Chapter 14
Affine Linear Geometry

14.1 Affine Spaces

Intuitively, an affine space is a vector space without a 'preferred origin', that is as a set of points such that at each of these there is associated a model (a reference) vector space.

Definition 14.1.1 The *real affine space of dimension n*, denoted by $\mathbb{A}^n(\mathbb{R})$ or simply $\mathbb{A}^n$, is the set $\mathbb{R}^n$ equipped with the map

$$\alpha : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}^n$$

given by

$$\alpha((a_1, \dots, a_n), (b_1, \dots, b_n)) = (b_1 - a_1, \dots, b_n - a_n).$$

Notice that the domain of α is the cartesian product of $\mathbb{R}^n \times \mathbb{R}^n$, while the range of α is the vector space $\mathbb{R}^n$. The notation $\mathbb{A}^n$ stresses the differences between an affine space structure and a vector space structure on the same set $\mathbb{R}^n$. The n-tuples of $\mathbb{A}^n$ are called *points*.

By $\mathbb{A}^1$ we have the *affine real line*, by $\mathbb{A}^2$ the *affine real plane*, by $\mathbb{A}^3$ the *affine real space*. There is an analogous notion of complex affine space $\mathbb{A}^n(\mathbb{C})$, modelled on the vector space $\mathbb{C}^n$.

Remark 14.1.2 The following properties for $\mathbb{A}^n$ easily follows from the above definition:

(p1) for any point $P \in \mathbb{A}^n$ and for any vector $v \in \mathbb{R}^n$, there exists a unique point Q in $\mathbb{A}^n$ such that $\alpha(P, Q) = v$,
(p2) for any triple P, Q, R of points in $\mathbb{A}^n$, it holds that $\alpha(P, Q) + \alpha(Q, R) = \alpha(P, R)$.

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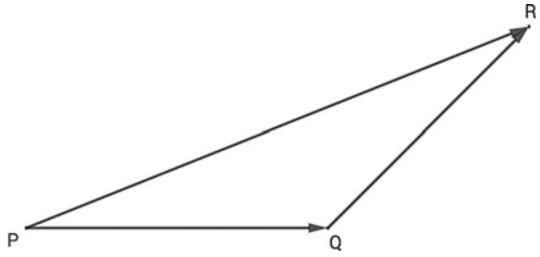

Fig. 14.1 The sum rule $(Q - P) + (R - Q) = R - P$

The property (p2) amounts to the sum rule of vectors (see Fig. 14.1).

Remark 14.1.3 Given the points $P, Q \in \mathbb{A}^n$ and the definition of the map α, the vector $\alpha(P, Q)$ will be also denoted by

$$v = \alpha(P, Q) = Q - P.$$

Then, from the property (p1), we shall write

$$Q = P + v.$$

And property (p2), the sum rule for vectors in $\mathbb{R}^n$, is written as

$$(Q - P) + (R - Q) = R - P.$$

Remark 14.1.4 Given an affine space $\mathbb{A}^n$, from (p2) we have that

(a) for any $P \in \mathbb{A}^n$ it is $\alpha(P, P) = 0_{\mathbb{R}^n}$ (setting $P = Q = R$),
(b) for any pair of points $P, Q \in \mathbb{A}^n$ it is (setting $R = P$), $\alpha(P, Q) = -\alpha(Q, P)$.

A reference system in an affine space is given by selecting a point $O \in \mathbb{A}^n$ so that from (p1) we have a bijection

$$\alpha_O : \mathbb{A}^n \to \mathbb{R}^n, \qquad \alpha_O(P) = \alpha(O, P) = P - O, \tag{14.1}$$

and then a basis $\mathcal{B} = (v_1, \dots, v_n)$ for $\mathbb{R}^n$.

Definition 14.1.5 The datum $(O, \mathcal{B})$ is called an *affine coordinate system* or an *affine reference system* for $\mathbb{A}^n$ with origin O and basis $\mathcal{B}$. With respect to a reference system $(O, \mathcal{B})$ for $\mathbb{A}^n$, if

$$P - O = (x_1, \dots, x_n)_{\mathcal{B}} = x_1 v_1 + \cdots + x_n v_n$$

we call $(x_1, \dots, x_n)$ the *coordinates* of the point $P \in \mathbb{A}^n$ and often write $P = (x_1, \dots, x_n)$. If $\mathcal{E}$ is the canonical basis for $\mathbb{R}^n$, then $(O, \mathcal{E})$ is the *canonical reference system* for $\mathbb{A}^n$.

Remark 14.1.6 Once an origin has been selected, the affine space $\mathbb{A}^n$ has the structures of $\mathbb{R}^n$ as a vector space. Given a reference system $(O, \mathcal{B})$ for $\mathbb{A}^n$, with $\mathcal{B} = (b_1, \dots, b_n)$, the points A_i in $\mathbb{A}^n$ given by

$$A_i = O + b_i$$

for $i = 1, \dots, n$, are called the *coordinate points* of $\mathbb{A}^n$ with respect to $\mathcal{B}$. They have coordinates

$$A_1 = (1, 0, \dots, 0)_{\mathcal{B}}, \quad A_2 = (0, 1, \dots, 0)_{\mathcal{B}}, \quad \dots \quad A_n = (0, 0, \dots, 1)_{\mathcal{B}}.$$

With the canonical basis $\mathcal{E} = (e_1, \dots, e_n)$, for $\mathbb{R}^n$ the coordinates points $A_i = O + e_i$ will have coordinates

$$A_1 = (1, 0, \dots, 0), \quad A_2 = (0, 1, \dots, 0), \quad \dots \quad A_n = (0, 0, \dots, 1).$$

Definition 14.1.7 With $w \in \mathbb{R}^n$, the map

$$T_w : \mathbb{A}^n \to \mathbb{A}^n, \qquad T_w(P) = P + w.$$

is called the *translation of* $\mathbb{A}^n$ *along* ws.

It is clear that T_w is a bijection between $\mathbb{A}^n$ and itself, since T_{-w} is the inverse map to T_w. Once a reference system has been introduced in $\mathbb{A}^n$, a translation can be described by a set of equations, as the following exercise shows.

Exercise 14.1.8 Let us fix the canonical cartesian coordinate system $(O, \mathcal{E})$ for $\mathbb{A}^3$, and consider the vector $w = (1, -2, 1)$. If $P = (x, y, z) \in \mathbb{A}^3$, then $P - O = xe_1 + ye_2 + ze_3$ and we write

$$\begin{aligned} T_w(P) - O &= (P + w) - O \\ &= (P - O) + w \\ &= (xe_1 + ye_2 + ze_3) + (e_1 - 2e_2 + e_3) \\ &= (x + 1)e_1 + (y - 2)e_2 + (z + 1)e_3, \end{aligned}$$

so $T_w\big((x, y, z)\big) = (x + 1, y - 2, z + 1\big)$.

Following this exercise, it is easy to obtain the equations for a generic translation.

Proposition 14.1.9 *Let $\mathbb{A}^n$ be an affine space with the reference system $(O, \mathcal{B})$. With a vector $w = (w_1, \ldots, w_n)_\mathcal{B}$ in $\mathbb{R}^n$, the translation T_w has the following equations*

$$T_w\big((x_1, \ldots, x_n)_\mathcal{B}\big) = (x_1 + w_1, \ldots, x_n + w_n)_\mathcal{B}.$$

Remark 14.1.10 The translation T_w induces an isomorphism of vector spaces $\phi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$P - O \quad \mapsto \quad T_w(P) - T_w(O).$$

It is easy to see that ϕ is the *identity* isomorphism. By fixing the orthogonal cartesian reference system $(O, \mathcal{E})$ for $\mathbb{A}^n$, with corresponding coordinates $(x_1, \ldots, x_n)$ for a point P, and a vector $w = w_1e_1 + \cdots + w_ne_n$, we can write

$$\mathbb{R}^n \ni P - O = x_1e_1 + \cdots + x_ne_n$$

and

$$T_w(P) = (x_1 + w_1, \ldots, x_n + w_n), \qquad T_w(O) = (w_1, \ldots, w_n),$$

so that we compute

$$\begin{aligned} T_w(P) - T_w(O) &= (T_w(P) - O) - (T_w(O) - O) \\ &= \big((x_1 + w_1)e_1 + \cdots (x_n + w_n)e_n\big) - (w_1e_1 + \cdots + w_ne_n) \\ &= x_1e_1 + \cdots + x_ne_n = P - O. \end{aligned}$$

More precisely, such an isomorphism is defined between two distinct copies of the vector space $\mathbb{R}^n$, those associated to the points O and $O' = T_w(O)$ in $\mathbb{A}^n$ thought of as the origins of two different reference systems for $\mathbb{A}^n$. This is depicted in Fig. 14.2.

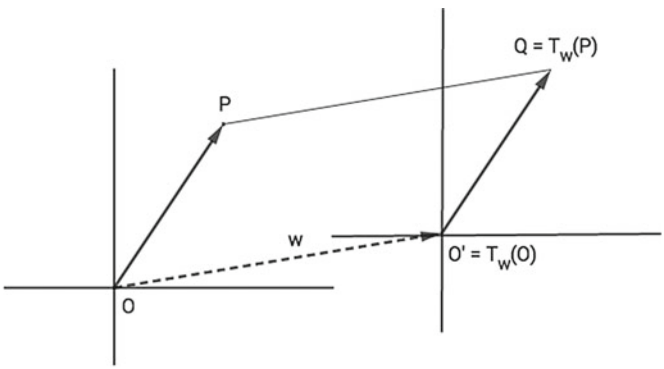

Fig. 14.2 The translation T_w

14.2 Lines and Planes

From the notion of vector line in $\mathbb{R}^2$, using the bijection $\alpha_O : \mathbb{A}^2 \mapsto \mathbb{R}^2$, given in (14.1), it is natural to define a *(straight) line by the origin* the subset in $\mathbb{A}^2$ that corresponds to $\mathcal{L}(v)$ in $\mathbb{R}^2$.

