# Chapter 5
# Infinite Sequences II

Finding the limit of a sequence is generally a difficult task. Sometimes, just determining whether a sequence has a limit is tough. Consider the sequence (18) in Example 4.1, that is, let $a_n$ be the $n$th digit in the decimal expansion of $\sqrt{2}$. We know that $(a_n)$ does not have a limit. But does the sequence $c_n = \sqrt[n]{a_n}$ have a limit? First of all, let us note that $a_n \geq 1$, and thus $c_n \geq 1$ for infinitely many $n$. Now if there are infinitely many zeros among the terms $a_n$, then $c_n = 0$ also holds for infinitely many $n$, so the sequence $(c_n)$ is divergent. However, if there are only finitely many zeros among the terms $a_n$, that is, $a_n \neq 0$ for all $n > n_0$, then $1 \leq a_n \leq 9$, and so $1 \leq c_n \leq \sqrt[n]{9}$ also holds if $n > n_0$. By Theorem 4.17, $\sqrt[n]{9} \to 1$. Thus for a given $\varepsilon > 0$, there is an $n_1$ such that $\sqrt[n]{9} < 1 + \varepsilon$ for all $n > n_1$. So if $n > \max(n_0, n_1)$, then $1 \leq c_n < 1 + \varepsilon$, and thus $c_n \to 1$.

This reasoning shows that the sequence $(c_n)$ has a limit if and only if there are only finitely many zeros among the terms $a_n$. However, *the question whether the decimal expansion of* $\sqrt{2}$ *has infinitely many zeros is a famous open problem in number theory.* Thus with our current knowledge, we are unable to determine whether the sequence $(c_n)$ has a limit.

Fortunately, the above example is atypical; we can generally determine the limits of sequences that we encounter in practice. In most cases, we use the method of decomposing the given sequence into simpler sequences whose limits we know. Of course, to determine the limit, we need to know how to find the limit of a sequence that is constructed from simpler sequences. In the following, this will be explored.

## 5.1 Basic Properties of Limits

**Definition 5.1.** For a sequence $(a_1, a_2, \ldots, a_n, \ldots)$, we say that a *subsequence* is a sequence of the form

$$(a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots),$$

where $n_1 < n_2 < \cdots < n_k < \ldots$ are positive integers.

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A subsequence, then, is formed by deleting some (possibly infinitely many) terms from the original sequence, keeping infinitely many.

**Theorem 5.2.** *If the sequence* $(a_n)$ *has a limit, then every subsequence* $(a_{n_k})$ *does too, and* $\lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} a_n$.

*Proof.* Suppose first that $(a_n)$ is convergent, and let $\lim_{n\to\infty} a_n = b$ be a finite limit. This means that for every positive $\varepsilon$, there are only finitely many terms of the sequence outside the interval $(b-\varepsilon, b+\varepsilon)$. Then clearly, the same holds for terms of the subsequence, which means exactly that $\lim_{k\to\infty} a_{n_k} = b$.

The statement can be proved similarly if $(a_n)$ tends to infinity or negative infinity. □

We should note that the existence of a limit for a subsequence does not imply the existence of a limit for the original sequence, as can be seen in sequences (3) and (16) in Example 4.1. However, if we already know that $(a_n)$ has a limit, then (by Theorem 5.2) every subsequence will have this same limit.

**Definition 5.3.** We say that the sequences $(a_n)$ and $(b_n)$ have *identical convergence behavior* if it is the case that $(a_n)$ has a limit if and only if $(b_n)$ has a limit, in which case the limits agree.

To determine limits of sequences, it is useful to inspect what changes we can make to a sequence that results in a new sequence whose convergence behavior is identical to that of the old one.

We will list a few such changes below:

I. We can "rearrange" a sequence, that is, we can change the order of its terms. A rearranged sequence contains the same numbers, and moreover, each one is listed the same number of times as in the previous sequence. (The formal definition is as follows: the sequence $(a_{n_1}, a_{n_2}, \ldots)$ is a rearrangement of the sequence $(a_1, a_2, \ldots)$ if the map $f(k) = n_k$ $(k \in \mathbb{N}^+)$ is a permutation of $\mathbb{N}^+$, which means that $f$ is a one-to-one correspondence from $\mathbb{N}^+$ to itself.)

II. We can repeat certain terms (possibly infinitely many) of a sequence finitely many times.

III. We can add finitely many terms to a sequence.

IV. We can take away finitely many terms from a sequence.

The above changes can, naturally, change the indices of the terms.

