Chapter 18
The Stieltjes Integral

In this chapter we discuss a generalization of the Riemann integral that is often used in both theoretical and applied mathematics. Stieltjes[1] originally introduced this concept to deal with infinite continued fractions,[2] but it was soon apparent that the concept is useful in other areas of mathematics—and thus in mathematical physics, probability, and number theory, independently of its role in continued fractions. We illustrate the usefulness of the concept with two simple examples.

Example 18.1. Consider a planar curve parameterized by $\gamma(t) = \big(x(t), y(t)\big)$ $(t \in [a,b])$, where the x-coordinate function is strictly monotone increasing and continuous, and the y-coordinate function is nonnegative on $[a,b]$. The problem is to find the area under the region bounded by the curve. If $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of the interval $[a,b]$ and $c_i \in [t_{i-1}, t_i]$ for all i, then the area can be approximated by the sum

$$\sum_{i=1}^{n} y(c_i)\big(x(t_i) - x(t_{i-1})\big).$$

We can expect the area to be the limit—in a suitable sense—of these sums.

Example 18.2. Consider a metal rod of negligible thickness but not negligible mass $M > 0$. Suppose that the rod lies on the interval $[a,b]$, and let the mass of the rod over the subinterval $[a,x]$ be $m(x)$ for all $x \in [a,b]$. Our task is to find the center of mass of the rod.

We know that if we place weights $m_1, \ldots, m_n$ at the points $x_1, \ldots, x_n$, then the center of mass of this system of points $\{x_1, \ldots, x_n\}$ is

$$\frac{m_1 x_1 + \cdots + m_n x_n}{m_1 + \cdots + m_n}.$$

[1] Thomas Joannes Stieltjes (1856–1894), Dutch mathematician.

[2] For more on continued fractions, see [5].

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Consider a partition $a = t_0 < t_1 < \cdots < t_n = b$ and choose points $c_i \in [t_{i-1}, t_i]$ for all i. If we suppose that the mass distribution of the rod is continuous (meaning that the mass of the rod at every single point is zero), then the mass of the rod over the interval $[t_{i-1}, t_i]$ is $m(t_i) - m(t_{i-1})$. Concentrating this weight at the point c_i, the center of mass of the system of points $\{c_1, \ldots, c_n\}$ is

$$\frac{c_1\big(m(t_1) - m(t_0)\big) + \cdots + c_n\big(m(t_n) - m(t_{n-1})\big)}{M}.$$

This approximates the center of mass of the rod itself, and once again, we expect that the limit of these numbers in a suitable sense will be the center of mass.

We can see that in both examples, a sum appears that depends on two functions. In these sums, we multiply the value of the first function (which, in Example 18.2, was the function x) at the inner points by the increments of the second function.

We use the following notation and naming conventions. Let $f, g \colon [a,b] \to \mathbb{R}$ be given functions, let $F \colon a = x_0 < x_1 < \cdots < x_n = b$ be a partition of the interval $[a,b]$, and let $c_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, n)$ be arbitrary inner points. Then the sum

$$\sum_{i=1}^{n} f(c_i) \cdot \big(g(x_i) - g(x_{i-1})\big)$$

is denoted by $\sigma_F\big(f, g; (c_i)\big)$, and is called the **approximating sum of f with respect to g**.

Definition 18.3. Let $f, g \colon [a,b] \to \mathbb{R}$ be given functions. We say that the *Stieltjes integral $\int_a^b f\,dg$ of f with respect to g exists and has value I* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $F \colon a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $[a,b]$ with mesh smaller than δ and $c_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, n)$ are arbitrary inner points, then

$$\left|\sigma_F\big(f, g; (c_i)\big) - I\right| < \varepsilon. \tag{18.1}$$

Remarks 18.4. **1.** Let $g(x) = x$ for all $x \in [a,b]$. It is clear that the Stieltjes integral $\int_a^b f\,dg$ exists exactly when the Riemann integral $\int_a^b f\,dx$ does, and in this case, their values agree.

2. If the function g is constant, then the Stieltjes integral $\int_a^b f\,dg$ always exists, and its value is zero. If the function f is constant and has value c, then the Stieltjes integral $\int_a^b f\,dg$ always exists and has value $c \cdot \big(g(b) - g(a)\big)$.