Exercise 14.2.1 Consider $v = (1, 2) \in \mathbb{R}^2$. The corresponding line by the origin in $\mathbb{A}^2$ is the set

$$\{P \in \mathbb{A}^2 \ : \ (P - O) \in \mathcal{L}(v)\} \ = \ \{(x, y) = \lambda(1, 2), \ \lambda \in \mathbb{R}\}.$$

Based on this, we have the following definition.

Definition 14.2.2 A (straight) line by the origin in $\mathbb{A}^n$ is the subset

$$r_O \ = \ \{P \in \mathbb{A}^n \ : \ (P - O) \in \mathcal{L}(v)\}$$

for a vector $v \in \mathbb{R}^n \setminus \{0\}$. The vector v is called the *direction vector* of r_O.

Using the identification between $\mathbb{A}^n$ and $\mathbb{R}^n$ given in (14.1) we write

$$r_O = \{P \in \mathbb{A}^n \ : \ P = \lambda v, \ \lambda \in \mathbb{R}\},$$

or even

$$r_O : \ P = \lambda v \ , \quad \lambda \in \mathbb{R}.$$

We call such an expression the *vector equation* for the line r_O. Once a reference system $(O, \mathcal{B})$ for $\mathbb{A}^n$ is chosen, via the identification of the components of $P - O$ with respect to $\mathcal{B}$ with the coordinates of a point P, we write the vector equation above as

$$r_O : \ (x_1, \ldots, x_n) = \lambda(v_1, \ldots, v_n) \ , \qquad \text{with} \quad \lambda \in \mathbb{R}$$

with $v = (v_1, \ldots, v_n)$ providing the direction of the line.

Remark 14.2.3 It is clear that the subset r_O coincides with $\mathcal{L}(v)$, although they belong to different spaces, that is $r_O \subset \mathbb{A}^n$ while $\mathcal{L}(v) \subset \mathbb{R}^n$. With such a caveat, these sets will be often identified.

Exercise 14.2.4 The line r_O in $\mathbb{A}^3$ with direction vector $v = (1, 2, 3)$ has the vector equation,

$$r_O : \ (x, y, z) = \lambda(1, 2, 3), \quad \lambda \in \mathbb{R}.$$

Exercise 14.2.5 Consider the affine space $\mathbb{A}^2$ with the orthogonal reference system $(O, \mathcal{E})$. The subset given by

$$r = \{(x, y) = (1, 2) + \lambda(0, 1), \ \lambda \in \mathbb{R}\}$$

clearly represents a line that runs parallel to the second reference axis. Under the translation T_u with $u = (-1, -2)$ we get the set

$$\begin{aligned} T_u(r) &= \{P + u,\ \ P \in r\} \\ &= \{(x, y) = \lambda(0, 1),\ \ \lambda \in \mathbb{R}\}, \end{aligned}$$

which is a line by the origin (indeed the second axis of the reference system). If $r_O = T_u(r)$, a line by the origin, it is clear that $r = T_w(r_O)$, with $w = -u$.

This exercise suggests the following definition.

Definition 14.2.6 A set $r \subset \mathbb{A}^n$ is called a *line* if there exist a translation T_w in $\mathbb{A}^n$ and a line r_O by the origin such that $r = T_w(r_O)$.

Being the sets r_O and $\mathcal{L}(v)$ in $\mathbb{R}^n$ coincident, we shall refer to $\mathcal{L}(v)$ as the *direction* of r, and we shall denote it by S_r (with the letter S referring to the fact that $\mathcal{L}(v)$ is a vector subspace in $\mathbb{R}^n$). Notice that, for a line, it is $\dim(S_r) = 1$.

The equation for an arbitrary line follows easily from that of a line by the origin. Let us consider a line by the origin,

$$r_O :\ P = \lambda v\ , \quad \lambda \in \mathbb{R},$$

and the translation T_w with $w \in \mathbb{R}^n$. If $w = Q - O$, the line $r = T_w(r_O)$ is given by

$$\begin{aligned} r &= \{P \in \mathbb{A}^n\ :\ P = T_w(P_O),\ \ P_O \in r_O\} \\ &= \{P \in \mathbb{A}^n\ :\ P = Q + \lambda v,\ \ \lambda \in \mathbb{R}\}, \end{aligned}$$

so we write

$$r :\ P = Q + \lambda v. \tag{14.2}$$

With respect to a reference system $(O, \mathcal{B})$, where $Q = (q_1, \ldots, q_n)_{\mathcal{B}}$ and $v = (v_1, \ldots, v_n)_{\mathcal{B}}$, the previous equation can be written as

$$r : \quad (x_1, \ldots, x_n) = (q_1, \ldots, q_n) + \lambda(v_1, \ldots, v_n), \tag{14.3}$$

or equivalently

$$r : \quad \begin{cases} x_1 = q_1 + \lambda v_1 \\ \vdots \\ x_n = q_n + \lambda v_n \end{cases}. \tag{14.4}$$

Definition 14.2.7 The expression (14.2) (or equivalently 14.3) is called the *vector equation* of the line r, while the expression (14.4) is called the *parametric equation* of r (stressing that λ is a real parameter).

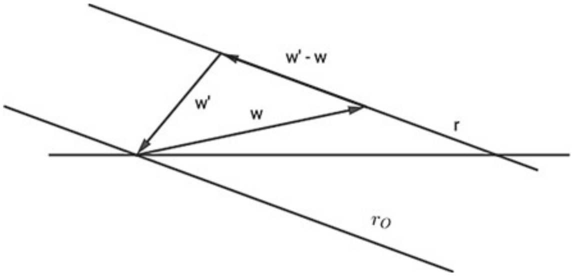

Fig. 14.3 The translation $T_{w'}$ with $w' - w \in \mathcal{L}(v)$ maps r into r_O

Remark 14.2.8 Consider the line whose vector equation is $r \;:\; P = Q + \lambda v$.

(a) We have a unique point in r for each value of λ, and selecting a point of r gives a unique value for λ. The point in r is Q if and only if $\lambda = 0$;
(b) The direction of r is clearly the vector line $\mathcal{L}(v)$. This means that the direction vector v is not uniquely determined by the equation, since each element $v' \in \mathcal{L}(v)$ is a direction vector for r. This arbitrariness can be re-absorbed by a suitable rescaling of the parameter λ: with a rescaling the equation for r can always be written in the given form with v its direction vector.
(c) The point Q is not unique. As the Fig. 14.3 shows, if $Q = O + w$ is a point in r, then any translation $T_{w'}$ with $w' - w \in \mathcal{L}(v)$ maps r into the same line by the origin.

Exercise 14.2.9 We check whether the following lines coincide:

$$\begin{aligned} r &: \quad (x, y) = (1, 2) + \lambda(1, -1), \\ r' &: \quad (x, y) = (2, 1) + \mu(1, -1). \end{aligned}$$

Clearly r and r' have the same direction, which is $S_r = S_{r'} = \mathcal{L}((1, -1)) = r_O$. If we consider $Q = (1, 2) \in r$ and $Q' = (2, 1) \in r'$ with $w = Q - O = (1, 2)$ and $w' = Q' - O = (2, 1)$ we compute,

$$r = T_w(r_O), \qquad r' = T_{w'}(r_O).$$

We have that r coincides with r': as described in the remark above, $w - w' = (-1, 1) \in \mathcal{L}((1, -1))$.

In analogy with the definition of affine lines, one defines planes in $\mathbb{A}^n$.

Definition 14.2.10 A *plane through the origin* in $\mathbb{A}^n$ is any subset of the form

$$\pi_O = \{P \in \mathbb{A}^n \;:\; (P - O) \in \mathcal{L}(u, v)\},$$

with two linearly independent vectors $u, v \in \mathbb{R}^n$.

With the usual identification of a point $P \in \mathbb{A}^n$ with its image $\alpha(P) \in \mathbb{R}^n$ (see 14.1), we write

$$\pi_O = \{P \in \mathbb{A}^n \ : \ P = \lambda u + \mu v, \quad \lambda, \mu \in \mathbb{R}\},$$

or also

$$\pi_O : \ P = \lambda u + \mu v$$

with λ, μ real parameters.

Definition 14.2.11 A subset $\pi \subset \mathbb{A}^n$ is called a *plane* if there exist a translation map T_w in $\mathbb{A}^n$ and a plane π_O through the origin such that $\pi = T_w(\pi_O)$. Since we can identify the elements in π_O with the vectors in $\mathcal{L}(u, v) \subset \mathbb{R}^n$, generalising the analogue Definition 14.2.6 for a line, we define the space $S_\pi = \mathcal{L}(u, v)$ to be the *direction* of π. Notice that $\dim(S_\pi) = 2$.

If $Q = T_w(O)$, that is $w = Q - O$, the points $P \in \pi$ are characterised by

$$P = Q + \lambda u + \mu v. \tag{14.5}$$

Let $(O, \mathcal{B})$ be a reference system for $\mathbb{A}^n$. If $Q = (q_1, \dots, q_n)_\mathcal{B} \in \mathbb{A}^n$, with $u = (u_1, \dots, u_n)_\mathcal{B}$ and $v = (v_1, \dots, v_n)_\mathcal{B} \in \mathbb{R}^n$, the above equation can be written as

$$\pi : \quad \begin{cases} x_1 = q_1 + \lambda u_1 + \mu v_1 \\ \vdots \\ x_n = q_n + \lambda u_v + \mu v_n \end{cases}. \tag{14.6}$$

The relation (14.5) is the *vector equation* of the plane π, while (14.6) is a parametric equation of π.

