Chapter 15
Euclidean Affine Linear Geometry

15.1 Euclidean Affine Spaces

In the previous chapter we have dealt with the (real and linear) affine space $\mathbb{A}^n$ as modelled on the vector space $\mathbb{R}^n$. In this chapter we study the additional structures on $\mathbb{A}^n$ that come when passing from $\mathbb{R}^n$ to the euclidean space E^n (see the Chap. 3). Taking into account the scalar product allows one to introduce metric notions (such as distances and angles) into an affine space.

Definition 15.1.1 The affine space $\mathbb{A}^n$ associated to the Euclidean vector space $E^n = (\mathbb{R}^n, \cdot)$ is called the *Euclidean affine space* and denoted $\mathbb{E}^n$. A reference system $(O, \mathcal{B})$ for $\mathbb{E}^n$ is called *cartesian orthogonal* if the basis $\mathcal{B}$ for E^n is orthonormal.

Recall that, if $\mathcal{B}$ is an orthonormal basis for E^n, the matrix of change of basis $M^{\mathcal{E},\mathcal{B}}$ (the matrix whose column vectors are the components of the vectors in $\mathcal{B}$ with respect to the canonical basis $\mathcal{E}$) is orthogonal by definition (see the Chap. 10, Definition 10.1.1), and thus $\det(M^{\mathcal{E},\mathcal{B}}) = \pm 1$.

In our analysis in this chapter we shall always consider cartesian orthogonal reference systems.

Exercise 15.1.2 Let r be the (straight) line in $\mathbb{E}^2$ with vector equation

$$(x, y) = (1, -2) + \lambda(1, -1).$$

We take $A = (1, -2)$ and $v = (1, -1)$. To determine a cartesian equation for r, in alternative to the procedure described at length in the previous chapter (that is removing the parameter λ), one observes that, since $\mathcal{L}(v)$ is the direction of r, and thus the vector $u = (1, 1)$ is orthogonal to v, we can write

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$$\begin{aligned} P = (x, y) \in r \quad &\iff \quad P - A \in \mathcal{L}(v) \\ &\iff \quad (P - A) \cdot u = 0 \\ &\iff \quad (x - 1, y + 2) \cdot (1, 1) = 0. \end{aligned}$$

This condition can be written as

$$x + y + 1 = 0,$$

yielding a cartesian equation Σ_r for r.

This exercise shows that, if r is a line in $\mathbb{E}^2$ whose vector equation is $r : P = A + \lambda v$, with u a vector orthogonal to v so that for the direction of r one has $S_r = \mathcal{L}(u)^\perp$, we have

$$P \in r \quad \iff \quad (P - A) \cdot u = 0.$$

This expression is called the *normal equation* for the line r.

We can generalise this example to any hyperplane.

Proposition 15.1.3 *Let $H \subset \mathbb{E}^n$ be a hyperplane, with $A \in H$. If $u \in \mathbb{R}^n$ is a non zero vector orthogonal to the direction S_H of the hyperplane, that is $\mathcal{L}(u) = (S_H)^\perp$, then it holds that*

$$P \in H \quad \iff \quad (P - A) \cdot u = 0.$$

Definition 15.1.4 The equation

$$\mathcal{N}_H : \quad (P - A) \cdot u = 0$$

is called the *normal equation of the hyperplane H* in $\mathbb{E}^n$. If $n = 2$, it yields the normal equation of a line; if $n = 3$, it yields the normal equation of a plane.

Remark 15.1.5 Notice that, as we already seen for a cartesian equation in the previous chapter (see the Remark 14.4.11), the normal equation $\mathcal{N}_H$ for a given hyperplane in $\mathbb{E}^n$ is not uniquely determined, since A can range in H and the vector u is given up to an arbitrary non zero scalar.

