Chapter 7
Rudiments of Infinite Series

If we add infinitely many numbers (more precisely, if we take the sum of an infinite sequence of numbers), then we get an infinite series.

Mathematicians in India were dealing with infinite series as early as the fifteenth century, while European mathematics caught up with them only in the seventeenth. But then, however, the study of infinite series underwent rapid development, because mathematicians realized that certain quantities and functions can be computed more easily if they are expressed as infinite series.

There were ideas outside of mathematics that led to infinite series as well. One of the earliest such ideas was Zeno's paradox about Achilles and the tortoise (as seen in our *brief historical introduction*). So-called "elementary" mathematics and recreational mathematics also give rise to questions that lead to infinite series:

1. What is the area of the shaded region in Figure 7.1, formed by infinitely many triangles?

If the area of the big triangle is 1, then the area we seek is clearly $(1/4)+(1/4^2)+(1/4^3)+\cdots$. On he other hand, the big triangle is the union of three copies of the shaded region, so the area of the region is $1/3$. We get that

$$\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots=\frac{1}{3}.$$

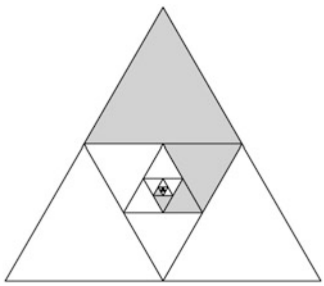

Fig. 7.1

2. A simple puzzle asks that if the weight of a brick is 1 pound plus the weight of half a brick, then how heavy is one brick? Since the weight of half of a brick is $1/2$ pound plus the weight of a quarter of a brick, while the weight of a quarter of a brick is $1/4$ pound plus the weight of an eighth of a brick, and so on, the weight of one brick is $1+(1/2)+(1/4)+(1/8)+\cdots$. On the other hand,

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if we subtract half of a brick from both sides of the equation 1 brick $=$ 1 pound $+\frac{1}{2}$ brick, we get that half a brick weighs 1 pound. Thus the brick weighs 2 pounds, and so

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2.$$

In the *brief historical introduction*, we saw several examples of how the sum of an infinite series can result in some strange or contradictory statements (for example, (1.4) and (1.5)). We can easily see that these strange results come from faulty reasoning, namely from the assumption that every infinite series has a well-defined "predestined" sum. This is false, because only the axioms can be assumed as "predestined" (once we accept them), and every other concept needs to be created by us. We have to give up on every infinite series having a sum: we have to decide ourselves which series should have a sum, and then decide what that sum should be. The concept that we create should satisfy certain expectations and should mirror any intuition we might already have for the concept.

To define infinite series, let us start with finite sums. Even sums of multiple terms are not "predestined," since the axioms talk only about sums of two numbers. We defined the sum of n numbers by adding parentheses to the sum (in the first appendix of Chapter 3), which simply means that we get the sum of n terms after $n-1$ summations. It is only natural to define the infinite sum $a_1+a_2+\cdots$ with the help of the running sums a_1, a_1+a_2, $a_1+a_2+a_3,\ldots$, which are called the partial sums.[1]

It is easy to check (and we will soon do so in Example 7.3) that the partial sums of the series $1+1/2+1/4+1/8+\cdots$ tend to 2, and the partial sums of $3/10+3/100+3/1000+\cdots$ tend to $1/3$. Both results seem to be correct; the second is simply the decimal expansion of the number $1/3$. On the other hand, the partial sums of the problematic series $1+2+4+\cdots$ do not tend to -1, while the partial sums of the series $1-1+1-1+\cdots$ does not tend to anything, but oscillate at infinity instead. By all of these considerations, the definition below follows naturally.

To simplify our formulas, we introduce the following notation for sums of multiple terms: $a_1+\cdots+a_n=\sum_{i=1}^n a_i$.

The infinite series $a_1+a_2+\cdots$ will also get new notation, which is $\sum_{n=1}^\infty a_n$.

Definition 7.1. The *partial sums* of the infinite series $\sum_{n=1}^\infty a_n$ are the numbers $s_n=\sum_{i=1}^n a_i$ $(n=1,2,\ldots)$. If the sequence of partial sums (s_n) is convergent with limit A, then we say that the *infinite series* $\sum_{n=1}^\infty a_n$ *is convergent, and its sum is* A. We denote this by $\sum_{n=1}^\infty a_n=A$.