Exercise 14.2.12 Given the linearly independent vectors $v_1 = (1, 0, 1)$ and $v_2 = (1, -1, 0)$ with respect to the basis $\mathcal{B}$ in $\mathbb{R}^3$, the plane π_O through the origin associated to them is the set of points $P \in \mathbb{A}^3$ given by the vector equation

$$P = \lambda_1 v_1 + \lambda_2 v_2, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

With the reference system $(O, \mathcal{B})$, with $P = (x, y, z)$ its parametric equations is

$$(x, y, z) = \lambda_1(1, 0, 1) + \lambda_2(1, -1, 0) \quad \Leftrightarrow \quad \pi : \begin{cases} x = \lambda_1 + \lambda_2 \\ y = -\lambda_2 \\ z = \lambda_1 \end{cases}.$$

Exercise 14.2.13 Given the translation T_w in $\mathbb{A}^3$ with $w = (1, -1, 2)$ in a basis $\mathcal{B}$, the plane π_O of the previous exercise is mapped into the plane $\pi = T_w(\pi_O)$ whose vector equation is

$$\pi : \quad P = Q + \lambda_1 v_1 + \lambda_2 v_2,$$

with $Q = T_w(O) = (1, -1, 2)$. We can equivalently represent the points in π as

$$\pi : \quad (x, y, z) = (1, -1, 2) + \lambda_1(1, 0, 1) + \lambda_2(1, -1, 0).$$

Exercise 14.2.14 Let us consider the vectors v_1, v_2 in $\mathbb{R}^4$ with the following components

$$v_1 = (1, 0, 1, 0), \quad v_2 = (2, 1, 0, -1)$$

in a basis $\mathcal{B}$, and the point Q in $\mathbb{A}^4$ with coordinates

$$Q = (2, 1, 1, 2).$$

in the corresponding reference system $(O, \mathcal{B})$. The plane $\pi \subset \mathbb{A}^4$ through Q whose direction is $S_\pi = \mathcal{L}(v_1, v_2)$ has the vector equation

$$\pi : \quad (x_1, x_2, x_3, x_4) = (2, 1, 1, 2) + \lambda_1(1, 0, 1, 0) + \lambda_2(2, 1, 0, -1)$$

and parametric equation

$$\pi : \quad \begin{cases} x_1 = 2 + \lambda_1 + 2\lambda_2 \\ x_2 = 1 + \lambda_2 \\ x_3 = 1 + \lambda_1 \\ x_4 = 2 - \lambda_2 \end{cases}.$$

Remark 14.2.15 The natural generalisation of the Remark 14.2.8 holds for planes as well. A vector equation for a given plane π is not unique. If

$$\pi : \ P = Q + \lambda u + \mu v$$
$$\pi' : \ P = Q' + \lambda u' + \mu v'$$

are two planes in $\mathbb{A}^n$, then

$$\pi = \pi' \quad \Leftrightarrow \quad \begin{cases} S_\pi = S_{\pi'} \quad \text{(that is } \mathcal{L}(u, v) = \mathcal{L}(u', v')) \\ Q - Q' \in S_\pi \end{cases}$$

Proposition 14.2.16 *Given two distinct points A, B in $\mathbb{A}^n$ (with $n \geq 2$), there is only one line through A and B. A vector equation is*

$$r_{AB} : \ P = A + \lambda(B - A).$$

Proof Being $A \neq B$, this vector equation gives a line since $B - A$ is a non zero vector and the set of points $P - A$ is a one dimensional vector space (that is the direction is one dimensional). The equation r_{AB} contains A (for $\lambda = 0$) and B (for $\lambda = 1$). This shows there exists a line through A and B.

Let us consider another line r_A through A. Its vector equation will be $P = A + \mu v$, with $v \in \mathbb{R}^n$ and μ a real parameter. The point B is contained in r_A if and only if there exists a value μ_0 of the parameter such that $B = A + \mu_0 v$, that is $B - A = \mu_0 v$. Thus the direction of r_A would be $S_{r_A} = \mathcal{L}(v) = \mathcal{L}(B - A) = S_{r_{AB}}$. The line r_A then coincides with r_{AB}. □

Exercise 14.2.17 The line in $\mathbb{A}^2$ through the points $A = (1, 2)$ and $B = (1, -2)$ has equation

$$P = (x, y) = (1, 2) + \lambda(0, -4).$$

Exercise 14.2.18 Let the points A and B in $\mathbb{A}^3$ have coordinates $A = (1, 1, 1)$ and $B = (1, 2, -2)$. The line r_{AB} through them has the vector

$$(x, y, z) = (1, 1, 1) + \lambda(0, 1, -3).$$

Does the point $P = (1, 0, 4)$ belong to r_{AB}? In order to answer this question we need to check whether there is a $\lambda \in \mathbb{R}$ that solves the linear system

$$\begin{cases} 1 = 1 \\ 0 = 1 + \lambda \\ 4 = 1 - 3\lambda. \end{cases}$$

It is evident that $\lambda = -1$ is a solution, so P is a point in r_{AB}.

An analogue of the Proposition 14.2.16 holds for three points in an affine space.

Proposition 14.2.19 *Let A, B, C be three points in an affine space $\mathbb{A}^n$ (with $n \geq 3$). If they are not contained in the same line, there exists a unique plane π_{ABC} through them, with a vector equation given by*

$$\pi_{ABC} : \quad P = A + \lambda(B - A) + \mu(C - A).$$

Proof The vectors $B - A$ and $C - A$ are linearly independent, since they are not contained in the same line. The direction of π_{ABC} is then two dimensional, with $S_{\pi_{ABC}} = \mathcal{L}(B - A, C - A)$. The point A is in π_{ABC}, corresponding to $P(\lambda = \mu = 0)$; the point B is in π_{ABC}, corresponding to $P(\lambda = 1, \mu = 0)$; the point C is in π_{ABC} corresponding to $P(\lambda = 0, \mu = 1)$.

We have then proven that a plane through A, B, C exists. Let us suppose that

$$\pi' : \; P = A + \lambda u + \mu v.$$

gives a plane through the points A, B, C (which are not on the same line) with u and v linearly independent (so its direction is given by $S_{\pi'} = \mathcal{L}(u, v)$). This means that

$B - A \in \mathcal{L}(u, v)$ and $C - A \in \mathcal{L}(u, v)$. Since the spaces are both two dimensional, this reads $\mathcal{L}(B - A, C - A) = \mathcal{L}(u, v)$, proving that π' coincides with π_{ABC}. □

Exercise 14.2.20 Let $A = (1, 2, 0)$, $B = (1, 1, 1)$ and $C = (0, 1, -1)$ be three points in $\mathbb{A}^3$. They are not on the same line, since the vectors $B - A = (0, -1, 1)$ and $C - A = (-1, -1, -1)$ are linearly independent. A vector equation of the plane π_{ABC} is

$$\pi : \quad (x, y, z) = (1, 2, 0) + \lambda(0, -1, 1) + \mu(-1, -1, -1).$$

14.3 General Linear Affine Varieties and Parallelism

The natural generalisation of (straight) lines and planes leads to the definition of a linear affine variety L in $\mathbb{A}^n$, where the direction of L is a subspace in $\mathbb{R}^n$ of dimension greater than 2.

Definition 14.3.1 A *linear affine variety of dimension k* in $\mathbb{A}^n$ is a set

$$L = \{P \in \mathbb{A}^n \ : \ (P - Q) \in V\},$$

where Q is a point in the affine space $\mathbb{A}^n$ and $V \subset \mathbb{R}^n$ is a vector subspace of dimension k in $\mathbb{R}^n$. The vector subspace V is called the *direction* of the variety L, and denoted by $S_L = V$. If $V = \mathcal{L}(v_1, \ldots, v_k)$, a *vector equation* for L is

$$L : \ P = Q + \lambda_1 v_1 + \cdots + \lambda_k v_k$$

for scalars $\lambda_1, \ldots, \lambda_k$ in $\mathbb{R}$.

Remark 14.3.2 It is evident that a line is a one dimensional linear affine variety, while a plane is a two dimensional linear affine variety.

Definition 14.3.3 An linear affine variety of dimension $n - 1$ in $\mathbb{A}^n$ is called a *hyperplane* in $\mathbb{A}^n$.

It is clear that a line is a hyperplane in $\mathbb{A}^2$, while a plane is a hyperplane in $\mathbb{A}^3$.

Exercise 14.3.4 We consider the affine space $\mathbb{A}^4$, the point Q with coordinates $Q = (2, 1, 1, 2)$ with respect to a given reference system $(O, \mathcal{B})$, and the vector subspace $S = \mathcal{L}(v_1, v_2, v_3)$ in $\mathbb{R}^4$ with generators $v_1 = (1, 0, 1, 0)$, $v_2 = (2, 1, 0, -1)$, $v_3 = (0, 0, -1, 1)$ with respect to $\mathcal{B}$. The vector equation of the linear affine variety L with direction $S_L = \mathcal{L}(v_1, v_2, v_3)$ and containing Q is

$$L \ : \ (x_1, x_2, x_3, x_4) = (2, 1, 1, 2) + \lambda_1(1, 0, 1, 0) + \lambda_2(2, 1, 0, -1) + \lambda_3(0, 0, -1, 1),$$

while its parametric equation reads

$$L\ :\quad \begin{cases} x_1 = 2 + \lambda_1 + 2\lambda_2 \\ x_2 = 1 + \lambda_2 \\ x_3 = 1 + \lambda_1 - \lambda_3 \\ x_4 = 2 - \lambda_2 + \lambda_3 \end{cases}.$$

Definition 14.3.5 Let L, L' be two linear affine varieties of the same dimension k in $\mathbb{A}^n$. We say that L is *parallel* to L' if they have the same directions, that is if $S_L = S_{L'}$.

