurring in the sequence (a_n) by $\{a_n\}$. Let us examine the relationship between (a_n) and $\{a_n\}$. On one hand, we know that for the set, a number is either an element of it or it is not, and there is no meaning in saying that it appears multiple times, while in a sequence, a number can occur many times. In fact, for the infinite sequence (a_n) in Example (3), the corresponding set $\{a_n\} = \{-1, 1\}$ is finite. (This distinction is further emphasized by talking about *elements* of sets, and *terms* of sequences.) Consider the following two properties:

I. For every $\varepsilon > 0$, there are finitely many terms of (a_n) outside the interval $(a-\varepsilon, a+\varepsilon)$.
II. For every $\varepsilon > 0$, there are finitely many elements of $\{a_n\}$ outside the interval $(a-\varepsilon, a+\varepsilon)$.

Property I. means that $\lim_{n\to\infty} a_n = a$. The same cannot be said of Property II., since we can take our example above, where $\{a_n\}$ had only finitely many elements outside every interval $(a-\varepsilon, a+\varepsilon)$, but the sequence was still divergent. Therefore, it is clear that I implies II, but II does not imply I. In other words, I is a stronger property than II.

Exercises

4.6. Find the limits of the following sequences, and find an n_0 (not necessarily the smallest) for a given $\varepsilon > 0$ as in Definition 4.3.

(a) $1/\sqrt{n}$;
(b) $(2n+1)/(n+1)$;
(c) $(5n-1)/(7n+2)$;
(d) $1/(n-\sqrt{n})$;
(e) $(1+\cdots+n)/n^2$;
(f) $(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n})/n^{4/3}$;
(g) $n\cdot\left(\sqrt{1+(1/n)}-1\right)$;
(h) $\sqrt{n^2+1}+\sqrt{n^2-1}-2n$;
(i) $\sqrt[3]{n+2}-\sqrt[3]{n-2}$;
(j) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\cdots+\dfrac{1}{(n-1)\cdot n}$.

4.7. Consider the definition of $a_n \to b$: $(\forall\varepsilon > 0)(\exists n_0)(\forall n \geq n_0)(|a_n - b| < \varepsilon)$. Permuting and changing the quantifiers yields the following statements:

(a) $(\forall\varepsilon > 0)(\exists n_0)(\exists n \geq n_0)(|a_n - b| < \varepsilon)$;
(b) $(\forall\varepsilon > 0)(\forall n_0)(\forall n \geq n_0)(|a_n - b| < \varepsilon)$;
(c) $(\forall\varepsilon > 0)(\forall n_0)(\exists n \geq n_0)(|a_n - b| < \varepsilon)$;

(d) $(\exists\varepsilon > 0)(\forall n_0)(\forall n \geq n_0)(|a_n - b| < \varepsilon)$;
(e) $(\exists\varepsilon > 0)(\forall n_0)(\exists n \geq n_0)(|a_n - b| < \varepsilon)$;
(f) $(\exists\varepsilon > 0)(\exists n_0)(\forall n \geq n_0)(|a_n - b| < \varepsilon)$;
(g) $(\exists\varepsilon > 0)(\exists n_0)(\exists n \geq n_0)(|a_n - b| < \varepsilon)$;
(h) $(\exists n_0)(\forall\varepsilon > 0)(\forall n \geq n_0)(|a_n - b| < \varepsilon)$;
(i) $(\exists n_0)(\forall\varepsilon > 0)(\exists n \geq n_0)(|a_n - b| < \varepsilon)$;
(j) $(\forall n_0)(\exists\varepsilon > 0)(\forall n \geq n_0)(|a_n - b| < \varepsilon)$;
(k) $(\forall n_0)(\exists\varepsilon > 0)(\exists n \geq n_0)(|a_n - b| < \varepsilon)$.

What properties of (a_n) do these statements express? For each property, give a sequence (if exists) with the given property.

4.8. Prove that a convergent sequence always has a smallest or a largest term.

4.9. Give examples such that $a_n - b_n \to 0$, but a_n/b_n does not tend to 1, as well as $a_n/b_n \to 1$ but $a_n - b_n$ does not tend to 0.