If the sequence of partial sums (s_n) is divergent, then we say that the *series* $\sum_{n=1}^\infty a_n$ *is divergent.*

If $\lim_{n\to\infty} s_n=\infty$ (or $-\infty$), then we say that *the sum of the series* $\sum_{n=1}^\infty a_n$ *is* ∞ *(or* $-\infty$*).* We denote this by $\sum_{n=1}^\infty a_n=\infty$ (or $-\infty$).

[1] This viewpoint does not treat an infinite sum as an expression whose value is already defined, but instead, the value is "gradually created" with our method. From a philosophical viewpoint, the infinitude of the series is viewed not as "actual infinity" but "potential infinity."

Remark 7.2. Strictly speaking, the expression $\sum_{n=1}^{\infty} a_n$ alone has no meaning. The phrase "consider the infinite series $\sum_{n=1}^{\infty} a_n$" simply means "consider the sequence (a_n)," with the difference that we usually are more concerned about the partial sums[2] $a_1 + \cdots + a_n$. For our purposes, the statement $\sum_{n=1}^{\infty} a_n = A$ is simply a shorthand way of writing that $\lim_{n\to\infty}(a_1 + \cdots + a_n) = A$.

Examples 7.3. **1.** The nth partial sum of the series $1 + 1/2 + 1/4 + 1/8 + \cdots$ is $s_n = \sum_{i=0}^{n-1} 2^{-i} = 2 - 2^{-n+1}$. Since $\lim_{n\to\infty} s_n = 2$, the series is convergent, and its sum is 2.
2. The nth partial sum of the series $3/10 + 3/100 + 3/1000 + \cdots$ is

$$s_n = \sum_{i=1}^{n} 3 \cdot 10^{-i} = \frac{3}{10} \cdot \frac{1 - 10^{-n}}{1 - (1/10)}.$$

Since $\lim_{n\to\infty} s_n = 3/9 = 1/3$, the series is convergent, and its sum is $1/3$.
3. The nth partial sum of the series $1 + 1 + 1 + \cdots$ is $s_n = n$. Since $\lim_{n\to\infty} s_n = \infty$, the sequence is divergent (and its sum is ∞).
4. The $(2k)$th partial sum of the series $1 - 1 + 1 - \ldots$ is zero, while the $(2k+1)$th partial sum is 1 for all $k \in \mathbb{N}$. Since the sequence (s_n) is oscillating at infinity, the series is divergent (and has no sum).
5. The kth partial sum of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{7.1}$$

is

$$s_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k-1} \cdot \frac{1}{k}.$$

If $n < m$, then we can see that

$$|s_m - s_n| = \left| \frac{1}{n+1} - \frac{1}{n+2} + \cdots + (-1)^{n-m+1} \cdot \frac{1}{m} \right| < \frac{1}{n}.$$

It follows that the sequence (s_n) satisfies Cauchy's criterion (Theorem 6.13), so it is convergent. This shows that the series (7.1) is convergent. We will later see that the sum of the series is equal to the natural logarithm of 2 (see Exercise 12.92 and Remark 13.16).

The second example above is a special case of the following theorem, which states that infinite decimal expansions can be thought of as convergent series.

Theorem 7.4. *Let the infinite decimal expansion of x be $m.a_1a_2\ldots$. Then the infinite series $m + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots$ is convergent, and its sum is x.*

[2] There are some who actually mean the series (s_n) when they write $\sum_{n=1}^{\infty} a_n$. We do not follow this practice, since then, the expression $\sum_{n=1}^{\infty} a_n = A$ would state equality between a sequence and a number, which is not a good idea.

Proof. By the definition of the decimal expansion,

$$m.a_1\dots a_n \le x \le m.a_1\dots a_n + \frac{1}{10^n}$$

for all n, so $\lim_{n\to\infty} m.a_1\dots a_n = x$. We see that $m.a_1\dots a_n$ is the $(n+1)$th partial sum of the series

$$m + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots ,$$

which makes the statement of the theorem clear. □

The sums appearing in Examples 7.3.1. and 7.3.2. are special cases of the following theorem.