Exercise 14.3.6 Let $L_O \subset \mathbb{A}^n$ be a line through the origin. A line L in $\mathbb{A}^n$ is parallel to L_O if and only if $L = T_w(L_O)$, for $w \in \mathbb{R}^n$. From the Remark 14.2.15 we know that $L = L_O$ if and only if $w \in S_L$.

Let us consider the line through the origin in $\mathbb{A}^2$ given by $L_O\,:\,(x, y) = \lambda(3, -2)$. A line will be parallel to L_O if and only if its vector equation is given by

$$L:\quad (x, y) = (\alpha, \beta) + \lambda(3, -2),$$

with $(\alpha, \beta) \in \mathbb{R}^2$. The line L is moreover distinct from L' if and only if $(\alpha, \beta) \notin S_L$.

Definition 14.3.7 Let us consider in $\mathbb{A}^n$ a linear affine variety L of dimension k and a second linear affine variety L' of dimension k', with $k > k'$. The variety L is said to be *parallel* to L' if $S_{L'} \subset S_L$, that is if the direction of L' is a subspace of the direction of L.

Exercise 14.3.8 Let us consider in $\mathbb{A}^3$ the plane given by

$$\pi:\quad (x, y, z) = (0, 2, -1) + \lambda_1(1, 0, 1) + \lambda_2(0, 1, 1).$$

We check whether the following lines,

$$\begin{aligned} r_1 &:\quad (x, y, z) = \lambda(1, -1, 0) \\ r_2 &:\quad (x, y, z) = (0, 3, 0) + \lambda(1, 1, 2) \\ r_3 &:\quad (x, y, z) = (1, -1, 1) + \lambda(1, 1, 1), \end{aligned}$$

are parallel to π.

If S_π denotes the direction of π, we clearly have that $S_\pi = \mathcal{L}(w_1, w_2) = \mathcal{L}((1, 0, 1), (0, 1, 1))$, while we denote by v_i a vector spanning the direction S_{r_i} of the line r_i, $i = 1, 2, 3$. To verify whether $S_{r_i} \subset S_\pi$ it is sufficient to compute the rank of the matrix whose rows are given by (w_1, w_2, v_i).

- For $i = 1$, after a reduction procedure we have,

$$\begin{pmatrix} w_1 \\ w_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Since this matrix has rank 2, we have that $v_1 \in \mathcal{L}(w_1, w_2)$, that is $S_{r_1} \subset S_\pi$. We conclude that r_1 is parallel to π. One also checks that $r_1 \not\subset \pi$, since $(0, 0, 0) \in r_1$ but $(0, 0, 0) \notin \pi$. To show this, one notices that the origin $(0, 0, 0)$ is contained in π if and only if the linear system

$$(0, 0, 0) = (0, 2, -1) + \lambda_1(1, 0, 1) + \lambda_2(0, 1, 1) \quad \Rightarrow \quad \begin{cases} 0 = \lambda_1 \\ 0 = 2 + \lambda_2 \\ 0 = -1 + \lambda_1 + \lambda_2 \end{cases}$$

has a solution. It is evident that such a solution does not exist.

- For $i = 2$ we proceed as above. The following reduction by rows

$$\begin{pmatrix} w_1 \\ w_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

shows that $v_2 \in \mathcal{L}(w_1, w_2)$, thus r_2 is parallel to π. Now $r_2 \subset \pi$: a point P is in r_2 if and only there exists a $\lambda \in \mathbb{R}$ such that $P = (\lambda, \lambda + 3, 2\lambda)$. For any value of λ, the linear system

$$(\lambda, \lambda + 3, 2\lambda) = (0, 2, -1) + \lambda_1(1, 0, 1) + \lambda_2(0, 1, 1) \quad \Rightarrow \quad \begin{cases} \lambda = \lambda_1 \\ \lambda + 3 = 2 + \lambda_2 \\ 2\lambda \end{cases}$$

has the unique solution $\lambda_1 = \lambda, \ \lambda_2 = \lambda + 1$.

- For $i = 3$ the following reduction by rows

$$\begin{pmatrix} w_1 \\ w_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

shows that the matrix ${}^t(w_1, w_2, v_3)$ has rank 3, so r_3 is not parallel to π.

Definition 14.3.9 Let $L, L' \subseteq \mathbb{A}^n$ two distinct linear affine varieties. We say that L and L' are *incident* if their intersection is non empty, while they are said to be *skew* if they are neither parallel nor incident.

Remark 14.3.10 It is easy to see that two lines (or a line and a plane) are incident if they have a common point. Two distinct planes in $\mathbb{A}^n$ (with $n \geq 3$) are incident if they have a common line.

Exercise 14.3.11 In the affine space $\mathbb{A}^3$ we consider the line r_3 and the plane π as in the Exercise 14.3.8. We know already that they are not parallel, and a point $P = (x, y, z)$ belongs to the intersection $r_3 \cap \pi$ if and only if there exists a λ such that

$P = (1 + \lambda, -1 + \lambda, 1 + \lambda) \in r_3$ and there exist scalars λ_1, λ_2 such that $P = (\lambda_1, 2 + \lambda_2, -1 + \lambda_1 + \lambda_2) \in \pi$. These conditions are equivalent to the linear system

$$\begin{cases} 1 + \lambda = \lambda_1 \\ -1 + \lambda = 2 + \lambda_2 \\ 1 + \lambda = -1 + \lambda_1 + \lambda_2 \end{cases}$$

that has the unique solution $(\lambda = 4, \lambda_1 = 5, \lambda_2 = 1)$. This corresponds to $P = (5, 3, 5) \in r_3 \cap \pi$.

Exercise 14.3.12 Consider again the lines r_1 and r_2 in the Exercise 14.3.8. We know they are not parallel, since $v_1 \notin \mathcal{L}(v_2)$. They are not incident: there are indeed no values of λ and μ such that a point $P = \lambda(1, -1, 0)$ in r_1 coincides with a point $P = (0, 3, 0) + \mu(1, 1, 2)$ in r_2, since the following linear system

$$\begin{cases} \lambda = \mu \\ -\lambda = 3 + \mu \\ 0 = 2\mu \end{cases}$$

has no solution. Thus r_1 and r_2 are skew.

Exercise 14.3.13 Given the planes

$$\begin{aligned} \pi : \quad (x, y, z) &= (0, 2, -1) + \lambda_1(1, 0, 1) + \lambda_2(0, 1, 1) \\ \pi' : \quad (x, y, z) &= (1, -1, 1) + \lambda_1(0, 0, 1) + \lambda_2(2, 1, -1) \end{aligned}$$

in $\mathbb{A}^3$, we determine all the lines which are parallel to *both* π *and* π'.

We denote by r a generic line satisfying such a condition. From the Definition 14.3.5, we require that $S_r \subseteq S_\pi \cap S_{\pi'}$ for the direction S_r of r. Since $S_\pi = \mathcal{L}((1, 0, 1), (0, 1, 1))$ while $S_{\pi'} = \mathcal{L}((0, 0, 1), (2, 1, -1))$, in order to compute $S_\pi \cap S_{\pi'}$ we write the condition

$$\alpha(1, 0, 1) + \beta(0, 1, 1) = \alpha'(0, 0, 1) + \beta'(2, 1, -1)$$

as the linear homogeneous system for $(\alpha, \beta, \alpha', \beta')$ given by $\Sigma : \; AX = 0$ with

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} \alpha \\ \beta \\ -\alpha' \\ -\beta' \end{pmatrix}.$$

The space of solution for such a linear system is easily found to be

$$S_\Sigma = \{(\alpha, \beta, -\alpha', -\beta') = t(2, 1, -4, -1) \; : \; t \in \mathbb{R}\},$$

so we have that the intersection $S_\pi \cap S_{\pi'}$ is one dimensional and spanned by the vector

$$2(1, 0, 1) + (0, 1, 1) = 4(0, 0, 1) + (2, 1, -1) = (2, 1, 3).$$

This gives that $S_r = \mathcal{L}((2, 1, 3))$, so we finally write

$$r : \quad (x, y, z) = (a, b, c) + \lambda(2, 1, 3).$$

for an arbitrary $(a, b, c) \in \mathbb{A}^3$.

14.4 The Cartesian Form of Linear Affine Varieties

In the previous sections we have seen that a linear affine variety can be described either with a vector equation or a parametric equation. In this section we relate linear affine varieties to systems of linear equations.

Proposition 14.4.1 *A linear affine variety $L \subseteq \mathbb{A}^n$ corresponds to the space of the solutions of an associated linear system with m equations in n unknowns, that is*

$$\Sigma_L : \quad AX = B, \qquad \text{for} \quad A \in \mathbb{R}^{m,n}. \tag{14.7}$$

Moreover, the space of solutions of the corresponding homogeneous linear system describes the direction space $S_L = L_O$ of L, that is

$$\Sigma_{L_O} : \quad AX = 0.$$

We say that the linear system Σ_L given in (14.7) is the *cartesian equation* for the linear affine variety L of dimension $n - \text{rk}(A)$. By computing the space of the solutions of Σ_L in terms of $n - \text{rk}(A)$ parameters, one gets the parametric equation for L. Conversely, given the parametric equation of L, its corresponding cartesian equation is given by consistently 'eliminating' all the parameters in the parametric equation. This linear affine variety can be represented both by a cartesian equation and by a parametric equation, which are related as

linear system $\Sigma : AX = B$ (*cartesian equation*)	$\Longleftrightarrow$	space of the solutions for $\Sigma : AX = B$ (*parametric equation*)

Notice that for a linear affine variety L a cartesian equation is not uniquely determined: any linear system Σ' which is equivalent to Σ_L (that is for which the spaces of the solutions for Σ_L and Σ' coincide) describe the same linear affine variety. An analogue result holds for the direction space of L, which is equivalently described by any homogenous linear system Σ'_O equivalent to Σ_{L_O}.

We avoid an explicit proof of the Proposition 14.4.1 in general, and analyse the equivalence between the two descriptions via the following examples.

