Chapter 8
Dual Spaces

8.1 The Dual of a Vector Space

Let us consider two finite dimensional real vector spaces V and W, and denote by $\mathrm{Lin}(V \to W)$ the collection of all linear maps $f : V \to W$. It is easy to show that $\mathrm{Lin}(V \to W)$ is itself a vector space over $\mathbb{R}$. This is with respect to a sum $(f_1 + f_2)$ and a product by a scalar (λf), for any $f_1, f_2, f \in \mathrm{Lin}(V \to W)$ and $\lambda \in \mathbb{R}$, defined *pointwise*, that is by

$$\begin{aligned}(f_1 + f_2)(v) &= f_1(v) + f_2(v)\\ (\lambda f)(v) &= \lambda f(v)\end{aligned}$$

for any $v \in V$. If $\mathcal{B}$ is a basis for V (of dimension n) and $\mathcal{C}$ a basis for W (of dimension m), the map $\mathrm{Lin}(V \to W) \to \mathbb{R}^{m,n}$ given by

$$f \quad \mapsto \quad M_f^{\mathcal{C},\mathcal{B}}$$

is an isomorphism of real vector spaces and the following relations

$$\begin{aligned}M_{f_1+f_2}^{\mathcal{C},\mathcal{B}} &= M_{f_1}^{\mathcal{C},\mathcal{B}} + M_{f_2}^{\mathcal{C},\mathcal{B}}\\ M_{\lambda f}^{\mathcal{C},\mathcal{B}} &= \lambda M_f^{\mathcal{C},\mathcal{B}}\end{aligned} \tag{8.1}$$

hold (see the Proposition 4.1.4). It is then clear that $\dim(\mathrm{Lin}(V \to W)) = mn$.

In particular, the vector space of linear maps from a vector space V to $\mathbb{R}$, that is the set of *linear forms* on V, deserves a name of its own.

Definition 8.1.1 Given a finite dimensional vector space V, the space of linear maps $\mathrm{Lin}(V \to \mathbb{R})$ is called the *dual space* to V and is denoted by $V^* = \mathrm{Lin}(V \to \mathbb{R})$.

The next result follows from the general discussion above.

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Proposition 8.1.2 *Given a finite dimensional real vector space V, its dual space V^* is a real vector space with* $\dim(V^*) = \dim(V)$.

Let $\mathcal{B} = (b_1, \ldots, b_n)$ be a basis for V. We define elements $\{\varphi_i\}_{i=1,\ldots,n}$ in V^* by

$$\varphi_i(b_j) = \delta_{ij} \quad \text{with} \quad \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,. \end{cases} \tag{8.2}$$

With $V \ni v = x_1 b_1 + \ldots + x_n b_n$, we have for the components that $x_i = \varphi_i(v)$. If $f \in V^*$ we write

$$\begin{aligned} f(v) &= f(b_1)\,x_1 + \ldots + f(b_n)\,x_n \\ &= f(b_1)\,\varphi_1(v) + \ldots + f(b_n)\,\varphi_n(v) \\ &= \big(f(b_1)\varphi_1 + \ldots + f(b_n)\varphi_n\big)(v). \end{aligned}$$

This shows that the action of f upon the vector v is the same as the action on v of the linear map $f = f(b_1)\varphi_1 + \ldots + f(b_n)\varphi_n$, that is we have that $V^* = \mathcal{L}(\varphi_1, \ldots, \varphi_n)$. It is indeed immediate to prove that, with respect to the linear structure in V^*, the linear maps φ_i are linearly independent, so they provide a basis for V^*. We have then sketched the proof of the following proposition.

Proposition 8.1.3 *Given a basis $\mathcal{B}$ for a n-dimensional real vector space V, the elements φ_i defined in (8.2) provide a basis for V^*. Such a basis, denoted $\mathcal{B}^*$, is called the* dual basis *to $\mathcal{B}$.*

We can also write

$$f(v) = (f(b_1)\ \ldots\ f(b_n)) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{8.3}$$

Referring to the Definition 7.1.12 (and implicitly fixing a basis for $W \cong \mathbb{R}$), the relation (8.3) provides us the single row matrix $M_f^{\mathcal{B}} = (f(b_1)\ \ldots\ f(b_n))$ associated to f with respect to the basis $\mathcal{B}$ for V. Its entries are the image under f of the basis elements in $\mathcal{B}$. The proof of the proposition above shows that such entries are the components of $f \in V^*$ with respect to the dual basis $\mathcal{B}^*$.