4.10. Prove that if (a_n) is convergent, then $(|a_n|)$ is convergent. Is this statement true the other way around?

4.11. If $a_n^2 \to a^2$, does it follow that $a_n \to a$? If $a_n^3 \to a^3$, does it follow that $a_n \to a$?

4.12. Prove that if $a_n \to a > 0$, then $\sqrt{a_n} \to \sqrt{a}$.

4.13. For the sequence (a_n), consider the corresponding sequence of arithmetic means, $s_n = (a_1 + \cdots + a_n)/n$. Prove that if $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} s_n = a$. Give a sequence for which (s_n) is convergent but (a_n) is divergent. (S)

4.2 Sequences That Tend to Infinity

It is easy to see that the sequences (3), (4), (7), (8), (9), (16), and (17) in Example 4.1 are divergent. Moreover, terms of the sequences (4), (8), (9), and (17)—aside from diverging—share the trend that for "large" n, the a_n terms are "large"; more precisely, for an arbitrarily large number P, there are only finitely many terms of the sequence that are smaller than P. We say that sequences like this "diverge to ∞." This is expressed precisely be the definition below. As in the case of convergent sequences, we give two definitions and then show that they are equivalent.

Definition 4.6. We say that the *limit* of the sequence (a_n) is ∞ (or that (a_n) tends to infinity) if for arbitrary P, there are only finitely many terms of the sequence outside the interval (P, ∞).[1]

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[1] That is, to the left of P on the number line.

Definition 4.7. We say that the *limit* of the sequence (a_n) is ∞ (or that (a_n) tends to infinity) if for arbitrary P, there exists a number n_0 (depending on P) for which the statement

$$a_n > P, \qquad \text{if} \qquad n > n_0 \tag{4.3}$$

holds.

We can show the equivalence of the above definitions in the following way. If there are no terms outside the interval (P,∞), then (4.3) holds for every n_0. If there are terms outside the interval (P,∞), but only finitely many, then among these indices, call the largest n_0, which will make (4.3) hold.

Conversely, if (4.3) holds, then there are at most n_0 terms of the sequence, finitely many, outside the interval (P,∞).

If the sequence (a_n) tends to infinity, then we denote this by $\lim_{n\to\infty} a_n = \infty$, or by $a_n \to \infty$ as $n \to \infty$, or just $a_n \to \infty$ for short. In this case, we say that the sequence (a_n) **diverges to infinity**.

We define the concept of tending to $-\infty$ in the same manner.

Definition 4.8. We say that the *limit* of the sequence (a_n) is $-\infty$ (or that (a_n) tends to negative infinity) if for arbitrary P, there are only finitely many terms of the sequence outside the interval $(-\infty, P)$.[2]

The following is equivalent.

Definition 4.9. We say that the *limit* of the sequence (a_n) is $-\infty$ (or that (a_n) tends to negative infinity) if for arbitrary P, there exists a number n_0 (dependent on P) for which if $n > n_0$, then $a_n < P$ holds.

If the sequence (a_n) tends to negative infinity, then we denote this by $\lim_{n\to\infty} a_n = -\infty$, or by $a_n \to -\infty$ if $n \to \infty$, or just $a_n \to -\infty$ for short. In this case, we say that the sequence (a_n) **diverges to negative infinity**.

There are several sequences in Example 4.1 that tend to infinity. It is clear that (4) and (8) are such sequences, for in both cases, if $n > P$, then $a_n > P$. The sequence (9) also tends to infinity, for if $n > P^2$, then $a_n > P$.

Now we show that the sequence (11) tends to infinity as well. Let P be given. Then Bernoulli's inequality (Theorem 2.5) implies

$$\left(1+\frac{1}{n}\right)^{n^2} > 1 + n^2 \cdot \frac{1}{n} = 1 + n,$$

so for arbitrary $n > P$, we have

$$\left(1+\frac{1}{n}\right)^{n^2} > 1 + n > P.$$

[2] That is, to the right of P on the number line.