*Examples 5.4.* Consider the following sequences:

(1) $a_n = n$: $(a_n) = (1, 2, \ldots, n, \ldots)$;

(2) $a_n = n+2$: $(a_n) = (3, 4, 5, \ldots)$;

(3) $a_n = n-2$: $(a_n) = (-1, 0, 1, \ldots)$;

(4) $a_n = k$, if $k(k-1)/2 < n \le k(k+1)/2$:
$(a_n) = (1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, k, \ldots)$;

(5) $a_n = 2n - 1$: $(a_n) = (1, 3, 5, 7 \ldots)$;

(6) $a_n = \begin{cases} 0, & \text{if } n = 2k+1, \\ k, & \text{if } n = 2k \end{cases}$: $(a_n) = (0, 1, 0, 2, 0, 3, \ldots)$;

(7) $a_n = \begin{cases} n+1, & \text{if } n = 2k+1, \\ n-1, & \text{if } n = 2k \end{cases}$: $(a_n) = (2, 1, 4, 3, 6, 5, \ldots)$.

In the above sequences, starting from the sequence (1), we can get

(2) by a type IV change,
(3) by a type III change,
(4) by a type II change,
(7) by a type I change.

Moreover, (6) cannot be a result of type I–IV changes applied to (1), since there is only one new term in (6), 0, but it appears infinitely many times. (Although we gained infinitely many new numbers in (4), too, we can get there using II.)

**Theorem 5.5.** *The sequences $(a_n)$ and $(b_n)$ have identical convergence behavior if one can be reached from the other by applying a finite combination of type I–IV changes.*

*Proof.* The property that there are finitely or infinitely many terms outside an interval remains unchanged by each one of the type I–IV changes. From this, by Definition 4.2 and Definition 4.6, the statement of the theorem is clear. □

## *Exercises*

**5.1.** Prove that if every subsequence of $(a_n)$ has a subsequence that tends to $b$, then $a_n \to b$.

**5.2.** Prove that if the sequence $(a_n)$ does not have a subsequence tending to infinity, then $(a_n)$ is bounded from above.

**5.3.** Prove that if $(a_{2n})$, $(a_{2n+1})$, and $(a_{3n})$ are convergent, then $(a_n)$ is as well.

**5.4.** Give an example for an $(a_n)$ that is divergent, but for each $k > 1$, the subsequence $(a_{kn})$ is convergent. (H)

## 5.2 Limits and Inequalities

First of all, we introduce some terminology. Let $(A_n)$ be a sequence of statements. We say that $A_n$ *holds for all n sufficiently large* if there exists an $n_0$ such that $A_n$ is true for all $n > n_0$. So, for example, we can say that $2^n > n^2$ for all $n$ sufficiently large, since this inequality holds for all $n > 4$.

**Theorem 5.6.** *If* $\lim_{n\to\infty} a_n = \infty$ *and* $b_n \geq a_n$ *for all n sufficiently large, then* $\lim_{n\to\infty} b_n = \infty$. *If* $\lim_{n\to\infty} a_n = -\infty$ *and* $b_n \leq a_n$ *for all n sufficiently large, then* $\lim_{n\to\infty} b_n = -\infty$.

*Proof.* Suppose that $b_n \geq a_n$ if $n > n_0$. If $a_n > P$ for all $n > n_1$, then $b_n \geq a_n > P$ holds for all $n > \max(n_0, n_1)$. From this, it is clear that when $\lim_{n\to\infty} a_n = \infty$ holds, $\lim_{n\to\infty} b_n = \infty$ also holds. The second statement follows in the same way. □

The following theorem is often known as the **squeeze theorem** (or **sandwich theorem**).

**Theorem 5.7.** *If* $a_n \leq b_n \leq c_n$ *for all n sufficiently large and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a,$$

*then* $\lim_{n\to\infty} b_n = a$.

*Proof.* By the previous theorem, it suffices to restrict ourselves to the case that $a$ is finite. Suppose that $a_n \leq b_n \leq c_n$ for all $n > n_0$. It follows from our assumption that for every $\varepsilon > 0$, there exist $n_1$ and $n_2$ such that

$$a - \varepsilon < a_n < a + \varepsilon, \quad \text{if} \quad n > n_1$$

and

$$a - \varepsilon < c_n < a + \varepsilon, \quad \text{if} \quad n > n_2.$$

Then for $n > \max(n_0, n_1, n_2)$,

$$a - \varepsilon < a_n \leq b_n \leq c_n < a + \varepsilon,$$

that is, $b_n \to a$. □

We often use the above theorem for the special case $a_n \equiv 0$: if $0 \leq b_n \leq c_n$ and $\lim_{n\to\infty} c_n = 0$, then $\lim_{n\to\infty} b_n = 0$.

The following theorems state that strict inequality between limits is inherited by terms of sufficiently large index, while not-strict inequality between terms is inherited by limits.

**Theorem 5.8.** *Let* $(a_n)$ *and* $(b_n)$ *be convergent sequences, and let* $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. *If* $a < b$, *then* $a_n < b_n$ *holds for all n sufficiently large.*

*Proof.* Let $\varepsilon = (b-a)/2$. We know that for suitable $n_1$ and $n_2$, $a_n < a + \varepsilon$ if $n > n_1$, and $b_n > b - \varepsilon$ if $n > n_2$. Let $n_0 = \max(n_1, n_2)$. If $n > n_0$, then both inequalities hold, that is, $a_n < a + \varepsilon = b - \varepsilon < b_n$. □

*Remark 5.9.* Note that from the weaker assumption $a \leq b$, we generally do not get that $a_n \leq b_n$ holds even for a single index. If, for example, $a_n = 1/n$ and $b_n = -1/n$, then $\lim_{n\to\infty} a_n = 0 \leq 0 = \lim_{n\to\infty} b_n$, but $a_n > b_n$ for all $n$.