Exercise 14.4.2 Let us consider the line $r \subset \mathbb{A}^2$ with parametric equation

$$r: \quad \begin{cases} x = 1 + \lambda \\ y = 2 - \lambda \end{cases}.$$

We can express the parameter λ in terms of x from the first relation, that is $\lambda = x - 1$, and replace this in the second relation, having

$$x + y - 3 = 0.$$

We set

$$s = \{(x, y) \in \mathbb{A}^2 : x + y - 3 = 0\}$$

and show that s coincides with r. Clearly $r \subseteq s$, since a point with coordinates $(1 + \lambda, 2 - \lambda) \in r$ solves the linear equation for s:

$$(1 + \lambda) + (2 - \lambda) - 3 = 0.$$

In order to prove that $s \subseteq r$, consider a point $P = (x, y) \in s$, so that $P = (x, y = 3 - x)$ for any value of x: this means considering x as a real parameter. By writing $\lambda = x - 1$, we have $P = (x = \lambda + 1, y = 2 - \lambda)$ for any $\lambda \in \mathbb{R}$, so $P \in r$. We have then $s = r$ as linear affine varieties.

Proposition 14.4.3 *Given a, b, c in $\mathbb{R}$ with $(a, b) \neq (0, 0)$, the solutions of the equation*

$$\Sigma_r : \quad ax + by + c = 0 \tag{14.8}$$

provide the coordinates of all the points $P = (x, y)$ of a line r in $\mathbb{A}^2$ whose direction $S_r = \mathcal{L}((-b, a))$ is given by the solutions of the associated linear homogenous equation

$$\Sigma_{r_O} : \quad ax + by = 0.$$

Moreover, if $r \subset \mathbb{A}^2$ is a line with direction $S_r = \mathcal{L}((-b, a))$, then there exists $c \in \mathbb{R}$ such that the cartesian form for the equation of r is given by (14.8).

Proof We start by showing that the solutions of (14.8) give the coordinates of the points representing the line with direction $\mathcal{L}((-b, a))$ in parametric form.

Let us assume $a \neq 0$. We can then write the space of the solutions for (14.8) as

$$(x, y) = (-\frac{b}{a}\mu - \frac{c}{a}, \mu)$$

where $\mu \in \mathbb{R}$ is a parameter. By rescaling the parameter, that is defining $\lambda = \mu/a$, we write the space of solutions as the points having coordinates,

$$\begin{aligned}(x, y) &= (-b\lambda - \frac{c}{a}, a\lambda) \\ &= (-\frac{c}{a}, 0) + \lambda(-b, a).\end{aligned}$$

This expression gives the vector (and the parametric) equation of a line through $(-c/a, 0)$ with direction $S_r = \mathcal{L}((-b, a))$.

If $a = 0$, we can write the space of the solutions for (14.8) as

$$(x, y) = (\mu, -\frac{c}{b})$$

where $\mu \in \mathbb{R}$ is a parameter. By rescaling the parameter, we can write

$$(x, y) = (-\lambda b, -\frac{c}{b}) = (0, -\frac{c}{b}) + \lambda(-b, 0),$$

giving the vector equation of a line through the point $(0, -c/b)$ with direction $S_r = \mathcal{L}((-b, 0))$.

Now let r be a line in $\mathbb{A}^2$ with direction $S_r = \mathcal{L}((-b, a))$. Its parametric equation is of the form

$$r: \quad \begin{cases} x = x_0 - b\lambda \\ y = y_0 + a\lambda \end{cases}$$

where (x_0, y_0) is an arbitrary point in $\mathbb{A}^2$. If $a \neq 0$, we can eliminate λ by setting

$$\lambda = \frac{y - y_0}{a}$$

from the second relation and then

$$x = x_0 - \frac{b}{a}(y - y_0),$$

resulting into the linear equation

$$ax + by + c = 0$$

with $c = -(ax_0 + by_0)$.

If $a = 0$ then $b \neq 0$, so by rescaling the parameter as $\mu = x_0 - \lambda b$, the points of the line r are $(x = \mu, y = y_0)$. This is indeed the set of the solutions of the linear equation

$$ax + by + c = 0$$

with $a = 0$ and $c = -by_0$. We have then shown that any line with a given direction has the cartesian form given by a suitable linear equation (14.8). □

The equation $ax + by + c = 0$ is called the *cartesian equation of a line* in $\mathbb{A}^2$.

Remark 14.4.4 As already mentioned, a line does not uniquely determine its cartesian equation. With $ax + by + c = 0$ the cartesian equation for r, any other linear equation

$$\rho ax + \rho by + \rho c = 0, \qquad \text{with} \quad \rho \neq 0$$

yields a cartesian equation for the same line, since

$$\rho ax + \rho by + \rho c = 0 \quad \Leftrightarrow \quad \rho(ax + by + c) = 0 \quad \Leftrightarrow \quad ax + by + c = 0.$$

Exercise 14.4.5 The line $\Sigma_r : \ 2x - y + 3 = 0$ in $\mathbb{A}^2$ has direction $\Sigma_{r_O} : \ 2x - y = 0$, or $S_r = \mathcal{L}((1, 2))$.

Exercise 14.4.6 We turn now to the description of a plane in the three dimensional affine space in terms of a cartesian equation. Let us consider the plane $\pi \subset \mathbb{A}^3$ with parametric equation

$$\pi : \quad \begin{cases} x = 1 + 2\lambda + \mu \\ y = 2 - \lambda - \mu \\ z = \mu \end{cases} .$$

We eliminate the parameter μ by setting $\mu = z$ from the third relation, and write

$$\pi : \quad \begin{cases} x = 1 + 2\lambda + z \\ y = 2 - \lambda - z \\ \mu = z \end{cases} .$$

We can then eliminate the parameter λ by using the second (for example) relation, so to have $\lambda = 2 - y - z$ and write

$$\pi : \quad \begin{cases} x = 1 + 2(2 - y - z) + z \\ \lambda = 2 - y - z \\ \mu = z \end{cases} .$$

Since these relations are valid for any choice of the parameters λ and μ, we have a resulting linear equation with three unknowns:

$$\Sigma_\pi : \quad x + 2y + z - 5 = 0.$$

Such an equation still represents π, since every point $P \in \pi$ solves the equation (as easily seen by taking $P = (1 + 2\lambda + \mu, 2 - \lambda - \mu, \mu)$) and the space of solutions of Σ_π coincides with the set π.

This example has a general validity for representing in cartesian form a plane in $\mathbb{A}^3$. A natural generalisation of the proof of the previous Proposition 14.4.3 allows one to show the following result.

Proposition 14.4.7 *Given* a, b, c, d *in* $\mathbb{R}$ *with* $(a, b, c) \neq (0, 0, 0)$*, the solutions of the equation*

$$\Sigma_\pi : \quad ax + by + cz + d = 0 \tag{14.9}$$

provide the coordinates of all the points $P = (x, y, z)$ *of a plane* π *in* $\mathbb{A}^3$ *whose direction* S_π *is given by the solutions of the associated linear homogenous equation*

$$\Sigma_{\pi_O} : \quad ax + by + cz = 0. \tag{14.10}$$

If $\pi \subset \mathbb{A}^3$ *is a plane with direction* $S_\pi = \mathbb{R}^2$ *given by the space of the solutions of* (14.10)*, then there exists* $d \in \mathbb{R}$ *such that the cartesian form for the equation of* π *is given by* (14.9).

The equation

$$ax + by + cz + d = 0$$

is called the *cartesian equation of a plane* in $\mathbb{A}^3$.

Remark 14.4.8 Analogously to what we noticed in the Remark 14.4.4, the cartesian equation of a plane π in $\mathbb{A}^3$ is not uniquely determined, since it can be again multiplied by a non zero scalar.

Exercise 14.4.9 We next look for a cartesian equation for a line in $\mathbb{A}^3$. As usual, by way of an example, we start by considering the parametric equation of the line $r \subset \mathbb{A}^3$ given by

$$r : \quad \begin{cases} x = 1 + 2\lambda \\ y = 2 - \lambda \\ z = \lambda \end{cases} .$$

By eliminating the parameter λ via (for example) the third relation $\lambda = z$ we have

$$r : \quad \begin{cases} x = 1 + 2z \\ y = 2 - z \\ \lambda = z \end{cases} .$$

Since the third relations formally amounts to redefine a parameter, we write

$$\Sigma_r : \quad \begin{cases} x - 2z - 1 = 0 \\ y + z - 2 = 0 \end{cases} ,$$

which is a linear system with three unknowns and rank 2, thus having ∞^1 solutions. In analogy with the procedure used above for the other examples, it is easy to show that the space of solutions of Σ_r coincides with the line r in $\mathbb{A}^3$.

The following result is the natural generalisation of the Propositions 14.4.3 and 14.4.7.

Proposition 14.4.10 *Given the (complete, see the Definition 6.1.5) matrix*

$$(A, B) = \begin{pmatrix} a_1 & b_1 & c_1 & -d_1 \\ a_2 & b_2 & c_2 & -d_2 \end{pmatrix} \in \mathbb{R}^{2,4}$$

with

$$\mathrm{rk}(A) = \mathrm{rk}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2,$$

the solutions of the linear system

$$\Sigma_\pi : AX = B \quad \Leftrightarrow \quad \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{cases} \tag{14.11}$$

provide the coordinates of all the points $P = (x, y, z)$ of a line r in $\mathbb{A}^3$ whose direction S_r is given by the solutions of the associated linear homogenous system

$$\Sigma_{r_o} : \quad AX = 0. \tag{14.12}$$

If $r \subset \mathbb{A}^3$ is a line whose direction $S_r = \mathbb{R}$ is given by the space of the solutions of the linear homogenous system (14.12) with $A \in \mathbb{R}^{3,2}$ and $\mathrm{rk}(A) = 2$, then there exists a vector $B = {}^t(-d_1, -d_2)$ such that the cartesian form for the equation of r is given by (14.11).