Remark 15.1.6 With a cartesian equation

$$\Sigma_H : \quad a_1 x_1 + \cdots + a_n x_n = b$$

for the hyperplane for H in $\mathbb{E}^n$, one has $S_H^\perp = \mathcal{L}((a_1, \ldots, a_n))$. This follows from the definition

$$\begin{aligned} S_H &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ a_1 x_1 + \cdots + a_n x_n = 0\} \\ &= \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ (a_1, \ldots, a_n) \cdot (x_1, \ldots, x_n) = 0\}. \end{aligned}$$

With A an arbitrary point in H, a normal equation for H is indeed given by

$$\mathcal{N}_H : \quad (P - A) \cdot (a_1, \ldots, a_n) = 0.$$

Exercise 15.1.7 We determine both a cartesian and a normal equation for the plane π in $\mathbb{A}^3$ whose direction is orthogonal to $u = (1, 2, 3)$ and that contains the point $A = (1, 0, -1)$. We have

$$\mathcal{N}_\pi : \quad (x - 1, y, z + 1) \cdot (1, 2, 3) = 0,$$

equivalent to the cartesian equation

$$\Sigma_\pi : \quad x + 2y + 3z + 2 = 0.$$

Exercise 15.1.8 Given the (straight) line r in $\mathbb{A}^2$ with cartesian equation

$$\Sigma_r : \quad 2x - 3y + 3 = 0$$

we look for its normal equation. We start by noticing (see the Remark 15.1.6) that the direction of r is orthogonal to the vector $u = (2, -3)$, and that the point $A = (0, 1)$ lays in r, so we can write

$$\mathcal{N}_r : \quad (P - (0, 1)) \cdot (2, -3) = 0 \quad \Leftrightarrow \quad (x, y - 1) \cdot (2, -3) = 0$$

as a normal equation for r.

From what discussed above, it is clear that there exist deep relations between cartesian and normal equations for an hyperplane in a Euclidean affine space. Moreover, as we have discussed in the previous chapter, a generic linear affine variety in $\mathbb{A}^n$ can be described as a suitable intersection of hyperplanes. Therefore it should come as no surprise that a linear affine variety can be described in a Euclidean affine space in terms of a suitable number of normal equations. The general case is illustrated by the following exercise.

Exercise 15.1.9 Let r be the line through the point $A = (1, 2, -3)$ in $\mathbb{E}^3$ which is orthogonal to the space $\mathcal{L}((1, 1, 0), (0, 1, -1))$. Its normal equation is given by

$$\mathcal{N}_r : \quad \begin{cases} (P - A) \cdot (1, 1, 0) = 0 \\ (P - A) \cdot (0, 1, -1) = 0 \end{cases},$$

that is

$$\mathcal{N}_r : \quad \begin{cases} (x - 1, y - 2, z + 3) \cdot (1, 1, 0) = 0 \\ (x - 1, y - 2, z + 3) \cdot (0, 1, -1) = 0 \end{cases}$$

yielding then the cartesian equation

$$\Sigma_r : \quad \begin{cases} x + y - 3 = 0 \\ y - z - 5 = 0 \end{cases} .$$

15.2 Orthogonality Between Linear Affine Varieties

In the Euclidean affine space $\mathbb{E}^n$ there is the notion of orthogonality. Thus, we have:

Definition 15.2.1 One says that

(a) the lines $r, r' \subset \mathbb{E}^n$ are *orthogonal* if $v \cdot v' = 0$ for any $v \in S_r$ and any $v' \in S_{r'}$,
(b) the planes $\pi, \pi' \subset \mathbb{E}^3$ are *orthogonal* if $u \cdot u' = 0$ for any $u \in S_\pi^\perp$ and any $u' \in S_{\pi'}^\perp$,
(c) the line r with direction v is *orthogonal* to the plane π in $\mathbb{E}^3$ if $v \in S_\pi^\perp$.

Exercise 15.2.2 We consider the following lines in $\mathbb{E}^2$,

$$\begin{aligned} \Sigma_{r_1} &: \quad 2x - 2y + 1 = 0, \\ \Sigma_{r_2} &: \quad x + y + 3 = 0, \\ r_3 &: \quad (x, y) = (1, -3) + \lambda(1, 1), \\ \mathcal{N}_{r_4} &: \quad (x + 1, y - 4) \cdot (1, 2) = 0 \end{aligned}$$

with directions spanned by the vectors

$$\begin{aligned} v_1 &= (2, 2), \\ v_2 &= (1, -1), \\ v_3 &= (1, 1), \\ v_4 &= (1, -2). \end{aligned}$$

It is immediate to show that the only orthogonal pairs of lines among them are $r_1 \perp r_2$ and $r_2 \perp r_3$.