3. The existence of the Stieltjes integral $\int_a^b f\,dg$ is not guaranteed by f and g being Riemann integrable in $[a,b]$. One can show that *if f and g share a point of discontinuity, then the Stieltjes integral $\int_a^b f\,dg$ does not exist.* (See Exercise 18.4.) Thus if f and g are bounded functions in $[a,b]$ and are continuous everywhere except at a common point, then they are both Riemann integrable in $[a,b]$, while the Stieltjes integral $\int_a^b f\,dg$ does not exist.

4. For the Stieltjes integral $\int_a^b f\,dg$ to exist, it is not even sufficient for f and g to be continuous in $[a,b]$. See Exercise 18.5.

Now we show that if g is strictly monotone and continuous, then the Stieltjes integral $\int_a^b f\,dg$ can be reduced to a Riemann integral.

Theorem 18.5. *If $g\colon [a,b]\to\mathbb{R}$ is strictly monotone increasing and continuous, then the Stieltjes integral $\int_a^b f\,dg$ exists if and only if the Riemann integral $\int_{g(a)}^{g(b)} f\circ g^{-1}\,dx$ does, and then*

$$\int_a^b f\,dg = \int_{g(a)}^{g(b)} f\circ g^{-1}\,dx.$$

Proof. If $F\colon a = x_0 < x_1 < \cdots < x_n = b$ is a partition and $c_i \in [x_{i-1}, x_i]$ $(i = 1,\ldots,n)$ are arbitrary inner points, then the points $t_i = g(x_i)$ $(i = 1,\ldots,n)$ give us a partition $\overline{F}$ of the interval $[g(a), g(b)]$, and we have $g(c_i) \in [g(x_{i-1}), g(x_i)]$ for all $i = 1,\ldots,n$. Then

$$\sum_{i=1}^{n} f(c_i)\cdot\big(g(x_i) - g(x_{i-1})\big) = \sum_{i=1}^{n} (f\circ g^{-1})\big(g(c_i)\big)\cdot\big(t_i - t_{i-1}\big) \tag{18.2}$$

is an approximating sum of the Riemann sum $\int_{g(a)}^{g(b)} f\circ g^{-1}\,dx$. Conversely, if we have $\overline{F}\colon g(a) = t_0 < t_1 < \cdots < t_n = g(b)$ as a partition of the interval $[g(a), g(b)]$ and $d_i \in [t_{i-1}, t_i]$ for all $i = 1,\ldots,n$, then the points $x_i = g^{-1}(t_i)$ $(i = 1,\ldots,n)$ create a partition F of the interval $[a,b]$. If $c_i = g^{-1}(d_i)$, then $c_i \in [x_{i-1}, x_i]$ for all $i = 1,\ldots,n$, and (18.2) holds.

By the uniform continuity of g, we know that for every $\delta > 0$, if the partition F has small enough mesh, then $\overline{F}$ has mesh smaller than δ. Thus comparing statement (iii) of Theorem 14.23 with equality (18.2), we get that if the Riemann integral $\int_{g(a)}^{g(b)} f\circ g^{-1}\,dx$ exists, then the Stieltjes integral $\int_a^b f\,dg$ also exists, and they are equal.

Since the function g^{-1} is also uniformly continuous, it follows that for every $\delta > 0$, if the partition $\overline{F}$ has small enough mesh, then F has mesh smaller than δ. Thus by the definition of the Stieltjes integral, by statement (iii) of Theorem 14.23, and by equality (18.2), it follows that if the Stieltjes integral $\int_a^b f\,dg$ exists, then the Riemann integral $\int_{g(a)}^{g(b)} f\circ g^{-1}\,dx$ exists as well, and they have the same value. □

The following statements can be deduced easily from the definition of the Stieltjes integral. We leave their proofs to the reader.

Theorem 18.6.