The linear system

$$\Sigma_r : \quad \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{cases}$$

with $\mathrm{rk}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$ is called the *cartesian equation of the line* r in $\mathbb{A}^3$.

Remark 14.4.11 We notice again that the cartesian form (14.11) is not uniquely determined by the line r, since any linear system Σ' which is equivalent to Σ_r describes the same line.

We now a few examples of linear affine varieties described by cartesian equations obtained via removing parameters in their parametric equations.

Exercise 14.4.12 We consider the hyperplane in $\mathbb{A}^4$ with parametric equation

$$H : \quad \begin{cases} x = 1 + \lambda + \mu + \nu \\ y = \lambda - \mu \\ z = \mu + \nu \\ t = \nu \end{cases} .$$

Let us eliminate the parameters: we start by eliminating μ via the fourth relations, then ν by the third relation and eventually λ via the second relation. We have then

$$H: \begin{cases} x = 1 + \lambda + \mu + t \\ y = \lambda - \mu \\ z = \mu + t \\ \nu = t \end{cases} \Leftrightarrow \begin{cases} x = 1 + \lambda + (z - t) + t \\ y = \lambda - (z - t) \\ \mu = z - t \\ \nu = t \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 1 + (y + z - t) + (z - t) + t \\ \lambda = y + z - t \\ \mu = z - t \\ \nu = t \end{cases}.$$

As we have noticed previously, since these relations are valid for each value of the parameters λ, μ, ν, the computations amount to a redefinition of the parameters to y, z, t, so we consider only the first relation, and write

$$\Sigma_H : \quad x - y - 2z + t - 1 = 0$$

as the cartesian equation of the hyperplane H in $\mathbb{A}^4$ with the starting parametric equation. The direction $S_H = \mathbb{R}^3$ of such a hyperplane is given by the vector space corresponding to the space of the solutions of the homogeneous linear equation

$$x - y - 2z + t = 0.$$

Exercise 14.4.13 We consider the plane π in $\mathbb{A}^3$ whose vector equation is given by

$$\pi : \quad P = Q + \lambda v_1 + \mu v_2,$$

with $Q = (2, 3, 0)$ and $v_1 = (1, 0, 1)$, $v_2 = (1, -1, 0)$. By denoting the coordinates $P = (x, y, z)$ we write

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

which reads as the parametric equation

$$\pi : \quad \begin{cases} x = 2 + \lambda + \mu \\ y = 3 - \mu \\ z = \lambda \end{cases}.$$

If we eliminate the parameters we write

$$H: \quad \begin{cases} \lambda = z \\ \mu = 3 - y \\ x = 2 + z + 3 - y \end{cases}$$

so to have the following cartesian equation for π:

$$\Sigma_\pi: \quad x + y - z - 5 = 0.$$

The direction $S_\pi = \mathbb{R}^2$ of the plane π is the space of the solutions of the homogeneous equation

$$x + y - z = 0,$$

and it is easy to see that $S_\pi = \mathcal{L}(v_1, v_2)$.

Exercise 14.4.14 We consider the line $r : P = Q + \lambda v$ in $\mathbb{A}^4$, with $Q = (1, -1, 2, 1)$ and direction vector $v = (1, 2, 2, 1)$. Its parametric equation is given by

$$r: \quad \begin{cases} x_1 = 1 + \lambda \\ x_2 = 2 - \lambda \\ x_3 = 2 + 2\lambda \\ x_4 = 1 + \lambda \end{cases}.$$

If we use the first relation to eliminate the parameter λ, we write

$$r: \quad \begin{cases} \lambda = x_1 - 1 \\ x_2 = 2 - (x_1 - 1) \\ x_3 = 2 + 2(x_1 - 1) \\ x_4 = 1 + (x_1 - 1) \end{cases}$$

which amounts to the following cartesian equation

$$\Sigma_r: \quad \begin{cases} x_1 + x_2 - 3 = 0 \\ 2x_1 + x_3 = 0 \\ x_1 + x_4 = 0 \end{cases}.$$

Again, the direction $S_r = \mathbb{R}$ of the line r is given by the space of the solutions for the homogeneous linear system

$$\begin{cases} x_1 + x_2 = 0 \\ 2x_1 + x_3 = 0 \\ x_1 + x_4 = 0 \end{cases}.$$

It is easy to see that $S_{r_o} = \mathcal{L}(v)$.

Exercise 14.4.15 We consider the plane $\pi \subseteq \mathbb{A}^3$ whose cartesian equation is

$$\Sigma_\pi : \quad 2x - y + z - 1 = 0.$$

By choosing as free unknowns x, y, we have $z = -2x + y + 1$, that is $P = (x, y, z) \in \pi$ if and only if

$$(x, y, z) = (a, b, -2a + b + 1) = (0, 0, 1) + a(1, 0, -2) + b(0, 1, 1)$$

for any choice of the real parameters a, b. The former relation is then the vector equation of π.

Exercise 14.4.16 We consider the line $r \subseteq \mathbb{A}^3$ with cartesian equation

$$\Sigma_r : \quad \begin{cases} x - y + z - 1 = 0 \\ 2x + y + 2 = 0 \end{cases}.$$

In order to have a vector equation for r we solve such a linear system, getting

$$\Sigma_r : \quad \begin{cases} y = -2x - 2 \\ z = -3x - 3 \end{cases}.$$

Then the space of the solutions for Σ_r is given by the elements

$$(x, y, z) = (a, -2a - 2, -3a - 3) = (0, -2, -3) + a(1, -2, -3).$$

This relation yields a vector equation for r.

We conclude this section by rewriting the Proposition 14.4.1, whose formulation should appear now clearer.

Proposition 14.4.17 *Given the matrix*

$$(A, B) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & -b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & -b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & -b_m \end{pmatrix} \in \mathbb{R}^{m,n}$$

with

$$\mathrm{rk}(A) = \mathrm{rk} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = m < n,$$

the solutions of the linear system

$$\Sigma_L : AX = B \quad \Leftrightarrow \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 = 0 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + b_m = 0 \end{cases} \tag{14.13}$$

give the coordinates of all points $P = (x_1, x_2, \dots, x_n)$ *of a linear affine variety* L *in* $\mathbb{A}^n$ *of dimension* $k = n - m$ *and whose direction* S_L *is given by the solutions of the associated linear homogenous system*

$$\Sigma_{L_o} : \quad AX = 0. \tag{14.14}$$

If $L \subset \mathbb{A}^n$ *is a linear affine variety of dimension* k*, whose direction* $S_L \cong \mathbb{R}^k$ *is the space of solutions of the linear homogenous system* $AX = 0$ *with* $A \in \mathbb{R}^{m,n}$ *and* $\mathrm{rk}(A) = m < n$*, then there is a vector* $B = {}^t(-b_1, \dots, -b_m)$ *such that the cartesian form for the equation of* L *is given by* (14.13).

14.5 Intersection of Linear Affine Varieties

In this section, by studying particular examples, we introduce some aspects of the general problem of the intersection (that is of the mutual position) of different linear affine varieties.

14.5.1 *Intersection of two lines in* $\mathbb{A}^2$
Let r and r' be the lines in $\mathbb{A}^2$ given by the cartesian equations

$$\Sigma_r : \quad ax + by + c = 0; \qquad \Sigma_{r'} : \quad a'x + b'y + c' = 0.$$

Their intersection is given by the solutions of the linear system

$$\Sigma_{r\cap r'} : \quad \begin{cases} ax + by = -c \\ a'x + b'y = -c' \end{cases}.$$

By defining

$$A = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}, \qquad (A, B) = \begin{pmatrix} a & b & -c \\ a' & b' & -c' \end{pmatrix}$$

the matrices associated to such a linear system, we have three different possibilities:

- if $\mathrm{rk}(A) = \mathrm{rk}((A, B)) = 1$, the system $\Sigma_{r\cap r'}$ is solvable, with the space of solutions $S_{\Sigma_{r\cap r'}}$ containing ∞^1 solutions. This means that $r = r'$, the two lines coincide;
- if $\mathrm{rk}(A) = \mathrm{rk}((A, B)) = 2$, the system $\Sigma_{r\cap r'}$ is solvable, with the space of solutions $S_{\Sigma_{r\cap r'}}$ made of only one solution, the point $P = (x_0, y_0)$ of intersection between the lines r and r';

- if $\mathrm{rk}(A) = 1$ and $\mathrm{rk}((A, B)) = 2$, the system $\Sigma_{r\cap r'}$ is not solvable, which means that $r \cap r' = \emptyset$; the lines r and r' are therefore parallel, with common direction given by $\mathcal{L}((-b, a))$.

We can summarise such cases as in the following table

$\mathrm{rk}(A)$	$\mathrm{rk}((A, B))$	$S_{\Sigma_{r\cap r'}}$	$r \cap r'$
1	1	∞^1	$r = r'$
2	2	1	$P = (x_0, y_0)$
1	2	$\emptyset$	$\emptyset$

The following result comes easily from the analysis above.

Corollary 14.5.2 *Given the lines r and r' in $\mathbb{A}^2$ with cartesian equations $\Sigma_r : ax + by + c = 0$ and $\Sigma_{r'} : a'x + b'y + c' = 0$, we have that*

$$r = r' \iff \mathrm{rk}\begin{pmatrix} a & b & -c \\ a' & b' & -c' \end{pmatrix} = 1.$$

Exercise 14.5.3 Given the lines r and s on $\mathbb{A}^2$ whose cartesian equations are

$$\Sigma_r : \quad x + y - 1 = 0, \qquad \Sigma_s : \quad x + 2y + 2 = 0,$$

we study their mutual position. We consider therefore the linear system

$$\Sigma_{r\cap s} : \quad \begin{cases} x + y = 1 \\ x + 2y = -2 \end{cases}.$$

The reduction

$$(A, B) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \end{pmatrix} = (A', B')$$

proves that $\mathrm{rk}(A, B) = \mathrm{rk}(A', B') = 2$ and $\mathrm{rk}(A) = \mathrm{rk}(A') = 2$. The lines r and s have a unique point of intersection, which is computed to be $r \cap s = \{(4, -3)\}$.