**Theorem 5.10.** *Let* $(a_n)$ *and* $(b_n)$ *be convergent sequences, and let* $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. *If* $a_n \leq b_n$ *holds for all sufficiently large n, then* $a \leq b$.

*Proof.* Suppose that $a > b$. By Theorem 5.8, it follows that $a_n > b_n$ for all $n$ sufficiently large, which contradicts our assumption. $\square$

*Remark 5.11.* Note that even the assumption $a_n < b_n$ does not imply $a < b$. If, for example, $a_n = -1/n$ and $b_n = 1/n$, then $a_n < b_n$ for all $n$, but $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$.

## *Exercises*

**5.5.** Prove that if $a_n \to a > 1$, then $(a_n)^n \to \infty$.

**5.6.** Prove that if $a_n \to a$, where $|a| < 1$, then $(a_n)^n \to 0$.

**5.7.** Prove that if $a_n \to a > 0$, then $\sqrt[n]{a_n} \to 1$.

**5.8.** $\lim_{n\to\infty} \sqrt[n]{2^n - n} = ?$

**5.9.** Prove that if $a_1, \ldots, a_k \geq 0$, then

$$\lim_{n\to\infty} \sqrt[n]{a_1^n + \cdots + a_k^n} = \max_{1\leq i\leq k} a_i. \qquad \text{(S)}$$

## 5.3 Limits and Operations

We say that the **sum of the sequences** $(a_n)$ and $(b_n)$ is the sequence $(a_n + b_n)$. The following theorem states that in most cases, the order of taking sums and taking limits can be switched, that is, the sum of the limits is equal to the limit of the sum of terms.

**Theorem 5.12.**

(i) *If the sequences $(a_n)$ and $(b_n)$ are convergent and $a_n \to a$, $b_n \to b$, then the sequence $(a_n + b_n)$ is convergent and $a_n + b_n \to a + b$.*
(ii) *If the sequence $(a_n)$ is convergent, $a_n \to a$ and $b_n \to \infty$, then $a_n + b_n \to \infty$.*
(iii) *If the sequence $(a_n)$ is convergent, $a_n \to a$ and $b_n \to -\infty$, then $a_n + b_n \to -\infty$.*
(iv) *If $a_n \to \infty$ and $b_n \to \infty$, then $a_n + b_n \to \infty$.*
(v) *If $a_n \to -\infty$ and $b_n \to -\infty$, then $a_n + b_n \to -\infty$.*

*Proof.* (i) Intuitively, if $a_n$ is close to $a$ and $b_n$ is close to $b$, then $a_n + b_n$ is close to $a + b$. Basically, we only need to make this idea precise using limits.

If $a_n \to a$ and $b_n \to b$, then for all $\varepsilon > 0$, there exist $n_1$ and $n_2$ for which $|a_n - a| < \varepsilon/2$ holds if $n > n_1$, and $|b_n - b| < \varepsilon/2$ holds if $n > n_2$. It follows from this, using the triangle inequality, that

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \varepsilon,$$

if $n > \max(n_1, n_2)$. Since $\varepsilon$ was arbitrary, this proves that $a_n + b_n \to a + b$.

(ii) If $(a_n)$ is convergent, then by Theorem 4.10, it is bounded. This means that for suitable $K > 0$, $|a_n| \le K$ for all $n$. Let $P$ be arbitrary. Since $b_n \to \infty$, there exists an $n_0$ such that when $n > n_0$, $b_n > P + K$. Then $a_n + b_n > (-K) + (P + K) = P$ for all $n > n_0$. Since $P$ was arbitrary, this proves that $a_n + b_n \to \infty$. Statement (iii) can be proved in the same way.
(iv) Suppose that $a_n \to \infty$ and $b_n \to \infty$. Let $P$ be arbitrary. Then there are $n_1$ and $n_2$ such that when $n > n_1$, $a_n > P/2$, and when $n > n_2$, $b_n > P/2$. If $n > \max(n_1, n_2)$, then $a_n + b_n > (P/2) + (P/2) = P$. Since $P$ was chosen arbitrarily, this proves that $a_n + b_n \to \infty$. Statement (v) can be proved in the same way. □

If $(a_n)$ is convergent and $a_n \to a$, then by applying statement (i) in Theorem 5.12 to the constant sequence $b_n = -a$, we get that $a_n - a \to 0$. Conversely, if $a_n - a \to 0$, then $a_n = (a_n - a) + a \to a$. This shows the following.

