

Chapter 15

Integration

In this chapter, we will familiarize ourselves with the most important methods for computing integrals, which will also make the link between definite and indefinite integrals clear.

15.1 The Link Between Integration and Differentiation

Examples 14.8 and 14.21 both provide equalities of the form $\int_a^b f(x) dx = F(b) - F(a)$ with the following cast:

$$\begin{aligned} f(x) &\equiv c, & F(x) &= c \cdot x; \\ f(x) &= 1/x, & F(x) &= \log x; \\ f(x) &= x^\alpha, & F(x) &= \frac{1}{\alpha+1} \cdot x^{\alpha+1} \quad (\alpha > 0); \\ f(x) &= e^x, & F(x) &= e^x; \\ f(x) &= \cos x, & F(x) &= \sin x. \end{aligned}$$

As we can see, in each example, $F' = f$, that is, F is a primitive function of f . These examples illustrate an important link between integration and differentiation, and are special cases of a famous general theorem.

Theorem 15.1 (Fundamental Theorem of Calculus). *Let f be integrable on $[a, b]$. If the function F is continuous on $[a, b]$, differentiable on (a, b) , and $F'(x) = f(x)$ for all $x \in (a, b)$ (that is, F is a primitive function of f on (a, b)), then*

$$\int_a^b f(x) dx = F(b) - F(a).$$

Proof. Let $a = x_0 < x_1 < \dots < x_n = b$ be an arbitrary partition of $[a, b]$. By the mean value theorem (Theorem 12.50), for all i , there exists a point $c_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)(x_i - x_{i-1})$$

holds. If we sum these equalities for all $i = 1, \dots, n$, then every term cancels out on the left-hand side except for the terms $F(x_n) = F(b)$ and $F(x_0) = F(a)$, and so we get that

$$F(b) - F(a) = \sum_{i=1}^n f(c_i)(x_i - x_{i-1}).$$

This means that for every partition, there exist inner points such that the Riemann sum with those points is equal to $F(b) - F(a)$. Thus the number $F(b) - F(a)$ lies between the lower and upper sums for every partition. Since f is integrable, there is only one such number: the integral of f . Thus $F(b) - F(a) = \int_a^b f(x) dx$. \square

Remark 15.2. While making clear the definition of differentiability back in Chapter 12, we concluded that if the function $s(t)$ defines the position of a moving point, then its instantaneous velocity is $v(t) = s'(t)$. Since $s(b) - s(a)$ is the distance the point travels during the time interval $[a, b]$, the physical interpretation of the fundamental theorem of calculus says that the distance traveled is equal to the integral of the velocity.

As we saw in Chapter 13, deciding whether a function has a primitive function is generally a hard task (see Remarks 13.27 and 13.45). However, if the function f is integrable, then deciding this question—with the help of the fundamental theorem of calculus—is quite easy. Suppose, for example, that f is integrable on $[a, b]$, and that F is a primitive function of f . We can assume that $F(a) = 0$, since if this does not hold, we can just consider the function $F(x) - F(a)$ instead of $F(x)$. Let $x \in [a, b]$, and apply the fundamental theorem of calculus to the interval $[a, x]$. We get that $\int_a^x f(t) dt = F(x) - F(a) = F(x)$, that is, $F(x) = \int_a^x f(t) dt$ for all $x \in [a, b]$. This means that if f has a primitive function, then the function $x \mapsto \int_a^x f(t) dt$ must also be a primitive function. We will introduce a name for this function.

Definition 15.3. Let f be integrable on $[a, b]$. The function

$$I(x) = \int_a^x f(t) dt \quad (x \in [a, b])$$

is called the *integral function* of f .

With the use of this new concept, we can summarize the results of our previous argument as follows.

Theorem 15.4. *An integrable function has a primitive function if and only if its integral function is its primitive function.*

The most important properties of the integral function are expressed by the following theorem.

Theorem 15.5. *Let f be integrable on $[a, b]$, and let $I(x)$ be its integral function.*

- (i) *The function I is continuous and even has the Lipschitz property on the interval $[a, b]$.*

- (ii) If f is continuous at the point $x_0 \in [a, b]$, then I is differentiable there, and $I'(x_0) = f(x_0)$.
- (iii) If f is continuous on $[a, b]$, then I is differentiable on $[a, b]$, and $I' = f$. It follows that if f is continuous on $[a, b]$, then it has a primitive function there.

Proof. (i) Let $|f(x)| \leq K$ for all $x \in [a, b]$. If $a \leq x < y \leq b$, then by Theorem 14.38, we have

$$I(y) - I(x) = \int_a^y f(t) dt - \int_a^x f(t) dt = \int_x^y f(t) dt,$$

so $|I(y) - I(x)| \leq K \cdot |y - x|$ by statement (iv) of Theorem 14.49.

(ii) Again by Theorem 14.38, we have

$$I(x) - I(x_0) = \int_a^x f(t) dt - \int_a^{x_0} f(t) dt = \int_{x_0}^x f(t) dt$$

so the difference quotient of the function I corresponding to the points x and x_0 is

$$\frac{I(x) - I(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^x f(t) dt.$$

Since f is continuous at x_0 , for arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(x_0) - \varepsilon < f(t) < f(x_0) + \varepsilon \quad \text{if } |t - x_0| < \delta.$$

First let $x_0 < x < x_0 + \delta$. For all such x , it follows from statement (iii) of Theorem 14.49 that

$$(f(x_0) - \varepsilon)(x - x_0) \leq \int_{x_0}^x f(t) dt \leq (f(x_0) + \varepsilon)(x - x_0)$$

holds, that is,

$$f(x_0) - \varepsilon \leq \frac{I(x) - I(x_0)}{x - x_0} \leq f(x_0) + \varepsilon.$$

The same can be said when $x < x_0$ by rearranging $(I(x_0) - I(x))/(x_0 - x)$, so we have

$$I'(x_0) = \lim_{x \rightarrow x_0} \frac{I(x) - I(x_0)}{x - x_0} = f(x_0).$$

Statement (iii) is clear from (ii). \square

Remarks 15.6. 1. We can see from the proof that if f is continuous from the right or the left at x_0 , then $I'_+(x_0) = f(x_0)$ or $I'_-(x_0) = f(x_0)$ respectively.

2. The proof of statement (ii) above uses an argument we have already seen before. In Example 10.7, when we determined the area under the graph of a nonnegative monotone increasing and continuous function $f: [a, b] \rightarrow \mathbb{R}$, we showed that if $T(x)$ denotes the area over the interval $[a, x]$, then $T'(x) = f(x)$. Statement (ii) of Theorem 15.5 is actually a rephrasing of this, in which we replace area—which we still have not clearly defined—with the integral, and the function $T(x)$ with the integral function.

3. The fundamental theorem of calculus implies that if a function F is continuously differentiable,¹ then differentiating F and integrating the derivative gives us F back (more precisely, its increment on the interval $[a, b]$). By statement (iii) of 15.5, if we integrate a continuous function f from a to x , and then we differentiate the integral function we get, then we obtain f . These two statements express that *integration and differentiation are inverse operations in some sense*.

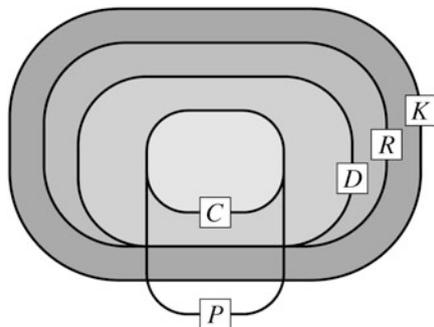


Fig. 15.1

In talking about the theory of integration, several different properties of functions came into play: boundedness, integrability, continuity, and the property of having a primitive function. We will use the following notation for functions that have the corresponding properties.

$$K[a, b] = \{f: [a, b] \rightarrow \mathbb{R} \text{ and } f \text{ is bounded in } [a, b]\},$$

$$R[a, b] = \{f: [a, b] \rightarrow \mathbb{R} \text{ and } f \text{ is Riemann integrable on } [a, b]\},$$

$$C[a, b] = \{f: [a, b] \rightarrow \mathbb{R} \text{ and } f \text{ is continuous on } [a, b]\},$$

$$P[a, b] = \{f: [a, b] \rightarrow \mathbb{R} \text{ and } f \text{ has a primitive function on } [a, b]\}.$$

We introduce separate notation for the set of integrable functions whose integral function is differentiable:

$$D[a, b] = \{f \in R[a, b] \text{ and the integral function of } f \text{ is differentiable in } [a, b]\}.$$

By Theorem 15.5 (and the proper definitions), the containment relations

$$C[a, b] \subset D[a, b] \subset R[a, b] \subset K[a, b] \quad \text{and} \quad C[a, b] \subset P[a, b] \quad (15.1)$$

¹ By this we mean that the function is differentiable and its derivative is continuous.

hold for these classes of functions. Moreover,

$$R[a, b] \cap P[a, b] \subset D[a, b]. \quad (15.2)$$

This is a straightforward corollary of Theorem 15.4 (Figure 15.1).

We now show that aside from what is listed in (15.1), no other containment relations exist between these classes of functions.

- Examples 15.7.*
1. $f \in K[0, 1] \not\Rightarrow f \in R[0, 1]$: The Dirichlet function is an example.
 2. $f \in R[0, 1] \not\Rightarrow f \in D[0, 1]$: Let $f(x) = 0$ if $0 \leq x < 1/2$, and $f(x) = 1$ if $1/2 \leq x \leq 1$.
 3. $f \in D[0, 1] \not\Rightarrow f \in C[0, 1]$: By (15.2), it suffices to give a function that is integrable, has a primitive function, but is not continuous. Let

$$f(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x=0. \end{cases}$$

Since f is bounded and continuous everywhere except at a point, it is integrable. The function f has a primitive function, too, namely the function

$$F(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

On the other hand, f is not continuous at 0.

4. $f \in D[0, 1] \not\Rightarrow f \in P[0, 1]$: Let $f(0) = 1$ and $f(x) = 0$ ($0 < x \leq 1$).
5. $f \in P[0, 1] \not\Rightarrow f \in K[0, 1]$: See Example 13.46.

We mention that there exists a bounded function that has a primitive function but is not integrable (that is, $f \in K[0, 1] \cap P[0, 1] \not\Rightarrow f \in R[0, 1]$). Constructing such a function is significantly more difficult than constructing the previous ones, so we will skip that for now.

Combining the continuity of integral functions with Abel's inequality yields an important result.

Theorem 15.8 (Second Mean Value Theorem for Integration).

