Chapter 16
Conic Sections

This chapter is devoted to conics. We shall describe at length their algebraic and geometric properties and their use in physics, notably for the Kepler laws for the motion of celestial bodies.

16.1 Conic Sections as *Geometric Loci*

The conic sections (or simply conics) are parabolæ, ellipses (with circles as limiting case), hyperbolæ. They are also known as geometric *loci*, that is collections of points $P(x, y) \in \mathbb{E}^2$ satisfying one or more conditions, or determined by such conditions. The following three relations, whose origins we briefly recall, should be well known

$$x^2 = 2py, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{16.1}$$

Definition 16.1.1 (*Parabolæ*) Given a straight line δ and a point F on the plane $\mathbb{E}^2$, the set (locus) of points P equidistant from δ and F is called *parabola*. The straight line δ is the *directrix* of the parabola, while the point F is the *focus* of the parabola. This is shown in Fig. 16.1.

Fix a cartesian orthogonal reference system $(O; x, y)$ for $\mathbb{E}^2$, with a generic point P having coordinates $P = (x, y)$. Consider the straight line δ given by the points with equation $y = -p/2$ and the focus $F = (0, p/2)$ (with $p > 0$). The parabola with directrix δ and focus F is the set of points fulfilling the condition

$$\mathrm{d}(P, \delta) = \mathrm{d}(P, F). \tag{16.2}$$

G. Landi and A. Zampini, *Linear Algebra and Analytic Geometry for Physical Sciences*, Undergraduate Lecture Notes in Physics,
https://doi.org/10.1007/978-3-319-78361-1_16

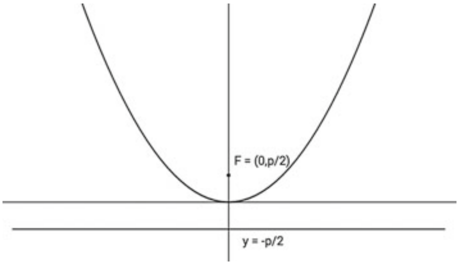

Fig. 16.1 The parabola $y = x^2/2p$

Since the point $P' = (x, -p/2)$ is the orthogonal projection of P over δ, with $\mathrm{d}(P, \delta) = \mathrm{d}(P, P')$, the condition (16.2) reads

$$\|P - P'\|^2 = \|P - F\|^2 \qquad \Rightarrow \qquad \|(0, y + p/2)\|^2 = \|(x, y - p/2)\|^2,$$

that is

$$(y + p/2)^2 = x^2 + (y - p/2)^2 \qquad \Rightarrow \qquad x^2 = 2py.$$

If C is a parabola with focus F and directrix δ then,

- the straight line through F which is orthogonal to δ is the *axis* of C,
- the point where the parabola C intersects its axis is the *vertex* of the parabola.

Definition 16.1.2 (*Ellipses*) Given two points F_1 ed F_2 on the plane $\mathbb{E}^2$, the set (locus) of points P for which the sum of the distances between P and the points F_1 and F_2 is constant is called *ellipse*. The points F_1 and F_2 are called the *foci* of the ellipse. This is shown in Fig. 16.2.

Fix a cartesian orthogonal reference system $(O; x, y)$ for $\mathbb{E}^2$, with a generic point P having coordinates $P = (x, y)$. Consider the points $F_1 = (-q, 0)$, $F_2 = (q, 0)$ (with $q \geq 0$) and k a real parameter such that $k > 2q$. The ellipse with foci F_1, F_2 and parameter k is the set of points $P = (x, y)$ fulfilling the condition

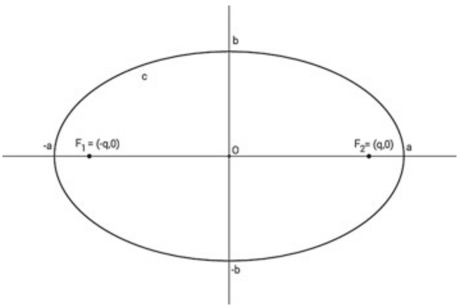

Fig. 16.2 The ellipse $x^2/a^2 + y^2/b^2 = 1$

$$\mathrm{d}(P, F_1) + \mathrm{d}(P, F_2) = k. \tag{16.3}$$

We denote by $A = (a, 0)$ and $B = (0, b)$ the intersection of the ellipse with the positive x-axis half-line and the positive y-axis half-line, thus $a > 0$ and $b > 0$. From $\mathrm{d}(A, F_1) + \mathrm{d}(A, F_2) = k$ we have that $k = 2a$; from $\mathrm{d}(B, F_1) + \mathrm{d}(B, F_2) = k$ we have that $2\sqrt{q^2 + b^2} = k$, so we write

$$k = 2a, \quad q^2 = a^2 - b^2,$$

with $a \geq b$. By squaring the condition (16.3) we have

$$\|(x + q, y)\|^2 + \|(x - q, y)\|^2 + 2\,\|(x + q, y)\|\,\|(x - q, y)\| = 4a^2,$$

that is

$$2(x^2 + y^2 + q^2) + 2\sqrt{(x^2 + y^2 + q^2 + 2qx)(x^2 + y^2 + q^2 - 2qx)} = 4a^2$$

that we write as

$$\sqrt{(x^2 + y^2 + q^2)^2 - 4q^2x^2} = 2a^2 - (x^2 + y^2 + q^2).$$

By squaring such a relation we have

$$-q^2x^2 = a^4 - a^2(x^2 + y^2 + q^2).$$

Since $q^2 = a^2 - b^2$, the equation of the ellipse depends on the real positive parameters a, b as follows

$$b^2x^2 + a^2y^2 = a^2b^2,$$

which is equivalent to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Notice that, if $q = 0$, that is if $a = b$, the foci F_1 ed F_2 coincide with the origin O of the reference system, and the ellipse reduces to a circle whose equation is

$$x^2 + y^2 = r^2$$

with radius $r = a = b > 0$.

If C is an ellipse with (distinct) foci F_1 and F_2, then

- the straight line passing through the foci is the *major axis* of the ellipse,
- the straight line orthogonally bisecting the segment $\overline{F_1F_2}$ is the *minor axis* of the ellipse,
- the midpoint of the segment $\overline{F_1F_2}$ is the *centre* of the ellipse,

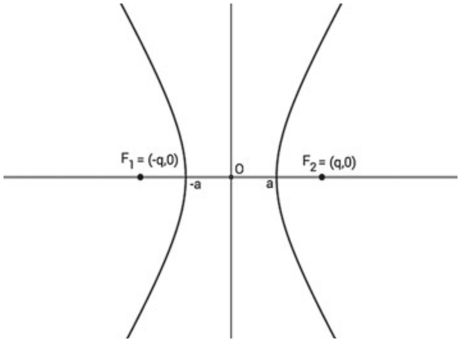

Fig. 16.3 The hyperbola $x^2/a^2 - y^2/b^2 = 1$

- the four points where the ellipse intersects its axes are the *vertices* of the ellipse,
- the distance between the centre of the ellipse and the vertices on the major axis (respectively on the minor axis) is called the *major semi-axis* (respectively *minor semi-axis*).

Definition 16.1.3 (*Hyperbolæ*) Given two points F_1 and F_2 on the plane $\mathbb{E}^2$, the set (locus) of points P for which the absolute difference of the distances $\mathrm{d}(P, F_1)$ and $\mathrm{d}(P, F_2)$ is constant, is the *hyperbola* with foci $F_1,\ F_2$. This is shown in Fig. 16.3.

Fix a cartesian orthogonal reference system $(O; x, y)$ for $\mathbb{E}^2$, with a generic point P having coordinates $P = (x, y)$. Consider the points $F_1 = (-q, 0)$, $F_2 = (q, 0)$ (with $q \geq 0$) and k a real parameter such that $k > 2q$. The hyperbola with foci F_1, F_2 and parameter k is the set of points $P = (x, y)$ fulfilling the condition

$$|\mathrm{d}(P, F_1) - \mathrm{d}(P, F_2)| = k. \tag{16.4}$$

Notice that, since $k > 0$, such a hyperbola does not intersect the y-axis, since the points on the y-axis are equidistant from the foci. By denoting by $A = (a, 0)$ (with $a > 0$) the intersection of the hyperbola with the x-axis, we have

$$k = |\mathrm{d}(A, F_1) - \mathrm{d}(A, F_2)| = \big|a + q - |a - q|\big|,$$

which yields $a < q$, since from $a > q$ it would follow that $|a - q| = a - q$, giving $k = 2q$. The previous condition then show that

$$k = |2a| = 2a.$$

By squaring the relation (16.4) we have

$$\|(x + q, y)\|^2 + \|(x - q, y)\|^2 - 2\ \|(x + q, y)\|\ \|(x - q, y)\| = 4a^2,$$

that is

$$2(x^2+y^2+q^2)-2\sqrt{(x^2+y^2+q^2+2qx)(x^2+y^2+q^2-2qx)}=4a^2$$

which we write as

$$\sqrt{(x^2+y^2+q^2)^2-4q^2x^2}=(x^2+y^2+q^2)-2a^2.$$

By squaring once more, we have

$$-q^2x^2=a^4-a^2(x^2+y^2+q^2),$$

that reads

$$(a^2-q^2)x^2+a^2y^2=a^2(a^2-q^2).$$

From $a<q$ we have $q^2-a^2>0$, so we set $q^2-a^2=b^2$ and write the previous relation as

$$-b^2x^2+a^2y^2=-a^2b^2,$$

which is equivalent to

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

If C is a hyperbola with foci F_1 and F_2, then

- the straight line through the foci is the *transverse axis* of the hyperbola,
- the straight line orthogonally bisecting the segment $\overline{F_1F_2}$ is the *axis* of the hyperbola,
- the midpoint of the segment $\overline{F_1F_2}$ is the *centre* of the hyperbola;
- the points where the hyperbola intersects its transverse axis are the *vertices* of the hyperbola,
- the distance between the centre of the hyperbola and its foci is the *transverse semi-axis* of the hyperbola.