Let $\mathcal{B}'$ be another basis for V, with elements $\{b'_i\}_{i=1,\ldots,n}$. With

$$v = x_1 b_1 + \ldots + x_n b_n = x'_1 b'_1 + \ldots + x'_n b'_n$$

we can write, following the Definition 7.9.3,

$$x'_k = \sum_{s=1}^{n} (M^{\mathcal{B}',\mathcal{B}})_{ks} x_s, \qquad b_i = \sum_{j=1}^{n} (M^{\mathcal{B}',\mathcal{B}})_{ji} b'_j$$

or, in a matrix notation,

$$\begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix} = M^{\mathcal{B}',\mathcal{B}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad (b_1 \dots b_n)\, M^{\mathcal{B},\mathcal{B}'} = (b'_1 \dots b'_n). \tag{8.4}$$

From the Theorem 7.9.9 we have the matrix associated to f with respect to $\mathcal{B}'$

$$M_f^{\mathcal{B}'} = M_f^{\mathcal{B}} M^{\mathcal{B},\mathcal{B}'},$$

which we write as

$$(f(b_1) \dots f(b_n))\, M^{\mathcal{B},\mathcal{B}'} = (f(b'_1) \dots f(b'_n)). \tag{8.5}$$

Since the entries of $M_f^{\mathcal{B}'}$ provide the components of the element $f \in V^*$ with respect to the basis $\mathcal{B}'^*$, a comparison between (8.4) and (8.5) shows that, under a change of basis $\mathcal{B} \mapsto \mathcal{B}'$ for V and the corresponding change of the dual basis in V^*, the components of a vector in V^* are transformed under a map which is the *inverse* of the map that transforms the components of a vector in V.

The above is usually referred to by saying that the transformation law for vectors in V^* is *contravariant* with respect to the *covariant* one for vectors in V. In Sect. 13.3 we shall describe these facts with an important physical example, the study of the electromagnetic field.

If we express $f \in V^*$ with respect to the dual bases $\mathcal{B}^*$ and $\mathcal{B}'^*$ as

$$f(v) = \sum_{i=1}^{n} f(b_i)\varphi_n = \sum_{k=1}^{n} f(v'_k)\varphi'_k$$

and consider the rule for the change of basis, we have

$$\sum_{k,i=1}^{n} (M^{\mathcal{B}',\mathcal{B}})_{ki}\, f(b'_k)\varphi_i = \sum_{k=1}^{n} f(v'_k)\varphi'_k.$$

Since this must be valid for any $f \in V^*$, we can write the transformation law $\mathcal{B}^* \mapsto \mathcal{B}'^*$:

$$\varphi'_k = \sum_{i=1}^{n} (M^{\mathcal{B}',\mathcal{B}})_{ki}\varphi_i \qquad \text{that is} \qquad \begin{pmatrix} \varphi'_1 \\ \vdots \\ \varphi'_n \end{pmatrix} = M^{\mathcal{B}',\mathcal{B}} \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}.$$

It is straightforward to extend to the complex case, *mutatis mutandis*, all the results of the present chapter given above. In particular, one has the following natural definition.

Definition 8.1.4 Let V be a finite dimensional complex vector space. The set $V^* = \mathrm{Lin}(V \to \mathbb{C})$ is called the *dual* space to V.

Indeed, the space V^* is a complex vector space, with $\dim(V^*) = \dim(V)$, and a natural extension of (8.2) to the complex case allows one to introduce a dual basis $\mathcal{B}^*$ for any basis $\mathcal{B}$ of V.

Also, we could consider linear maps between finite dimensional *complex* vector spaces. In the next section we shall explicitly consider linear transformations of the complex vector space $\mathbb{C}^n$.