**Corollary 5.13.** *A sequence $(a_n)$ tends to a finite limit $a$ if and only if $a_n - a \to 0$.*

The statements of Theorem 5.12 can be summarized in the table below.

lim $b_n$

| | | $b$ | $\infty$ | $-\infty$ |
|---|---|---|---|---|
| | $a$ | $a+b$ | $\infty$ | $-\infty$ |
| lim $a_n$ | $\infty$ | $\infty$ | $\infty$ | ? |
| | $-\infty$ | $-\infty$ | ? | $-\infty$ |

The question marks appearing in the table mean that the given values of $\lim a_n$ and $\lim b_n$ do not determine the value of $\lim(a_n + b_n)$. Specifically, if $\lim a_n = \infty$ and $\lim b_n = -\infty$ (or the other way around), then using only this information, we cannot say what value $\lim(a_n + b_n)$ takes. Let us see a few examples.

$$\begin{array}{lll} a_n = n + c, & b_n = -n, & a_n + b_n = c \to c \in \mathbb{R} \\ a_n = 2n, & b_n = -n, & a_n + b_n = n \to \infty \\ a_n = n, & b_n = -2n, & a_n + b_n = -n \to -\infty \\ a_n = n + (-1)^n, & b_n = -n, & a_n + b_n = (-1)^n \text{ oscillates at infinity.} \end{array}$$

We see that $(a_n + b_n)$ can be convergent, can diverge to positive or negative infinity, and can oscillate at infinity. We express this by saying that the limit $\lim(a_n + b_n)$ is a **critical limit** if $\lim a_n = \infty$ and $\lim b_n = -\infty$. More concisely, we can say that limits of the type $\infty - \infty$ are critical.

Now we look at the limit of a product. We call the **product of two sequences** $(a_n)$ and $(b_n)$ the sequence $(a_n \cdot b_n)$.

**Theorem 5.14.**

(i) *If the sequences $(a_n)$ and $(b_n)$ are convergent and $a_n \to a$, $b_n \to b$, then the sequence $(a_n \cdot b_n)$ is convergent, and $a_n \cdot b_n \to a \cdot b$.*

(ii) *If the sequence* $(a_n)$ *is convergent,* $a_n \to a$ *where* $a > 0$*, and* $b_n \to \pm\infty$*, then* $a_n \cdot b_n \to \pm\infty$.

(iii) *If the sequence* $(a_n)$ *is convergent,* $a_n \to a$ *where* $a < 0$*, and* $b_n \to \pm\infty$*, then* $a_n \cdot b_n \to \mp\infty$.

(iv) *If* $a_n \to \pm\infty$ *and* $b_n \to \pm\infty$*,then* $a_n \cdot b_n \to \infty$.

(v) *If* $a_n \to \pm\infty$ *and* $b_n \to \mp\infty$*, then* $a_n \cdot b_n \to -\infty$.

**Lemma 5.15.** *If* $a_n \to 0$ *and* $(b_n)$ *is bounded, then* $a_n \cdot b_n \to 0$.

*Proof.* Since $(b_n)$ is bounded, there is a $K > 0$ such that $|b_n| \le K$ for all $n$. Let $\varepsilon > 0$ be given. From the assumption that $a_n \to 0$, it follows that $|a_n| < \varepsilon/K$ for all $n$ sufficiently large. Thus $|a_n \cdot b_n| < (\varepsilon/K) \cdot K = \varepsilon$ for all $n$ sufficiently large. Since $\varepsilon$ was chosen arbitrarily, this proves that $a_n \cdot b_n \to 0$. □

*Proof (Theorem 5.14).* (i) Since by our assumption, $(a_n)$ and $(b_n)$ are convergent, by Theorem 4.10, both are bounded. If $a_n \to a$ and $b_n \to b$, then by Corollary 5.13, $a_n - a \to 0$ and $b_n - b \to 0$. Moreover,

$$a_n \cdot b_n - a \cdot b = (a_n - a) \cdot b_n + a \cdot (b_n - b). \tag{5.1}$$

By Lemma 5.15, both terms on the right-hand side tend to 0, so by Theorem 5.12, the limit of the right-hand side is 0. Then $a_n \cdot b_n - a \cdot b \to 0$, so by Corollary 5.13, $a_n \cdot b_n \to a \cdot b$.

(ii) Suppose that $a_n \to a > 0$ and $b_n \to \infty$. Let $P$ be an arbitrary positive number. Since $a/2 < a$, there exists an $n_1$ such that $a_n > a/2$ for all $n > n_1$. By the assumption that $b_n \to \infty$, there exists an $n_2$ such that $b_n > 2P/a$ for all $n > n_2$. Then for $n > \max(n_1, n_2)$, $a_n \cdot b_n > (a/2) \cdot (2P/a) = P$. Since $P$ was chosen arbitrarily, this proves that $a_n \cdot b_n \to \infty$. It can be shown in the same way that if $a_n \to a > 0$ and $b_n \to -\infty$, then $a_n \cdot b_n \to -\infty$. Statement (iii) can also be proved in the same manner.