Exercises

4.14. For a fixed P, find an n_0 (not necessarily the smallest one) satisfying Definition 4.7 for the following sequences:

(a) $n-\sqrt{n}$;

(b) $(1+\cdots+n)/n$;

(c) $(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n})/n$;

(d) $\dfrac{n^2-10n}{10n+100}$;

(e) $2^n/n$.

4.15. Consider the definition of $a_n \to \infty$: $(\forall P)(\exists n_0)(\forall n \ge n_0)(a_n > P)$. Permuting or changing the quantifiers yields the following statements:

(a) $(\forall P)(\exists n_0)(\exists n \ge n_0)(a_n > P)$;
(b) $(\forall P)(\forall n_0)(\forall n \ge n_0)(a_n > P)$;
(c) $(\forall P)(\forall n_0)(\exists n \ge n_0)(a_n > P)$;
(d) $(\exists P)(\forall n_0)(\forall n \ge n_0)(a_n > P)$;
(e) $(\exists P)(\forall n_0)(\exists n \ge n_0)(a_n > P)$;
(f) $(\exists P)(\exists n_0)(\forall n \ge n_0)(a_n > P)$;
(g) $(\exists P)(\exists n_0)(\exists n \ge n_0)(a_n > P)$;
(h) $(\exists n_0)(\forall P)(\forall n \ge n_0)(a_n > P)$;
(i) $(\exists n_0)(\forall P)(\exists n \ge n_0)(a_n > P)$;
(j) $(\forall n_0)(\exists P)(\forall n \ge n_0)(a_n > P)$;
(k) $(\forall n_0)(\exists P)(\exists n \ge n_0)(a_n > P)$.

What properties of (a_n) do these statements express? For each property, give a sequence (if exists) with the given property.

4.16. Prove that a sequence tending to infinity always has a smallest term.

4.17. Find the limit of $(n^2+1)/(n+1) - an$ for each a.

4.18. Find the limit of $\sqrt{n^2-n+1} - an$ for each a.

4.19. Find the limit of $\sqrt{(n+a)(n+b)} - n$ for each a, b.

4.20. Prove that if $a_{n+1} - a_n \to c$, where $c > 0$, then $a_n \to \infty$.

4.3 Uniqueness of Limit

If the sequence (a_n) is convergent, or tends to plus or minus infinity, then we say that (a_n) **has a limit**. Instead of saying that a sequence is *convergent*, we can say that **the sequence has a finite limit.** If (a_n) doesn't have a limit, we can say that it **oscillates at infinity**. The following table illustrates the classification above.

$$\begin{array}{lll} \text{convergent} & a_n \to b \in \mathbb{R} & \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{has limit} \\ \text{divergent}\left\{\begin{array}{l} \\ \\ \\ \end{array}\right. & \begin{array}{l} a_n \to \infty \\ a_n \to -\infty \\ \text{has no limit} \end{array} & \text{oscillates at infinity} \end{array}$$

To justify the above classification, we need to show that the properties in the middle column are mutually exclusive. We will show more than this: in Theorem 4.14, we will show that a sequence can have at most one limit. For this, we need the following two theorems.

Theorem 4.10. *If the sequence (a_n) is convergent, then it is a bounded sequence.*[3]

Proof. Let $\lim_{n\to\infty} a_n = b$. Pick, according to the notation in Definition 4.2, ε to be 1. We get that there are only finitely many terms of the sequence outside the interval $(a-1, a+1)$. If there are no terms greater than $a+1$, then it is an upper bound. If there are terms greater than $a+1$, there are only finitely many of them. Out of these, the largest one is the largest term of the whole sequence, and as such, is an upper bound too. A bound from below can be found in the same fashion. □

Remark 4.11. The converse of the statement above is not true: the sequence $(-1)^n$ is bounded, but not convergent.