8.2 The Dirac's Bra-Ket Formalism

Referring to Sect. 3.4 let us denote by $H^n = (\mathbb{C}^n, \cdot)$ the canonical hermitian vector space. Following Dirac (and by now a standard practice in textbooks on quantum mechanics), the hermitian product is denoted as

$$\langle \, | \, \rangle \, : \, \mathbb{C}^n \times \mathbb{C}^n \, \to \, \mathbb{C}', \qquad \langle z|w\rangle = \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n,$$

for any $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. Thus its properties (see the Proposition 3.4.2) are written as follows. For any $z, w, v \in \mathbb{C}^n$ and $a, b \in \mathbb{C}$,

(i) $\langle w|z\rangle = \overline{\langle z|w\rangle}$,
(ii) $\langle az + bw|v\rangle = \bar{a}\langle z|v\rangle + \bar{b}\langle w|v\rangle$ while $\langle v|az + bw\rangle = a\langle v|z\rangle + b\langle v|w\rangle$,
(iii) $\langle z|z\rangle \geq 0$,
(iv) $\langle z|z\rangle = 0 \quad \Leftrightarrow \quad z = (0, \ldots, 0) \in \mathbb{C}^n$.

Since the hermitian product is bilinear (for the sum), for any fixed $w \in H^n$, the mapping

$$f_w \, : \, v \, \mapsto \, \langle w|z\rangle$$

provides indeed a linear map from $\mathbb{C}^n$ to $\mathbb{C}$, that is f_w is an element of the dual space $(\mathbb{C}^n)^*$. Given a hermitian basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ for H^n, with $w = (w_1, \ldots, w_n)_{\mathcal{B}}$ and $z = (z_1, \ldots, z_n)_{\mathcal{B}}$, one has

$$f_w(z) \; = \; \bar{w}_1 z_1 + \ldots + \bar{w}_n z_n.$$

The corresponding dual basis $\mathcal{B}^* = \{\varepsilon_1, \ldots, \varepsilon_n\}$ for $(\mathbb{C}^n)^*$ is defined in analogy to (8.2) for the real case by taking $\varepsilon_i(e_j) = \delta_{ij}$. In terms of the hermitian product, these linear maps can be defined as $\varepsilon_i(z) \; = \; \langle e_i|z\rangle$. Then, to any $w = (w_1, \ldots, w_n)_{\mathcal{B}}$ we can associate an element $f_w = \bar{w}_1\varepsilon_1 + \ldots + \bar{w}_n\varepsilon_n$ in $(\mathbb{C}^n)^*$, whose action on $\mathbb{C}^n$ can be written as

$$f_w(v) = \langle w|v\rangle.$$

Thus, *via the hermitian product*, to any vector $w \in \mathbb{C}^n$ one associates a unique dual element $f_w \in (\mathbb{C}^n)^*$; viceversa, to any element $f \in (\mathbb{C}^n)^*$ one associates a unique element $w \in \mathbb{C}^n$ in such a way that $f = f_w$:

$$w = w_1 e_1 + \ldots + w_n e_n \quad \leftrightarrow \quad f_w = \bar{w}_1 \varepsilon_1 + \ldots + \bar{w}_n \varepsilon_n.$$

Remark 8.2.1 Notice that this bijection between $\mathbb{C}^n$ and $(\mathbb{C}^n)^*$ is anti-linear (for the product by complex numbers), since we have to complex conjugate the components of the vectors in order to satisfy the defining requirement of the hermitian product in H^n, that is

$$f_{\lambda w} = \bar{\lambda} f_w, \qquad \text{for} \quad \lambda \in \mathbb{C},\, w \in \mathbb{C}^n.$$

For the canonical euclidean space E^n one could proceed in a similar manner and in such a case the bijection between E^n and its dual $(E^n)^*$ given by the euclidean product is linear.

Given the bijection above, Dirac's idea was to *split* the hermitian product bracket. Any element $w \in H^n$ provides a *ket* element $|w\rangle$ and a *bra* element $\langle w| \in (\mathbb{C}^n)^*$. A basis for H^n is then written as made of elements $|e_j\rangle$ while the bra elements $\langle e_j|$ form the dual basis for $(\mathbb{C}^n)^*$, with

$$\begin{aligned} w = w_1 e_1 + \ldots + w_n e_n \quad &\leftrightarrow \quad |w\rangle = w_1 |e_1\rangle + \ldots + w_n |e_n\rangle, \\ f_w = \bar{w}_1 \varepsilon_1 + \ldots + \bar{w}_n \varepsilon_n \quad &\leftrightarrow \quad \langle w| = \bar{w}_1 \langle e_1| + \ldots + \bar{w}_n \langle e_n| \,. \end{aligned}$$