Remark 16.1.4 The above analysis shows that, if C is a parabola with equation

$$x^2=2py,$$

then its directrix is the line $y=-p/2$ and its focus is the point $(0, p/2)$, while the equation

$$y^2=2px$$

is a parabola C with directrix $x=-p/2$ and focus $(p/2,0)$.

If C is an ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(and $a \geq b$), then its foci are the points $F_{\pm} = (\pm\sqrt{a^2 - b^2}, 0)$.

If C is a hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

then its foci are the points $F_{\pm} = (\pm\sqrt{a^2 + b^2}, 0)$.

We see that the definition of a parabola requires one single focus and a straight line (not containing the focus), while the definition of an ellipse and of a hyperbola requires two distinct foci and a suitable distance k. This apparent diversity can be reconciled. If F is a point is $\mathbb{E}^2$ and δ a straight line with $F \notin \delta$, then one can consider the *locus* given by points P in $\mathbb{E}^2$ fulfilling the condition

$$\mathrm{d}(P, F) = e\,\mathrm{d}(P, \delta) \tag{16.5}$$

with $e > 0$. It is clear that, if $e = 1$, this relation defines a parabola with focus F and directrix δ. We shall show later on (in Sect. 16.4 and then Sect. 16.7) that the relation above gives an ellipse for $0 < e < 1$ and a hyperbola if $e > 1$. The parameter $e > 0$ is called the *eccentricity* of the conic.

Since symmetry properties of conics do not depend on the reference system, when dealing with symmetries or geometric properties of conics one can refer to the Eqs. (16.1).

Remark 16.1.5 With the symmetry notions given in the Sect. 15.5, the y-axis is a symmetry axis for the parabola C whose equation is $y = 2px^2$. If $P = (x_0, y_0) \in C$, the symmetric point P' to P with respect to the y-axis is $P' = (-x_0, y_0)$, which belongs to C since $2py_0 = (-x_0^2) = x_0^2$. Furthermore, the axis of a parabola is a symmetry axis and its vertex is equidistant from the focus and the directrix if the parabola.

In a similar way one shows that the axes of an ellipse or of a hyperbola, are symmetry axes and the centre is a symmetry centre in both cases. For an ellipse with equation $\alpha x^2 + \beta y^2 = 1$ or a hyperbola with equation $\alpha x^2 - \beta y^2 = 1$ the centre coincided with the origin of the reference system.

16.2 The Equation of a Conic in Matrix Form

In the previous section we have shown how, in a given reference system, a parabola, an ellipse and a hyperbola are described by one of equations in (16.1). But evidently such equations are not the most general ones for the loci we are considering, since they have particular positions with respect to the axes of the reference system.

A common feature of the Eqs. (16.1) is that they are formulated as quadratic polynomials in x and y. In the present section we study general quadratic polynomial equations in two variables.

Since to a large extent one does not make use of the euclidean structure given by the scalar product in $\mathbb{E}^n$, one can consider the affine plane $\mathbb{A}^2(\mathbb{R})$. By taking complex coordinates, with the canonical inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{C}^2$, one enlarges the real affine plane to the complex one,

$$\mathbb{A}^2(\mathbb{R}) \hookrightarrow \mathbb{A}^2(\mathbb{C}).$$

Definition 16.2.1 A *conic section* (or simply a conic) is the set of points (locus) whose coordinates (x, y) satisfy a quadratic polynomial equation in the variables x, y, that is

$$a_{11}\,x^2 + 2\,a_{12}\,xy + a_{22}\,y^2 + 2\,a_{13}\,x + 2\,a_{23}\,y + a_{33} = 0 \qquad (16.6)$$

with coefficients $a_{ij} \in \mathbb{R}$.

Remark 16.2.2 We notice that

(a) The equations of conics considered in the previous section are particular case of the general Eq. (16.6). As an example, for a parabola we have

$$a_{11} = 1, \quad a_{23} = -2p, \quad a_{12} = a_{22} = a_{13} = a_{33} = 0.$$

Notice also that in all the equations considered in the previous section for a parabola or an ellipse or a hyperbola we have $a_{12} = 0$.

(b) There are polynomial equations like (16.6) which do not describe any of the conics presented before: neither a parabola, nor an ellipse or a hyperbola. Consider for example the equation $x^2 - y^2 = 0$, which is factorised as $(x + y)(x - y) = 0$. The set of solutions for such an equation is the union of the two lines with cartesian equations $x + y = 0$ and $x - y = 0$.
Any quadratic polynomial equation (16.6) that can be factorised as

$$(ax + by + c)(a'x + b'y + c') = 0$$

describes the union of two lines. Such lines are not necessarily real. Consider for example the equation $x^2 + y^2 = 0$. Its set of solutions is given only by the point $(0, 0)$ in $\mathbb{A}^2(\mathbb{R})$, while in $\mathbb{A}^2(\mathbb{C})$ we can write $x^2 + y^2 = (x + \mathrm{i}y)(x - \mathrm{i}y)$, so the conic is the union of the two conjugate lines with cartesian equation $x + \mathrm{i}y = 0$ and $x - \mathrm{i}y = 0$.

Definition 16.2.3 A conic is called *degenerate* if it is the union of two lines. Such lines can be either real (coincident or distinct) or complex (in such a case they are also conjugate).

The polynomial equation (16.6) can be written in a more succinct form by means of two symmetric matrices associated with a conic. We set

$$\mathbb{R}^{2,2} \ni A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \qquad \mathbb{R}^{3,3} \ni B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

By introducing these matrices, we write the left end side of the Eq. (16.6) as

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = a_{11}\, x^2 + 2\, a_{12}\, xy + a_{22}\, y^2 + 2\, a_{13}\, x + 2\, a_{23}\, y + a_{33}. \tag{16.7}$$

The quadratic homogeneous part of the polynomial defining (16.6) and (16.7), is written as

$$F_C(x, y) = a_{11}\, x^2 + 2\, a_{12}\, xy + a_{22}\, y^2 = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Such an F_C is a quadratic form, called the *quadratic form* associated to the conic C.

Definition 16.2.4 Let C be the conic given by the equation

$$a_{11}\, x^2 + 2\, a_{12}\, xy + a_{22}\, y^2 + 2\, a_{13}\, x + 2\, a_{23}\, y + a_{33} = 0.$$

The matrices

$$B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

are called respectively the *matrix of the coefficients* and the *matrix of the quadratic form* of C.

Exercise 16.2.5 The matrices associated to the parabola with equation $y = 3x^2$ are,

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & -1/2 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}.$$

Remark 16.2.6 Notice that the six coefficients a_{ij} in (16.6) determine a conic, but a conic is not described by a single array of six coefficients since the equation

$$ka_{11}\, x^2 + 2\, ka_{12}\, xy + ka_{22}\, y^2 + 2\, ka_{13}\, x + 2\, ka_{23}\, y + ka_{33} = 0$$

defines the same locus for any $k \in \mathbb{R} \setminus \{0\}$.

16.3 Reduction to Canonical Form of a Conic: Translations

A natural question arises. Given a non degenerate conic with equation written as in (16.6) with respect to a reference frame, does there exist a new reference system with respect to which the equation for the conic has a form close to one of those given in (16.1)?

Definition 16.3.1 We call *canonical form* of a non degenerate conic C one of the following equations for C in a given reference system $(O; x, y)$.

(i) A *parabola* has equation

$$x^2 = 2py \qquad \text{or} \qquad y^2 = 2px. \tag{16.8}$$

(ii) A *real ellipse* has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{16.9}$$

while an *imaginary ellipse* has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1. \tag{16.10}$$

(iii) A *hyperbola* has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{or} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1. \tag{16.11}$$

A complete answer to the question above is given in two steps.

One first considers only conics whose equation, in a given reference system, $(O; x, y)$ has coefficient $a_{12} = 0$, that is conics whose equation lacks the mixed term xy. The reference system $(O'; X, Y)$ for a canonical form is obtained with a translation from $(O; x, y)$.

The general case of a conic whose equation in a given reference system $(O; x, y)$ may have the mixed term xy will require the composition of a rotation and a translation from $(O; x, y)$ to obtain the reference system $(O'; X, Y)$ for a canonical form.

Exercise 16.3.2 Let $\Gamma : \; y = 2x^2$ describe a parabola in the canonical form, and let us define the following translation on the plane

$$T_{(x_0, y_0)} : \quad \begin{cases} x = X + x_0 \\ y = Y + y_0 \end{cases}.$$

The equation for the conic Γ with respect to the reference system $(O'; X, Y)$ is then

$$Y = 2X^2 + 4x_0 X + 2x_0^2 - y_0.$$

Exercise 16.3.3 Let $\Gamma' : \; x^2 + 2y^2 = 1$ be an ellipse in the canonical form. Under the translation of the previous example, the equation for Γ' with respect to the reference system $(O'; X, Y)$ is

$$X^2 + 2Y^2 + 2x_0 X + 4y_0 Y + x_0^2 + 2y_0^2 - 1 = 0.$$

Notice that, after the translation by $T_{(x_0, y_0)}$, the equations for the conics Γ and Γ' are no longer in canonical form, but both still lack the mixed term xy. We prove now, with a constructive method, that the converse holds as well.