Theorem 4.12. *If the sequence (a_n) tends to infinity, then it is bounded from below and not bounded from above. If the sequence (a_n) tends to negative infinity, then it is bounded from above and not bounded from below.*

Proof. Let $\lim_{n\to\infty} a_n = \infty$. Comparing Definition 4.6 with the definition of being bounded from above (3.14) clearly shows that (a_n) cannot be bounded from above.

Pick, according to the notation in Definition 4.6, P to be 0. The sequence has only finitely many terms outside the interval $(0, \infty)$. If there are no terms outside the interval, then 0 is a lower bound. If there are terms outside the interval, then there are only finitely many. Out of these, the smallest one is the smallest term of the whole sequence, and thus a lower bound too. This shows that (a_n) is bounded from below. The case $a_n \to -\infty$ can be dealt with in a similar way. □

Remark 4.13. The converses of the statements of the theorem are not true. It is clear that in Example 4.1, the sequence (16) is bounded from below (the number 1 is a lower bound), not bounded from above, but the sequence doesn't tend to infinity.

Theorem 4.14. *Every sequence has at most one limit.*

Proof. By Theorems 4.10 and 4.12, it suffices to show that every convergent sequence has at most one limit. Suppose that $a_n \to b$ and $a_n \to c$ both hold, where b and c are distinct real numbers. Then for each $\varepsilon > 0$, only finitely many terms of the sequence lie outside the interval $(b-\varepsilon,\ b+\varepsilon)$, so there are infinitely many terms inside $(b-\varepsilon,\ b+\varepsilon)$. Let ε be so small that the intervals $(b-\varepsilon,\ b+\varepsilon)$ and $(c-\varepsilon,\ c+\varepsilon)$ are disjoint, that is, do not have any common points. (Such is, for

[3] By this, we mean that the set $\{a_n\}$ is bounded.

example, $\varepsilon = |c-b|/2$.) Then there are infinitely many terms of the sequence outside the interval $(c-\varepsilon,\ c+\varepsilon)$, which is impossible, since then c cannot be a limit of a_n. □

Exercises

4.21. Let

S be the set of all sequences;
C be the set of convergent sequences;
D be the set of divergent sequences;
D_∞ be the set of sequences diverging to ∞;
$D_{-\infty}$ be the set of sequences diverging to $-\infty$;
O be the set of sequences oscillating at infinity;
K be the set of bounded sequences.

Prove the following statements:

(a) $S = C \cup D$.
(b) $D = D_\infty \cup D_{-\infty} \cup O$.
(c) $C \subset K$.
(d) $K \cap D_\infty = \emptyset$.

4.22. Give an example for each possible behavior of a sequence (a_n) (convergent, tending to infinity, tending to negative infinity, oscillating at infinity), while $a_{n+1} - a_n \to 0$ also holds. (H)

4.23. Give an example for each possible behavior of a sequence (a_n) (convergent, tending to infinity, tending to negative infinity, oscillating at infinity), while $a_{n+1}/a_n \to 1$ also holds.

4.24. Give an example of a sequence (a_n) that

(a) is convergent,
(b) tends to infinity,
(c) tends to negative infinity,

while $a_n < (a_{n-1} + a_{n+1})/2$ holds for each $n > 1$.

4.25. Prove that if $a_n \to \infty$ and (b_n) is bounded, then $(a_n + b_n) \to \infty$.

4.26. Is it true that if (a_n) oscillates at infinity and is unbounded, and (b_n) is bounded, then $(a_n + b_n)$ oscillates at infinity and is unbounded?

4.27. Let (a_n) be sequence (18) from Example 4.1, that is, let a_n be the nth term in the decimal expansion of $\sqrt{2}$. Prove that the sequence (a_n) oscillates at infinity. (H)

4.4 Limits of Some Specific Sequences

Theorem 4.15.

(i) For every fixed integer p,

$$\lim_{n\to\infty} n^p = \begin{cases} \infty, & \text{if } p > 0, \\ 1, & \text{if } p = 0, \\ 0, & \text{if } p < 0. \end{cases} \tag{4.4}$$

(ii) If $p > 0$, then $\lim_{n\to\infty} \sqrt[p]{n} = \infty$.