The action of a bra element on a ket element is just given as a *bra-ket juxtapposition*, with

$$f_w(z) = \langle w|z\rangle \in \mathbb{C}.$$

We are now indeed allowed to define a *ket-bra juxtaposition*, that is we have elements $T = |z\rangle\langle w|$. The action of such a T from the left upon a $|u\rangle$, is then defined as

$$T \,:\, |u\rangle \;\mapsto\; |z\rangle\langle w|u\rangle.$$

Since $\langle w|u\rangle$ is a complex number, we see that for this action the element T maps a ket vector *linearly* into a ket vector, so T is a linear map from H^n to H^n.

Definition 8.2.2 With $z, w \in H^n$, the ket-bra element $T = |z\rangle\langle w|$ is the linear operator whose action is defined as $v \mapsto T(v) = \langle w|v\rangle z = (w \cdot v) z$.

It is then natural to consider linear combination of the form $T = \sum_{k,s=1}^{n} T_{ks} |e_k\rangle\langle e_s|$ with $T_{ks} \in \mathbb{C}$ the entries of a matrix $T \in \mathbb{C}^{n,n}$ so to compute

$$\begin{aligned} T|e_j\rangle &= \sum_{k,s=1}^{n} T_{ks} |e_k\rangle\langle e_s|e_j\rangle = \sum_{k=1}^{n} T_{kj} |e_k\rangle \\ T_{kj} &= \langle e_k|T(e_j)\rangle. \end{aligned} \tag{8.6}$$

In order to relate this formalism to the one we have already developed in this chapter, consider a linear map $\phi : H^n \to H^n$ and its associated matrix $M_\phi^{\mathcal{B},\mathcal{B}} = (a_{ks})$ with respect to a given hermitian basis $\mathcal{B} = (e_1, \ldots, e_n)$. From the Propositions 7.1.13 and 7.1.14 it is easy to show that one has

$$a_{kj} = e_k \cdot (A(e_j)) = \langle e_k | A(e_j) \rangle. \tag{8.7}$$

The analogy between (8.6) and (8.7) shows that, for a fixed basis of H^n, the action of a linear map ϕ with associated matrix $A = M_\phi^{\mathcal{B},\mathcal{B}} = (a_{ks})$ is equivalently written as the action of the operator

$$T_A (= T_\phi) = \sum_{k,s=1}^{n} a_{ks} |e_k\rangle\langle e_s|$$

in the Dirac's notation. The association $A \to T_A$ is indeed an isomorphism of (complex) vector space of dimension n^2.

Next, let ϕ, ψ be two linear maps on H^n with associated matrices A, B with respect to the hermitian basis $\mathcal{B}$. They correspond to the operators that we write as $T_A = \sum_{r,s=1}^{n} a_{rs} |e_r\rangle\langle e_s|$ and $T_B = \sum_{j,k=1}^{n} b_{jk} |e_j\rangle\langle e_k|$. With a natural juxtaposition we write the composition of the linear maps as

$$\begin{aligned}\phi \circ \psi &= \sum_{r,s=1}^{n} \sum_{j,k=1}^{n} a_{rs} b_{jk} |e_r\rangle\langle e_s | e_j\rangle\langle e_k| \\ &= \sum_{r,k=1}^{n} (\sum_{j=1}^{n} a_{rj} b_{jk}) |e_r\rangle\langle e_k| \,.\end{aligned}$$

We see that the matrix associated, via the isomorphism $A \to T_A$ above, to the composition $\phi \circ \psi$ has entries (r, k) given by $\sum_{j=1}^{n} a_{rj} b_{jk}$, thus coinciding with the row by column product between the matrices A and B associated to ϕ and ψ, that is

$$T_{AB} = T_A T_B.$$

Thus, the Proposition 7.8.3 for composition of matrices associated to linear maps is valid when we represent linear maps on H^n using the Dirac's notation.

All of this section has clearly a real version and could be repeated for the (real) euclidean space E^n with its linear maps and associated real matrices $T \in \mathbb{R}^{n,n}$.