- (i) Let f be monotone decreasing and nonnegative, and let g be integrable in $[a, b]$. Then there exists a $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x) dx = f(a) \cdot \int_a^\xi g(x) dx. \quad (15.3)$$

- (ii) Let f be monotone and let g be integrable in $[a, b]$. Then there exists a $\xi \in [a, b]$ such that

$$\int_a^b f(x)g(x) dx = f(a) \cdot \int_a^\xi g(x) dx + f(b) \cdot \int_\xi^b g(x) dx. \quad (15.4)$$

Proof. (i) The integral function $G(x) = \int_a^x g(t) dt$ is continuous in $[a, b]$ by Theorem 15.5, so its range in $[a, b]$ has a smallest and a greatest element. Let $m = \min G[a, b]$, $M = \max G[a, b]$, and $I = \int_a^b fg dx$. Then $f(a) \cdot m \leq I \leq f(a) \cdot M$ by Theorem 14.53. Since $f(a) \cdot G$ takes on every value between $f(a) \cdot m$ and $f(a) \cdot M$ by the Bolzano–Darboux theorem, there exists a $\xi \in [a, b]$ such that $f(a) \cdot G(\xi) = I$, which is exactly (15.3).

(ii) We can suppose that f is monotone decreasing, since otherwise, we can switch to the function $-f$. Then $f - f(b)$ is monotone decreasing and nonnegative in $[a, b]$, so by (i), there exists a $\xi \in [a, b]$ such that

$$\int_a^b (f(x) - f(b))g(x) dx = (f(a) - f(b)) \cdot \int_a^\xi g(x) dx,$$

from which we get

$$\begin{aligned} \int_a^b f(x)g(x) dx &= (f(a) - f(b)) \cdot \int_a^\xi g(x) dx + f(b) \cdot \int_a^b g(x) dx = \\ &= f(a) \cdot \int_a^\xi g(x) dx + f(b) \cdot \int_\xi^b g(x) dx. \end{aligned} \quad \square$$

Exercises

15.1. Give every primitive function, integral function, indefinite integral (see Definition 13.24), and definite integral of the functions below over the interval $[-2, 3]$:

(a) $|x|$;

(b) $\operatorname{sgn}(x)$;

(c) $f(x) = \begin{cases} 1 + x^2, & \text{if } x \geq 0, \\ 1 - x^2, & \text{if } x < 0. \end{cases}$

15.2. Let $f(x) = |x| - 2$ ($x \in [-2, 1]$). Does there exist a function whose integral function is f ? Decide the same for the function $g(x) = [x]$ ($x \in [-2, 1]$).

15.3. Does there exist a function on $[0, 1]$ whose integral function is \sqrt{x} ? (H)

15.4. Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded, and let F be a primitive function of f . Prove that

$$\int_a^b f(x) dx \leq F(b) - F(a) \leq \overline{\int}_a^b f(x) dx. \quad (\text{H})$$

15.5. Let

$$G(x) = \int_0^{x^4} e^{t^3} \cdot \sin t dt \quad (x \in \mathbb{R}).$$

Determine the derivative of G .

15.6. Prove that there are only two continuous functions defined on $[a, b]$ that satisfy

$$\int_a^x f(t) dt = \int_a^x f^2(t) dt$$

for all $x \in [a, b]$.

15.7. Prove that if f is continuous in $[0, 1]$ and $f(x) < 1$ for all $x \in [0, 1]$, then the equation

$$2x - \int_0^x f(t) dt = 1$$

has exactly one root in $[0, 1]$.

15.8. For which values of x is the value of

$$\int_0^x \frac{\sin t}{\sqrt{t}} dt$$

maximized?

15.9. Let f be integrable on $[a, b]$, and let the integral function of f be I . Is it possible for I to be differentiable everywhere and $I'(x) \neq f(x)$ for all $x \in [a, b]$? (H)

15.10. Prove that

$$\lim_{n \rightarrow \infty} n \left(\frac{1}{1+n^2} + \frac{1}{2^2+n^2} + \cdots + \frac{1}{n^2+n^2} \right) = \frac{\pi}{4}.$$

15.11. Determine the limits of the following sequences:

(a) $a_n = \sum_{k=1}^{2n} \frac{n}{k^2 + n^2},$

(b) $a_n = \sum_{k=n}^{2n} \frac{n}{k(n+k)},$

(c) $a_n = \sum_{k=1}^n \frac{k}{k^2 + n^2},$

(d) $a_n = \sum_{k=2n}^{3n} \frac{k}{n^2} e^{\frac{k}{n}},$

(e) $a_n = \left(\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \cdots \left(1 + \frac{n}{n}\right) \right)^{\frac{1}{n}}.$

15.12. Let

$$G(x) = \int_x^{2x} \frac{dt}{t} \quad (x > 0).$$

Determine $G'(x)$ without using the fundamental theorem of calculus. How can we interpret the result?

15.2 Integration by Parts

The fundamental theorem of calculus is significant not only from a theoretical standpoint (in which it outlines a link between differentiation and integration), but in terms of applications as well, since it tells us the value of the definite integral whenever we know a primitive function of the function we are integrating. Thus we can use the methods for computing indefinite integrals to compute definite integrals. In order to have the formulas in a more concise form, we introduce the following notation: if the function F is defined on the interval $[a, b]$, then we denote the difference $F(b) - F(a)$ by $[F]_a^b$.

We now extend our toolkit from Chapter 13 (Theorems 13.28, 13.30, and 13.33) with two new methods that greatly increase the number of integrals we can compute.

Theorem 15.9 (Integration by Parts). *Suppose the functions f and g are differentiable on the interval I , and fg' has a primitive function there. Then $f'g$ has a primitive function on I as well, and*

$$\int f'g \, dx = fg - \int fg' \, dx. \quad (15.5)$$

Proof. Let $F \in \int fg' \, dx$. Since $(fg)' = f'g + fg'$, we have

$$(fg - F)' = f'g + fg' - fg' = f'g,$$

which is exactly (15.5). □

Examples 15.10. A few examples of integration by parts follow.

1. $\int x \cdot \cos x \, dx = \int x \cdot (\sin x)' \, dx = x \sin x - \int x' \cdot \sin x \, dx = x \sin x - \int 1 \cdot \sin x \, dx = x \sin x + \cos x + C.$
2. $\int x \cdot e^x \, dx = \int x \cdot (e^x)' \, dx = x \cdot e^x - \int x' \cdot e^x \, dx = x \cdot e^x - \int 1 \cdot e^x \, dx = (x - 1)e^x + C.$
3. $\int x \cdot \log x \, dx = \int \left(\frac{x^2}{2}\right)' \cdot \log x \, dx = \frac{x^2}{2} \cdot \log x - \int \frac{x^2}{2} \cdot (\log x)' \, dx = \frac{x^2}{2} \cdot \log x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \frac{x^2}{2} \log x - \frac{x^2}{4} + C \quad (x > 0).$
4. $\int e^x \cdot \cos x \, dx = \int e^x \cdot (\sin x)' \, dx = e^x \sin x - \int (e^x)' \cdot \sin x \, dx = e^x \sin x - \int e^x \sin x \, dx = e^x \sin x - \int e^x (-\cos x)' \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx,$
so $\int e^x \cos x \, dx = \frac{1}{2} \cdot (e^x \sin x + e^x \cos x) + C.$
5. $\int \log x \, dx = \int x' \cdot \log x \, dx = x \log x - \int x \cdot (\log x)' \, dx = x \log x - \int x \cdot \frac{1}{x} \, dx = (x \cdot \log x) - x + C \quad (x > 0).$
6. $\int \arctg x \, dx = \int x' \cdot \arctg x \, dx = x \cdot \arctg x - \int x \cdot (\arctg x)' \, dx = x \cdot \arctg x - \int \frac{x}{1+x^2} \, dx = x \cdot \arctg x - \frac{1}{2} \log(1+x^2) + C.$

Applying integration by parts repeatedly allows us to compute various integrals such as the following:

$$\int x^k \cos x dx, \int x^k \sin x dx, \int x^k e^x dx,$$

$$\int x^k \log^n x dx, \int x^k e^x \cos x dx, \int x^k e^x \sin x dx.$$

The following theorem gives us integration by parts for definite integrals.

Theorem 15.11. *Suppose f and g are differentiable functions, while f' and g' are integrable over $[a, b]$. Then*

$$\int_a^b f'g dx = [fg]_a^b - \int_a^b fg' dx. \quad (15.6)$$

Proof. Since f and g are differentiable, they are continuous, so by Theorem 14.26, they are also integrable on $[a, b]$. Thus $f'g$ and fg' are both integrable on $[a, b]$ by Theorem 14.33. Since $(fg)' = f'g + fg'$, we have

$$\int_a^b (f'g + fg') dx = [fg]_a^b$$

by the fundamental theorem of calculus. Applying Theorem 14.31 and rearranging what we get yields (15.6). \square

As an interesting application of the previous theorem, we get the following formulas.

Theorem 15.12.

$$\int_0^\pi \sin^{2n} x dx = \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \cdot \pi \quad (n \in \mathbb{N}^+), \quad (15.7)$$

and

$$\int_0^\pi \sin^{2n+1} x dx = \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n+1)} \cdot 2 \quad (n \in \mathbb{N}). \quad (15.8)$$

Proof. Let $I_k = \int_0^\pi \sin^k x dx$ for all $k \in \mathbb{N}$. Then $I_0 = \pi$ and $I_1 = \cos 0 - \cos \pi = 2$. If $k \geq 1$, then

$$\begin{aligned} I_{k+1} &= \int_0^\pi \sin^2 x \cdot \sin^{k-1} x dx = \int_0^\pi (1 - \cos^2 x) \cdot \sin^{k-1} x dx = \\ &= \int_0^\pi [\sin^{k-1} x - \cos^2 x \cdot \sin^{k-1} x] dx = \\ &= I_{k-1} - \int_0^\pi \cos x \cdot [\sin^{k-1} x \cdot \cos x] dx. \end{aligned} \quad (15.9)$$

Now, using integration by parts, we get that

$$\begin{aligned} \int_0^\pi \cos x \cdot [\sin^{k-1} x \cdot \cos x] dx &= \int_0^\pi \cos x \cdot \left(\frac{1}{k} \cdot \sin^k x \right)' dx = \\ &= \left[\cos x \cdot \frac{1}{k} \cdot \sin^k x \right]_0^\pi - \int_0^\pi \frac{1}{k} \cdot \sin^k x \cdot (-\sin x) dx = \\ &= 0 + \frac{1}{k} \cdot I_{k+1}. \end{aligned}$$

Combining this with (15.9), we obtain $I_{k+1} = I_{k-1} - \frac{1}{k} \cdot I_{k+1}$, so $I_{k+1} = \frac{k}{k+1} \cdot I_{k-1}$. Thus

$$I_{2n} = \frac{2n-1}{2n} \cdot I_{2n-2} = \cdots = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot I_0,$$

which is exactly (15.7). Similarly,

$$I_{2n+1} = \frac{2n}{2n+1} \cdot I_{2n-1} = \cdots = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot I_1,$$

which is (15.8). □

The equations above make possible the proof of a fundamentally important identity that expresses the number π as the limit of a simple product.