Exercise 15.2.3 Consider the lines $r, r' \subset \mathbb{E}^3$ given by

$$r : \quad (x, y, z) = (1, 2, 1) + \lambda(3, 0, -1) , \qquad r' : \quad \begin{cases} x = 3 + \mu \\ y = 2 - 2\mu \\ z = 3\mu \end{cases} .$$

We have $S_r = \mathcal{L}((3, 0, -1))$ and $S_{r'} = \mathcal{L}((1, -2, 3))$. Since

$$(3, 0, -1) \cdot (1, -2, 3) = 0$$

we conclude that r is orthogonal to r'.

Exercise 15.2.4 Let π be the plane in $\mathbb{E}^3$ whose cartesian equation is

$$\Sigma_\pi : \quad x - y + 2z - 3 = 0.$$

In order to find an equation for the line r through $A = (1, 2, 1)$ which is orthogonal to π we notice from the Remark 15.1.6, that it is $S_\pi^\perp = \mathcal{L}((1, -1, 2))$: we can then write

$$r : \quad (x, y, z) = (1, 2, 1) + \lambda(1, -1, 2).$$

Exercise 15.2.5 Consider in $\mathbb{E}^3$ the line given by

$$\Sigma_r : \quad \begin{cases} x - 2y + z - 1 = 0 \\ x + y = 0 \end{cases}.$$

We seek to determine:

(1) a cartesian equation for the plane π through the point $A = (-1, -1, -1)$ and orthogonal to r,
(2) the intersection between r and π.

We proceed as follows.

(1) From the cartesian equation Σ_r we have that

$$S_r^\perp = \mathcal{L}((1, -2, 1), (1, 1, 0))$$

and this subspace yields the direction S_π. Since $A \in \pi$, a vector equation for π is given by

$$\pi : \quad (x, y, z) = -(1, 1, 1) + \lambda(1, -2, 1) + \mu(1, 1, 0).$$

By noticing that $S_\pi = \mathcal{L}((1, -1, -3))$, a normal equation for π is given by

$$\mathcal{N}_\pi : \quad (P - A) \cdot (1, -1, -3) = 0$$

yielding the cartesian equation

$$\Sigma_\pi : \quad x - y - 3z - 3 = 0.$$

(2) The intersection $\pi \cap r$ is clearly given by the unique solution of the linear system

$$\Sigma_{\pi \cap r} : \quad \begin{cases} x - 2y + z - 1 = 0 \\ x + y = 0 \\ x - y - 3z - 3 = 0, \end{cases},$$

which is $P = \frac{1}{11}(6, -6, -7)$.

Exercise 15.2.6 We consider again the lines r and r' in $\mathbb{E}^3$ from the Exercise 15.2.3. We know that $r \perp r'$. We determine the plane π which is orthogonal to r' and such that $r \subset \pi$. Since $\mathcal{L}((1,-2,3))$ is the direction of r', we can write from the Remark 15.1.6 that

$$\Sigma_\pi : \quad x - 2y + 3z + d = 0$$

with d a real parameter. The line r is in π if and only if the coordinates of every of its points $P = (1+3\lambda, 2, 1-\lambda) \in r$ solve the equation Σ_π, that is, if and only if the equation

$$(1+3\lambda) - 2(2) + 3(1-\lambda) + d = 0$$

has a solution for each value of λ. This is true if and only if $d = 0$, so a cartesian equation for π is

$$\Sigma_\pi : \quad x - 2y + 3z = 0.$$

Exercise 15.2.7 For the planes

$$\Sigma_\pi : \quad 2x + y - z - 3 = 0, \qquad \Sigma_{\pi'} : \quad x + y + 3z - 1 = 0$$

in $\mathbb{E}^3$ we have $S_\pi^\perp = \mathcal{L}((2,1,-1))$ and $S_{\pi'}^\perp = \mathcal{L}((1,1,3))$. We conclude that π is orthogonal to π', since $(2,1,-1)\cdot(1,1,3) = 0$. Notice that

$$(2,1,-1) \in S_{\pi'} = \{(a,b,c,) \; : \; a + b + 3c = 0\},$$

that is $S_\pi^\perp \subset S_{\pi'}$. We can analogously show that $S_{\pi'}^\perp \subset S_\pi$. This leads to the following remark.