Theorem 7.5. *The series* $1 + x + x^2 + \cdots$ *is convergent if and only if* $|x| < 1$*, and then its sum is* $1/(1-x)$*.*

Proof. We already saw that in the case $x = 1$, the series is divergent, so we can assume that $x \neq 1$. Then the nth partial sum of the series is $s_n = \sum_{i=0}^{n-1} x^i = (1-x^n)/(1-x)$. If $|x| < 1$, then $x^n \to 0$ and $s_n \to 1/(1-x)$. Thus the series is convergent with sum $1/(1-x)$.

If $x > 1$, then $s_n \to \infty$, so the series is divergent (and its sum is ∞). If, however, $x \le -1$, then the sequence (s_n) oscillates at infinity, so the series is also divergent (with no sum). □

The next theorem outlines an important property of convergent series.

Theorem 7.6. *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let the sum of the series be A. Since

$$a_n = (a_1 + \cdots + a_n) - (a_1 + \cdots + a_{n-1}) = s_n - s_{n-1},$$

we have $a_n \to A - A = 0$. □

Remark 7.7. The theorem above states that for $\sum_{n=1}^{\infty} a_n$ to be convergent, it is necessary for $a_n \to 0$ to hold. It is important to note that this condition is in no way sufficient, since there are many divergent sequences whose terms tend to zero. A simple example: The terms of the series $\sum_{i=0}^{\infty}\left(\sqrt{i+1} - \sqrt{i}\right)$ tend to zero by Example 4.4.3. On the other hand, the nth partial sum is $\sum_{i=0}^{n-1}\left(\sqrt{i+1} - \sqrt{i}\right) = \sqrt{n} \to \infty$ as $n \to \infty$, so the series is divergent.

Another famous example of a divergent series whose terms tend to zero is the series $\sum_{n=1}^{\infty} 1/n$, which is called the **harmonic series**.[3]

[3] The name comes from the fact that the wavelengths of the overtones of a vibrating string of length h are h/n $(n = 2, 3, \dots)$. The wavelengths $h/2, h/3, \dots, h/8$ correspond to the octave, octave plus a fifth, second octave, second octave plus a major third, second octave plus a fifth, second octave plus a seventh, and the third octave, respectively. Thus the series $\sum_{n=1}^{\infty}(h/n)$ contains the tone and its overtones, which are often called harmonics.

Theorem 7.8. *The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.*

We give two proofs of the statement.

Proof. **1.** If the nth partial sum of the series is s_n, then

$$\begin{aligned} s_{2n} - s_n &= \left(1 + \frac{1}{2} + + \cdots + \frac{1}{2n}\right) - \left(1 + \frac{1}{2} + + \cdots + \frac{1}{n}\right) = \\ &= \left(\frac{1}{n+1} + + \cdots + \frac{1}{2n}\right) \geq \\ &\geq n \cdot \frac{1}{2n} = \frac{1}{2} \end{aligned}$$

for all n. Suppose that the series is convergent and its sum is A. Then if $n \to \infty$, then $s_{2n} - s_n \to A - A = 0$, which is impossible.
2. If $n > 2^k$, then

$$\begin{aligned} s_n &\geq 1 + \frac{1}{2} + + \cdots + \frac{1}{2^k} = \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right) \geq \\ &\geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \cdots + 2^{k-1} \cdot \frac{1}{2^k} = \\ &= 1 + k \cdot \frac{1}{2}. \end{aligned}$$

Thus $\lim_{n\to\infty} s_n = \infty$, so the series is divergent, and its sum is ∞. □

Remark 7.9. Since the harmonic series contains the reciprocal of every positive integer, one could expect the behavior of the series to have number-theoretic significance. This is true. Using the divergence of the harmonic series, we can give a new proof of the fact that *there exist infinitely many prime numbers.* Suppose that this were not true, and that there were only finitely many prime numbers. Let these be $p_1, \ldots, p_k$. For all i and N, the relations

$$1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^N} = \frac{1 - p_i^{-(N+1)}}{1 - \frac{1}{p_i}} < \frac{1}{1 - \frac{1}{p_i}}$$

hold. Multiplying these together, we get that

$$\prod_{i=1}^{k} \left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^N}\right) < \prod_{i=1}^{k} \frac{1}{1 - \frac{1}{p_i}} \tag{7.2}$$

for all N. (Here we use the notation $\prod_{i=1}^{k} a_i = a_1 \cdots a_k$.) If we expand the multiplication on the left-hand side, then we get the reciprocal of every number whose prime factorization does not contain any prime with power greater than or equal to N

(since we assumed that there are no prime numbers other than $p_1, \ldots, p_k$). It is clear that every number up to N is there, so $\sum_{n=1}^{N} 1/n = s_N$ is smaller than the right-hand side of (7.2). This, however, is impossible, since $s_N \to \infty$ as $N \to \infty$.