Theorem 15.13 (Wallis' Formula²).

$$\pi = \lim_{n \rightarrow \infty} \left[\frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)} \right]^2 \cdot \frac{1}{n}.$$

Proof. Since $\sin^{2n-1} x \geq \sin^{2n} x \geq \sin^{2n+1} x$ for all $x \in [0, \pi]$, we have $I_{2n-1} \geq I_{2n} \geq I_{2n+1}$. Thus

$$\frac{2 \cdot 4 \cdots (2n-2)}{1 \cdot 3 \cdots (2n-1)} \cdot 2 \geq \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \cdot \pi \geq \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n+1)} \cdot 2,$$

which gives

$$\left[\frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)} \right]^2 \cdot \frac{1}{n} \geq \pi \geq \left[\frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)} \right]^2 \cdot \frac{2}{2n+1}.$$

Let W_n denote the product $[(2 \cdot 4 \cdots 2n)/(1 \cdot 3 \cdots (2n-1))]^2 \cdot 1/n$. Then $W_n \geq \pi \geq W_n \cdot 2n/2n+1$, that is, $\pi \leq W_n \leq \pi \cdot 2n+1/2n$. Thus, by the squeeze theorem, $\lim_{n \rightarrow \infty} W_n = \pi$. □

² John Wallis (1616–1703), English mathematician.

Remark 15.14. Since

$$\frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)} = \frac{(2 \cdot 4 \cdots 2n)^2}{1 \cdot 2 \cdots (2n)} = \frac{[2^n \cdot n!]^2}{(2n)!} = \frac{4^n}{\binom{2n}{n}},$$

Wallis's formula gives

$$\lim_{n \rightarrow \infty} \frac{4^n}{\binom{2n}{n} \sqrt{n}} = \sqrt{\pi}.$$

With our asymptotic notation, we can express this by saying that

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{n\pi}}. \quad (15.10)$$

By the binomial theorem, the sum of the binomial coefficients

$$\binom{2n}{0}, \binom{2n}{1}, \dots, \binom{2n}{2n} \quad (15.11)$$

is 4^n , so their mean is $4^n/(2n+1)$. Now (15.10) says that the middle term in (15.11) (which is the largest term as well) is about $c \cdot \sqrt{n}$ times the mean, where $c = 2/\sqrt{\pi}$.

We now prove an important theorem as an application of Wallis's formula that itself has many applications in many fields of mathematics, especially in probability.

Theorem 15.15 (Stirling's Formula). $n! \sim \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$.

Proof. First we show that the sequence $a_n = (n/e)^n \sqrt{2\pi n}/n!$ is strictly monotone increasing and bounded. A simple computation yields

$$\frac{a_{n+1}}{a_n} = \frac{\left(1 + \frac{1}{n}\right)^{n+(1/2)}}{e},$$

so

$$\log a_{n+1} - \log a_n = \left(n + \frac{1}{2}\right) \cdot \log\left(1 + \frac{1}{n}\right) - 1 \quad (15.12)$$

for all n . We know that for $x > 0$, we have $\log(1+x) > 2x/(x+2)$ (as seen in Example 12.56). Apply this for $x = 1/n$, then multiply through by $n + (1/2)$ to get that

$$\left(n + \frac{1}{2}\right) \cdot \log\left(1 + \frac{1}{n}\right) > 1.$$

This proves that (a_n) is strictly monotone increasing by (15.12).

Next, we will use the inequality

$$\log(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} \quad (x > 0) \quad (15.13)$$

to find an upper bound for the right-hand side of (15.12). (We refer to Exercise 12.91 or (13.26) to justify (15.13).) If we substitute $x = 1/n$ into (15.13) and multiply through by $n + (1/2)$, then with the help of (15.12), we get

$$\log a_{n+1} - \log a_n \leq \frac{1}{12n^2} + \frac{1}{6n^3} \leq \frac{1}{12n^2} + \frac{1}{6n^2} = \frac{1}{4n^2}. \quad (15.14)$$

Thus

$$\log a_n - \log a_1 = \sum_{i=1}^{n-1} (\log a_{i+1} - \log a_i) \leq \sum_{i=1}^{n-1} \frac{1}{4i^2} < \frac{1}{2}$$

for all n , so it is clear that the sequence (a_n) is bounded. Since we have shown that (a_n) is monotone increasing and bounded, it must be convergent. Let $\lim_{n \rightarrow \infty} a_n = a$. Since every term of the sequence is positive, $a > 0$. It is clear that $a_n^2/a_{2n} \rightarrow a$. On the other hand, a simple computation gives

$$\frac{a_n^2}{a_{2n}} = \frac{\binom{2n}{n} \cdot \sqrt{\pi n}}{4^n}$$

for all n . Thus by Wallis's formula (or more precisely by (15.10)), $a_n^2/a_{2n} \rightarrow 1$, so $a = 1$. \square

Remark 15.16. One can show that

$$\left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} < n! < \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} \cdot e^{1/(12n)}$$

for every positive integer n . A somewhat weaker statement is presented in Exercise 15.24.

Exercises

15.13. Compute the following integrals:

- | | |
|---------------------------------------------------|---------------------------------------------|
| (a) $\int_0^1 \sqrt{x} \cdot e^{\sqrt{x}} dx$; | (b) $\int_2^3 \frac{\sqrt{\log x}}{x} dx$; |
| (c) $\int_0^{\pi^2} \sin \sqrt{x} dx$; | (d) $\int_0^1 \operatorname{arctg} x dx$; |
| (e) $\int_0^1 \operatorname{arctg} \sqrt{x} dx$; | (f) $\int_0^1 \log(1+x^2) dx$; |
| (g) $\int_0^1 \sqrt{x^3+x^2} dx$; | (h) $\int e^{ax} \cos(bx) dx$. |

15.14. Apply integration by parts to get the equation

$$\begin{aligned} \int \frac{1}{x} \cdot \frac{1}{\log x} dx &= \int (\log x)' \cdot \frac{1}{\log x} dx = \\ &= \log x \cdot \frac{1}{\log x} - \int \log x \cdot \frac{1}{x} \cdot \frac{-1}{\log^2 x} dx = 1 + \int \frac{1}{x} \cdot \frac{1}{\log x} dx. \end{aligned}$$

Thus $0 = 1$. Where did we make a mistake?

15.15. Prove that if f is strictly monotone and differentiable in the interval I , $\varphi = f^{-1}$, and $\int f(x) dx = F(x) + c$, then

$$\int \varphi(y) dy = y\varphi(y) - F(\varphi(y)) + c.$$

15.16. Check the correctness of the following computation:

$$\begin{aligned} 2n \cdot \int \frac{x^2}{(x^2+1)^{n+1}} dx &= - \int x \cdot \left(\frac{1}{(x^2+1)^n} \right)' dx = \\ &= - \frac{x}{(x^2+1)^n} + \int \frac{dx}{(x^2+1)^n} + c. \end{aligned}$$

15.17. Prove that if f and g are n times continuously differentiable in an interval I , then

$$\begin{aligned} \int f g^{(n)} dx &= \\ &= f g^{(n-1)} - f' g^{(n-2)} + \dots + (-1)^{n-1} f^{(n-1)} g + (-1)^n \int f^{(n)} g dx. \quad (\text{H}) \end{aligned}$$

15.18. Prove that if p is a polynomial of degree n , then

$$\int e^{-x} p(x) dx = -e^{-x} \cdot [p(x) + p'(x) + \dots + p^{(n)}(x)] + c.$$

15.19. Prove that if f is twice differentiable and f'' is integrable in $[a, b]$, then

$$\int_a^b x f''(x) dx = (b f'(b) - f(b)) - (a f'(a) - f(a)).$$

15.20. Prove that

$$\int_0^1 x^m (1-x)^n dx = \frac{m! n!}{(m+n+1)!} \quad (m, n \in \mathbb{N}).$$

15.21. Compute the value of $\int_0^1 (1-x^2)^n dx$ for every $n \in \mathbb{N}$.

15.22. Prove that if

$$f_1(x) = \int_0^x f(t) dt, \quad f_2(x) = \int_0^x f_1(t) dt, \quad \dots, \quad f_k(x) = \int_0^x f_{k-1}(t) dt$$

then

$$f_k(x) = \frac{1}{(k-1)!} \cdot \int_0^x f(t) (x-t)^{k-1} dt.$$

15.23. (a) Prove that for every $n \in \mathbb{N}$, there exist integers a_n and b_n such that $\int_0^1 x^n e^x dx = a_n \cdot e + b_n$ holds.

(b) Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 x^n e^x dx = 0.$$

(c) Prove that e is irrational.

15.24. Prove the following stronger version of Stirling's formula:

$$\left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} < n! \leq \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n} \cdot e^{1/(4(n-1))} \quad (15.15)$$

for every integer $n > 1$. (S)

15.3 Integration by Substitution

We obtained the formulas in Theorems 13.30 and 13.33 by differentiating $f(ax + b)$, $f(x)^{\alpha+1}$, and $\log f(x)$, and using the differentiation rules for compositions of functions. These formulas are special cases of the following theorem, which is called **integration by substitution**.

Theorem 15.17. *Suppose the function g is differentiable on the interval I , f is defined on the interval $J = g(I)$, and f has a primitive function on J .³ Then the function $(f \circ g) \cdot g'$ also has a primitive function on I , and*

$$\int f(g(t)) \cdot g'(t) dt = F(g(t)) + c, \quad (15.16)$$

where $\int f dx = F(x) + c$.