Remark 15.2.8 The planes $\pi, \pi' \subset \mathbb{E}^3$ are orthogonal if and only if $S_\pi^\perp \subset S_{\pi'}$ (or equivalently if and only if $S_{\pi'}^\perp \subset S_\pi$).

In order to recap the results we described in the previous pages, we consider the following example.

Exercise 15.2.9 Consider the point $A = (1,0,1)$ in $\mathbb{E}^3$ and the lines r, s with equations

$$r : \quad (x,y,z) = (1,2,1) + \lambda(3,0,-1), \qquad \Sigma_s : \quad \begin{cases} x - y + z + 2 = 0 \\ x - z + 1 = 0 \end{cases}.$$

We seek to determine:

(a) the set $\mathcal{F}$ of lines through A which are orthogonal to r,
(b) the line $l \in \mathcal{F}$ which is parallel to the plane π given by $\Sigma_\pi : \; x - y + z + 2 = 0$,
(c) the line $l' \in \mathcal{F}$ which is orthogonal to s,
(d) the lines $q \subset \pi'$ with $\Sigma_{\pi'} : \; y - 2 = 0$ which are orthogonal to r.

For these we proceed as follows.

(a) A line u through A has a vector equation

$$(x, y, z) = (1, 0, 1) + \lambda(a, b, c)$$

with arbitrary direction $S_u = \mathcal{L}((a, b, c))$. Since u is to be orthogonal to r, we have the condition $(a, b, c) \cdot (3, 0, 1) = 3a - c = 0$. The set $\mathcal{F}$ is then given by the union $\mathcal{F} = \{r_\alpha\}_{\alpha\in\mathbb{R}} \cup \{\bar{r}\}$ with

$$r_\alpha : \quad (x, y, z) = (1, 0, 1) + \mu(1, \alpha, 3) \qquad \text{for} \quad a \neq 0,$$

and

$$\bar{r} : \quad (x, y, z) = (1, 0, 1) + \mu(0, 1, 0) \qquad \text{for} \quad a = c = 0.$$

(b) Since the direction S_π of the plane π is given by the subspace orthogonal to $\mathcal{L}((1, -1, 1))$, it is clear from $(0, 1, 0) \cdot (1, -1, 1) \neq 0$ that the line $\bar{r}$ is not parallel to π. This means that the line l must be found within the set $\{r_\alpha\}_{\alpha\in\mathbb{R}}$. If we impose that $(1, \alpha, 3) \cdot (1, -1, 1) = 0$, we have $\alpha = 4$, so the line l is given by

$$l : \quad (x, y, z) = (1, 0, 1) + \mu(1, 4, 3).$$

(c) A cartesian equation for s is given by solving the linear system Σ_s in terms of one free unknown. It is immediate to show that

$$s : \quad (x, y, z) = (-1 + \eta, 1 + 2\eta, \eta) = (-1, 1, 0) + \eta(1, 2, 1).$$

The condition $r_\alpha \perp s$ is equivalent to $(1, \alpha, 3) \cdot (1, 2, 1) = 0$, reading $\alpha = -2$, so we have

$$l' : \quad (x, y, z) = (1, 0, 1) + \mu(1, -2, 3).$$

This is the unique solution to the problem: we directly inspect that $\bar{r}$ is not orthogonal to s, since $(0, 1, 0) \cdot (1, 2, 1) = 2 \neq 0$.

(d) A plane π_h is orthogonal to r if and only if

$$\Sigma_{\pi_h} : \quad 3x - z + h = 0.$$

The lines q_h are then given by the intersection

$$\Sigma_{q_h} = \Sigma_{\pi_h \cap \pi'} : \quad \begin{cases} 3x - z + h = 0 \\ y - 2 = 0 \end{cases} \qquad \text{with} \quad h \in \mathbb{R}.$$

15.3 The Distance Between Linear Affine Varieties

It is evident that the distance between two points A and B on a plane is defined to be the length of the line segment whose endpoints are A and B. This definition can be consistently formulated in a Euclidean affine space.