Exercise 16.3.4 (Completing the squares) Let C be a non degenerate conic whose equation reads, with respect to the reference system $(O; x, y)$,

$$a_{11}\, x^2 + a_{22}\, y^2 + 2\, a_{13}\, x + 2\, a_{23}\, y + a_{33} = 0. \qquad (16.12)$$

Since the polynomial must be quadratic, there are two possibilities. Either both a_{11} and a_{22} different from zero, or one of them is zero. We then consider:

(I) It is $a_{11} = 0,\; a_{22} \neq 0$ (the case $a_{11} \neq 0$ and $a_{22} = 0$ is analogue). The Eq. (16.12) is then

$$a_{22}\, y^2 + 2\, a_{23}\, y + a_{33} + 2\, a_{13}\, x = 0. \qquad (16.13)$$

From the algebraic identities:

$$\begin{aligned} a_{22}\, y^2 + 2\, a_{23}\, y &= a_{22}\left(y^2 + 2\,\frac{a_{23}}{a_{22}}\, y\right) \\ &= a_{22}\left[\left(y + \frac{a_{23}}{a_{22}}\right)^2 - \left(\frac{a_{23}}{a_{22}}\right)^2\right] \\ &= a_{22}\left(y + \frac{a_{23}}{a_{22}}\right)^2 - \frac{a_{23}^2}{a_{22}} \end{aligned}$$

we write the Eq. (16.13) as

$$a_{22}\left(y + \frac{a_{23}}{a_{22}}\right)^2 - \frac{a_{23}^2}{a_{22}} + a_{33} + 2\, a_{13}\, x = 0. \qquad (16.14)$$

Since C is not degenerate, we have $a_{13} \neq 0$ so we write (16.14) as

$$a_{22}\left(y + \frac{a_{23}}{a_{22}}\right)^2 + 2\, a_{13}\left(x + \frac{a_{33}a_{22} - a_{23}^2}{2\, a_{22}a_{13}}\right) = 0$$

which reads

$$\left(y+\frac{a_{23}}{a_{22}}\right)^2 = -\frac{2\,a_{13}}{a_{22}}\left(x+\frac{a_{33}a_{22}-a_{23}^2}{2\,a_{22}a_{13}}\right).$$

Under the translation

$$\begin{cases} X = x + (a_{33}a_{22}-a_{23}^2)/2\,a_{22}a_{13} \\ Y = y + a_{23}/a_{22} \end{cases}$$

we get

$$Y^2 = 2\,p\,X \tag{16.15}$$

with $p=-a_{13}/a_{22}$. This is the canonical form (16.8).

If we drop the hypothesis that the conics C is non degenerate, we have $a_{13}=0$ in the Eq. (16.13). Notice that, for the case $a_{11}=0$ we are considering, $\det B=-a_{13}^2a_{22}$. Thus the condition of non degeneracy can be expressed as a condition on the determinant of the matrix of the coefficients, since

$$a_{13}=0 \qquad \Leftrightarrow \qquad \det B = -a_{13}^2a_{22}=0.$$

The Eq. (16.14) is then

$$\left(y+\frac{a_{23}}{a_{22}}\right)^2 = \frac{a_{23}^2-a_{33}a_{22}}{a_{22}^2}$$

and with the translation

$$\begin{cases} X = x \\ Y = y + a_{23}/a_{22} \end{cases}$$

it reads

$$Y^2 = q \tag{16.16}$$

with $q=(a_{23}^2-a_{33}a_{22})/a_{22}^2$.

(II) It is $a_{11}\neq 0,\ a_{22}\neq 0$.

With algebraic manipulation as above, we can write

$$a_{11}\,x^2+2\,a_{13}\,x = a_{11}\left(x+\frac{a_{13}}{a_{11}}\right)^2 - \frac{a_{13}^2}{a_{11}},$$
$$a_{22}\,y^2+2\,a_{23}\,y = a_{22}\left(y+\frac{a_{23}}{a_{22}}\right)^2 - \frac{a_{23}^2}{a_{22}}.$$

So the Eq. (16.12) is written as

$$a_{11}\left(x+\frac{a_{13}}{a_{11}}\right)^2+a_{22}\left(y+\frac{a_{23}}{a_{22}}\right)^2+a_{33}-\frac{a_{13}^2}{a_{11}}-\frac{a_{23}^2}{a_{22}}=0. \qquad (16.17)$$

If we consider the translation given by

$$\begin{cases} X = x + a_{13}/a_{11} \\ Y = y + a_{23}/a_{22} \end{cases}$$

the conic C has the equation

$$a_{11}X^2+a_{22}Y^2=h, \qquad \text{with} \quad h=-a_{33}+\frac{a_{13}^2}{a_{11}}+\frac{a_{23}^2}{a_{22}}, \qquad (16.18)$$

and $h \neq 0$ since C is non degenerate. The coefficients a_{11} and a_{22} can be either concordant or not. Up to a global factor (-1), we can take $a_{11} > 0$. So we have the following cases.

(IIa) It is $a_{11} > 0$ and $a_{22} > 0$. One distinguish according to the sign of the coefficient h:

- If $h > 0$, the Eq. (16.18) is equivalent to

$$\frac{a_{11}}{h}X^2+\frac{a_{22}}{h}Y^2=1.$$

  Since $a_{11}/h > 0$ and $a_{22}/h > 0$, we have (positive) real numbers a, b by defining $h/a_{11} = a^2$ and $h/a_{22} = b^2$. The Eq. (16.18) is written as

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1, \qquad (16.19)$$

  which is the canonical form of a real ellipse (16.9).
- If $h < 0$, we have $-a_{11}/h > 0$ and $-a_{22}/h > 0$, we can again introduce (positive) real numbers a, b by $-h/a_{11} = a^2$ and $-h/a_{22} = b^2$. The Eq. (16.18) can be written as

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=-1, \qquad (16.20)$$

  which is the canonical form of an imaginary ellipse (16.10).
- If $h = 0$ (which means that C is degenerate), we set $1/a_{11} = a^2$ and $1/a_{22} = b^2$ with real number a, b, so to get from (16.18) the expression

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=0. \qquad (16.21)$$

(IIb) It is $a_{11} > 0$ and $a_{22} < 0$. Again depending on the sign of the coefficient h we have:

- If $h > 0$, the Eq. (16.18) is

$$\frac{a_{11}}{h} X^2 + \frac{a_{22}}{h} Y^2 = 1.$$

Since $a_{11}/h > 0$ and $a_{22}/h < 0$, we can define $h/a_{11} = a^2$ and $-h/a_{22} = b^2$ with a, b positive real numbers. The Eq. (16.18) becomes

$$\frac{X^2}{a^2} - \frac{X^2}{b^2} = 1, \tag{16.22}$$

which the first canonical form in (16.11).

- If $h < 0$, we have $-a_{11}/h > 0$ and $-a_{22}/h < 0$, so we can define $-h/a_{11} = a^2$ and $h/a_{22} = 1/b^2$ with a, b positive real numbers. The Eq. (16.18) becomes

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = -1, \tag{16.23}$$

which is the second canonical form in (16.11).

- If $h = 0$ (that is C is degenerate), we set $1/a_{11} = a^2$ and $-1/a_{22} = b^2$ with a, b real number, so to get from (16.18) the expression

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 0. \tag{16.24}$$

Once again, with B the matrix of the coefficients for C, the identity

$$\det B = a_{11}a_{22}\, h$$

shows that the condition of non degeneracy of the conic C is equivalently given by $\det B \neq 0$.

The analysis done for the cases of degenerate conics makes it natural to introduce the following definition, which has to be compared with the Definition 16.3.1.

We call canonical form of a degenerate conic C one of the following equations for C in a given reference system $(O; x, y)$.

(i) A degenerate parabola has equation

$$x^2 = q \qquad \text{or} \qquad y^2 = q. \tag{16.25}$$

(ii) A degenerate ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0. \tag{16.26}$$

(iii) A degenerate hyperbola has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \tag{16.27}$$

Remark 16.3.5 With the definition above, we have that

(i) The conic C with equation $x^2 = q$ is the union of the lines with cartesian equations $x = \pm\sqrt{q}$. If $q > 0$ the lines are real and distinct, if $q < 0$ the lines are complex and conjugate. If $q = 0$ the conic C is the y-axis counted twice. Analogue cases are obtained for the equation $y^2 = q$.

(ii) The equation $b^2 x^2 + a^2 y^2 = 0$ has the unique solution $(0, 0)$ if we consider real coordinates. On the complex affine plane $\mathbb{A}^2(\mathbb{C})$ the solutions to such equations give a degenerate conic C which is the union of two complex conjugate lines, since we can factorise

$$b^2 x^2 + a^2 y^2 = (b\,x + \mathrm{i}\,a\,y)(b\,x - \mathrm{i}\,a\,y).$$

(iii) The solutions to the equation $b^2 x^2 - a^2 y^2 = 0$ give the union of two real and distinct lines, since we can factorise as follows

$$b^2 x^2 - a^2 y^2 = (b\,x + a\,y)(b\,x - a\,y).$$

What we have studied up to now is the proof of the following theorem.