Proof. The theorem is rather clear from the differentiation rule for compositions of functions. \square

We can use equation (15.16) in both directions. We use it “left to right” when we need to compute an integral of the form $\int f(g(t)) \cdot g'(t) dt$. Then the following formal procedure automatically changes the integral we want to compute to the right-hand side of (15.16):

$$g(t) = x; \quad g'(t) = \frac{dx}{dt}; \quad g'(t)dt = dx; \quad \int f(g(t)) \cdot g'(t)dt = \left(\int f(x) dx \right)_{x=g(t)}.$$

Examples 15.18. 1. The integral $\int t \cdot e^{t^2} dt$ is changed into the form $\int f(g(t)) \cdot g'(t) dt$ if we divide it and multiply it by 2 at the same time:

$$\int t \cdot e^{t^2} dt = \frac{1}{2} \cdot \int e^{t^2} \cdot (2t) dt = \frac{1}{2} \cdot \int e^{t^2} \cdot (t^2)' dt = F(t^2) + c,$$

³ Here we use the fact that the image of an interval under a continuous function is also an interval; see Corollary 10.58 and the remark following it.

where $F(x) = \int e^x dx = e^x + c$. Thus the integral is equal to $(1/2) \cdot e^{t^2} + c$. With the help of the formalism above, we can get the same result faster: $x = t^2$, $dx/dt = 2t$, $2t dt = dx$,

$$\int t \cdot e^{t^2} dt = \int \frac{1}{2} \cdot e^{t^2} 2t dt = \int \frac{1}{2} \cdot e^x dx = \frac{1}{2} \cdot e^x + c = \frac{1}{2} \cdot e^{t^2} + c.$$

2. $\int \operatorname{tg} t dt = \int (\sin t / \cos t) dt = -\int (1/\cos t) \cdot (\cos t)' dt = -F(\cos t) + c$, where $F(x) = \int (1/x) dx = \log|x| + c$. Thus the integral is equal to $-\log|\cos t| + c$. The same result can be obtained with the formal procedure we introduced above: $\cos t = x$, $dx/dt = -\sin t$, $-\sin t dt = dx$,

$$\int \frac{\sin t}{\cos t} dt = \int -\frac{dx}{x} = -\log|x| + c = -\log|\cos t| + c.$$

Let us see some examples when we apply (15.16) “right to left” that is, when we want to determine an integral of the form $\int f dx$, and we are looking for a g with which we can compute the left-hand side of (15.16), that is, the integral $\int f(g(t)) \cdot g'(t) dt$. To achieve this goal, we usually look for a function g for which $f \circ g$ is simpler than f (and then we hope that the $g'(t)$ factor does not make our integral too complicated). If $\int f(g(t)) \cdot g'(t) dt = G(t) + c$, then by (15.16), the primitive function F of the function f we seek satisfies $G(t) = F(g(t)) + c$. Therefore, we have $\int f dx = F(x) + c = G(g^{-1}(x)) + c$, assuming that g has an inverse.

Examples 15.19. 1. We can attempt to solve the integral $\int dx/(1 + \sqrt{x})$ with the help of the function $g(t) = t^2$, since then for $t > 0$, we have $f(g(t)) \cdot g'(t) = 2t/(1 + t)$, whose integral can be easily computed. If this is $G(t) + c$, then the integral we seek is $g(\sqrt{x}) + c$, since the inverse of g is the function \sqrt{x} . With the formalism above, the computation looks like this: $x = t^2$, $dx/dt = 2t$, $dx = 2t dt$,

$$\begin{aligned} \int \frac{dx}{1 + \sqrt{x}} &= \int \frac{1}{1 + t} \cdot 2t dt = 2 \cdot \int \left(1 - \frac{1}{1 + t}\right) dt = \\ &= 2t - 2\log(1 + t) + c = 2\sqrt{x} - 2\log(1 + \sqrt{x}) + c. \end{aligned}$$

Here we get the last inequality by substituting $t = \sqrt{x}$, that is, $g(t) = t^2$.

2. We can use the substitution $e^x = t$, i.e., $x = \log t$, for the integral $\int e^{2x}/(e^x + 1) dx$. We get that $dx/dt = 1/t$, $dx = dt/t$,

$$\begin{aligned} \int \frac{e^{2x}}{e^x + 1} dx &= \int \frac{t^2}{t + 1} \cdot \frac{1}{t} dt = \int \frac{t}{t + 1} dt = \int \left(1 - \frac{1}{t + 1}\right) dt = \\ &= t - \log(t + 1) + c = e^x - \log(e^x + 1) + c. \end{aligned}$$

3. Let us compute the integral $\int \sqrt{1 - x^2} dx$. Let $x = \sin t$, where $t \in [-\pi/2, \pi/2]$. Then $dx/dt = \cos t$, $dx = \cos t dt$,

$$\begin{aligned} \int \sqrt{1-x^2} dx &= \int \sqrt{1-\sin^2 t} \cdot \cos t dt = \\ &= \int \cos^2 t dt = \int \frac{1+\cos 2t}{2} dt = \frac{t}{2} + \frac{\sin 2t}{4} + c = \\ &= \frac{\arcsin x}{2} + \frac{\sin(2\arcsin x)}{4} + c. \end{aligned}$$

Here the second term can be simplified if we notice that $\sin 2t = 2 \sin t \cdot \cos t$, and so $\sin(2\arcsin x) = 2x \cdot \cos(\arcsin x) = 2x\sqrt{1-x^2}$. In the end, we get that

$$\int \sqrt{1-x^2} dx = \frac{1}{2} \cdot \arcsin x + \frac{1}{2} \cdot x\sqrt{1-x^2} + c. \quad (15.17)$$

Examples 15.20. **1.** Let $r > 0$. By (15.17) and applying Theorem 13.30, we get

$$\int \sqrt{r^2-x^2} dx = \frac{r^2}{2} \cdot \arcsin \frac{x}{r} + \frac{rx}{2} \cdot \sqrt{1-\left(\frac{x}{r}\right)^2} + c.$$

Thus by the fundamental theorem of calculus,

$$\int_{-r}^r \sqrt{r^2-x^2} dx = \left[\frac{r^2}{2} \cdot \arcsin \frac{x}{r} + \frac{rx}{2} \cdot \sqrt{1-\left(\frac{x}{r}\right)^2} \right]_{-r}^r = r^2 \cdot \arcsin 1 = \frac{r^2 \pi}{2},$$

that is, the area of the semicircle with radius r is $r^2\pi/2$. (Recall that we defined π to be the circumference of the unit semicircle on p. 163.) We have recovered the well-known formula for the area of a circle with radius r (namely $r^2\pi$), which Archimedes stated as *the area of a circle agrees with the area of the right triangle whose legs (sides adjacent to the right angle) are equal to the radius and the circumference of the circle*.

2. With the help of the integral (15.17), we can determine the area of an ellipse as well. The equation of an ellipse with axes a and b is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

so the graph of the function $f(x) = b \cdot \sqrt{1-\left(\frac{x}{a}\right)^2}$ ($x \in [-a, a]$) bounds half of the area of the ellipse. Now by (15.17) and Theorem 13.30,

$$\int b \cdot \sqrt{1-\left(\frac{x}{a}\right)^2} dx = \frac{ba}{2} \cdot \arcsin \frac{x}{a} + \frac{bx}{2} \cdot \sqrt{1-\left(\frac{x}{a}\right)^2} + c,$$

so by the fundamental theorem of calculus,

$$\int_{-a}^a b \cdot \sqrt{1-\left(\frac{x}{a}\right)^2} dx = ba \cdot \arcsin 1 = \frac{ab\pi}{2},$$

that is, *the area of the ellipse is $ab\pi$* .

The following theorem gives us a version of integration by substitution for definite integrals.

Theorem 15.21. *Suppose that g is differentiable and g' is integrable on the interval $[a, b]$. If f is continuous on the image of g , that is, on the interval⁴ $g([a, b])$, then*

$$\int_a^b f(g(t)) \cdot g'(t) dt = \int_{g(a)}^{g(b)} f(x) dx. \quad (15.18)$$

Proof. Since g is differentiable, it is continuous. Thus $f \circ g$ is also continuous, so it is integrable on $[a, b]$, which implies that $(f \circ g) \cdot g'$ is also integrable on $[a, b]$. On the other hand, statement (iii) of Theorem 15.5 ensures that f has a primitive function. If $F' = f$, then by the fundamental theorem of calculus, the right-hand side of (15.18) is $F(g(b)) - F(g(a))$. Now by the differentiation rules for compositions of functions, $(F \circ g)' = (f \circ g) \cdot g'$, so by applying the fundamental theorem of calculus again, we get that the left-hand side of (15.18) is also $F(g(b)) - F(g(a))$, meaning that (15.18) is true. \square

We note that in the theorem above, we can relax the condition of continuity of f and assume only that f is integrable on the image of g . In other words, the following theorem also holds.

Theorem 15.22. *Suppose that g is differentiable and g' is integrable on the interval $[a, b]$. If f is integrable on the image of g , that is, on the interval $g([a, b])$, then $(f \circ g) \cdot g'$ is integrable on $[a, b]$, and (15.18) holds.*

This more general theorem is harder to prove, since the integrability of $(f \circ g) \cdot g'$ does not follow as easily as in the case of Theorem 15.21, and the fundamental theorem of calculus cannot be applied either. The proof can be found in the appendix of this chapter.

Exercises

15.25. Prove that

$$\int_0^\pi \frac{\sin 2kx}{\sin x} dx = 0$$

holds for every integer k . (H)

15.26. Prove that if f is integrable on $[0, 1]$, then

$$\int_0^\pi f(\sin x) \cos x dx = 0.$$

⁴ Corollary 10.58 ensures that $g([a, b])$ is an interval.

15.4 Integrals of Elementary Functions

In Chapter 11, we became acquainted with the elementary functions. These are the polynomials, rational, exponential, power, and logarithmic functions, trigonometric and hyperbolic functions, their inverses, and every function that can be expressed from these using a finite sequence of basic operations and compositions.⁵ We will familiarize ourselves with methods that allow us to determine the indefinite integrals of numerous elementary functions.

15.4.1 Rational Functions

Definition 15.23. We define *elementary rational functions* to be

- (i) quotients of the form $A/(x-a)^n$, where $n \in \mathbb{N}^+$ and $A, a \in \mathbb{R}$; as well as
- (ii) quotients of the form $(Ax+B)/(x^2+ax+b)^n$, where $n \in \mathbb{N}^+$ and $A, B, a, b \in \mathbb{R}$ are constants such that $a^2 - 4b < 0$ holds.

(This last condition is equivalent to saying that $x^2 + ax + b$ does not have any real roots.)

We will first determine the integrals of the elementary rational functions. The first type does not give us any trouble, since $\int \frac{A}{(x-a)} dx = A \cdot \log|x-a| + c$, and $n > 1$ implies $\int A/(x-a)^n dx = (A/(1-n))/(x-a)^{n-1} + c$.