Proof. (i) Let first $p > 0$. For arbitrary $P > 0$, if $n > P$, then $n^p \geq n > P$. Then $n^p \to \infty$. If $p = 0$, then $n^p = 1$ for every n, so $n^p \to 1$. Finally, if p is a negative integer and $\varepsilon > 0$, then $n > 1/\varepsilon$ implies $0 < n^p \leq 1/n < \varepsilon$, which proves that $n^p \to 0$. (ii) If $n > P^p$, then $\sqrt[p]{n} \geq P$, so $\sqrt[p]{n} \to \infty$. □

Theorem 4.16. *For every fixed real number a,*

$$\lim_{n\to\infty} a^n = \begin{cases} \infty, & \text{if } a > 1, \\ 1, & \text{if } a = 1, \\ 0, & \text{if } |a| < 1. \end{cases} \tag{4.5}$$

If $a \leq -1$, then (a^n) oscillates at infinity.

Proof. If $a > 1$, then by Bernoulli's inequality, we have that

$$a^n = (1 + (a-1))^n \geq 1 + n \cdot (a-1)$$

holds for every n. So for an arbitrary real number P, if $n > (P-1)/(a-1)$, then $a^n > P$, so $a^n \to \infty$.

If $a = 1$, then $a^n = 1$ for each n, and so $a^n \to 1$.

If $|a| < 1$, then $1/|a| > 1$. If an $\varepsilon > 0$ is given, then by the already proved statement, there is an n_0 such that for $n > n_0$,

$$\frac{1}{|a|^n} = \left(\frac{1}{|a|}\right)^n > \frac{1}{\varepsilon}$$

holds, that is, $|a^n| = |a|^n < \varepsilon$. This means that $a^n \to 0$ in this case.

Finally, if $a \leq -1$, then for even n, we have $a^n \geq 1$, while for odd n, we have $a^n \leq -1$. It is clear that a sequence with these properties can have neither a finite nor an infinite limit. □

Theorem 4.17.

(i) *For every fixed positive real number a, we have* $\lim_{n\to\infty} \sqrt[n]{a} = 1$.
(ii) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. Let $a > 0$ be fixed. If $0 < \varepsilon \le 1$, then by Theorem 4.16, $(1+\varepsilon)^n \to \infty$ and $(1-\varepsilon)^n \to 0$. Thus there exist n_1 and n_2 such that when $n > n_1$, we have $(1+\varepsilon)^n > a$, and when $n > n_2$, we have $(1-\varepsilon)^n < a$. So if $n > \max(n_1, n_2)$, then $(1-\varepsilon)^n < a < (1+\varepsilon)^n$, that is, $1-\varepsilon < \sqrt[n]{a} < 1+\varepsilon$. This shows that if $0 < \varepsilon \le 1$, then there is an n_0 such that when $n > n_0$ holds, we also have $|\sqrt[n]{a} - 1| < \varepsilon$. It follows from this that for arbitrary positive ε, we can find an n_0, since if $\varepsilon \ge 1$, then the n_0 corresponding to 1 also works. We have shown (i).

(ii) Let $0 < \varepsilon < 1$ be fixed. If $n > 4/\varepsilon^2$ is even, then by Bernoulli's inequality, we have $(1+\varepsilon)^{n/2} > n\varepsilon/2$, so

$$(1+\varepsilon)^n > \left(\frac{n}{2}\varepsilon\right)^2 > n.$$

If $n > 16/\varepsilon^2$ is odd, then $(1+\varepsilon)^{(n-1)/2} > (n-1)\varepsilon/2 > n\varepsilon/4$, which gives

$$(1+\varepsilon)^n > (1+\varepsilon)^{n-1} > \left(\frac{n-1}{2}\varepsilon\right)^2 > \left(\frac{n}{4}\varepsilon\right)^2 > n.$$

Therefore, if $n > 16/\varepsilon^2$, then we have $\sqrt[n]{n} < 1+\varepsilon$. □