(i) *If the Stieltjes integrals $\int_a^b f_1\,dg$ and $\int_a^b f_2\,dg$ exist, then for all $c_1, c_2 \in \mathbb{R}$, the Stieltjes integral $\int_a^b (c_1 f_1 + c_2 f_2)\,dg$ exists as well, taking on the value $c_1\cdot\int_a^b f_1\,dg + c_2\cdot\int_a^b f_2\,dg$.*

(ii) *If the Stieltjes integrals $\int_a^b f\,dg_1$ and $\int_a^b f\,dg_2$ exist, then for all $c_1, c_2 \in \mathbb{R}$, the Stieltjes integral $\int_a^b f\,d(c_1 g_1 + c_2 g_2)$ exists as well, and has value $c_1\cdot\int_a^b f\,dg_1 + c_2\cdot\int_a^b f\,dg_2$.*

The following theorem gives us a necessary and sufficient condition for the existence of Stieltjes integrals. For the proof of the theorem, see Exercise 18.6.

Theorem 18.7 (Cauchy's Criterion). *The Stieltjes integral $\int_a^b f\,dg$ exists if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if F_1 and F_2 are partitions of $[a,b]$ with mesh smaller than δ, then*

$$\left|\sigma_{F_1}\big(f,g;(c_i)\big) - \sigma_{F_2}\big(f,g;(d_j)\big)\right| < \varepsilon$$

with an arbitrary choice of the inner points c_i and d_j.

With the help of Cauchy's criterion, it is easy to prove the following theorem. We leave the proof to the reader once again.

Theorem 18.8. *If the Stieltjes integral $\int_a^b f\,dg$ exists, then for all $a < c < b$, the integrals $\int_a^c f\,dg$ and $\int_c^b f\,dg$ also exist, and $\int_a^b f\,dg = \int_a^c f\,dg + \int_c^b f\,dg$.*

We should note that the existence of the Stieltjes integrals $\int_a^c f\,dg$ and $\int_c^b f\,dg$ alone does not imply the existence of $\int_a^b f\,dg$ (see Exercise 18.2). If, however, at least one of f and g is continuous at c, and the other function is bounded then the existence of $\int_a^b f\,dg$ already follows (see Exercise 18.3).

The following important theorem shows that the roles of f and g in the Stieltjes integral are symmetric in some sense. We remind our readers that $[F]_a^b$ denotes the difference $F(b) - F(a)$.

Theorem 18.9 (Integration by Parts). *If the Stieltjes integral $\int_a^b f\,dg$ exists, then $\int_a^b g\,df$ also exists, and $\int_a^b f\,dg + \int_a^b g\,df = [f \cdot g]_a^b$.*

Proof. The proof relies on Abel's rearrangement (see equation (14.31)).

Let $F\colon a = x_0 < x_1 < \cdots < x_n = b$ be a partition, and let $c_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, n)$ be inner points. If we apply Abel's rearrangement to the approximating sum $\sigma_F\big(g,f;(c_i)\big)$, we get that

$$\begin{aligned}\sigma_F\big(g,f;(c_i)\big) &= \sum_{i=1}^{n} g(c_i)\cdot\big(f(x_i) - f(x_{i-1})\big) = \\ &= f(b)g(b) - f(a)g(a) - \sum_{i=0}^{n} f(x_i)\big(g(c_{i+1}) - g(c_i)\big), \qquad (18.3)\end{aligned}$$

where $c_0 = a$ and $c_{n+1} = b$. Since $a = c_0 \le c_1 \le \ldots \le c_{n+1} = b$ and $x_i \in [c_i, c_{i+1}]$ for all $i = 0, \ldots, n$, we have

$$\sum_{i=0}^{n} f(x_i)\big(g(c_{i+1}) - g(c_i)\big) = \sigma_{F'}\big(f,g;(d_i)\big), \qquad (18.4)$$

where F' denotes the partition defined by the points c_i $(0 \le i \le n)$ with the corresponding inner points. (We list each of the c_i points only once. Note that on the left-hand side of (18.4), we can leave out the terms in which $c_i = c_{i+1}$.) Then by (18.3),

$$\sigma_F\big(g,f;(c_i)\big) = [f\cdot g]_a^b - \sigma_{F'}\big(f,g;(d_i)\big).$$

Let $\varepsilon > 0$ be given, and suppose that $\delta > 0$ satisfies the condition of Definition 18.3. It is easy to see that if the mesh of the partition F is smaller than $\delta/2$, then the mesh of F' is smaller than δ, and so $\left|\sigma_{F'}\left(f,g;(d_i)\right) - I\right| < \varepsilon$, where $I = \int_a^b f\,dg$. This shows that if the mesh of F is smaller than $\delta/2$, then $\left|\sigma_F\left(g,f;(c_i)\right) - ([fg]_a^b - I)\right| < \varepsilon$ for an arbitrary choice of the inner points. It follows that the integral $\int_a^b g\,df$ exists, and that its value is $[fg]_a^b - I$. □

Since Cauchy's criterion (Theorem 18.7) is hard to apply in deciding whether a specific Stieltjes integral exists, we need other conditions guaranteeing the existence of the Stieltjes integral that can be easier to check. The simplest such condition is the following.