To find the integrals of the second type of elementary rational functions we will first show that computing the integral of $(Ax+B)/(x^2+ax+b)^n$ can be reduced to computing the integral $\int dx/(x^2+1)^n$. This can be seen as

$$\begin{aligned} \int \frac{Ax+B}{(x^2+ax+b)^n} dx &= \int \frac{(2x+a) \cdot (A/2) + (B - (aA/2))}{(x^2+ax+b)^n} = \\ &= \frac{A}{2} \cdot \int \frac{(x^2+ax+b)'}{(x^2+ax+b)^n} dx + \left(B - \frac{aA}{2}\right) \cdot \int \frac{dx}{(x^2+ax+b)^n}. \end{aligned}$$

Here

$$\int \frac{(x^2+ax+b)'}{(x^2+ax+b)^n} dx = \log(x^2+ax+b) + c,$$

and if $n > 1$, then

$$\int \frac{(x^2+ax+b)'}{(x^2+ax+b)^n} dx = \frac{1}{1-n} \cdot \frac{1}{(x^2+ax+b)^{n-1}} + c.$$

⁵ What we call elementary functions is partially based on history and tradition, partially based on usefulness, and partially based on a deeper reason that comes to light through complex analysis. In some investigations, it proves to be reasonable to list algebraic functions among the elementary functions. (Algebraic functions were defined in Exercise 11.45.)

Moreover,

$$\begin{aligned}\int \frac{dx}{(x^2 + ax + b)^n} &= \int \frac{dx}{\left(\left(x + (a/2)\right)^2 + b - (a^2/4)\right)^n} = \\ &= \int \frac{dx}{\left(\left(x + (a/2)\right)^2 + d^2\right)^n},\end{aligned}$$

where $d = \sqrt{b - (a^2/4)}$. (By the conditions, $b - (a^2/4) > 0$.) Now if

$$\int \frac{dx}{(x^2 + 1)^n} = F_n(x) + c,$$

then by Theorem 13.30,

$$\int \frac{dx}{\left(\left(x + (a/2)\right)^2 + d^2\right)^n} = \frac{1}{d^{2n}} \cdot \int \frac{dx}{\left(\left(\frac{x}{d} + \frac{a}{2d}\right)^2 + 1\right)^n} = \frac{1}{d^{2n-1}} \cdot F_n\left(\frac{x}{d} + \frac{a}{2d}\right) + c.$$

As for the functions F_n , we know that $F_1(x) = \arctg x + c$. On the other hand, for every $n \geq 1$, the equality

$$F_{n+1} = \frac{1}{2n} \cdot \frac{x}{(x^2 + 1)^n} + \frac{2n-1}{2n} \cdot F_n + c \quad (15.19)$$

holds. This is easy to check by differentiating both sides (see also Exercise 15.16). Applying the recurrence formula (15.19) repeatedly gives us the functions F_n . So for example,

$$\begin{aligned}\int \frac{dx}{(x^2 + 1)^2} &= \frac{1}{2} \cdot \frac{x}{x^2 + 1} + \frac{1}{2} \cdot \arctg x + c, \\ \int \frac{dx}{(x^2 + 1)^3} &= \frac{1}{4} \cdot \frac{x}{(x^2 + 1)^2} + \frac{3}{8} \cdot \frac{x}{x^2 + 1} + \frac{3}{8} \cdot \arctg x + c.\end{aligned} \quad (15.20)$$

By the following theorem every rational function can be expressed as the sum of a polynomial and finitely many elementary rational functions, and if we know how to integrate these, we know how to determine the integral of any rational function (at least theoretically).

We will need the concept of divisibility for polynomials. We say that the polynomial p is **divisible** by the polynomial q , and we denote this by $q \mid p$, if there exists a polynomial r such that $p = q \cdot r$. It is known that if the polynomials q_1 and q_2 do not have a nonconstant common divisor, then there exist polynomials p_1 and p_2 such that $p_1 q_1 + p_2 q_2 \equiv 1$. (We can find such a p_1 and p_2 by repeatedly applying the Euclidean algorithm to q_1 and q_2 .)

Moreover, we will use the fundamental theorem of algebra (see page 201) and the corollary that every polynomial with real coefficients can be written as the product of polynomials with real coefficients of degree one and two.

Theorem 15.24 (Partial Fraction Decomposition). *Every rational function R can be written as the sum of a polynomial and finitely many elementary rational functions such that the denominators of these elementary rational functions all divide the denominator of R .*

Proof. Let $R = p/q$, where p and $q \neq 0$ are polynomials. Let the degree of q be n ; we will prove the theorem by induction on n . If $n = 0$, that is, q is constant, then R is a polynomial, and the statement holds (without any elementary rational functions in the decomposition).

Let $n > 0$, and assume the statement is true for every rational function whose denominator has degree smaller than n . Factor q into a product of polynomials of degree one and two. We can assume that the degree-two polynomials here do not have any real roots, since otherwise, we could factor them further. We distinguish three cases.

1. Two different terms appear in the factorization of the polynomial q . Then q can be written as q_1q_2 , where $q_1 \neq c$, $q_2 \neq c$, and the polynomials q_1, q_2 do not have any nonconstant common divisors. Then there exist polynomials p_1 and p_2 such that $p_1q_1 + p_2q_2 \equiv 1$;

$$\frac{p}{q} = \frac{pp_1q_1 + pp_2q_2}{q_1q_2} = \frac{pp_1}{q_2} + \frac{pp_2}{q_1}.$$

Here we can apply the inductive hypothesis to both pp_1/q_2 and pp_2/q_1 , which immediately gives us the statement of the theorem.

2. $q = c(x-a)^n$. In this case, let us divide p by $(x-a)$ with remainder: $p = p_1(x-a) + A$, from which

$$\frac{p}{q} = \frac{p_1}{c(x-a)^{n-1}} + \frac{A/c}{(x-a)^n}$$

follows. Here $(A/c)/(x-a)^n$ is an elementary rational function, and we can apply the induction hypothesis to the term $p_1/(c(x-a)^{n-1})$.

3. $q = c \cdot (x^2 + ax + b)^k$, where $a^2 - 4b < 0$ and $n = 2k$. Then dividing p by $(x^2 + ax + b)$ with remainder gives us $p = p_1 \cdot (x^2 + ax + b) + (Ax + B)$, from which

$$\frac{p}{q} = \frac{p_1}{c(x^2 + ax + b)^{k-1}} + \frac{\frac{A}{c}x + \frac{B}{c}}{(x^2 + ax + b)^k}$$

follows. Here we can apply the inductive hypothesis to the first term, while the second term is an elementary rational function, proving the theorem. \square

Remarks 15.25. 1. One can show that the decomposition of rational functions in Theorem 15.24 is unique (see Exercise 15.32). This is the **partial fraction decomposition** of a rational function.

2. If the degree of p is smaller than q , then only elementary rational functions appear in the partial fraction decomposition of p/q (and no polynomial). This is because then $\lim_{x \rightarrow \infty} p/q = 0$. Since every elementary rational function tends to 0 at ∞ , this

must hold for the polynomial appearing in the decomposition as well, which implies it must be identically zero.

How can we find a partial fraction decomposition? We introduce three methods.

1. Follow the proof of the theorem. If, for example,

$$\frac{p}{q} = \frac{x+2}{x(x^2+1)^2},$$

then $1 \cdot (x^2+1)^2 - (x^3+2x) \cdot x = 1$ and

$$\begin{aligned} \frac{p}{q} &= \frac{x+2}{x} - \frac{x^4+2x^3+2x^2+4x}{(x^2+1)^2} = \\ &= 1 + \frac{2}{x} - \frac{(x^2+1)(x^2+2x+1)+2x-1}{(x^2+1)^2} = \\ &= \frac{2}{x} - \frac{2x}{x^2+1} - \frac{2x-1}{(x^2+1)^2}. \end{aligned}$$

2. The method of indeterminate coefficients. From the theorem and by remark 15.25, we know that

$$\frac{p}{q} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}. \quad (15.21)$$

Bringing this to a common denominator yields

$$x+2 = A(x^4+2x^2+1) + (Bx+C)(x^2+1)x + (Dx+E)x.$$

This gives us a system of equalities for the unknown coefficients: $A+B=0$, $C=0$, $2A+B+D=0$, $C+E=1$, and $A=2$. Then we can compute that $A=2$, $B=-2$, $C=0$, $D=-2$, and $E=1$.

3. If a term of the form $A/(x-a)^n$ appears in the decomposition, then (assuming that n is the largest such power for a) we can immediately determine A if we multiply everything by $(x-a)^n$ and we substitute $x=a$ into the equation. So for example, for (15.21) we have

$$A = \frac{x+2}{(x^2+1)^2} \Big|_{x=0} = 2.$$

If we subtract the known terms from both sides, we reduce the question to finding the partial fraction decomposition of a simpler rational function:

$$\frac{x+2}{x(x^2+1)^2} - \frac{2}{x} = \frac{x+2-2x^4-4x^2-2}{x(x^2+1)^2} = \frac{-2x^3-4x+1}{(x^2+1)^2}.$$

Here $-2x^3-4x+1 = (-2x)(x^2+1) + (-2x+1)$, so we get the same decomposition.

15.4.2 Integrals Containing Roots

From now on, $R(u, v)$ will denote a two-variable rational function. This means that $R(u, v)$ is constructed from the variables u and v and from constants by the four basic operations. One can easily show that this holds exactly when

$$R(u, v) = \frac{\sum_{i=0}^n \sum_{j=0}^n a_{ij} u^i v^j}{\sum_{i=0}^n \sum_{j=0}^n b_{ij} u^i v^j}, \quad (15.22)$$

where $n \geq 0$ is an integer and a_{ij} and b_{ij} are constants.

1. We show that the integral

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$$

can be reduced to the integral of a (one-variable) rational function with the substitution $\sqrt[n]{(ax+b)/(cx+d)} = t$. Clearly, with this substitution, $(ax+b)/(cx+d) = t^n$, $ax+b = ct^n x + dt^n$ and $x = (dt^n - b)/(a - ct^n)$, so dx/dt is a rational function.

Example 15.26. Compute the integral $\int x^{-2} \cdot \sqrt[3]{x+1/x} dx$. With the substitution $\sqrt[3]{x+1/x} = t$, $x+1 = t^3 x$, $x = 1/(t^3 - 1)$, and $dx/dt = -3t^2/(t^3 - 1)^2$, so

$$\begin{aligned} \int \frac{1}{x^2} \cdot \sqrt[3]{\frac{x+1}{x}} dx &= \int (t^3 - 1)^2 \cdot t \cdot \frac{-3t^2}{(t^3 - 1)^2} dt = \\ &= \int -3t^3 dt = -\frac{3}{4}t^4 + c = -\frac{3}{4} \cdot \left(\frac{x+1}{x}\right)^{4/3} + c. \end{aligned}$$

2. The integral $R(x, \sqrt{ax^2 + bx + c}) dx$ (where $a \neq 0$) can also be reduced to an integral of a rational function with a suitable substitution.

a. If $ax^2 + bx + c$ has a root, then $ax^2 + bx + c = a(x - \alpha)(x - \beta)$, so

$$\sqrt{ax^2 + bx + c} = \sqrt{a(x - \alpha)(x - \beta)} = |x - \alpha| \sqrt{\frac{a(x - \beta)}{x - \alpha}},$$

and this leads us back to an integral in the previous part.

b. If $ax^2 + bx + c$ does not have real roots, then it must be positive everywhere, since otherwise, the integrable function is not defined anywhere. Thus $a > 0$ and $c > 0$. In this case, we can use the substitution $\sqrt{ax^2 + bx + c} - \sqrt{a} \cdot x = t$. We get that

$$ax^2 + bx + c = t^2 + 2t\sqrt{a} \cdot x + ax^2,$$

$x = (c - t^2)/(2\sqrt{a} \cdot t - b)$, and so dx/dt is also a rational function.