Definition 15.3.1 Let A and B be a pair of points in $\mathbb{E}^n$. The *distance* $\mathrm{d}(A, B)$ between them is defined as

$$\mathrm{d}(A, B) = \|B - A\| = \sqrt{(B - A) \cdot (B - A)}.$$

Exercise 15.3.2 If $A = (1, 2, 0, -1)$ and $B = (0, -1, 2, 2)$ are points in $\mathbb{E}^4$, then

$$\mathrm{d}(A, B) = \|(-1, -3, 2, 3)\| = \sqrt{23}.$$

The well known properties of a Euclidean distance function are a consequence of the corresponding properties of the scalar product.

Proposition 15.3.3 *For any A, B, C points in $\mathbb{E}^n$ the following properties hold.*

(1) $\mathrm{d}(A, B) \geq 0$,
(2) $\mathrm{d}(A, B) = 0$ *if and only if* $A = B$,
(3) $\mathrm{d}(A, B) = \mathrm{d}(B, A)$.
(4) $\mathrm{d}(A, B) + \mathrm{d}(B, C) \geq \mathrm{d}(A, C)$.

In order to introduce a notion of distance between a point and a linear affine variety, we start by looking at an example. Let us consider in $\mathbb{E}^2$ the point $A = (0, 0)$ and the line r whose vector equation is $(x, y) = (1, 1) + \lambda(1, -1)$. By denoting $P_\lambda = (1 + \lambda, 1 - \lambda)$ a generic point in r, we compute

$$\mathrm{d}(A, P_\lambda) = \sqrt{2 + 2\lambda^2}.$$

It is immediate to verify that, as a function of λ, the quantity $\mathrm{d}(A, P)$ ranges between $\sqrt{2}$ and $+\infty$: it is therefore natural to consider the minimum of this range as the distance between A and r. We have then $\mathrm{d}(A, r) = \sqrt{2}$.

Definition 15.3.4 If L is a linear affine variety and A is a point in $\mathbb{E}^n$, the *distance* $\mathrm{d}(A, L)$ between A and L is defined to be

$$\mathrm{d}(A, L) = \min\{\mathrm{d}(A, B) \ : \ B \in L\}.$$

Remark 15.3.5 It is evident from the definition above that $\mathrm{d}(A, L) = 0$ if and only if $A \in L$. We shall indeed prove that, given a point A and a linear affine variety L in $\mathbb{E}^n$, there always exists a point $A_0 \in L$ such that $\mathrm{d}(A, L) = \mathrm{d}(A_0, L)$, thus showing that the previous definition is well posed.

Proposition 15.3.6 *Let L be a linear affine variety and $A \notin L$ a point in $\mathbb{E}^n$. It holds that*

$$\mathrm{d}(A, L) = \mathrm{d}(A, A_0) \quad \textit{with} \quad A_0 = L \cap (A + S_L^{\perp}).$$

Here the set $A + S_L^{\perp}$ denotes the linear affine variety through A whose direction is $S_L^{\perp}$. The point A_0 is called the orthogonal projection *of A on L.*

Proof Since the linear system $\Sigma_{L\cap(A+S_L^{\perp})}$ given by the cartesian equations Σ_L and $\Sigma_{A+S_L^{\perp}}$ is of rank n with n unknowns, the intersection $L \cap (A + S_L^{\perp})$ consists of a single point that we denote by A_0.

Let B be an arbitrary point in L. We can decompose

$$A - B = (A - A_0) + (A_0 - B),$$

with $A_0 - B \in S_L$ (since both A_0 and B are in L) and $A - A_0 \in S_L^{\perp}$ (since both A and A_0 are points in the linear affine variety $A + S_L^{\perp}$). We have then

$$(A - A_0) \cdot (A_0 - B) = 0$$

and we write

$$\begin{aligned}(\mathrm{d}(A, B))^2 = \|A - B\|^2 &= \|(A - A_0) + (A_0 - B)\|^2 \\ &= \|A - A_0\|^2 + \|A_0 - B\|^2.\end{aligned}$$