Theorem 18.10. *If one of the functions f and g defined on the interval $[a,b]$ is continuous, while the other is of bounded variation, then the Stieltjes integrals $\int_a^b f\,dg$ and $\int_a^b g\,df$ exist.*

Proof. By Theorem 18.9, it suffices to prove the existence of the integral $\int_a^b f\,dg$, and we can also assume that f is continuous and g is of bounded variation. By Theorems 17.8 and 18.6, it suffices to consider the case that g is monotone increasing.

For an arbitrary partition $F\colon\ a = x_0 < x_1 < \cdots < x_n = b$, let

$$s_F = \sum_{i=1}^n m_i \cdot \left(g(x_i) - g(x_{i-1})\right) \quad \text{and} \quad S_F = \sum_{i=1}^n M_i \cdot \left(g(x_i) - g(x_{i-1})\right),$$

where $m_i = \min\{f(x)\colon x \in [x_{i-1}, x_i]\}$ and $M_i = \max\{f(x)\colon x \in [x_{i-1}, x_i]\}$ for all $i = 1, \dots, n$. Since $g(x_i) - g(x_{i-1}) \ge 0$ for all i,

$$s_F \le \sigma_F\left(f,g;(c_i)\right) \le S_F \tag{18.5}$$

with any choice of inner points c_i.

It is easy to see that $s_{F_1} \le S_{F_2}$ for any partitions F_1 and F_2 (by repeating the proofs of Lemmas 14.3 and 14.4, using that g is monotone increasing). Thus the set of "lower sums" s_F is nonempty and bounded from above. If I denotes the supremum of this set, then $s_F \le I \le S_F$ for every partition F.

Now we show that $\int_a^b f\,dg$ exists, and its value is I. Let $\varepsilon > 0$ be given. By Heine's theorem, f is uniformly continuous on $[a,b]$, so there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a,b]$ and $|x - y| < \delta$. Let $F\colon\ a = x_0 < x_1 < \cdots < x_n = b$ be an arbitrary partition with mesh smaller than δ. By Weierstrass's theorem, for each i, there are points $c_i, d_i \in [x_{i-1}, x_i]$ such that $f(c_i) = m_i$ and $f(d_i) = M_i$. Then $|d_i - c_i| \le x_i - x_{i-1} < \delta$, so by our choice of δ, we have $M_i - m_i = f(d_i) - f(c_i) < \varepsilon$. Thus

$$\begin{aligned} S_F - s_F &= \sum_{i=1}^n (M_i - m_i) \cdot \left(g(x_i) - g(x_{i-1})\right) \le \varepsilon \cdot \sum_{i=1}^n \left(g(x_i) - g(x_{i-1})\right) = \\ &= \left(g(b) - g(a)\right) \cdot \varepsilon. \end{aligned}$$

Now using (18.5), we get that

$$I - \big(g(b) - g(a)\big) \cdot \varepsilon < s_F \le \sigma_F\big(f, g; (c_i)\big) \le S_F < I + \big(g(b) - g(a)\big) \cdot \varepsilon$$

for arbitrary inner points c_i. This shows that $\int_a^b f\,dg$ exists and that its value is I. □

Remark 18.11. One can show that the class of continuous functions and the class of functions of bounded variation are "dual classes" in the sense that a function f is continuous if and only if $\int_a^b f\,dg$ exists for all functions g that are of bounded variation, and a function g is of bounded variation if and only if $\int_a^b f\,dg$ exists for every continuous function f. (See Exercises 18.8 and 18.9.)

The following theorem can often be applied to computing Stieltjes integrals.