We can apply the substitution $\sqrt{ax^2+bx+c} - \sqrt{c} = tx$ as well. This gives us $ax^2+bx+c = x^2t^2 + 2\sqrt{ct}x + c$, $ax+b = xt^2 + 2\sqrt{ct}$, $x = (2\sqrt{ct} - b)/(a - t^2)$, so dx/dt is a rational function.

Example 15.27. Compute the integral $\int \sqrt{x^2+1} dx$. Substituting $\sqrt{x^2+1} - x = t$, we obtain $x = (1-t^2)/(2t)$, $\sqrt{x^2+1} = x+t = (1+t^2)/(2t)$, $dx/dt = -(1+t^2)/(2t^2)$, so

$$\begin{aligned} \int \sqrt{x^2+1} dx &= \int \frac{1+t^2}{2t} \cdot \frac{-(1+t^2)}{2t^2} dt = -\frac{1}{4} \int \frac{1+2t^2+t^4}{t^3} dt = \\ &= \frac{1}{8} \cdot \frac{1}{t^2} - \frac{1}{2} \log|t| - \frac{t^2}{8} + c = \\ &= \frac{1}{8} \cdot \frac{1}{(\sqrt{x^2+1}-x)^2} - \frac{1}{2} \log(\sqrt{x^2+1}-x) - \frac{(\sqrt{x^2+1}-x)^2}{8} + c = \\ &= \frac{1}{2} \cdot x \cdot \sqrt{x^2+1} - \frac{1}{2} \log(\sqrt{x^2+1}-x) + c. \end{aligned} \quad (15.23)$$

15.4.3 Rational Functions of e^x

Let $R(x)$ be a one-variable rational function. To compute the integral $\int R(e^x) dx$, let us use the substitution $e^x = t$, $x = \log t$, $dx/dt = 1/t$. The integral becomes the integral of a rational function. See, e.g., Example 15.19.

15.4.4 Trigonometric Functions

- a. Integration of an expression of the form $R(\sin x, \cos x)$ can always be done with the substitution $\operatorname{tg}(x/2) = t$. Indeed, $\sin^2 x + \cos^2 x = 1$ gives $\operatorname{tg}^2 x + 1 = 1/\cos^2 x$, and thus

$$\cos x = \frac{1}{\pm\sqrt{1+\operatorname{tg}^2 x}}, \quad \sin x = \frac{\operatorname{tg} x}{\pm\sqrt{1+\operatorname{tg}^2 x}}, \quad (15.24)$$

so

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \operatorname{tg}(x/2)}{1+\operatorname{tg}^2(x/2)} \quad \text{and} \quad \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \frac{1-\operatorname{tg}^2(x/2)}{1+\operatorname{tg}^2(x/2)}.$$

Thus with the substitution $\operatorname{tg}(x/2) = t$, we have

$$\sin x = 2t/(1+t^2), \quad \cos x = (1-t^2)/(1+t^2), \quad x = 2 \operatorname{arctg} t,$$

and $dx/dt = 2/(1+t^2)$.

Example 15.28.

$$\int \frac{dx}{\sin x} = \int \frac{1+t^2}{2t} \cdot \frac{2}{1+t^2} dt = \int \frac{dt}{t} = \log |t| + c = \log \left| \operatorname{tg} \frac{x}{2} \right| + c.$$

Then

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin(\frac{\pi}{2} - x)} = -\log \left| \operatorname{tg} \left(\frac{\pi}{4} - \frac{x}{2} \right) \right| + c.$$

b. In some cases, the substitution $\operatorname{tg} x = t$ will also lead us to our goal. By (15.24),

$$\sin x = \frac{t}{\pm\sqrt{1+t^2}}, \quad \cos x = \frac{1}{\pm\sqrt{1+t^2}} \quad \text{and} \quad \frac{dx}{dt} = \frac{1}{1+t^2},$$

so this substitution also leads to a rational function if the exponents of $\sin x$ and $\cos x$ are of the same parity in every term of the denominator and every term of the numerator.

Example 15.29.

$$\begin{aligned} \int \frac{dx}{1+\cos^2 x} &= \int \frac{1}{1+1/(1+t^2)} \cdot \frac{dt}{1+t^2} = \int \frac{dt}{2+t^2} = \frac{1}{2} \cdot \int \frac{dt}{1+(t/\sqrt{2})^2} = \\ &= \frac{\sqrt{2}}{2} \cdot \operatorname{arc} \operatorname{tg} \frac{t}{\sqrt{2}} + c = \frac{1}{\sqrt{2}} \cdot \operatorname{arc} \operatorname{tg} \left(\frac{1}{\sqrt{2}} \operatorname{tg} x \right) + c. \end{aligned}$$

c. Applying the substitution $\sin x = t$ on the interval $[-\pi/2, \pi/2]$ gives us

$$\cos x = \sqrt{1-t^2}, \quad x = \operatorname{arcsin} t \quad \text{and} \quad \frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}},$$

so we get a rational function if the power of $\cos x$ is even in the numerator and odd in the denominator, or vice versa.

Example 15.30.

$$\begin{aligned} \int \frac{dx}{\cos x} &= \int \frac{1}{\sqrt{1-t^2}} \cdot \frac{dt}{\sqrt{1-t^2}} = \int \frac{dt}{1-t^2} = \\ &= \frac{1}{2} \cdot \log \left| \frac{1+t}{1-t} \right| + c = \frac{1}{2} \cdot \log \left| \frac{1+\sin x}{1-\sin x} \right| + c. \end{aligned}$$

(Check that this agrees with the result from Example 15.28. Also check that the right-hand side is the primitive function of $1/\cos x$ in every interval where $\cos x \neq 0$, not just in $(-\pi/2, \pi/2)$.)

d. The substitution $\cos x = t$ also leads us to the integral of a rational function if the power of $\sin x$ is even in the numerator and odd in the denominator, or vice versa.

Let us note that the integrals of the form $\int R(x, \sqrt{ax^2 + bx + c}) dx$ can also be computed using a method different from the one seen on page 354. With a linear substitution, we can reduce the integral to one of the integrals

$$\int R(x, \sqrt{1-x^2}) dx, \quad \int R(x, \sqrt{x^2-1}) dx, \quad \text{or} \quad \int R(x, \sqrt{x^2+1}) dx.$$

In the first case, the substitution $x = \sin t$ gives us an integral that we encountered in the last section. In the second case, the substitution $x = \operatorname{ch} t$ gives us the integral $\int R(\operatorname{ch} t, \operatorname{sh} t) \operatorname{sh} t$, which we can tackle as in the third section if we recall the definitions of $\operatorname{ch} t$ and $\operatorname{sh} t$. The third integral can be computed with the substitution $x = \operatorname{sh} t$.

Exercises

15.27. Compute the following definite integrals.

- | | |
|---------------------------------------------------------|-----------------------------------------------------------|
| (a) $\int_0^1 \frac{x}{x^4 + 1} dx;$ | (b) $\int_1^2 \frac{e^x + 2}{e^x + e^{2x}} dx;$ |
| (c) $\int_1^2 \frac{dx}{4x - 2^x} dx;$ | (d) $\int_{\pi/4}^{\pi/2} \frac{dx}{\sin x(2 + \cos x)};$ |
| (e) $\int_0^1 \sqrt{2^x - 1} dx;$ | (f) $\int_0^{\pi/4} (\operatorname{tg} x)^2 dx;$ |
| (g) $\int_1^2 \arcsin(1/x) dx;$ | (h) $\int_2^3 x \cdot \log(x^2 - x) dx;$ |
| (i) $\int_0^{\pi/4} \frac{dx}{\cos^{10} x} dx;$ | (j) $\int_2^4 \frac{x^2}{\sqrt{x^2 - 1}} dx;$ |
| (k) $\int_0^{\pi/4} \frac{dx}{\sin^4 x + \cos^4 x} dx.$ | |

15.28. Compute the following indefinite integrals.

- | | |
|---------------------------------------------------|--------------------------------------------------------|
| (a) $\int \frac{2x + 3}{x^2 - 5x + 6} dx;$ | (b) $\int \frac{x^3 - 2x^2 + 5x + 1}{x^2 + 1} dx;$ |
| (c) $\int \frac{x^{100}}{x-1} dx;$ | (d) $\int \frac{dx}{x^3 + 8} dx;$ |
| (e) $\int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}} dx;$ | (f) $\int \frac{1 + \sqrt[3]{x}}{1 - \sqrt[3]{x}} dx;$ |
| (g) $\int \frac{dx}{\log \log x};$ | (h) $\int \log(x + \sqrt{1+x^2}) dx;$ |

- (i) $\int \frac{e^x}{\sqrt{1+e^x}} dx$; (j) $\int \frac{dx}{1+\sin x} dx$;
- (k) $\int \frac{dx}{1+\cos x} dx$; (l) $\int \frac{\sin x \cos x}{1+\sin^2 x} dx$;
- (m) $\int \frac{dx}{1+\operatorname{tg} x}$; (n) $\int \sin x \cdot \log(\operatorname{tg} x) dx$.

15.29. Compute the integral $\int \sqrt{x^2+1} dx$ with the substitution $x = \operatorname{sht} t$, and compare the result with (15.23).

15.30. Compute the integral $\int \sqrt{x^2-1} dx$ with the substitution $x = \operatorname{cht} t$. (S)

15.31. The radius of a regular cylindrical container filled with water is r . The container is lying horizontally, that is, the curved part is on the ground. What force does the water exert on the vertical flat circular sides of the container due to pressure if the pressure at depth x is ρx ?

15.32. Show that the partial fraction decomposition of rational functions is unique. (H)

15.33. Suppose that p and q are polynomials, $a \in \mathbb{R}$, and $q(a) \neq 0$. We know that terms of the form $A_k/(x-a)^k$ appear in the partial fraction decomposition of the rational function $p(x)/(q(x) \cdot (x-a)^n)$ for all $k = 0, \dots, n$.