With a refinement of the proof above, one can show that the series consisting of the reciprocals of all primes is divergent. We also know more precisely that the sum of reciprocals of primes smaller than x is greater than $\log\log x - 1$ for all $x \geq 2$ (see Chapter 5 of [2] or Corollary 18.16 and Theorem 18.17 of this book).

The second proof of Theorem 7.8 seems to tell us more than the first, because it says not only that the series is divergent, but that its sum is infinite too. By the following simple theorem, we see that the sum of a divergent series consisting of nonnegative terms is always infinite.

Theorem 7.10.

(i) *A series consisting of nonnegative terms is convergent if and only if the sequence of its partial sums is bounded (from above).*
(ii) *If a series consisting of nonnegative terms is divergent, then its sum is infinite.*

Proof. By the assumption that the terms of the series are nonnegative, we clearly get that the sequence of partial sums of the series is monotone increasing. If this sequence is bounded from above, then it is convergent by Theorem 6.2. Then the series in question is convergent. If, however, the sequence of partial sums is not bounded from above, then by Theorem 6.3, it tends to infinity, so the series will be divergent, and its sum will be infinity. □

We emphasize that by the above theorem, *a series consisting of nonnegative terms always has a sum: either a finite number (if the series converges) or infinity (if the series diverges).*

Examples 7.11. **1.** The series $\sum_{i=1}^{\infty} 1/i^2$ is convergent, because its nth partial sum is

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 1 + \sum_{i=2}^{n} \frac{1}{(i-1)\cdot i} = 1 + \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i} \right) = 2 - \frac{1}{n} < 2.$$

During the seventeenth century and the beginning of the eighteenth, many mathematicians tried to determine the value of the series $\sum_{i=1}^{\infty} 1/i^2$. Finally, Johann Bernoulli[4] and Euler independently found that $\sum_{i=1}^{\infty} 1/i^2 = \pi^2/6$. This fact now has dozens of proofs (see http://mathworld.wolfram.com/RiemannZetaFunction Zeta2.html). For a relatively elementary proof, see Exercise 11.36.

2. By part (b) of Exercise 2.7, the partial sums of the series $\sum_{i=1}^{\infty} 1/i^{3/2}$ are less than 3. Thus the series is convergent, and its sum is at most 3.

3. We generally call series of the form $\sum_{i=1}^{\infty} 1/i^c$ **hyperharmonic series.** It is easy to see that if $b > 0$, then

$$1 + \frac{1}{2^{b+1}} + \cdots + \frac{1}{n^{b+1}} \leq 1 + \frac{1}{b} - \frac{1}{b \cdot n^b} \tag{7.3}$$

[4] Johann Bernoulli (1667–1748) Swiss mathematician, brother of Jacob Bernoulli.

for all n (see Exercise 7.5). It then follows that *the hyperharmonic series* $\sum_{i=1}^{\infty} 1/i^c$ *is convergent for all* $c > 1$ (see also Exercise 7.6). We denote the sum of the series by $\zeta(c)$. By equality (7.3), it follows that every partial sum of the series is less than $c/(c-1)$, and so $1 < \zeta(c) \le c/(c-1)$ for all $c > 1$.

The theorems of Johann Bernoulli and Euler can be summarized using this notation as $\zeta(2) = \pi^2/6$.

Remark 7.12. The series $\sum_{n=1}^{\infty} 1/n^2$ is an example of the rare occurrence whereby we can specifically determine the sum of a series (such as the series appearing in Exercises 7.2 and 7.3). These can be thought of as exceptions, for we generally cannot express the sum of an arbitrary series in closed form. The sums $\sum_{n=1}^{\infty} 1/n^c$ (that is, the values of $\zeta(c)$) have closed formulas only for some special values of c. We already saw that $\zeta(2) = \pi^2/6$. Bernoulli and Euler proved that if k is an even positive integer, then $\zeta(k)$ is equal to a rational multiple of π^k. However, to this day, we do not know whether this is also true if k is an odd integer. In fact, *nobody has found closed expressions for the values of* $\zeta(3)$, $\zeta(5)$, *and so on for the past 300 years*, and it is possible that no such closed form exists.