Theorem 18.12. *If f is Riemann integrable, g is differentiable, and g' is Riemann integrable on $[a,b]$, then the Stieltjes integral of f with respect to g exists, and*

$$\int_a^b f\,dg = \int_a^b f \cdot g'\,dx. \tag{18.6}$$

Proof. Since f and g' are integrable on $[a,b]$, the Riemann integral on the right-hand side of (18.6) exists. Let its value be I. We want to show that for an arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that for every partition $a = x_0 < x_1 < \cdots < x_n = b$ with mesh smaller than δ, (18.1) holds with an arbitrary choice of inner values $c_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, n)$.

Let $\varepsilon > 0$ be given. Since f is Riemann integrable on $[a,b]$, there must exist a $\delta_1 > 0$ such that $\Omega_F(f) < \varepsilon$ whenever the partition F has mesh smaller than δ_1. By Theorem 14.23, there exists a $\delta_2 > 0$ such that whenever $a = x_0 < x_1 < \cdots < x_n = b$ is a partition with mesh smaller than δ_2,

$$\left| I - \sum_{i=1}^{n} f(d_i) g'(d_i)(x_i - x_{i-1}) \right| < \varepsilon \tag{18.7}$$

holds for arbitrary inner points $d_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, n)$.

Let $\delta = \min(\delta_1, \delta_2)$, and consider an arbitrary partition $F\colon a = x_0 < x_1 < \cdots < x_n = b$ with mesh smaller than δ. Let $c_i \in [x_{i-1}, x_i]$ $(i = 1, \ldots, n)$ be arbitrary inner points.

Since g is differentiable on $[a,b]$, by the mean value theorem,

$$\sum_{i=1}^{n} f(c_i)\big(g(x_i) - g(x_{i-1})\big) = \sum_{i=1}^{n} f(c_i) g'(d_i)(x_i - x_{i-1}) \tag{18.8}$$

for suitable numbers $d_i \in [x_{i-1}, x_i]$. Let K denote an upper bound of $|g'|$ on $[a,b]$. Then

$$\left| I - \sum_{i=1}^{n} f(c_i)\big(g(x_i) - g(x_{i-1})\big) \right| = \left| I - \sum_{i=1}^{n} f(c_i) g'(d_i)(x_i - x_{i-1}) \right| \le$$

$$\le \left| I - \sum_{i=1}^{n} f(d_i) g'(d_i)(x_i - x_{i-1}) \right| + \left| \sum_{i=1}^{n} \big(f(c_i) - f(d_i)\big) g'(d_i)(x_i - x_{i-1}) \right| <$$

$$< \varepsilon + K \cdot \sum_{i=1}^{n} \omega(f; [x_{i-1}, x_i]) \cdot (x_i - x_{i-1}) = \varepsilon + K \cdot \Omega_F(f) < (1+K) \cdot \varepsilon,$$

which concludes the proof of the theorem. □

Remarks 18.13. **1.** The conditions for the existence of the integral in Theorem 18.12 can be significantly weakened. So for example, the integral $\int_a^b f\,dg$ is guaranteed to exist if f is Riemann integrable and g is Lipschitz (see Exercise 18.11).

2. In the statement above, the Lipschitz property of g can be weakened further. We say that the function $f\colon [a,b] \to \mathbb{R}$ is **absolutely continuous** if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $[a_1,b_1], \ldots, [a_n,b_n]$ are nonoverlapping subintervals of $[a,b]$ such that $\sum_{i=1}^{n} (b_i - a_i) < \delta$, then $\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \varepsilon$. One can show that *if f is Riemann integrable and g is absolutely continuous in $[a,b]$, then the Stieltjes integral $\int_a^b f\,dg$ exists.*

3. The class of Riemann integrable functions and the class of absolutely continuous functions are also dual: *a function f is Riemann integrable if and only if $\int_a^b f\,dg$ exists for every absolutely continuous function g, and a function g is absolutely continuous if and only if $\int_a^b f\,dg$ exists for every Riemann integrable function f.* The proof of this theorem, however, uses concepts from measure theory that we do not deal with in this book.

A number-theoretic application. In the introduction of the chapter we mentioned that Stieltjes integrals pop up in many areas of mathematics, such as in number theory. Dealing with an important problem—namely the distribution of the prime numbers—we often need to approximate sums that consist of the values of specific functions at the prime numbers. For example, $L(x) = \sum_{p \le x} (\log p)/p$ and $R(x) = \sum_{p \le x} 1/p$ are such sums. In these sums, we need to add the numbers $(\log p)/p$ or $1/p$ for all prime numbers p less than or equal to x. Transforming these sums (often using Abel's rearrangement) can be efficiently done with the help of the Stieltjes integral, as shown by the following theorem.