- (a) Prove that $A_n = p(a)/q(a)$.
 (b) Express the rest of the A_k with p and q as well.

15.34. Prove (15.19) with the help of exercise 15.16.

15.5 Nonelementary Integrals of Elementary Functions

Not all integrals of elementary functions can be computed with the help of the methods above. It might sound surprising at first, but there are some elementary functions whose primitive functions cannot be expressed by elementary functions.⁶ This is a significant difference between differentiation and integration, since—as we saw in Chapter 12—the derivative of an elementary function is always elementary. But if an operation (in this case differentiation) does not lead us out of a class of objects, why should we expect its inverse operation (integration) to do the same? For example, addition keeps us within the class of positive numbers, but subtraction does not; the set of integers is closed under multiplication but not under division; squaring numbers keeps us in the realm of rational numbers but taking roots does not. It appears that inverse operations are more complicated.

This phenomenon can be observed with differentiation and integration as well if we consider subclasses of the elementary functions. For example, the derivatives of

⁶ Some examples are e^x/x , $1/\log x$, and e^{-x^2} ; see Example 15.32.

rational functions are rational functions. However, the integral of a rational function is not necessarily a rational function:

$$\int \frac{dx}{x} = \log|x| + c,$$

and $\log|x|$ is not a rational function (which can easily be shown from the fact that $\lim_{x \rightarrow \infty} x^\beta \cdot \log x = \infty$ and $\lim_{x \rightarrow \infty} x^{-\beta} \cdot \log x = 0$ for all $\beta > 0$; see Example 13.2 regarding this last statement).

The same is the case with rational functions formed from trigonometric functions: the derivative of a function of the form $R(\cos x, \sin x)$ has the same form, but the integral can be different. This is clear from the fact that every function of the form $R(\cos x, \sin x)$ is periodic with period 2π , while, for example,

$$\int (1 + \cos x) dx = x + \sin x + c$$

is not periodic. A less trivial example:

$$\int \frac{dx}{\sin x} = \log \left| \operatorname{tg} \frac{x}{2} \right| + c,$$

and here the right-hand side cannot be expressed as a rational function of $\cos x$ and $\sin x$.

Thinking about it more carefully, we see that it is not that surprising that the integrals of some elementary functions are not elementary functions. It is a different matter that a rigorous proof of this is surprisingly hard. Joseph Liouville⁷ was the first to show—with the help of complex-analytic methods—that such elementary functions exist. In fact, Liouville proved that if the integral of an elementary function is elementary, then the formula for that integral cannot be much more complicated than the original function. Unfortunately, even the precise expression of this is achieved with great difficulty. Some special cases are easier to express, as is the following theorem of Liouville (but the proof still exceeds the scope of this book).

Theorem 15.31. *Let f and g be rational functions, and suppose that f is not constant. If $\int e^f g dx$ can be expressed as an elementary function, then $\int e^f g dx = e^f h + c$, where h is a rational function and c is a constant.*

With the help of this theorem, we can find several functions that do not have an elementary integral.

Examples 15.32. 1. Let us show that $\int \frac{e^x}{x} dx$ cannot be expressed in terms of elementary functions.

If $\int (e^x/x) dx$ could be expressed in terms of elementary functions, then by Liouville's theorem, there would exist a rational function S such that $(S \cdot e^x)' = e^x/x$. Let $S = p/q$, where the polynomials p and q do not have any nonconstant common divisors. Then

⁷ Joseph Liouville (1809–1882), French mathematician.

$$\left(\frac{p}{q} \cdot e^x\right)' = \frac{p'q - pq'}{q^2} \cdot e^x + \frac{p}{q} \cdot e^x = \frac{e^x}{x},$$

so $x(p'q - pq' + pq) = q^2$. We will show that this is impossible. First of all, q must be divisible by x . Let $q = x^k q_1$, where q_1 is not divisible by x . Then p cannot be divisible by x either, since $x \mid q$, and we assumed that p and q do not have any nonconstant common divisors. Thus the polynomial

$$P = p'q - pq' + pq = p'x^k q_1 - p(kx^{k-1} q_1 + x^k q_1') + px^k q_1$$

is not divisible by x^k , since every term on the right-hand side except for one is divisible by x^k . On the other hand, $P = q^2/x = x^{2k-1} q_1^2$ is divisible by x^k , since $2k - 1 \geq k$. This is a contradiction, which shows that $\int(e^x/x)dx$ cannot be elementary.

2. We can immediately deduce that the integral $\int(1/\log x)dx$ cannot be expressed in terms of elementary functions either. With the substitution $e^x = t$, we get $x = \log t$, $dx = dt/t$, so

$$\int \frac{e^x}{x} dx = \int \frac{dt}{\log t}.$$

Thus if $\int(1/\log t)dt$ were elementary, then so would $\int(e^x/x)dx$, which is impossible.

The integral $\int(1/\log t)dt$ appears often in various fields of mathematics, so it has its own notation:

$$\text{Li } x = \int_2^x \frac{dt}{\log t} \quad (x \geq 2).$$

The function $\text{Li } x$ (which is called the **logarithmic integral function**, or sometimes the **integral logarithm**) plays an important role in number theory. Let $\pi(x)$ denote the number of primes less than or equal to x . According to the **prime number theorem**,

$$\pi(x) \sim \frac{x}{\log x} \quad \text{if } x \rightarrow \infty,$$

that is, the function $x/\log x$ approximates $\pi(x)$ asymptotically well. One can show that the function $\text{Li } x$ is an even better approximation, in that

$$|\pi(x) - \text{Li } x| = o\left(\frac{x}{\log^k x}\right)$$

for all k , while

$$\left|\pi(x) - \frac{x}{\log x}\right| > \frac{cx}{\log^2 x}$$

for a suitable constant $c > 0$. The logarithmic integral function is just one of many important functions that are defined as integrals of elementary functions but are not elementary themselves.

3. Another such function of great importance in probability theory is $\Phi(x) = \int_0^x e^{-t^2} dt$, the function that describes the so-called normal distribution.

By Liouville's theorem above, we can easily deduce that this function is not an elementary function (see Exercise 15.38).

4. Another example is the **elliptic integrals**. In Remark 15.20, we saw that the area of an ellipse with axes a and b is $ab\pi$. Determining the circumference of an ellipse is a harder problem. To simplify computation, assume that $a = 1$ and $b < a$. The graph of the function $f(x) = b \cdot \sqrt{1-x^2}$ over the interval $[0, 1]$ gives us the portion of the ellipse with axes 1 and b lying in the quadrant $\{(x, y) : x, y \geq 0\}$. The arc length of this graph is thus a quarter of the circumference of the ellipse. The function f is monotone decreasing on the interval $[0, 1]$, so by Theorem 10.79, the graph is rectifiable there. Let $s(x)$ denote the arc length of the graph over the interval $[0, x]$. Since the derivative of f ,

$$f'(x) = b \cdot \frac{-2x}{2 \cdot \sqrt{1-x^2}} = -\frac{bx}{\sqrt{1-x^2}},$$

is continuous on $[0, 1]$, s is differentiable on this interval by Theorem 13.41, and $s'(x) = \sqrt{1+(f'(x))^2}$ for all $0 \leq x < 1$. Let us introduce the notation $k = \sqrt{1-b^2}$. Then $0 < k < 1$, and

$$1+(f'(x))^2 = 1 + \frac{b^2x^2}{1-x^2} = \frac{1-(1-b^2)x^2}{1-x^2} = \frac{1-k^2x^2}{1-x^2}.$$

Thus s is a primitive function of $\sqrt{(1-k^2t^2)/(1-t^2)}$ in the interval $(0, 1)$, that is,

$$s = \int \frac{\sqrt{1-k^2 \cdot t^2}}{\sqrt{1-t^2}} dt = \int \frac{1-k^2 \cdot t^2}{\sqrt{(1-t^2)(1-k^2t^2)}} dt. \quad (15.25)$$

Liouville showed that the integral appearing in (15.25) cannot be expressed with elementary functions, so the circumference of the ellipse cannot generally be expressed in a "closed" form (not containing an integral symbol).

Let us substitute $t = 1 - (1/u)$ in the integral above. We get that $dt = du/u^2$ and

$$\begin{aligned} \int \frac{\sqrt{1-k^2 \cdot t^2}}{\sqrt{1-t^2}} dt &= \int \frac{\sqrt{(1-k^2) + (2k^2/u) - (k^2/u^2)}}{\sqrt{(2/u) - (1/u^2)}} \cdot \frac{1}{u^2} du = \\ &= \int \frac{b^2u^2 + 2k^2u - k^2}{u^2 \cdot \sqrt{(2u-1)(b^2u^2 + 2k^2u - k^2)}} du. \end{aligned} \quad (15.26)$$

Of course, we still cannot express this integral in terms of elementary functions as in (15.25), but it appears simpler than that one, since here, the polynomial inside the square root has degree three (and not four). These integrals motivate the following naming convention.

5. An **elliptic integral** is an integral of the form $\int R(x, \sqrt{f}) dx$, where f is a polynomial of degree three or four and $R(x, \sqrt{f})$ is a rational function with arguments x and

\sqrt{f} . If the degree of f is greater than four, then we call the integral a **hyperelliptic integral**.

The (hyper)elliptic integrals usually cannot be expressed in terms of elementary functions (but they can in some special cases, for example,

$$\int \frac{f'}{\sqrt{f}} dx = 2\sqrt{f} + c$$

for every positive f).

Since (hyper)elliptic integrals appear often in various applications, it is best to reduce them to simpler integrals. Write $R(u, v)$ in the form seen in (15.22), and then replace u by x and v by $\sqrt{f(x)}$. After taking powers and combining all that we can, we get that

$$R(x, \sqrt{f}) = \frac{A + B\sqrt{f}}{C + D\sqrt{f}},$$

where A , B , C , and D are polynomials. If we multiply both the numerator and denominator by $C - D\sqrt{f}$ here, then we get an expression of the form $(E + F\sqrt{f})/G$. Since we already know the integrals of rational functions, it suffices to find the integral of

$$\frac{F\sqrt{f}}{G} = \frac{Ff}{G\sqrt{f}}.$$

Now apply Theorem 15.24 and decompose the rational function Ff/G into the sum of a polynomial and finitely many elementary rational functions. If we divide this decomposition by \sqrt{f} , then we deduce that it suffices to find the integrals

$$I_k = \int \frac{x^k}{\sqrt{f(x)}} dx \quad \text{and} \quad J_r = \int \frac{r(x)}{\sqrt{f(x)}} dx,$$

where $k \in \mathbb{N}$ and r is an arbitrary elementary rational function. One can show that if the degree of f is n , then every I_k can be expressed as a linear combination of elementary functions and the integrals I_0, I_1, \dots, I_{n-2} . A similar recurrence holds for the integrals J_r (see Exercise 15.39).