By the transcendence of the number π, we know that the values $\zeta(2k)$ are irrational for every positive integer k. In the 1970s, it was proven that the value $\zeta(3)$ is also irrational. *Whether the numbers* $\zeta(5)$, $\zeta(7)$, *and so on are rational or not, however, is still an open question.*

In the general case, the following theorem gives us a precise condition for the convergence of a series.

Theorem 7.13 (Cauchy's Criterion). *The infinite series* $\sum_{n=1}^{\infty} a_n$ *is convergent if and only if for all* $\varepsilon > 0$, *there exists an index* N *such that for all* $N \le n < m$,

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \varepsilon.$$

Proof. Since $a_{n+1} + a_{n+2} + \cdots + a_m = s_m - s_n$, the statement is clear by Cauchy's criterion for sequences (Theorem 6.13). □

Exercises

7.1. For a fixed $\varepsilon > 0$, give threshold indices above which the partial sums of the following series differ from their actual sum by less than ε.

(a) $\sum_{n=0}^{\infty} 1/2^n$;
(b) $\sum_{n=0}^{\infty} (-2/3)^n$;
(c) $1 - 1/2 + 1/3 - 1/4 + \cdots$;
(d) $1/1\cdot 2 + 1/2\cdot 3 + 1/3\cdot 4 + \cdots$.

7.2.
(a) $\sum_{n=1}^{\infty} 1/(n^2+2n) = ?$
(b) $\sum_{n=1}^{\infty} 1/(n^2+4n+3) = ?$
(c) $\sum_{n=2}^{\infty} 1/(n^3-n) = ?$ (H S)

7.3. Give a general method for determining the sums $\sum_{n=a}^{\infty} p(n)/q(n)$ in which p and q are polynomials, $\operatorname{gr} p \le \operatorname{gr} q - 2$, and $q(x) = (x-a_1)\cdots(x-a_k)$, where $a_1,\ldots,a_k$ are distinct integers smaller than a. (H)

7.4. Are the following series convergent?

(a) $\sum_1^{\infty} 1/\sqrt[n]{2}$; (b) $\sum_2^{\infty} 1/\sqrt[n]{\log n}$;
(c) $\sum_2^{\infty} n\log((n+1)/n)$; (d) $\sum_1^{\infty} (3^{\sqrt{n}} - 2^n)/(3^{\sqrt{n}} + 2^n)$.

7.5. Prove inequality (7.3) for all $b > 0$. (H S)

7.6. Show that

$$\left(1 + \frac{1}{2^c} + \cdots + \frac{1}{n^c}\right)\left(1 - \frac{2}{2^c}\right) < 1$$

for all $n = 1,2,\ldots$ and $c > 0$. Deduce from this that the series $\sum_{n=1}^{\infty} 1/n^c$ is convergent for all $c > 1$.

7.7. Prove that $\lim_{n\to\infty} \zeta(1 + (1/n)) = \infty$.

7.8. Let $a_1, a_2, \ldots$ be a listing of the positive integers that do not contain the digit 7 (in decimal representation). Prove that $\sum_{n=1}^{\infty} 1/a_n$ is convergent. (H)

7.9. Let $\sum_{n=1}^{\infty} a_n$ be convergent. $\lim_{n\to\infty} (a_{n+1} + a_{n+2} + \cdots + a_{n^2}) = ?$

7.10. Let the series (a_n) satisfy

$$\lim_{n\to\infty} |a_n + a_{n+1} + \cdots + a_{n+2^n}| = 0.$$

Does it then follow that the series $\sum_{n=1}^{\infty} a_n$ is convergent? (H)

7.11. Let the series (a_n) satisfy

$$\lim_{n\to\infty} |a_n + a_{n+1} + \cdots + a_{n+i_n}| = 0$$

for every sequence of positive integers (i_n). Does it then follow that the series $\sum_{n=1}^{\infty} a_n$ is convergent? (H)

7.12. Prove that if $|a_{n+1} - a_n| < 1/n^2$ for all n, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

7.13. Suppose that $a_n \le b_n \le c_n$ for all n. Prove that if the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$ are convergent, then the series $\sum_{n=1}^{\infty} b_n$ is convergent as well.

7.14. Prove that if the series $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = 0. \text{ (S)}$$