Theorem 16.3.6 *Let C be a conic whose equation, with respect to a reference system $(O; x, y)$ lacks the monomial xy. There exists a reference system $(O'; X, Y)$, obtained from $(O; x, y)$ by a translation, with respect to which the equation for the conic C has a canonical form.*

Exercise 16.3.7 We consider the conic C with equation

$$x^2 + 4y^2 + 2x - 12y + 3 = 0.$$

We wish to determine a reference system $(O'; X, Y)$ with respect to which the equation for C is canonical. We complete the squares as follows:

$$x^2 + 2x = (x + 1)^2 - 1,$$
$$4y^2 - 12y = 4\left(y - \tfrac{3}{2}\right)^2 - 9$$

and write

$$x^2 + 4y^2 + 2x - 12y + 3 = (x + 1)^2 + 4\left(y - \tfrac{3}{2}\right)^2 - 7.$$

With the translation

$$\begin{cases} X = x + 1 \\ Y = y - \frac{3}{2} \end{cases}$$

the equation for C reads

$$X^2 + 4Y^2 = 7 \quad \Rightarrow \quad \frac{X^2}{7} + \frac{Y^2}{7/4} = 1.$$

This is an ellipse with centre $(X = 0, Y = 0) = (x = -1, y = 3/2)$, with axes given by the lines $X = 0$ and $Y = 0$ which are $x = -1$ and $y = 3/2$, and semi-axes given by $\sqrt{7}$, $\sqrt{7}/2$.

16.4 Eccentricity: Part 1

We have a look now at the relation (16.5) for a particular class of examples. Consider the point $F = (a_x, a_y)$ in $\mathbb{E}^2$ and the line δ whose points satisfy the equation $x = u$, with $u \neq a_x$. The relation $\mathrm{d}(P, F) = e\,\mathrm{d}(P, \delta)$ (with $e > 0$) is satisfied by the points $P = (x, y)$ whose coordinates are the solutions of the equation

$$(y - a_y)^2 + (1 - e^2)x^2 + 2(ue^2 - a_x)x + a_x^2 - u^2e^2 = 0. \tag{16.28}$$

We have different cases, depending on the parameter e.

(a) We have already mentioned that for $e = 1$ we are describing the parabola with focus F and directrix δ. Its equation from (16.28) is given by

$$(y - a_y)^2 + 2(u - a_x)x + a_x^2 - u^2 = 0. \tag{16.29}$$

(b) Assume $e \neq 1$. Using the results of the Exercise 16.3.4, we complete the square and write

$$(y - a_y)^2 + (1 - e^2)x^2 + 2(ue^2 - a_x)x + a_x^2 - u^2e^2 = 0$$

$$\text{or} \quad (y - a_y)^2 + (1 - e^2)\left(x + \frac{ue^2 - a_x}{1 - e^2}\right)^2 - \frac{e^2(u - a_x)^2}{1 - e^2} = 0. \tag{16.30}$$

Then the translation given by

$$\begin{cases} Y = y - a_y \\ X = x + (ue^2 - a_x)/(1 - e^2) \end{cases}$$

allows us to write, with respect to the reference system $(O'; X, Y)$, the equation as

$$Y^2 + (1 - e^2)X^2 = \frac{e^2(u - a_x)^2}{1 - e^2}.$$

Depending on the value of e, we have the following possibilities.

(b1) If $0 < e < 1$, all the coefficients of the equation are positive, so we have the ellipse

$$\left(\frac{1 - e^2}{e(u - a_x)}\right)^2 X^2 + \frac{1 - e^2}{e^2(u - a_x)^2} Y^2 = 1.$$

An easy computation shows that its foci are given by

$$F_\pm = (\pm \frac{e^2(u - a_x)}{1 - e^2}, a_y)$$

with respect to the reference system $(O'; X, Y)$ and then clearly by

$$F_+ = (a_x, a_y), \qquad F_- = (\frac{a_x + e^2 a_x - 2ue^2}{1 - e^2}, a_y)$$

with respect to $(O; x, y)$. Notice that $F_+ = F$, the starting point.

(b2) If $e > 1$ the equation

$$\left(\frac{1 - e^2}{e(u - a_x)}\right)^2 X^2 - \frac{e^2 - 1}{e^2(u - a_x)^2} Y^2 = 1.$$

represents a hyperbola with foci again given by the points $F_\pm$ written before.

Remark 16.4.1 Notice that, if $e = 0$, the relation (16.28) becomes

$$(y - a_y)^2 + (x - a_x)^2 = 0,$$

that is a degenerate imaginary conic, with

$$(y - a_y + \mathrm{i}(x - a_x))(y - a_y - \mathrm{i}(x - a_x)) = 0.$$

If we fix $e^2(u - a_x)^2 = r^2 \neq 0$ and consider the limit $e \to 0$, the Eq. (16.28) can be written as

$$(x - a_x)^2 + (y - a_y)^2 = r^2.$$

This is another way of viewing a circle as a limiting case of a sequence of ellipses.

The case for which the point $F \in \delta$ also gives a degenerate conic. In this case $u = a_x$ and the Eq. (16.28) is

$$(y - a_x)^2 + (1 - e^2)(x - 2u)^2 = 0$$

which is the union of two lines either real (if $1 < e$) or imaginary (if $1 > e$).

16.5 Conic Sections and Kepler Motions

Via the notion of eccentricity it is easier to describe a fundamental relation between the conic sections and the so called *Keplerian* motions.

If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ describe the motion in $\mathbb{E}^3$ of two point masses m_1 and m_2, and the only force acting on them is the mutual gravitational attraction, the equations of motions are given by

$$\begin{aligned} m_1\ddot{\mathbf{x}}_1 &= -Gm_1m_2\,\frac{\mathbf{x}_1 - \mathbf{x}_2}{\|\mathbf{x}_1 - \mathbf{x}_2\|^3} \\ m_2\ddot{\mathbf{x}}_2 &= -Gm_1m_2\,\frac{\mathbf{x}_2 - \mathbf{x}_1}{\|\mathbf{x}_1 - \mathbf{x}_2\|^3}\,. \end{aligned}$$

Here G is a constant, the gravitational constant. We know from physics that the centre of mass of this system moves with no acceleration, while for the relative motion $\mathbf{r}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$ the Newton equations are

$$\mu\,\ddot{\mathbf{r}}(t) = -Gm_1m_2\,\frac{\mathbf{r}}{r^3} \tag{16.31}$$

with the norm $r = \|\mathbf{x}\|$ and $\mu = m_1m_2/(m_1 + m_2)$ the so called *reduced mass* of the system. A qualitative analysis of this motion can be given as follows.

With a cartesian orthogonal reference system $(O; x, y, z)$ in $\mathbb{E}^3$, we can write $\mathbf{r}(t) = (x(t), y(t), z(t))$ and $\dot{\mathbf{r}}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$ for the vector representing the corresponding velocity. From the Newton equations (16.31) the angular momentum (recall its definition and main properties from Sects. 1.3 and 11.2) with respect to the origin O,

$$\frac{\mathrm{d}\mathbf{L}_O}{\mathrm{d}t} = \mu\,\{\dot{\mathbf{r}} \wedge \dot{\mathbf{r}} + \mathbf{r} \wedge \ddot{\mathbf{r}}\} = 0,$$

is a constant of the motion, since $\ddot{\mathbf{r}}$ is parallel to $\mathbf{r}$ from (16.31). This means that both vectors $\mathbf{r}(t)$ and $\dot{\mathbf{r}}(t)$ remain orthogonal to the direction of $\mathbf{L}_O$, which is constant: if the initial velocity $\dot{\mathbf{r}}(t = 0)$ is not parallel to the initial position $\mathbf{r}(t = 0)$, the motion stays at any time t on the plane orthogonal to $\mathbf{L}_O(t = 0)$.

We can consider the plane of the motion as $\mathbb{E}^2$, and fix a cartesian orthogonal reference system $(O; x, y)$, so that the angular momentum conservation can be written as

$$\mu\,(\dot{x}y - \dot{y}x) = l$$

with the constant l fixed by the initial conditions. We also know that the gravitational force is conservative, thus the total energy

$$\frac{1}{2}\,\mu\,\|\dot{\mathbf{r}}\|^2 - Gm_1m_2\,\frac{1}{r} = E.$$

is also a constant of the motion. It is well known that the Eq. (16.31) can be completely solved. We omit the proof of this claim, and mention that the possible trajectories of such motions are conic sections, with focus $F = (0, 0) \in \mathbb{E}^2$ and directrix δ given by the equation $x = \tilde{l}/e$ with

$$\tilde{l} = \frac{l^2}{Gm_1m_2\mu}.$$

and eccentricity parameter given by

$$e = \sqrt{1 + \frac{2\mu El^2}{(Gm_1m_2\mu)^2}}.$$

One indeed shows that

$$\frac{2\mu El^2}{(Gm_1m_2\mu)^2} > -1$$

for any choice of initial values for position and velocity.

This result is one of the reasons why conic sections deserve a special attention in affine geometry. From the analysis of the previous section, we conclude that for $E < 0$, since $0 < e < 1$, the trajectory of the motion is elliptic. If the point mass m_2 represents the Sun, while m_1 a planet in our solar system, this result gives the well observed fact that planet orbits are plane elliptic and the Sun is one of the foci of the orbit (Kepler law).

The Sun is also the focus of hyperbolic orbits ($E > 0$) or parabolic ones ($E = 0$), orbits that are travelled by comets and asteroids.