Exercises

15.35. Prove that $\text{Li } x \sim \frac{x}{\log x}$ if $x \rightarrow \infty$.

15.36. Prove that for all $n \in \mathbb{N}^+$,

$$\text{Li } x = \frac{x}{\log x} + \frac{x}{\log^2 x} + \cdots + (n-1)! \frac{x}{\log^n x} + n! \int_2^x \frac{dt}{\log^{n+1} t} + c_n \quad (x \geq 2)$$

with a suitable constant c_n .

15.37. Prove that for all $n \in \mathbb{N}^+$,

$$\left| \operatorname{Li} x - \sum_{k=1}^n (k-1)! \frac{x}{\log^k x} \right| = o\left(\frac{x}{\log^n x}\right) \quad \text{if } x \rightarrow \infty. \quad (\text{H})$$

15.38. Prove that the function $\Phi(x) = \int_0^x e^{-t^2} dt$ is not elementary.

15.39. Let $I_k = \int (x^k / \sqrt{f(x)}) dx$ ($k \in \mathbb{N}$), where f is a polynomial of degree n . Prove that for all $k > n - 2$, I_k can be expressed as a linear combination of an elementary function and the integrals I_0, I_1, \dots, I_{n-2} . (S)

15.6 Appendix: Integration by Substitution for Definite Integrals (Proof of Theorem 15.22)

Proof (Theorem 15.22). **I.** First, we assume that g is monotone increasing. If g is constant, then on the one hand, $g' = 0$, so the left-hand side of (15.18) is zero, and on the other hand, $g(a) = g(b)$, so the right-hand side of (15.18) is also zero. Thus we can assume that g is not constant, that is, $g(a) < g(b)$.

By our assumptions, f is integrable on the image of g , that is, on the interval $[g(a), g(b)]$ (which must be the image of g by the Bolzano–Darboux theorem). Let $\varepsilon > 0$ be fixed, and let F be a partition of the interval $[g(a), g(b)]$ such that $\Omega_F(f) < \varepsilon$. Since g' is integrable on $[a, b]$, we can choose a partition $\Phi : a = t_0 < t_1 < \dots < t_n = b$ such that $\Omega_\Phi(g') < \varepsilon$. By adding new base points (which does not increase the value of $\Omega_\Phi(g')$), we can ensure that the points $g(t_0), \dots, g(t_n)$ include every base point of F . Let $h = (f \circ g) \cdot g'$. We will show that if $c_i \in [t_{i-1}, t_i]$ ($i = 1, \dots, n$) are arbitrary inner points, then the approximating sum

$$\sigma_\Phi(h; (c_i)) = \sum_{i=1}^n f(g(c_i)) \cdot g'(c_i) \cdot (t_i - t_{i-1})$$

is close to the value of $I = \int_{g(a)}^{g(b)} f dx$.

Let F_1 denote the partition with the base points $g(t_0), \dots, g(t_n)$. Then F_1 is a refinement of F . Let us introduce the notation $g(t_i) = u_i$, where ($i = 0, \dots, n$). Then the points $g(a) = u_0 \leq u_1 \leq \dots \leq u_n = g(b)$ list the base points of F_1 , possibly more than once (if g is not strictly monotone). By the mean value theorem, for every $i = 1, \dots, n$, there exists a point $d_i \in (t_{i-1}, t_i)$ such that

$$u_i - u_{i-1} = g'(d_i) \cdot (t_i - t_{i-1}).$$

Then

$$\sum_{i=1}^n f(g(c_i))(u_i - u_{i-1}) = \sum_{i=1}^n f(g(c_i)) \cdot g'(d_i) \cdot (t_i - t_{i-1}). \quad (15.27)$$

If we drop the terms where $u_{i-1} = u_i$ (and are thus zero) from the left-hand side of (15.27), then we get the approximating sum for the function f corresponding to the partition F_1 , since $g(c_i) \in [g(t_{i-1}), g(t_i)] = [u_{i-1}, u_i]$ for all i . Since $\Omega_{F_1}(f) \leq \Omega_F(f) < \varepsilon$, every such approximating sum must differ from I by less than ε . Thus by (15.27), we get that

$$\left| \sum_{i=1}^n f(g(c_i)) \cdot g'(d_i) \cdot (t_i - t_{i-1}) - I \right| < \varepsilon. \quad (15.28)$$

Let $\omega_i(g')$ denote the oscillation of the function g' over the interval $[t_{i-1}, t_i]$. Then $|g'(c_i) - g'(d_i)| \leq \omega_i(g')$ for all i , so

$$\begin{aligned} |\sigma_{\Phi}(h; (c_i)) - I| &\leq \left| \sum_{i=1}^n f(g(c_i)) (g'(c_i) - g'(d_i)) (t_i - t_{i-1}) \right| + \\ &\quad + \left| \sum_{i=1}^n f(g(c_i)) \cdot g'(d_i) \cdot (t_i - t_{i-1}) - I \right| < \\ &< K \cdot \sum_{i=1}^n \omega_i(g') (t_i - t_{i-1}) + \varepsilon = K \cdot \Omega_{\Phi}(g') + \varepsilon < \\ &< (K + 1)\varepsilon, \end{aligned}$$

where K denotes an upper bound of $|f|$ on the interval $[g(a), g(b)]$. Since this holds for an arbitrary choice of inner points c_i , by Theorem 14.19, $(f \circ g) \cdot g'$ is integrable on $[a, b]$, and its integral is I there. This proves (15.18) for the case that g is monotone increasing.

II. If g is monotone decreasing, then the proof goes the same way, using the fact that $g(a) \geq g(b)$ implies

$$\int_{g(a)}^{g(b)} f dx = - \int_{g(b)}^{g(a)} f dx.$$

III. Now consider the general case. Let $\varepsilon > 0$ be given, and let $\Phi: a = t_0 < t_1 < \dots < t_n = b$ be a partition such that $\Omega_{\Phi}(g') < \varepsilon$. Let $g(t_i) = u_i$ ($i = 0, \dots, n$). If $I = \int_{g(a)}^{g(b)} f dx$ and $I_i = \int_{u_{i-1}}^{u_i} f dx$ for all $i = 1, \dots, n$, then $I_1 + \dots + I_n = I$ by Theorem 14.41.

Let J_1 denote the set of indices i such that the function g is monotone on the interval $[t_{i-1}, t_i]$. If $i \in J_1$, then by the previous cases that we have already proved, $(f \circ g) \cdot g'$ is integrable on $[t_{i-1}, t_i]$, and

$$\int_{t_{i-1}}^{t_i} f(g(t)) \cdot g'(t) dt = \int_{u_{i-1}}^{u_i} f dx = I_i.$$

Thus the interval $[t_{i-1}, t_i]$ has a partition Φ_i such that

$$I_i - (\varepsilon/n) < s_{\Phi_i} \leq S_{\Phi_i} < I_i + (\varepsilon/n), \quad (15.29)$$

where s_{Φ_i} and S_{Φ_i} denote the lower and upper sums of the function $(f \circ g) \cdot g'$ restricted to the interval $[t_{i-1}, t_i]$ over the partition Φ_i . Let Φ' be the union of the partitions Φ and Φ_i ($i \in J_1$). We will show that the lower and upper sums $s_{\Phi'}$ and $S_{\Phi'}$ of $(f \circ g) \cdot g'$ over the partition Φ' are close to I . Consider first the upper sum. Clearly,

$$S_{\Phi'} = \sum_{i \in J_1} S_{\Phi_i} + \sum_{i \in J_2} M_i \cdot (t_i - t_{i-1}), \quad (15.30)$$

where $J_2 = \{1, \dots, n\} \setminus J_1$ and $M_i = \sup\{f(g(x))g'(x) : x \in [t_{i-1}, t_i]\}$. If $i \in J_2$, then g is not monotone on the interval $[t_{i-1}, t_i]$, and so there exists a point $d_i \in [t_{i-1}, t_i]$ such that $g'(d_i) = 0$. If this weren't the case, then by Darboux's theorem (Theorem 13.44), g' would have a constant sign on the interval $[t_{i-1}, t_i]$, and so by Theorem 12.54, g would be monotone there, which is a contradiction to what we just assumed.

Let $\omega_i(g')$ denote the oscillation of the function g' over the interval $[t_{i-1}, t_i]$. Then for arbitrary inner points $c_i \in [t_{i-1}, t_i]$, we have

$$|g'(c_i)| = |g'(c_i) - g'(d_i)| \leq \omega_i(g'), \quad (15.31)$$

so $|M_i| \leq K \cdot \omega_i(g')$, where K is an upper bound of $|f|$ on the image of g , that is, on the interval $g([a, b])$. Then using (15.29) and (15.30), we get that

$$\begin{aligned} |S_{\Phi'} - I| &\leq \sum_{i \in J_1} |S_{\Phi_i} - I_i| + \sum_{i \in J_2} |M_i| \cdot (t_i - t_{i-1}) + \sum_{i \in J_2} |I_i| < \\ &< n \cdot (\varepsilon/n) + \sum_{i \in J_2} K \cdot \omega_i(g') (t_i - t_{i-1}) + \sum_{i \in J_2} |I_i| \leq \\ &\leq \varepsilon + K \cdot \Omega_{\Phi}(g') + \sum_{i \in J_2} |I_i| < \\ &< (K+1)\varepsilon + \sum_{i \in J_2} |I_i|. \end{aligned} \quad (15.32)$$

Now by the mean value theorem, for all $i = 1, \dots, n$, there exists a point $c_i \in (t_{i-1}, t_i)$ such that

$$u_i - u_{i-1} = g'(c_i) \cdot (t_i - t_{i-1}).$$

Thus if $i \in J_2$ then by (15.31),

$$|u_i - u_{i-1}| \leq \omega_i(g') \cdot (t_i - t_{i-1}),$$

so by statement (iv) of Theorem 14.49,

$$|I_i| \leq K \cdot |u_i - u_{i-1}| \leq K \cdot \omega_i(g') \cdot (t_i - t_{i-1}).$$

Then

$$\sum_{i \in J_2} |I_i| \leq \sum_{i \in J_2} K \cdot \omega_i(g') \cdot (t_i - t_{i-1}) \leq K \cdot \Omega_{\Phi}(g') < K\varepsilon.$$

Comparing this with (15.32), we find that $|S_{\Phi'} - I| < (2K + 1)\varepsilon$. The same argument gives that $|s_{\Phi'} - I| < (2K + 1)\varepsilon$. Since this inequality holds for all ε , $(f \circ g) \cdot g'$ is integrable, and its integral is I . \square