(iv) If $a_n \to \infty$ and $b_n \to \infty$, then for all $P > 0$, there exist $n_1$ and $n_2$ such that for all $n > n_1$, $a_n > P$, and for all $n > n_2$, $b_n > 1$. Then for $n > \max(n_1, n_2)$, we have $a_n \cdot b_n > P \cdot 1 = P$. It can be shown in the same way that if $a_n \to -\infty$ and $b_n \to -\infty$, then $a_n \cdot b_n \to \infty$. Statement (v) can also be proved in the same manner. □

The statements of Theorem 5.14 are summarized in the table below.

lim $b_n$

| | | $b > 0$ | $0$ | $b < 0$ | $\infty$ | $-\infty$ |
|---|---|---|---|---|---|---|
| lim $a_n$ | $a > 0$ | $a \cdot b$ | $0$ | $a \cdot b$ | $\infty$ | $-\infty$ |
| | $0$ | $0$ | $0$ | $0$ | ? | ? |
| | $a < 0$ | $a \cdot b$ | $0$ | $a \cdot b$ | $-\infty$ | $\infty$ |
| | $\infty$ | $\infty$ | ? | $-\infty$ | $\infty$ | $-\infty$ |
| | $-\infty$ | $-\infty$ | ? | $\infty$ | $-\infty$ | $\infty$ |

The question marks indicate the critical limits again. As the examples below show, $\lim(a_n \cdot b_n)$ is critical if $a_n \to 0$ and $b_n \to \infty$. (In short, the limit of type $0 \cdot \infty$ is critical.)

$$\begin{array}{lll}
a_n = c/n, & b_n = n, & a_n \cdot b_n = c \to c \in \mathbb{R} \\
a_n = 1/n, & b_n = n^2, & a_n \cdot b_n = n \to \infty \\
a_n = -1/n, & b_n = n^2, & a_n \cdot b_n = -n \to -\infty \\
a_n = (-1)^n/n, & b_n = n, & a_n \cdot b_n = (-1)^n \text{ oscillates at infinity.}
\end{array}$$

Similar examples show that the limit of type $0 \cdot (-\infty)$ is also critical.

We now turn to defining quotient limits. Suppose that $b_n \neq 0$ for all $n$. We sometimes call the sequence $(a_n/b_n)$ the **quotient sequence**.

**Theorem 5.16.** *Suppose that the sequences $(a_n)$ and $(b_n)$ have limits, and that $b_n \neq 0$ for all $n$. Then the limit of the sequence $(a_n/b_n)$ is given by the table below.*

$\lim b_n$

| | | $b>0$ | $0$ | $b<0$ | $\infty$ | $-\infty$ |
|---|---|---|---|---|---|---|
| | $a>0$ | $a/b$ | ? | $a/b$ | $0$ | $0$ |
| | $0$ | $0$ | ? | $0$ | $0$ | $0$ |
| $\lim a_n$ | $a<0$ | $a/b$ | ? | $a/b$ | $0$ | $0$ |
| | $\infty$ | $\infty$ | ? | $-\infty$ | ? | ? |
| | $-\infty$ | $-\infty$ | ? | $\infty$ | ? | ? |

**Lemma 5.17.** *If $(b_n)$ is convergent and $b_n \to b \neq 0$, then $1/b_n \to 1/b$.*

*Proof.* Let $\varepsilon > 0$ be given; we need to show that $|1/b_n - 1/b| < \varepsilon$ if $n$ is sufficiently large. Since

$$\frac{1}{b_n} - \frac{1}{b} = \frac{b - b_n}{b \cdot b_n},$$

we have to show that if $n$ is large, then $b - b_n$ is very small, while $b \cdot b_n$ is not too small. By the assumption that $b_n \to b$, there exists an $n_1$ such that for $n > n_1$, $|b_n - b| < \varepsilon b^2/2$. Since $|b|/2 > 0$, we can find an $n_2$ such that $|b_n - b| < |b|/2$ for $n > n_2$. Then for $n > n_2$, $|b_n| > |b|/2$, since if $b > 0$, then $b_n > b - (b/2) = b/2$, while if $b < 0$, then $b_n < b + (|b|/2) = b/2 = -|b|/2$. Then for $n > \max(n_1, n_2)$,

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b \cdot b_n|} < \frac{\varepsilon b^2/2}{|b| \cdot |b|/2} = \varepsilon.$$

Since $\varepsilon$ was arbitrary, this proves that $1/b_n \to 1/b$. □

**Lemma 5.18.** *If $|b_n| \to \infty$, then $1/b_n \to 0$.*

*Proof.* Let $\varepsilon > 0$ be given. Since $|b_n| \to \infty$, there is an $n_0$ such that for $n > n_0$, $|b_n| > 1/\varepsilon$. Then when $n > n_0$, $|1/b_n| = 1/|b_n| < \varepsilon$, so $1/b_n \to 0$. □