Exercise 14.5.4 Consider the lines r and s_α given by their cartesian equations

$$\Sigma_r : \quad x + y - 1 = 0, \qquad \Sigma_{s_\alpha} : \quad x + \alpha y + 2 = 0$$

with $\alpha \in \mathbb{R}$ a parameter. We study the mutual position of r and s_α as depending on the value of α. We therefore study the linear system

$$\Sigma_{r\cap s_\alpha} : \quad \begin{cases} x + y = 1 \\ x + \alpha y = -2 \end{cases}.$$

We use the reduction

$$(A, B) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & -2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 0 & \alpha - 1 & -3 \end{pmatrix} = (A', B'),$$

proving that $\mathrm{rk}(A, B) = \mathrm{rk}(A', B') = 2$ for any value of α, while $\mathrm{rk}(A) = \mathrm{rk}(A') = 2$ if and only if $\alpha \neq 1$. This means that r is parallel to s_α if and only if $\alpha = 1$ (being in such a case $\Sigma_{s_1} : x + y + 2 = 0$), while for any $\alpha \neq 1$ the two lines intersects in one point, whose coordinates are computed to be

$$r \cap s_\alpha = (\frac{\alpha + 2}{\alpha - 1}, \frac{3}{1 - \alpha}).$$

The following examples show how to study the mutual position of two lines which are not given in the cartesian form. They present different methods without the need to explicitly transforming a parametric or a vector equation into its cartesian form.

Exercise 14.5.5 We consider the line r in $\mathbb{A}^2$ with vector equation

$$r : \quad (x, y) = (1, 2) + \lambda(1, -1),$$

and the line s whose cartesian equation is

$$\Sigma_s : \quad 2x - y - 6 = 0.$$

These line intersect for each value of the parameter λ giving a point in r whose coordinates solve the equation Σ_s. From

$$r : \quad \begin{cases} x = 1 + \lambda \\ y = 2 - \lambda \end{cases}$$

we have

$$2(1 + \lambda) - (2 - \lambda) - 6 = 0 \quad \Leftrightarrow \quad \lambda = 2.$$

This means that r and s intersects in one point, the one with coordinates $(x = 3, y = 0)$.

Exercise 14.5.6 As in the exercise above we consider the line r given by the vector equation

$$r : \quad (x, y) = (1, -1) + \lambda(2, -1)$$

and the line s given by the cartesian equation

$$\Sigma_s : \quad x + 2y - 3 = 0.$$

Their intersections correspond to the value of the parameter λ which solve the equation

$$(1 + 2\lambda) + 2(-1 - \lambda) - 3 = 0 \quad \Leftrightarrow \quad -4 = 0.$$

This means that $r \cap s = \emptyset$; these two lines are parallel.

Exercise 14.5.7 Consider the lines r and s in $\mathbb{A}^2$ both given by a vector equation, for example

$$r: \quad (x, y) = (1, 0) + \lambda(1, -2), \qquad s: \quad (x, y) = (1, -1) + \mu(-1, 1).$$

The intersection $r \cap s$ corresponds to values of the parameters λ and μ for which the coordinates of a point in r coincide with those of a point in s. We have then to solve the linear system

$$\begin{cases} 1 + \lambda = 1 - \mu \\ -2\lambda = -1 + \mu \end{cases} \quad \Leftrightarrow \quad \begin{cases} \lambda = -\mu \\ 2\mu = -1 + \mu \end{cases} \quad \Leftrightarrow \quad \begin{cases} \lambda = 1 \\ \mu = -1 \end{cases}.$$

Having such a linear system one solution, the intersection $s \cap r = P$ where the point P corresponds to the value $\lambda = 1$ in r or equivalently to the value $\mu = -1$ in s. Then $r \cap s = (2, -2)$.

Exercise 14.5.8 As in the previous exercise, we study the intersection of the lines

$$r: \quad (x, y) = (1, 1) + \lambda(-1, 2), \qquad s: \quad (x, y) = (1, 2) + \mu(1, -2).$$

We proceed as above, and consider the linear system

$$\begin{cases} 1 - \lambda = 1 + \mu \\ 1 + 2\lambda = 2 - 2\mu \end{cases} \quad \Leftrightarrow \quad \begin{cases} -\lambda = \mu \\ 1 - 2\mu = 2 - 2\mu \end{cases} \quad \Leftrightarrow \quad \begin{cases} -\lambda = \mu \\ 1 = 2 \end{cases}.$$

Since this linear system is not solvable, we conclude that r does not intersect s, and since the direction of r and s coincide, we have that r is parallel to s.

14.5.9 *Intersection of two planes in* $\mathbb{A}^3$

Consider the planes π and π' in $\mathbb{A}^3$ with cartesian equations given by

$$\Sigma_\pi: \quad ax + by + cz + d = 0, \qquad \Sigma_{\pi'}: \quad a'x + b'y + c'z + d' = 0.$$

Their intersection is given by the solutions of the linear system

$$\Sigma_{\pi\cap\pi'}: \quad \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases}$$

which is characterized by the matrices

$$A = \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}, \qquad (A, B) = \begin{pmatrix} a & b & c & -d \\ a' & b' & c' & -d' \end{pmatrix}.$$

We have the following possible cases.

$\mathrm{rk}(A)$	$\mathrm{rk}((A, B))$	$S_{\Sigma_{\pi\cap\pi'}}$	$\pi \cap \pi'$
1	1	∞^2	$\pi = \pi'$
2	2	∞^1	line
1	2	$\emptyset$	$\emptyset$

Notice that the case $\pi \cap \pi' = \emptyset$ corresponds to π parallel to π'.

The following corollary parallels the one in Corollary 14.5.2.

Corollary 14.5.10 *Consider two planes π and π' in $\mathbb{A}^3$ having cartesian equations $\Sigma_\pi : ax + by + cz + d = 0$ and $\Sigma_{\pi'} : a'x + b'y + c'z + d' = 0$. One has*

$$\pi = \pi' \iff \mathrm{rk}\begin{pmatrix} a & b & c & -d \\ a' & b' & c' & -d' \end{pmatrix} = 1.$$

Exercise 14.5.11 We consider the planes π and π' in $\mathbb{A}^3$ whose cartesian equations are

$$\Sigma_\pi : \quad x - y + 3z + 2 = 0 \qquad \Sigma_{\pi'} : \quad x - y + z + 1 = 0.$$

The intersection is given by the solutions of the system

$$\Sigma_{\pi\cap\pi'} : \quad \begin{cases} x - y + 3z = -2 \\ x - y + z = -1 \end{cases}.$$

By reducing the complete matrix of such a linear system,

$$(A, B) = \begin{pmatrix} 1 & -1 & 3 & -2 \\ 1 & -1 & 1 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{pmatrix},$$

we see that $\mathrm{rk}(A, B) = \mathrm{rk}(A) = 2$, so the linear system has ∞^1 solutions. The intersection $\pi \cap \pi'$ is therefore a line with cartesian equation given by $\Sigma_{\pi\cap\pi'}$.

Exercise 14.5.12 We consider the planes π and π' in $\mathbb{A}^3$ given by

$$\Sigma_\pi : \quad x - y + z + 2 = 0 \qquad \Sigma_{\pi'} : \quad 2x - 2y + 2z + 1 = 0.$$

As in the previous exercise, we reduce the complete matrix of the linear system $\Sigma_{\pi\cap\pi'}$,

$$(A, B) = \begin{pmatrix} 1 & -1 & 1 & -2 \\ 2 & -2 & 2 & -1 \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix},$$

to get $\mathrm{rk}(A) = 1$ while $\mathrm{rk}(A, B) = 2$, so $\pi \cap \pi' = \emptyset$. Since these planes are in $\mathbb{A}^3$, they are parallel.

Exercise 14.5.13 We consider the planes π, π', π'' in $\mathbb{A}^3$ whose cartesian equations are given by

$$\begin{aligned} \Sigma_\pi : &\quad x - 2y - z + 1 = 0 \\ \Sigma_{\pi'} : &\quad x + y - 2 = 0 \\ \Sigma_{\pi''} : &\quad 2x - 4y - 2z - 5 = 0\,. \end{aligned}$$

For the mutual positions of the pairs π, π' and π, π'', we start by considering the linear system

$$\Sigma_{\pi\cap\pi'} : \quad \begin{cases} x - 2y - z = -1 \\ x + y = 2 \end{cases}.$$

For the complete matrix

$$(A, B) = \begin{pmatrix} 1 & -2 & -1 & -1 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

we easily see that $\mathrm{rk}(A) = \mathrm{rk}(A, B) = 2$, so the intersection $\pi \cap \pi'$ is the line whose cartesian equation is the linear system $\Sigma_{\pi\cap\pi'}$.

For the intersections of π with π'' we consider the linear system

$$\Sigma_{\pi\cap\pi''} : \quad \begin{cases} x - 2y - z = -1 \\ 2x - 4y - 2z = 5 \end{cases}.$$

The complete matrix

$$(A, B) = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 2 & -4 & -2 & 5 \end{pmatrix},$$

has $\mathrm{rk}(A) = 1$ and $\mathrm{rk}(A, B) = 2$. This means that $\Sigma_{\pi\cap\pi''}$ has no solutions, that is the planes π and π' are parallel, having the same direction given by the vector space solutions of $S_{\pi_O} : x - 2y - z = 0$.