16.6 Reduction to Canonical Form of a Conic: Rotations

Let us consider two reference systems $(O; x, y)$ and $(O; X, Y)$ having the same origin and related by a rotation by an angle of α,

$$\begin{cases} x = \cos\alpha\, X + \sin\alpha\, Y \\ y = -\sin\alpha\, X + \cos\alpha\, Y \end{cases}.$$

With respect to $(O; x, y)$, consider the parabola Γ: $y = x^2$. In the rotated system $(O; X, Y)$ the equation for Γ is easily found to be

$$-\sin\alpha\, X + \cos\alpha\, Y = (\cos\alpha\, X + \sin\alpha\, Y)^2$$

$$\Rightarrow$$

$$\cos\alpha^2\, X^2 + \sin 2\alpha\, XY + \sin\alpha^2\, Y^2 + \sin\alpha\, X - \cos\alpha\, Y = 0.$$

We see that as a consequence of the rotation, there is a mixed term XY in the quadratic polynomial equation for the parabola Γ. It is natural to wonder whether such a behaviour can be reversed.

Example 16.6.1 With respect to $(O; x, y)$, consider the conic C: $xy = k$ for a real parameter k. Clearly, for $k = 0$ this is degenerate (the union of the coordinate axes x and y). On the other hand, the rotation to the system $(O; X, Y)$ by an angle $\alpha = \frac{\pi}{4}$,

$$\begin{cases} x = \frac{1}{\sqrt{2}}(X + Y) \\ y = \frac{1}{\sqrt{2}}(X - Y) \end{cases},$$

transforms the equation of the conic to

$$X^2 - Y^2 = 2k.$$

This is a hyperbola with foci $F_{\pm} = (\pm 2\sqrt{k}, 0$ when $k > 0$ or $F_{\pm} = (0, \pm 2\sqrt{|k|})$ when $k < 0$.

In general, if the equation of a conic has a mixed term, does there exist a reference system with respect to which the equation for the given conic does not have the mixed term?

It is clear that the answer to such a question is in the affirmative if and only if there exists a reference system with respect to which the quadratic form of the conic is diagonal. On the other hand, since the quadratic form associated to a conic is symmetric, we know from the Chap. 10 that it is always possible to diagonalise it with a suitable orthogonal matrix.

Let us first study how the equation in (16.7) for a conic changes under a general change of the reference system of the affine euclidean plane we are considering.

Definition 16.6.2 With a *rotation of the plane* we mean a change in the reference system from $(O; x, y)$ to $(O; x', y')$ that is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} x' \\ y' \end{pmatrix}, \tag{16.32}$$

with $P \in \mathrm{SO}(2)$ a special orthogonal matrix, referred to as the *rotation matrix*. If we write

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix},$$

the transformation above reads

$$\begin{cases} x = p_{11}x' + p_{12}y' \\ y = p_{21}x' + p_{22}y' \end{cases} \tag{16.33}$$

These relations give the *equations of the rotation*.

A translation from the reference system $(O; x', y')$ to another $(O'; X, Y)$ is described by the relations

$$\begin{cases} x' = X + x_0 \\ y' = Y + y_0 \end{cases} \tag{16.34}$$

where $(-x_0, -y_0)$ are the coordinates of the point O with respect to $(O'; X, Y)$ and, equivalently, (x_0, y_0) are the coordinates of the point O' with respect to $(O; x', y')$.

A *proper rigid transformation* on the affine euclidean plane $\mathbb{E}^2$ is a change of the reference system given by a rotation followed by a translation. We shall refer to a proper rigid transformation also under the name of *roto-translation*.

Let us consider the composition of the rotation given by (16.33) followed by the translation given by (16.34), so to map the reference system $(O; x, y)$ into $(O'; X, Y)$. The equation describing such a transformation are easily found to be

$$\begin{cases} x = p_{11}X + p_{12}Y + a \\ y = p_{21}X + p_{22}Y + b \end{cases} \tag{16.35}$$

where

$$\begin{cases} a = p_{11}\,x_0 + p_{12}\,y_0 \\ b = p_{21}\,x_0 + p_{22}\,y_0 \end{cases}$$

are the coordinates of O' with respect to $(O; x, y)$. The transformation (16.35) can be written as

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & a \\ p_{21} & p_{22} & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}, \tag{16.36}$$

and we call

$$Q = \begin{pmatrix} p_{11} & p_{12} & a \\ p_{21} & p_{22} & b \\ 0 & 0 & 1 \end{pmatrix} \tag{16.37}$$

the matrix of (associated to) the proper rigid transformation (roto-translation).

Remark 16.6.3 A rotation matrix P is special orthogonal, that is ${}^tP = P^{-1}$ and $\det(P) = 1$. A roto-translation matrix Q as in (16.37), although satisfies the identity $\det(Q) = 1$, is not orthogonal.

Clearly, with a transposition, the action (16.32) of a rotation matrix also gives $\begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} x' & y' \end{pmatrix} {}^tP$, while the action (16.36) of a roto-translation can be written as $\begin{pmatrix} x & y & 1 \end{pmatrix} = \begin{pmatrix} X & Y & 1 \end{pmatrix} {}^tQ$.

Let us then describe how the matrices associated to the equation of a conic are transformed under a roto-translation of the reference system. Then, let us consider a conic C described, with respect to the reference system $(O; x, y)$, by

$$\begin{pmatrix} x & y & 1 \end{pmatrix} B \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0, \qquad F_C(x, y) = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Under the roto-translation transformation (16.36) the equation of the conic C with respect to the reference system $(O'; X, Y)$ is easily found to becomes

$$\begin{pmatrix} X & Y & 1 \end{pmatrix} {}^tQ\, B\, Q \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = 0.$$

Also, under the same transformations, the quadratic form for C reads

$$F_C(x', y') = \begin{pmatrix} x' & y' \end{pmatrix} {}^tP\, A\, P \begin{pmatrix} x' \\ y' \end{pmatrix}$$

with respect to the reference system $(O; x', y')$ obtained from $(O; x, y)$ under the action of only the rotation P. Such a claim is made clearer by the following proposition.

Proposition 16.6.4 *The quadratic form associated to a conic C does not change for a translation of the reference system with respect to which it is defined.*

Proof Let us consider, with respect to the reference system $(O; x', y')$, the conic with quadratic form

$$F_C(x', y') = \begin{pmatrix} x' & y' \end{pmatrix} A' \begin{pmatrix} x' \\ y' \end{pmatrix} = a_{11}\,(x')^2 + 2\,a_{12}\,x'y' + a_{22}\,(y')^2.$$

Under the translation (16.34) we have $x' = X - x_0$ e $y' = Y - y_0$, that is

$$a_{11}\,X^2 + 2\,a_{12}\,XY + a_{22}\,Y^2 + \{\text{monomials of order} \le 1\}.$$

The quadratic form associated to C, with respect to the reference system $(O'; X, Y)$, is then

$$F_C(X, Y) = a_{11}\,X^2 + 2\,a_{12}\,XY + a_{22}\,Y^2 = \begin{pmatrix} X & Y \end{pmatrix} A' \begin{pmatrix} X \\ Y \end{pmatrix},$$

with the same matrix A'. □

Given the quadratic form F_C associated to the conic C in $(O; x', y')$, we have then the following:

$$F_C(x', y') = \begin{pmatrix} x' & y' \end{pmatrix} {}^tP\, A\, P \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \Rightarrow \quad F_C(X, Y) = \begin{pmatrix} X & Y \end{pmatrix} {}^tP\, A\, P \begin{pmatrix} X \\ Y \end{pmatrix}.$$

All of the above proves the following theorem.

Theorem 16.6.5 *Let C be a conic with associated matrix of the coefficients B and matrix of the quadratic form A with respect to the reference system $(O; x, y)$. If Q is the matrix of the roto-translation mapping the reference system $(O; x, y)$ to $(O'; X, Y)$, with P the corresponding rotation matrix, the matrix of the coefficients associated to the conic C with respect to $(O'; X, Y)$ is*

$$B' = {}^tQ\ B\ Q,$$

while the matrix of the canonical form is

$$A' = {}^tP\ A\ P = P^{-1}\ A\ P.$$

In light of the Definition 13.1.4, the matrices A and A' are quadratically equivalent. □

Exercise 16.6.6 Consider the conic C whose equation, in the reference system $(O; x, y)$ is

$$x^2 - 2xy + y^2 + 4x + 4y - 1 = 0.$$

Its associated matrices are

$$B = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We first diagonalise the matrix A. Its characteristic polynomial is

$$p_A(T) = |A - TI| = \begin{vmatrix} 1-T & -1 \\ -1 & 1-T \end{vmatrix} = (1-T)^2 - 1 = T(T-2).$$

The eigenvalues are $\lambda = 0$ and $\lambda = 2$ with associated eigenspaces,

$$V_0 = \ker(f_A) = \{(x, y) \in \mathbb{R}^2 \ :\ x - y = 0\} = \mathcal{L}((1, 1)),$$
$$V_2 = \ker(f_{A-2I}) = \{(x, y) \in \mathbb{R}^2 \ :\ x + y = 0\} = \mathcal{L}((1, -1)).$$

It follows that the special orthogonal matrix P giving the change of the basis is

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

and eigenvectors ordered so that $\det(P) = 1$. This rotated the reference system to $(O; x', y')$ with

$$\begin{cases} x = \frac{1}{\sqrt{2}} (x' + y') \\ y = \frac{1}{\sqrt{2}} (-x' + y') \end{cases} .$$