Let a sequence $a_1 < a_2 < \cdots$ that tends to infinity and the function φ defined at the numbers a_n be given. Let $A(x) = \sum_{a_n \le x} \varphi(a_n)$. (If (a_n) is the sequence of prime numbers and $\varphi(x) = (\log x)/x$, then $A(x) = L(x)$, and if $\varphi(x) = 1/x$, then $A(x) = R(x)$.)

Theorem 18.14. *Suppose that f is differentiable and f' is integrable on $[a,b]$, where $a < a_1 \le b$. Then*

$$\sum_{a_n \le b} f(a_n) \cdot \varphi(a_n) = f(b) \cdot A(b) - \int_a^b A(x) \cdot f'(x)\,dx. \tag{18.9}$$

Proof. We show that $\sum_{a_n \le b} f(a_n) \cdot \varphi(a_n) = \int_a^b f\,dA$. The function $A(x)$ is constant on the interval $[a_{n-1}, a_n)$, and has a jump discontinuity at the point a_n, and there, $A(a_n) - \lim_{x \to a_n - 0} A(x) = \varphi(a_n)$. Thus if we take a partition of the interval $[a,b]$ with mesh small enough, then any approximating sum of the Stieltjes integral $\int_a^b f\,dA$ will consist of mostly zero terms, except for the terms that correspond to a subinterval containing one of $a_n \le b$, and the nth such term will be close to $f(a_n) \cdot \varphi(a_n)$ by the continuity of f.

Now integration by parts (Theorem 18.9) gives

$$\sum_{a_n \le b} f(a_n) \cdot \varphi(a_n) = \int_a^b f\,dA = f(b) \cdot A(b) - \int_a^b A\,df = f(b) \cdot A(b) - \int_a^b A \cdot f'\,dx,$$

also using Theorem 18.12. □

If (a_n) is the sequence of prime numbers and $\varphi \equiv 1$, then the value of $A(x)$ is the sum of prime numbers up to x, that is $\pi(x)$. This gives us the following corollary.

Corollary 18.15. *Suppose that f is differentiable and f' is integrable on the interval $[1,x]$ $(x \ge 2)$. Then*

$$\sum_{p \le x} f(p) = f(x) \cdot \pi(x) - \int_1^x \pi(t) \cdot f'(t)\,dt. \tag{18.10}$$

If, for example, $f(x) = 1/x$, then we get that $\sum_{p \le x} 1/p \ge \int_2^x (\pi(t)/t^2)\,dt$. Now there exists a constant $c > 0$ such that $\pi(x) \ge c \cdot x/\log x$. (See [5], Corollary 8.6.) Since $\int_e^x dt/(t \cdot \log t) = \log\log x$, we get that $\sum_{p \le x} 1/p \ge c \cdot \log\log x$. This proves the following theorem.

Corollary 18.16.

$$\sum_p \frac{1}{p} = \infty.$$

We can get a much better approximation for the partial sums if we use the fact that the difference between the function $L(x) = \sum_{p \le x} (\log p)/p$ and $\log x$ is bounded. (A proof of this fact can be found in [5, Theorem 8.8(b)].) Let $\eta(x) = L(x) - \log x$.

Let (a_n) be the sequence of prime numbers, and apply (18.9) with the choices $\varphi(x) = (\log x)/x$ and $f(x) = 1/\log x$. Then $A(x) = L(x)$, and so

$$\begin{aligned}\sum_{p \le x} \frac{1}{p} &= \frac{L(x)}{\log x} + \int_2^x \frac{L(t)}{t \cdot \log^2 t}\,dt = \\ &= 1 + \frac{\eta(x)}{\log x} + \int_2^x \frac{1}{t \cdot \log t}\,dt + \int_2^x \frac{\eta(t)}{t \cdot \log^2 t}\,dt = \\ &= \log\log x - \log\log 2 + 1 + \frac{\eta(x)}{\log x} + \int_2^x \frac{\eta(t)}{t \cdot \log^2 t}\,dt.\end{aligned}$$

Here if $x \to \infty$, then $\eta(x)/\log x$ tends to zero, and we can also show that the integral $\int_2^x \big(\eta(t)/(t \cdot \log^2 t)\big)\,dt$ has a finite limit as $x \to \infty$; this follows easily by the theory of improper integrals; see the next chapter. Comparing all of this, we get the following.