**Corollary 5.19.** *If $b_n \to \infty$ or $b_n \to -\infty$, then $1/b_n \to 0$.*

*Proof.* It is easy to check that if $b_n \to \infty$ or $b_n \to -\infty$, then $|b_n| \to \infty$. □

*Proof (Theorem 5.16).* Suppose first that $a_n \to a \in \mathbb{R}$ and $b_n \to b \in \mathbb{R}$, $b \neq 0$. By Theorem 5.14 and Lemma 5.17,

$$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \to a \cdot \frac{1}{b} = \frac{a}{b}.$$

If $(a_n)$ is convergent and $b_n \to \infty$ or $b_n \to -\infty$, then by Theorem 5.14 and Corollary 5.19,

$$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \to a \cdot 0 = 0.$$

Now suppose that $a_n \to \infty$ and $b_n \to b \in \mathbb{R},\ b > 0$. Then by Theorem 5.14 and Lemma 5.17,

$$\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n} \to \infty,$$

since $1/b_n \to 1/b > 0$. It can be seen similarly that in the case of $a_n \to \infty$ and $b_n \to b < 0$, we have $a_n/b_n \to -\infty$; in the case of $a_n \to -\infty$ and $b_n \to b > 0$, we have $a_n/b_n \to -\infty$; while in the case of $a_n \to -\infty$ and $b_n \to b < 0$, we have $a_n/b_n \to \infty$. With this, we have justified every (non-question-mark) entry in the table. □

The question marks in the table of Theorem 5.16 once more denote the critical limits. Here, however, we need to distinguish two levels of criticality. As the examples below show, the limit of type $0/0$ is critical in the same sense that, for example, the limit of type $0 \cdot \infty$ is.

$$\begin{array}{lll}
a_n = c/n, & b_n = 1/n, & a_n/b_n = c \to c \in \mathbb{R} \\
a_n = 1/n, & b_n = 1/n^2, & a_n/b_n = n \to \infty \\
a_n = -1/n, & b_n = 1/n^2, & a_n/b_n = -n \to -\infty \\
a_n = (-1)^n/n, & b_n = 1/n, & a_n/b_n = (-1)^n \text{ oscillates at infinity.}
\end{array}$$

We can see that if $a_n \to 0$ and $b_n \to 0$, then $(a_n/b_n)$ can be convergent, can tend to infinity or negative infinity, and can oscillate at infinity as well.

The situation is different with the other question marks in the table of theorem 5.16. Consider the case that $a_n \to a > 0$ and $b_n \to 0$. The examples $a_n = 1$, $b_n = 1/n$; $a_n = 1$, $b_n = -1/n$; and $a_n = 1$, $b_n = (-1)^n/n$ show that $a_n/b_n$ can tend to infinity or negative infinity, but can oscillate at infinity as well. However, we do not find an example in which $a_n/b_n$ is convergent. This follows immediately from the following theorem.

**Theorem 5.20.**

(i) *Suppose that $b_n \to 0$ and $b_n \neq 0$ for all $n$. Then $1/|b_n| \to \infty$.*
(ii) *Suppose that $a_n \to a \neq 0$, $b_n \to 0$, and $b_n \neq 0$ for all $n$. Then $|a_n/b_n| \to \infty$.*

*Proof.* It is enough to prove (ii). Let $P > 0$ be given. There exists an $n_0$ such that for $n > n_0$, $|a_n| > |a|/2$ and $|b_n| < |a|/(2P)$. Then for $n > n_0$, $|a_n/b_n| > P$, that is, $|a_n/b_n| \to \infty$. □

Finally let us consider the case that $a_n \to \infty$ and $b_n \to \infty$. The examples $a_n = b_n = n$ and $a_n = n^2$, $b_n = n$ show that $a_n/b_n$ can be convergent and can tend to infinity as well. Now let

$$a_n = \begin{cases} n^2, & \text{if } n \text{ is even,} \\ n, & \text{if } n \text{ is odd} \end{cases}$$

and $b_n = n$. It is clear that $a_n \to \infty$, and $a_n/b_n$ agrees with sequence (16) from Example 4.1, which oscillates at infinity. However, we cannot find an example in which $a_n/b_n$ tends to negative infinity. Since both $a_n \to \infty$ and $b_n \to \infty$, $a_n$ and $b_n$ are both positive for all sufficiently large $n$. Thus $a_n/b_n$ is positive for all sufficiently large $n$, so it cannot tend to negative infinity. Similar observations can be made for the three remaining cases, when $(a_n)$ and $(b_n)$ tend to infinity or negative infinity.