Without translation, the roto-translation matrix is

$$Q' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

and from the Theorem 16.6.5, the matrix associated to C with respect to the reference system $(O; x', y')$ is $\widetilde{B} = {}^tQ' \, B \, Q'$. We have then

$$\widetilde{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & -1 \end{pmatrix},$$

so that the equation for C reads

$$2(x')^2 + 4\sqrt{2}y' - 1 = 0.$$

By completing the square at the right hand side, we write this equation as

$$(x')^2 = -2\sqrt{2} \left(y' - \frac{\sqrt{2}}{8} \right).$$

With the translation

$$\begin{cases} X = x' \\ Y = y' - \frac{\sqrt{2}}{8} \end{cases}$$

we see that C is a parabola with the canonical form

$$X^2 = -2\sqrt{2}\, Y$$

and the associated matrices

$$B' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \qquad A' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Rather than splitting the reduction to canonical form into a first step given by a rotation and a second step given by a translation, we can reduce the equation for C with respect to $(O; x, y)$ to its canonical form by a proper rigid transformation with a matrix Q encoding both a rotation and a translation. Such a composition is given by

$$\begin{cases} x = \frac{1}{\sqrt{2}}(x' + y') \\ y = \frac{1}{\sqrt{2}}(-x' + y') \end{cases} \quad \Rightarrow \quad \begin{cases} x = \frac{1}{\sqrt{2}}(X + Y + \frac{\sqrt{2}}{8}) \\ y = \frac{1}{\sqrt{2}}(-X + Y + \frac{\sqrt{2}}{8}) \end{cases}$$

which we write as

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = Q \begin{pmatrix} X \\ Y \\ 1 \end{pmatrix}$$

with

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & \sqrt{2}/8 \\ -1 & 1 & \sqrt{2}/8 \\ 0 & 0 & 1 \end{pmatrix}.$$

We end this example by checking that the matrix associated to the conic C with respect to the reference system $(O'; X, Y)$ can be computed as it is described in the Theorem 16.6.5, that is

$${}^{t}Q\, B\, Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 2\sqrt{2} & 0 \end{pmatrix} = 2B'.$$

We list the main steps of the method we described in order to reduce a conic to its canonical form as the proof of the following results.

Theorem 16.6.7 *Given a conic C whose equation is written in the reference system $(O; x, y)$, there always exists a reference system $(O'; X, Y)$, obtained with a roto-translation from $(O; x, y)$, with respect to which the equation for C is canonic.*

Proof Let C be a conic, with associated matrices A (of the quadratic form) and B (of the coefficients), with respect to the reference system $(O; x, y)$. Then,

(a) Diagonalise A, computing an orthonormal basis with eigenvectors $v_1 = (p_{11}, p_{21}),\ v_2 = (p_{12}, p_{22})$, given by the rotation

$$\begin{cases} x = p_{11}\, x' + p_{12}\, y' \\ y = p_{21}\, x' + p_{22}\, y' \end{cases} \tag{16.38}$$

and define

$$Q' = \begin{pmatrix} p_{11} & p_{12} & 0 \\ p_{21} & p_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

With respect to the reference system $(O; x', y')$, the conic C has matrix $B' = {}^tQ'\, B\, Q'$, and the corresponding quadratic equation, which we write as

$$\begin{pmatrix} x' & y' & 1 \end{pmatrix} B' \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = 0, \tag{16.39}$$

lacks the monomial term $x'y'$.

(b) Complete the square so to transform, by the corresponding translation, the reference system $(O; x', y')$ to $(O'; X, Y)$, that is

$$\begin{cases} X = x' + a \\ Y = y' + b \end{cases}. \tag{16.40}$$

From this, we can express the Eq. (16.39) for C with respect to the reference system $(O'; X, Y)$. The resulting equation is canonical for C.

(c) The equations for the roto-translation from $(O; x, y)$ to $(O'; X, Y)$ are given by substituting the translation transformation (16.40) into (16.38).

Corollary 16.6.8 *Given a degree-two polynomial equation in the variable x and y, the set (locus) of zeros of such equation is one of the following loci: ellipse, hyperbola, parabola, union of lines (either coincident or distinct).*

The proof of the Proposition 16.3.4 together with the result of the Theorem 16.6.5, which give the transformation relations for the matrices associated to a given conic C under a proper rigid transformation, allows one to prove the next proposition.

Proposition 16.6.9 *A conic C whose associated matrices are A and B with respect to a given orthonormal reference system $(O; x, y)$ is degenerate if and only if* $\det B = 0$. *Depending on the values of the determinant of A the following cases are possible*

$$\begin{array}{lcl} \det\ A < 0 & \Leftrightarrow & C \ \textit{hyperbola} \\ \det\ A = 0 & \Leftrightarrow & C \ \textit{parabola} \\ \det\ A > 0 & \Leftrightarrow & C \ \textit{ellipse}\,. \end{array}$$

The relative signs of $\det(A)$ *and* $\det B$ *determine whether the conic is real or imaginary.*

Exercise 16.6.10 As an example, we recall the results obtained in the Sect. 16.4. For the conic $\mathrm{d}(P, F) = e\,\mathrm{d}(P, \delta)$ with focus $F = (a_x, a_y)$ and directrix $\delta : x = u$, the matrix of the coefficients associated to the Eq. (16.28) is

$$B = \begin{pmatrix} 1 - e^2 & 0 & ue^2 - a_x \\ 0 & 1 & a_y \\ ue^2 - a_x & a_y & a_x^2 + a_y^2 - u^2e^2 \end{pmatrix}$$

with then $\det B = -e^2(a_x - u)^2$. We recover that the sign of $(1 - e^2)$ determines whether the conic C is an ellipse, or a parabola, or a hyperbola. We notice also that the conic is degenerate if and only if at least one of the conditions $e = 0$ or $a_x = u$ is met.

16.7 Eccentricity: Part 2

We complete now the analysis of the conics defined by the relation

$$\mathrm{d}(P, F) = e\,\mathrm{d}(P, \delta)$$

in terms of the eccentricity parameter. In Sect. 16.4 we have studied this equation with an arbitrary F and δ parallel to the y-axis, when it becomes the Eq. (16.28). In general, for a given eccentricity the previous relation depends only on the distance between F and δ. Using a suitable roto-translation as in the previous section, we have the following result.

Proposition 16.7.1 *Given a point F and a line δ in $\mathbb{E}^2$ such that $F \notin \delta$, there exists a cartesian orthogonal coordinate system $(O'; X, Y)$ with $F = O'$ and with respect to which the equation $\mathrm{d}(P, F) = e\,\mathrm{d}(P, \delta)$ (with $e > 0$) is written as*

$$Y^2 + X^2 - e^2(X - u)^2 = 0.$$

Proof Given a point F and a line $\delta \not\ni F$, it is always possible to roto-translate the starting coordinate system $(O; x, y)$ to a new one $(O'; X, Y)$ in such a way that $O' = F$ and the line δ is given by the equation $X = u \neq 0$. The result then follows from (16.28) being $a_X = a_Y = 0$. □

We know from the Sect. 16.4 that if $e = 1$, the equation represents a parabola with directrix $X = u \neq 0$ and focus $F = (0, 0)$. If $1 \neq e$, the equation represents either an ellipse $(0 < e < 1)$ or a hyperbola $(e > 1)$ with foci $F_+ = (0, 0)$ and $F_- = (-\frac{2ue^2}{1-e^2}, 0)$. Also, $e = 0$ yields the degenerate conic $X^2 + Y^2 = 0$, while $u = 0$ (that is $F \in \delta$) gives the degenerate conic $Y^2 + (1 - e^2)X^2 = 0$.

We can conclude that the Eq. (16.5) represents a conic whose type depends on the values of the eccentricity parameter. Its usefulness resides in yielding a constructive method to write the equation in canonical form, even for the degenerate cases.

We address the inverse question: given a non degenerate conic C with equation

$$a_{11}\,x^2 + 2\,a_{12}\,xy + a_{22}\,y^2 + 2\,a_{13}\,x + 2\,a_{23}\,y + a_{33} = 0$$

is it possible to determine its eccentricity and its directrix?

We give a constructive proof of the following theorem.

Theorem 16.7.2 *Given the non degenerate conic C whose equation is*

$$a_{11}\,x^2 + 2\,a_{12}\,xy + a_{22}\,y^2 + 2\,a_{13}\,x + 2\,a_{23}\,y + a_{33} = 0,$$

there exists a point F and a line δ with $F \notin \delta$ such that a point $P \in C$ if and only if

$$\mathrm{d}(P, F) \;=\; e\,\mathrm{d}(P, \delta)$$

for a suitable value $e > 0$ of the eccentricity parameter.