Theorem 18.17. *The limit* $\lim_{x\to\infty}\left(\sum_{p\le x}\frac{1}{p} - \log\log x\right)$ *exists and is finite.*

Exercises

18.1. Let

$$\alpha(x) = \begin{cases} 0, & (0 \le x < 1) \\ 1, & (x = 1) \end{cases}; \ \beta(x) = \begin{cases} 1, & (x = 0) \\ 0, & (0 < x \le 1) \end{cases}; \ \gamma(x) = \begin{cases} 0, & (x \ne 1/2) \\ 1, & (x = 1/2) \end{cases}.$$

Show that the following Stieltjes integrals exist, and compute their values.

(a) $\int_0^1 \sin x\, d\alpha$; (b) $\int_0^1 \alpha\, d\sin x$; (c) $\int_0^1 e^x\, d\beta$;

(d) $\int_0^1 \beta\, de^x$; (e) $\int_0^1 x^2\, d\gamma$; (f) $\int_0^1 \gamma\, dx^2$;

(g) $\int_0^2 e^x\, d[x]$; (h) $\int_0^2 [x]\, de^x$.

18.2. Let

$$f(x) = \begin{cases} 0, & \text{if } -1 \le x \le 0 \\ 1, & \text{if } \ \ 0 < x \le 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0, & \text{if } -1 \le x < 0 \\ 1, & \text{if } \ \ 0 \le x \le 1. \end{cases}$$

Prove that the Stieltjes integrals $\int_{-1}^0 f\,dg$ and $\int_0^1 f\,dg$ exist, but $\int_{-1}^1 f\,dg$ does not.

18.3. Prove that if the Stieltjes integrals $\int_a^c f\,dg$ and $\int_c^b f\,dg$ exist, at least one of f or g is continuous at c, and the other function is bounded then the Stieltjes integral $\int_a^b f\,dg$ also exists.

18.4. Prove that if the functions f and g share a point of discontinuity, then the Stieltjes integral $\int_a^b f\,dg$ does not exist. (H S)

18.5. Let $f(x) = \sqrt{x} \cdot \sin(1/x)$ if $x \ne 0$, and $f(0) = 0$. Prove that the Stieltjes integral $\int_0^1 f\,df$ does not exist. (H S)

18.6. Prove Theorem 18.7. (H)

18.7. Prove that if $\int_a^b f\,df$ exists, then its value is $\big(f(b)^2 - f(a)^2\big)/2$.

18.8. Prove that if $\int_a^b f\,dg$ exists for every function g that is of bounded variation, then f is continuous. (H)

18.9. Prove that if $\int_a^b f\,dg$ exists for every continuous function f, then g is of bounded variation. (H)

18.10. Let $\mathscr{F}$ be an arbitrary set of functions defined on the interval $[a,b]$. Let $\mathscr{G}$ be the set of those functions $g\colon [a,b] \to \mathbb{R}$ for which the integral $\int_a^b f\,dg$ exists for all $f \in \mathscr{F}$. Furthermore, let $\mathscr{H}$ denote the set of functions $h\colon [a,b] \to \mathbb{R}$ whose integral $\int_a^b h\,dg$ exists for all $g \in \mathscr{G}$. Show that $\mathscr{H}$ and $\mathscr{G}$ are dual classes, that is, a function h satisfies $h \in \mathscr{H}$ if and only if $\int_a^b h\,dg$ exists for all $g \in \mathscr{G}$, and a function g satisfies $g \in \mathscr{G}$ if and only if $\int_a^b h\,dg$ exists for all $h \in \mathscr{H}$.

18.11. Prove that if f is Riemann integrable and g is Lipschitz on $[a,b]$, then the Stieltjes integral $\int_a^b f\,dg$ exists. (H)

18.12. Show that

(a) if f is Lipschitz, then it is absolutely continuous;
(b) if f is absolutely continuous, then it is continuous and of bounded variation.