### *Exercises*

**5.10.** Prove that if $(a_n + b_n)$ is convergent and $(b_n)$ is divergent, then $(a_n)$ is divergent.

**5.11.** Is it true that if $(a_n \cdot b_n)$ is convergent and $(b_n)$ is divergent, then $(a_n)$ is also divergent?

**5.12.** Is it true that if $(a_n/b_n)$ is convergent and $(b_n)$ is divergent, then $(a_n)$ is also divergent?

**5.13.** Prove that if $\lim\limits_{n\to\infty}(a_n - 1)/(a_n + 1) = 0$, then $\lim\limits_{n\to\infty} a_n = 1$.

**5.14.** Let $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Prove that $\max(a_n, b_n) \to \max(a, b)$.

**5.15.** Prove that if $a_n < 0$ and $a_n \to 0$, then $1/a_n \to -\infty$.

## 5.4 Applications

First of all, we need generalizations of 5.12 (i) and 5.14 (i) for more summands and terms, respectively.

**Theorem 5.21.** *Let* $\left(a_n^1\right), \ldots, \left(a_n^k\right)$ *be convergent sequences,*[1] *and let* $\lim_{n\to\infty} a_n^i = b_i$ *for all* $i = 1, \ldots, k$. *Then the sequences* $\left(a_n^1 + \cdots + a_n^k\right)$ *and* $\left(a_n^1 \cdot \ldots \cdot a_n^k\right)$ *are also convergent, and their limits are* $b_1 + \cdots + b_k$ *and* $b_1 \cdot \ldots \cdot b_k$ *respectively.*

The statement can be proved easily by induction on $k$, using that the $k = 2$ case was already proved in Theorems 5.12 and 5.14.

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[1] here $a_n^i$ denotes the $n$th term of the $i$th sequence

It is important to note that Theorem 5.21 *holds only for a fixed number of sequences*, that is, the assumptions of the theorem do not allow the number of sequences ($k$) to depend on $n$. Consider

$$1 = \frac{1}{n} + \cdots + \frac{1}{n},$$

if the number of summands on the right-hand side is exactly $n$. Despite $1/n$ tending to 0, the sum on the left-hand side—the constant-1 sequence—still tends to 1. Similarly,

$$2 = \sqrt[n]{2} \cdot \ldots \cdot \sqrt[n]{2},$$

if the number of terms on the right side is exactly $n$. Even though $\sqrt[n]{2}$ tends to 1, the product—the constant-2 sequence—still tends to 2.

As a first application, we will determine the limits of sequences that can be obtained from the index $n$ and constants, using the four elementary operations.

**Theorem 5.22.** *Let*

$$c_n = \frac{a_0 + a_1 n + \cdots + a_k n^k}{b_0 + b_1 n + \cdots + b_\ell n^\ell} \quad (n = 1, 2, \ldots),$$

*where $a_k \neq 0$ and $b_\ell \neq 0$. Then*

$$\lim_{n\to\infty} c_n = \begin{cases} 0, & \text{if } \ell > k, \\ \infty, & \text{if } \ell < k \text{ and } a_k/b_\ell > 0, \\ -\infty, & \text{if } \ell < k \text{ and } a_k/b_\ell < 0, \\ a_k/b_\ell, & \text{if } \ell = k. \end{cases}$$

*Proof.* If we take out an $n^k$ from the numerator and $n^\ell$ from the denominator of the fraction representing $c_k$, then we get that

$$c_n = \frac{n^k}{n^\ell} \cdot \frac{a_k + \dfrac{a_{k-1}}{n} + \cdots + \dfrac{a_0}{n^k}}{b_\ell + \dfrac{b_{\ell-1}}{n} + \cdots + \dfrac{b_0}{n^\ell}}. \tag{5.2}$$

By Theorem 4.15, in the second term, except for the first summands, everything in both the numerator and the denominator tends to 0.

Then by applying Theorems 5.21 and 5.16, we get that the second term on the right-hand side of (5.2) tends to $a_k/b_\ell$. From this, based on the behavior of $n^{k-\ell}$ as $n \to \infty$ (Theorem 4.15), the statement immediately follows. □

An important sufficient condition for convergence to 0 is given by the next theorem.

**Theorem 5.23.** *Suppose that there exists a number $q < 1$ such that $a_n \neq 0$ and $|a_{n+1}/a_n| \leq q$ for all $n$ sufficiently large. Then $a_n \to 0$.*

*Proof.* If $a_n \neq 0$ and $|a_{n+1}/a_n| \leq q$ for every $n \geq n_0$, then

$$\begin{aligned} |a_{n_0+1}| &\leq q \cdot |a_{n_0}|, \\ |a_{n_0+2}| &\leq q \cdot |a_{n_0+1}| \leq q^2 \cdot |a_{n_0}|, \\ |a_{n_0+3}| &\leq q \cdot |a_{n_0+2}| \leq q^3 \cdot |a_{n_0}|, \end{aligned} \tag{5.3}$$

and so on; all inequalities $|a_n| \leq q^{n-n_0} \cdot |a_{n_0}|$ for $n > n_0$ hold. Since $q^n \to 0$ by Theorem 4.16, $a_n \to 0$. □

**Corollary 5.24.** *Suppose that $a_n \neq 0$ for all sufficiently large n, and $a_{n+1}/a_n \to c$, where $|c| < 1$. Then $a_n \to 0$.*

*Proof.* Fix a number $q$, for which $|c| < q < 1$. Since $|a_{n+1}/a_n| \to |c|$, $|a_{n+1}/a_n| < q$ for all sufficiently large $n$, so we can apply Theorem 5.23. □

*Remark 5.25.* The assumptions required for Theorem 5.23 and Corollary 5.24 are generally *not necessary conditions* to deduce $a_n \to 0$. The sequence $a_n = 1/n$ tends to zero, but $a_{n+1}/a_n = n/(n+1) \to 1$, so neither the assumptions of Theorem 5.23 nor the assumptions of Corollary 5.24 are satisfied.