Proof As in the example Exercise 16.6.6 we firstly diagonalise the matrix of the quadratic form of C finding a cartesian orthogonal system $(O; x', y')$ with respect to which the equation for C is written as

$$\alpha_{11}(x')^2 + \alpha_{22}(y')^2 + 2\alpha_{13}x' + 2\alpha_{23}y' + \alpha_{33} = 0,$$

with α_{11}, α_{22} the eigenvalues of the quadratic form. This is the equation of the conic in the form studied in the Proposition 16.3.4, whose proof we now use. We have the following cases

(a) One of the eigenvalues of the quadratic form is zero, say $\alpha_{11} = 0$ (the case $\alpha_{22} = 0$ is analogous).
Up to a global (-1) factor that we can rescale, the equation for C is

$$\alpha_{22}(y')^2 + 2\alpha_{13}x' + 2\alpha_{23}y' + \alpha_{33} = 0,$$

with $\alpha_{22} > 0$ and $\alpha_{13} \neq 0$ (non degeneracy of C). Since there is no term $(x')^2$, this equation is of the form (16.28) only if $e = 1$. Thus it is of the form (16.29) written as

$$(y - a_y)^2 + 2(u - a_x)\big(x - \tfrac{1}{2}\,(u + a_x)\big) = 0.$$

The two expression are the same if and only if we have $e = 1$, and

$$a_y = -\frac{\alpha_{23}}{\alpha_{22}} \qquad \text{and} \qquad \begin{cases} u - a_x \;=\; \alpha_{13}/\alpha_{22} \\ u + a_x \;=\; (\alpha_{23}^2 - \alpha_{33}\alpha_{22})/\alpha_{13}\alpha_{22} \end{cases}.$$

These say that C is the parabola with focus and directrix given, with respect to $(O; x', y')$, by

$$F \;=\; \Big(\frac{\alpha_{23}^2 - \alpha_{33}\alpha_{22} - \alpha_{13}^2}{2\alpha_{13}\alpha_{22}}, -\frac{\alpha_{23}}{\alpha_{22}}\Big), \qquad x' \;=\; \frac{\alpha_{13}^2 + \alpha_{23}^2 - \alpha_{33}\alpha_{22}}{2\alpha_{13}\alpha_{22}}.$$

With the translation

$$\begin{cases} X = x' + (\alpha_{33}\alpha_{22} - \alpha_{23}^2)/2\alpha_{22}\alpha_{13} \\ Y = y' + \alpha_{23}/\alpha_{22} \end{cases}$$

it can indeed be written as

$$Y^2 + 2\,\frac{\alpha_{13}}{\alpha_{22}}\,X = 0.$$

If $\alpha_{22} = 0$ and $\alpha_{11} \neq 0$ the result would be similar with the x'-axis and y' axis interchanged.

(b) Assume $\alpha_{11} \neq 0$ and $\alpha_{22} \neq 0$. We write the equation for C as in (16.17),

$$\frac{\alpha_{11}}{\alpha_{22}}\left(x + \frac{\alpha_{13}}{\alpha_{11}}\right)^2 + \left(y + \frac{\alpha_{23}}{\alpha_{22}}\right)^2 - \frac{1}{\alpha_{22}}\left(-\alpha_{33} + \frac{\alpha_{13}^2}{\alpha_{11}} + \frac{\alpha_{23}^2}{\alpha_{22}}\right) = 0, \tag{16.41}$$

and compare it with (16.30)

$$(1 - e^2)\left(x + \frac{ue^2 - a_x}{1 - e^2}\right)^2 + (y - a_y)^2 - \frac{e^2(u - a_x)^2}{1 - e^2} = 0. \tag{16.42}$$

Notice that with this choice (that the directrix be parallel to the y-axis, $x = u$) we are not treating the axes x and y in an equivalent way. We would have a similar analysis when exchanging the role of the axes x and y. The conditions to satisfy are

$$\begin{cases} 1 - e^2 = \alpha_{11}/\alpha_{22} \\ a_y = -\alpha_{23}/\alpha_{22} \end{cases} \quad \text{and} \quad \begin{cases} \frac{e^2(u-a_x)^2}{1-e^2} = \frac{h}{\alpha_{22}} \\ \frac{ue^2-a_x}{1-e^2} = \frac{\alpha_{13}}{\alpha_{11}} \end{cases} \quad \text{with} \quad h = -\alpha_{33} + \frac{\alpha_{13}^2}{\alpha_{11}} + \frac{\alpha_{23}^2}{\alpha_{22}}\,. \tag{16.43}$$

We see that $h = 0$ would give a degenerate conic with either $e = 0$ or $u = a_x$, that is the focus is on the directrix. As before, up to a global (-1) factor we may assume $\alpha_{22} > 0$. And as in Sect. 16.3 we have two possibilities according to the sign of α_{11}.

(b1) The eigenvalues have the same sign: $\alpha_{22} > 0$ and $\alpha_{11} > 0$. From the first condition in (16.43) we need $\alpha_{22} > \alpha_{11}$ and we get that $e < 1$. Then the last condition requires that the parameter $h > 0$ be positive. This means that C is a real ellipse. The case $\alpha_{22} < \alpha_{11}$ also results into a real ellipse but requires that the role of the axes x and y be exchanged. (The condition $\alpha_{11} = \alpha_{22}$ would give a circle and result in $e = 0$ which we are excluding.)

(b2) The eigenvalues α_{11} and α_{22} are discordant. Now the conditions (16.43) requires $e > 1$ and the parameter h to be negative. This means that C is a hyperbola of the second type in (16.11). To get the other type in (16.11), once again one needs to exchange the axes x and y.

As mentioned, the previous analysis is valid when the directrix is parallel to the y-axis. For the case when the directrix is parallel to the x-axis (the equation $y = u$), one has a similar analysis with the relations analogous to (16.43) now written as

$$\begin{cases} 1 - e^2 = \alpha_{22}/\alpha_{11} \\ \\ a_x = -\alpha_{13}/\alpha_{11} \end{cases} \quad \text{and} \quad \begin{cases} \frac{e^2(u-a_y)^2}{1-e^2} = \frac{h}{\alpha_{11}} \\ \frac{ue^2-a_y}{1-e^2} = \frac{\alpha_{23}}{\alpha_{22}} \end{cases} \text{with} \quad h = -\alpha_{33} + \frac{\alpha_{13}^2}{\alpha_{11}} + \frac{\alpha_{23}^2}{\alpha_{22}} \ . \tag{16.44}$$

In particular for $0 < \alpha_{22} < \alpha_{22}$ these are the data of a real ellipse, while for $\alpha_{11} > 0$ and $\alpha_{22} < 0$ (and $h < 0$) this are the data for a hyperbola of the first type in (16.11). □

In all cases above, the parameters e, u, a_x, a_y are given in terms of the conic coefficients by the relations (16.43) or (16.44). Being these quite cumbersome, we omit to write the complete solutions for these relations and rather illustrate with examples the general methods we developed.

Exercise 16.7.3 Consider the hyperbolas

$$y^2 - x^2 + k = 0, \qquad k = \pm 1 \,.$$

If $k = 1$, the relations (16.43) easily give the foci

$$F_{\pm} = (\pm\sqrt{2}, 0)$$

and corresponding directrix $\delta_{\pm}$ with equation

$$x = \pm\tfrac{\sqrt{2}}{2}.$$

On the other hand, for $k = 1$, the relations (16.44) now give the foci

$$F_{\pm} = (0, \pm\sqrt{2})$$

and corresponding directrix $\delta_{\pm}$,

$$y = \pm\tfrac{\sqrt{2}}{2}.$$

Exercise 16.7.4 Consider the C of the example Exercise 16.3.7, whose equation we write as

$$x^2 + 4y^2 + 2x - 12y + 3 \;=\; (x+1)^2 + 4(y - \tfrac{3}{2})^2 - 7 = 0.$$

It is easy now to compute that this ellipse has eccentricity $e = \frac{\sqrt{3}}{4}$ and foci

$$F_{\pm} = (-1 \pm \tfrac{\sqrt{21}}{2}, \ \tfrac{3}{2}).$$

The directrix $\delta_\pm$ corresponding to the focus $F_\pm$ is given by the line

$$x = -1 \pm \tfrac{2\sqrt{21}}{3}.$$

Exercise 16.7.5 Consider the conic C with equation

$$x^2 - ky^2 - 2x - 2 = 0$$

with a parameter $k \in \mathbb{R}$. By completing the square, we write this equation as

$$(x-1)^2 - ky^2 - 3 = 0.$$

Depending on the value of k, we have different cases.

(i) If $k < -1$, it is evident that C is a real ellipse with $\alpha_{11} < \alpha_{22}$, and the condition (16.43) gives eccentricity $e = \sqrt{1 + \frac{1}{k}}$, with foci

$$F_\pm = (1 \pm \sqrt{\tfrac{3(1+k)}{k}}, 0) \qquad (16.45)$$

and corresponding directrix $\delta_\pm$ with equation

$$x = 1 \pm \sqrt{\tfrac{3}{k(1+k)}}\,. \qquad (16.46)$$

(ii) If $-1 < k < 0$ the conic C is again a real ellipse, whose major axis is parallel to the y-axis, so $\alpha_{11} > \alpha_{22}$. Now the relations (16.44) yield eccentricity $e = \sqrt{1+k}$, with foci

$$F_\pm = (1, \pm\sqrt{-3\left(1 + \tfrac{1}{k}\right)})$$

and corresponding directrix $\delta_\pm$ given by the lines with equation

$$y = \pm\sqrt{\tfrac{3}{-k(k+1)}}\,.$$

(iii) If $k = 0$ the conic C is degenerate.

(iv) If $k > 0$, the conic C is a hyperbola. It is easy to compute the eccentricity to be $e = \sqrt{1 + \frac{1}{k}}$ (the same expression as for $k < -1$), with the foci and the directrix given by (16.45) and (16.46).

The matrix of the coefficients of this conic C is given by

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -k & 0 \\ -1 & 0 & -2 \end{pmatrix},$$

with det $A = -k$ and det $B = -3k$. By the Proposition 16.6.9 we recover the listed results: C is degenerate if and only if $k = 0$; it is a hyperbola if and only if $k > 0$; an ellipse if and only if $k < 0$.

16.8 Why Conic Sections

We close the chapter by explaining where the loci on the affine euclidean plane $\mathbb{E}^2$ that we have described, the *conic* sections, get their name from. This will also be related to finding solutions to a non-linear problem in $\mathbb{E}^3$.

Fix a line γ and a point $V \in \gamma$ in $\mathbb{E}^3$. A (double) cone with axis γ and vertex V is the bundle of lines through V whose direction vectors form, with respect to γ, an angle of fixed width.