14.5.14 *Intersection of a line with a plane in* $\mathbb{A}^3$
We consider the line r and the plane π in $\mathbb{A}^3$ given by the cartesian equations

$$\Sigma_r: \quad \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{cases}, \qquad \Sigma_\pi: \quad ax + by + cz + d = 0.$$

Again, their intersection is given by the solutions of the linear system

$$\Sigma_{\pi\cap r}: \quad \begin{cases} a_1x + b_1y + c_1z = -d_1 \\ a_2x + b_2y + c_2z = -d_2 \\ ax + by + cz = -d \end{cases},$$

with its associated matrices

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a & b & c \end{pmatrix}, \qquad (A, B) = \begin{pmatrix} a_1 & b_1 & c_1 & -d_1 \\ a_2 & b_2 & c_2 & -d_2 \\ a & b & c & -d \end{pmatrix}.$$

Since the upper two row vectors of both A and (A, B) matrices are linearly independent, because the corresponding equations represent a line in $\mathbb{A}^3$, only the following cases are possible.

$\mathrm{rk}(A)$	$\mathrm{rk}((A,B))$	$S_{\Sigma_{\pi\cap r}}$	$\pi \cap r$
2	2	∞^1	r
3	3	∞^0	point
2	3	$\emptyset$	$\emptyset$

Notice that, when $\mathrm{rk}(A) = \mathrm{rk}(A, B) = 2$, it is $r \subset \pi$, while, if $\mathrm{rk}(A) = 2$ and $\mathrm{rk}(A, B) = 3$, then r is parallel to π. Indeed, when $\mathrm{rk}(A) = 2$, then $S_r \subset S_\pi$, the direction of r is a subspace in the direction of π. In order to show this, we consider the linear systems for the directions S_r and S_π,

$$\Sigma_{r_o}: \quad \begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{cases}, \qquad \Sigma_{\pi_o}: \quad ax + by + cz = 0.$$

Since $\mathrm{rk}(A) = 2$ and the upper two row vectors are linearly independent, we can write

$$(a, b, c) = \lambda_1(a_1, b_1, c_1) + \lambda_2(a_2, b_2, c_2).$$

If $P = (x_0, y_0.z_0)$ is a point in S_r, then $a_ix_0 + b_iy_0 + c_iz_0 = 0$ for $i = 1, 2$. We can then write

$$\begin{aligned} ax_0 + by_0 + cz_0 &= (\lambda_1a_1 + \lambda_2a_2)x_0 + (\lambda_1b_1 + \lambda_2b_2)y_0 + (\lambda_1c_1 + \lambda_2c_2)z_0 \\ &= \lambda_1(a_1x_0 + b_1y_0 + c_1z_0) + \lambda_2(a_2x_0 + b_2y_0 + c_2z_0) \\ &= 0 \end{aligned}$$

and this proves that $P \in S_\pi$, that is the inclusion $S_r \subset S_\pi$.

Exercise 14.5.15 Given in $\mathbb{A}^3$ the line r and the plane π with cartesian equations

$$\Sigma_r : \quad \begin{cases} x - 2y - z + 1 = 0 \\ x + y - 2 = 0 \end{cases}, \qquad \Sigma_\pi : \quad 2x + y - 2z - 5 = 0,$$

their intersection is given by the solutions of the linear system $\Sigma_{\pi \cap r} : AX = B$ whose associated complete matrix, suitably reduced, reads

$$(A, B) = \begin{pmatrix} 1 & -2 & -1 & -1 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & -2 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -2 & -1 & -1 \\ 1 & 1 & 0 & 2 \\ 0 & 5 & 0 & 7 \end{pmatrix} = (A', B').$$

Then $\mathrm{rk}(A) = 3$ and $\mathrm{rk}(A, B) = 3$, so the linear system $\Sigma_{\pi \cap r}$ has a unique solution, which corresponds to the unique point P of intersection between r and π. The coordinates of P are easily computed to be $P = (\frac{3}{5}, \frac{7}{5}, -\frac{6}{5})$.

Exercise 14.5.16 We consider in $\mathbb{A}^3$ the line r and the plane π_h with equations

$$\Sigma_r : \quad \begin{cases} x - 2y - z + 1 = 0 \\ x + y - 2 = 0 \end{cases}, \qquad \Sigma_\pi : \quad 2x + hy - 2z - 5 = 0,$$

where h is a real parameter. The complete matrix of to the linear system $\Sigma_{\pi_h \cap r} : AX = B$ giving the intersection of π_h and r is

$$(A_h, B) = \begin{pmatrix} 1 & -2 & -1 & -1 \\ 1 & 1 & 0 & 2 \\ 2 & h & -2 & 5 \end{pmatrix}.$$

We notice that the rank of A_h is at least 2, with $\mathrm{rk}(A_h) = 3$ if and only if $\det(A_h) \neq 0$. It is $\det(A_h) = -h - 4$, so $\mathrm{rk}(A_h) = 3$ if and only if $h \neq -4$. In such a case $\mathrm{rk}(A_h) = 3 = \mathrm{rk}(A_h, B)$, and this means that r and $\pi_{h \neq -4}$ have a unique point of intersection.

If $h = -4$, then $\mathrm{rk}(A_{-4}) = 2$: the reduction

$$(A_{-4}, B) = \begin{pmatrix} 1 & -2 & -1 & -1 \\ 1 & 1 & 0 & 2 \\ 2 & -4 & -2 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -2 & -1 & -1 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

shows that $\mathrm{rk}(A_{-4}, B) = 3$, so the linear system $A_{-4}X = B$ has no solutions, and r is parallel to π.

Exercise 14.5.17 As in the Exercise 14.5.15 we study the intersection of a plane π (represented by a cartesian equation) and a line r in $\mathbb{A}^3$ (represented by a parametric equation). Consider for instance,

$$r: \quad (x, y, z) = (3, -1, 5) + \lambda(1, -1, 2), \qquad \Sigma_\pi: \quad x + y - z + 1 = 0.$$

As before, the intersection $\pi \cap r$ corresponds to the values of the parameter λ for which the coordinates $P = (3 + \lambda, -1 - \lambda, 5 + 2\lambda)$ of a point in r solve the cartesian equation for π, that is

$$(3 + \lambda) + (-1 - \lambda) - (5 + 2\lambda) + 1 = 0 \quad \Rightarrow \quad -2\lambda - 2 = 0 \quad \Rightarrow \quad \lambda = -1.$$

We have then $r \cap \pi = (2, 0, 3)$.

14.5.18 *Intersection of two lines in* $\mathbb{A}^3$
We consider a line r and a line r' in $\mathbb{A}^3$ with cartesian equations

$$\Sigma_r: \quad \begin{cases} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{cases}, \qquad \Sigma'_r: \quad \begin{cases} a'_1x + b'_1y + c'_1z + d'_1 = 0 \\ a'_2x + b'_2y + c'_2z + d'_2 = 0 \end{cases}.$$

The intersection is given by the linear system $\Sigma_{r\cap r'}$ whose associated matrices are

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a'_1 & b'_1 & c'_1 \\ a'_2 & b'_2 & c'_2 \end{pmatrix}, \qquad (A, B) = \begin{pmatrix} a_1 & b_1 & c_1 & -d_1 \\ a_2 & b_2 & c_2 & -d_2 \\ a'_1 & b'_1 & c'_1 & -d'_1 \\ a'_2 & b'_2 & c'_2 & -d'_2 \end{pmatrix}.$$

Once again, different possibilities depending on the mutual ranks of these. As we stressed in the previous case 14.5.14, since r and r' are lines, the upper two row vectors R_1 and R_2 of both A and (A, B) are linearly independent, as are the last two row vectors, R_3 and R_4. Then,

rk(A)	rk$((A, B))$	$S_{\Sigma_{r\cap r'}}$	$r \cap r'$
2	2	∞^1	r
3	3	∞^0	point
2	3	$\emptyset$	$\emptyset$
3	4	$\emptyset$	$\emptyset$

In the first case, with $\text{rk}(A) = \text{rk}(A, B) = 2$, the lines r, r' coincide, while in the second case, with $\text{rk}(A) = \text{rk}(A, B) = 3$, they have a unique point of intersection, whose coordinates are given by the solution of the system $AX = B$.

In the third and the fourth case, the condition $\mathrm{rk}(A) \neq \mathrm{rk}(A, B)$ means that the two lines do not intersect. If $\mathrm{rk}(A) = 2$, then the row vectors R_3 and R_4 of A are both linearly dependent of R_1 and R_2, and therefore the homogeneous linear systems

$$\Sigma_{r_o} : \begin{cases} a_1 x + b_1 y + c_1 z = 0 \\ a_2 x + b_2 y + c_2 z = 0 \end{cases}, \qquad \Sigma_{r'_o} : \begin{cases} a'_1 x + b'_1 y + c'_1 z = 0 \\ a'_2 x + b'_2 y + c'_2 z = 0 \end{cases},$$

are equivalent. We have then that $S_r = S_{r'}$, the direction of r coincide with that of r', that is r is parallel to r'. If $\mathrm{rk}(A) = 3$ (the fourth case in the table above) the lines are not parallel and do not intersect, so they are skew.

Exercise 14.5.19 We consider the line r and r' in $\mathbb{A}^3$ whose cartesian equations are

$$\Sigma_r : \begin{cases} x - y + 2z + 1 = 0 \\ x + z - 1 = 0 \end{cases}, \qquad \Sigma_{r'} : \begin{cases} y - z + 2 = 0 \\ x + y + z = 0 \end{cases}.$$

We reduce the complete matrix associated to the linear system $\Sigma_{r \cap r'}$, that is

$$(A, B) = \begin{pmatrix} 1 & -1 & 2 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 1 & 1 & 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -1 & 1 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{pmatrix} = (A', B').$$

Since $\mathrm{rk}(A') = 3$ and $\mathrm{rk}(A', B') = 4$, the two lines are skew.