Corollary 5.24 can often be applied to sequences that are given as a product. The following theorem introduces two important special cases. The notation $n!$ in the second statement denotes the product $1 \cdot 2 \cdot \ldots \cdot n$. We call this product $n$ **factorial**.

**Theorem 5.26.**

(i) *For an arbitrary real number $a > 1$ and positive integer k, we have $n^k/a^n \to 0$.*
(ii) *For an arbitrary real number a, $a^n/n! \to 0$.*

*Proof.* (i) Let $a_n = n^k/a^n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^k \cdot a^n}{n^k \cdot a^{n+1}} = \frac{\left(1+\frac{1}{n}\right)^k}{a}.$$

Here the numerator tends to 1 by Theorem 5.21, and so $a_{n+1}/a_n \to 1/a$. Since $a > 1$ by our assumption, $0 < 1/a < 1$, so we can apply Corollary 5.24.
(ii) If $b_n = a^n/n!$, then

$$\frac{b_{n+1}}{b_n} = \frac{a^{n+1} \cdot n!}{a^n \cdot (n+1)!} = \frac{a}{n+1}.$$

Then $b_{n+1}/b_n \to 0$, so $b_n \to 0$ by Corollary 5.24. □

By Theorem 4.16, $a^n \to \infty$ if $a > 1$. We also know that $n^k \to \infty$ if $k$ is a positive integer. By statement (i) of the above theorem, we can determine that $a^n/n^k \to \infty$ (see Theorem 5.20), which means that the sequence $(a^n)$ is "much bigger" than the sequence $(n^k)$. To make this phenomenon clear, we introduce new terminology and notation below.

**Definition 5.27.** Let $(a_n)$ and $(b_n)$ be sequences tending to infinity. We say that the sequence $(a_n)$ *tends to infinity faster* than the sequence $(b_n)$ if $a_n/b_n \to \infty$. We can also express this by saying that $(a_n)$ *has a larger order of magnitude than* $(b_n)$, and we denote this by $(b_n) \prec (a_n)$.

By Theorems 4.15, 4.16, and 5.26, we can conclude that the following order-of-magnitude relations hold:

$$(n) \prec (n^2) \prec (n^3) \prec \ldots \prec (2^n) \prec (3^n) \prec (4^n) \prec \ldots \prec (n!) \prec n^n.$$

**Definition 5.28.** If $a_n \to \infty$, $b_n \to \infty$, and $a_n/b_n \to 1$, then we say that the sequences $(a_n)$ and $(b_n)$ are *asymptotically equivalent*, and we denote this by $a_n \sim b_n$.

Thus, for example, $(n^2+n) \sim n^2$, since $(n^2+n)/n^2 = 1+(1/n) \to 1$.

## *Exercises*

**5.16.** Prove that if $a_n > 0$ and $a_{n+1}/a_n > q > 1$, then $a_n \to \infty$.

**5.17.** Prove that if $a_n > 0$ and $a_{n+1}/a_n \to c$, where $c > 1$, then $a_n \to \infty$.

**5.18.** Show that if $a_n > 0$ and $a_{n+1}/a_n \to q$, then $\sqrt[n]{a_n} \to q$.

**5.19.** Give an example of a positive sequence $(a_n)$ for which $\sqrt[n]{a_n} \to 1$, but $a_{n+1}/a_n$ does not tend to 1.

**5.20.** Prove that $\lim_{n\to\infty} 2^{\sqrt{n}}/n^k = \infty$ for all $k$.

**5.21.** Let $(a_n^1), (a_n^2), \ldots$ be an arbitrary sequence of sequences tending to infinity. (Here $a_n^k$ denotes the $n$th term of the $k$th sequence.) Prove that there exists a sequence $b_n \to \infty$, whose order of magnitude is larger than the order of magnitude of every $(a_n^k)$. (H)

**5.22.** Suppose that

$$(a_n^1) \prec (a_n^2) \prec \ldots \prec (b_n^2) \prec (b_n^1).$$

Prove that there exists a sequence $(c_n)$ for which $(a_n^k) \prec (c_n) \prec (b_n^k)$ for all $k$.

**5.23.** Let $p(n) = a_0 + a_1 n + \cdots + a_k n^k$, where $a_k > 0$. Prove that $p(n+1) \sim p(n)$.