Consider now a plane $\pi \subset \mathbb{E}^3$ which does not contain the vertex of the cone. We show that, depending on the relative orientation of π with the axis of the cone, the intersection $\pi \cap \mathcal{C}$ — a *conic section* — is a non degenerate ellipse, or a parabola, or a hyperbola.

Let $(O, \mathcal{E}) = (O; x, y, z)$ be an orthonormal reference frame for $\mathbb{E}^3$, with $\mathcal{E}$ an orthonormal basis for E^3. To be definite, we take the z-axis as the axis of a cone $\mathcal{C}$, its vertex to be $V = O$ and its width an angle $0 < \theta < \pi/2$. It is immediate to see that the cone $\mathcal{C}$ is given by the points $P = (x, y, z)$ of the lines whose normalised direction vectors are

$$E^3 \ni u(\alpha) = (\sin\theta \cos\alpha, \sin\theta \sin\alpha, \cos\theta)$$

with $\alpha \in [0, 2\pi)$. The parametric equation for these lines (see the Definition 14.2.7) is then

$$r(\alpha) = \begin{cases} x = \lambda \sin\theta \cos\alpha \\ y = \lambda \sin\theta \sin\alpha \\ z = \lambda \cos\theta \end{cases} .$$

with λ a real parameter. This expression provides a vector equation for the cone $\mathcal{C}$. By eliminating the parameter, one gets a cartesian equation for $\mathcal{C}$ as given by the relation

$$\Sigma_{r(\alpha)} : \quad x^2 + y^2 - (\tan^2\theta)z^2 = 0.$$

Without loss of generality, we may intersect the cone $\mathcal{C}$ with a plane π which is orthogonal to the yz coordinate plane and meeting the z axis at the point $A = (0, 0, k > 0)$. If $\beta \in (0, \pi/2)$ is the angle between the axis of the cone (the z axis) and (its projection on) the plane π, the direction S_π of the plane is orthogonal to the normalised vector $v = (0, \cos\beta, \sin\beta)$. We know from Chap. 15 that the cartesian equation for the plane π is then

$$\Sigma_\pi : \quad (\cos\beta)y + (\sin\beta)(z - k) = 0.$$

The intersection $\mathcal{C} \cap \pi$ is then given by the solution of the system

$$\begin{cases} x^2 + y^2 - (\tan^2\theta)z^2 = 0 \\ (\cos\beta)y + (\sin\beta)(z-k) = 0 \end{cases} . \tag{16.47}$$

This is the only problem in this textbook which is formulated in terms of a system of *non-linear* equations. By inserting the second equation in the first one, elementary algebra gives, for the projection on the plane xy of the intersection $\mathcal{C} \cap \pi$, the equation,

$$x^2 + (1 - \tan^2\theta \cot^2\beta)\, y^2 + 2k\tan^2\theta \cot\beta\, y - k^2\tan^2\theta. \tag{16.48}$$

From what we have described above in this chapter, this equation represents a conic.

Its matrix of the coefficients is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \tan^2\theta \cot^2\beta & k\tan^2\theta\cot\beta \\ 0 & k\tan^2\theta\cot\beta & -k^2\tan^2\theta \end{pmatrix},$$

while the matrix of the quadratic form is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \tan^2\theta\cot^2\beta \end{pmatrix}.$$

One then computes

$$\det(A) = 1 - \tan^2\theta\cot^2\beta, \qquad \det B = -k^2\tan^2\theta\,.$$

Having excluded the cases $k = 0$ and $\tan\theta = 0$, we know from the Proposition 16.6.9 that the intersection $\mathcal{C} \cap \pi$ represents a non degenerate real conic. Some algebra indeed shows that:

$$\begin{aligned} \det(A) > 0 \quad &\Leftrightarrow \quad \tan^2\beta > \tan^2\theta \quad \Leftrightarrow \quad \beta > \theta, \\ \det(A) = 0 \quad &\Leftrightarrow \quad \tan^2\beta = \tan^2\theta \quad \Leftrightarrow \quad \beta = \theta, \\ \det(A) < 0 \quad &\Leftrightarrow \quad \tan^2\beta < \tan^2\theta \quad \Leftrightarrow \quad \beta < \theta, \end{aligned} \tag{16.49}$$

thus giving an ellipse, a parabola, a hyperbola respectively. These are shown in Figs. 16.4 and 16.5.

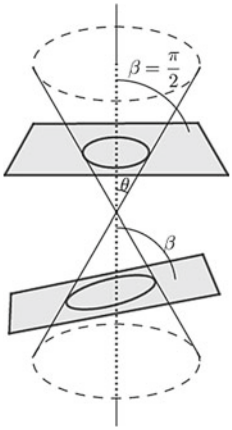

Fig. 16.4 The ellipse with the limit case of the circle

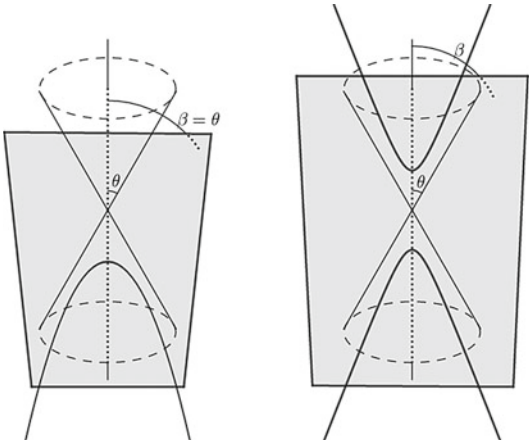

Fig. 16.5 The parabola and the hyperbola

Remark 16.8.1 As a particular case, if we take $\beta = \frac{\pi}{2}$, from (16.48) we see that $\mathcal{C} \cap \pi$ is a circle with radius $R = k \tan\theta$. On the other hand, with $k = 0$, that is π contains the vertex of the cone, one has $\det B = 0$. In such a case, the (projected) Eq. (16.48) reduces to

$$x^2 + (1 - \tan^2\theta \cot^2\beta)\, y^2 = 0.$$

Such equation represents:

2a. the union of two complex conjugate lines for $\beta > \theta$,
2b. the points $(x = 0, y)$, that is the y-axis for $\beta = \theta$,
2c. the union of two real lines for $\beta < \theta$.

We conclude giving a more transparent, in a sense, description of the intersection $\mathcal{C} \cap \pi$ by using a new reference system $(O; x', y', z')$, by a rotation around the x-axis where the plane π is orthogonal to the axis z'-axis. adapted to π. From Chap. 11, the transformation we consider is given in terms of the matrix in SO(3),

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\beta & -\cos\beta \\ 0 & \cos\beta & \sin\beta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

With respect to the new reference system, the system of Eq. (16.47) becomes

$$\begin{cases} (x')^2 + \big((\sin\beta)y' + (\cos\beta)z'\big)^2 - (\tan^2\theta)\big((\sin\beta)z' - (\cos\beta)y'\big)^2 = 0 \\ z' - k\,\sin\beta = 0 \end{cases}.$$

It is then easy to see that the solutions of this system of equations are the points having coordinates $z' = k\,\sin\beta$ and (x', y') satisfying the equation

$$(x')^2 + (\sin^2\beta - \tan^2\theta\,\cos^2\beta)(y')^2 + 2k\cos\beta\,\sin^2\beta(1+\tan^2\theta)y' + (\cos^2\beta - \tan^2\theta\sin^2\rho)\,k^2\sin^2\beta = 0. \tag{16.50}$$

Clearly, this equation represents a conic on the plane $z' = k\,\sin\beta$ with respect to the orthonormal reference system $(O; x', y')$. Its matrix of the coefficients is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\beta(1 - \tan^2\theta\,\cot^2\beta) & k\,\cos\beta\,\sin^2\beta(1+\tan^2\theta) \\ 0 & k\,\cos\beta\,\sin^2\beta(1+\tan^2\theta) & k^2\sin^2\beta\cos^2\beta(1 - \tan^2\theta\tan^2\beta) \end{pmatrix},$$

while the matrix of the quadratic form is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\beta(1 - \tan^2\theta\,\cot^2\beta) \end{pmatrix}.$$

One then computes

$$\det(A) = \sin^2\beta(1 - \tan^2\theta\,\cot^2\beta), \qquad \det B = -k^2\,\sin^2\beta\,\tan^2\theta\,.$$

With $k \neq 0$ and $\tan\theta \neq 0$, clearly also in this case the relations (16.49) are valid. And as particular cases, if we take $\beta = \pi/2$, one has that $\mathcal{C} \cap \pi$ is a circle with radius $R = k\,\tan\theta$. On the other hand, for $k = 0$, (that is π contains the vertex of the cone) so that $\det B = 0$, the Eq. (16.50) reduces to

$$(x')^2 + (\sin^2\beta - \tan^2\theta\,\cos^2\beta)(y')^2 = 0.$$

Such equation as before represents: the union of two complex conjugate lines for $\beta > \theta$; the points $x' = 0$ for $\beta = \theta$; the union of two real lines for $\beta < \theta$.

Remark 16.8.2 We remark that both Eqs. (16.48) and (16.50) describe the same type of conic, depending on the relative width of the angles β and θ. What differs is their eccentricity. The content of the Sect. 16.7 allows us to compute that the eccentricity of the conic in (16.48) is $e^2 = \tan^2\theta\cot^2\beta$, while for the conic in (16.50) we have $e^2 = (1 + \tan^2\theta)\cos^2\beta$.