As a consequence,

$$(\mathrm{d}(A, B))^2 \geq \|A - A_0\|^2 = (\mathrm{d}(A, A_0))^2$$

for any $B \in L$, and this proves the claim. □

Exercise 15.3.7 Let us compute the distance between the line $r :\ 2x + y + 4 = 0$ and the point $A = (1, -1)$ in $\mathbb{E}^2$. We start by finding the line $s_A = A + S_r^{\perp}$ through A which is orthogonal to r. The direction $S_r^{\perp}$ is spanned by the vector $(2, 1)$, so we have

$$s_A : \quad (x, y) = (1, -1) + \lambda(2, 1).$$

The intersection $A_0 = r \cap s_A$ is then given by the value of the parameter λ that solves the equation

$$2(1 + 2\lambda) + (-1 + \lambda) + 4 = 0,$$

that is $\lambda = -1$ giving $A_0 = (-1, -2)$. Therefore we have

$$\mathrm{d}(A, r) = \mathrm{d}(A, A_0) = \|(2, 1)\| = \sqrt{5}.$$

Exercise 15.3.8 Let us consider in $\mathbb{E}^3$ the point $A = (1, -1, 0)$ and the line r with vector equation $r : (x, y, z) = (1, 2, 1) + \lambda(1, -1, 2)$. In order to compute the distance between A and r we first determine the plane $\pi_A := A + S_r^{\perp}$. Since the direction of r must be orthogonal to π_A, from the Remark 15.1.6 the cartesian equation for π_A is given by

$$\Sigma_{\pi_A} : \quad x - y + 2z + d = 0,$$

with $d \in \mathbb{R}$. The value of d if fixed by asking that $A \in \pi_A$, that is $1 + 1 + d = 0$ giving $d = -2$. We then have

$$\Sigma_{\pi_A} : \quad x - y + 2z - 2 = 0.$$

The point A_0 is now the intersection $r \cap \pi_A$, which is given for the value of $\lambda = \frac{1}{6}$ which solves,

$$(1 + \lambda) - (2 - \lambda) + 2(1 + 2\lambda) - 2 = 0.$$

It is therefore $A_0 = (\frac{7}{6}, \frac{11}{6}, \frac{4}{3})$, with

$$\mathrm{d}(A, r) = \mathrm{d}(A, A_0) = \|(\frac{1}{6}, \frac{17}{6}, \frac{4}{3})\| = \sqrt{\frac{59}{6}}.$$

The next theorem yields a formula which allows one to compute more directly the distance $\mathrm{d}(Q, H)$ between a point Q and an hyperplane H in $\mathbb{E}^n$.

Theorem 15.3.9 *Let H be a hyperplane and Q a point in $\mathbb{E}^n$ with $\Sigma_H : a_1x_1 + \cdots + a_nx_n + b = 0$ and $Q = (x_1', \ldots, x_n')$. The distance between Q and H is given by*

$$\mathrm{d}(Q, H) = \frac{|a_1x_1' + \cdots + a_nx_n' + b|}{\sqrt{a_1^2 + \cdots + a_n^2}}.$$

Proof If we consider $X = (x_1, \ldots, x_n)$ and $A = (a_1, \ldots, a_n)$ as vectors in $\mathbb{R}^n$, using the scalar product in $\mathbb{E}^n$, the cartesian equation for H can be written as

$$\Sigma_H : \quad A \cdot X + b = 0.$$

We know that $A \in S_H^{\perp}$, so the line through A which is orthogonal to H is made of the points P such that

$$r : \quad P = Q + \lambda A.$$

The intersection point $Q_0 = r \cap H$ is given by replacing X in Σ_H with such a P, that is

$$\begin{aligned} A \cdot (Q + \lambda A) + b = 0 \quad &\Rightarrow \quad A \cdot Q + \lambda A \cdot A + b = 0 \\ &\Rightarrow \quad \lambda = -\frac{A \cdot Q + b}{A \cdot A}. \end{aligned}$$

The equation for r gives then

$$Q_0 = Q - \frac{A \cdot Q + b}{\|A\|^2}\, A.$$

We can now easily compute

$$\begin{aligned} \|Q - Q_0\|^2 &= \left\| \frac{A \cdot Q + b}{\|A\|^2}\, A \right\|^2 \\ &= \frac{|A \cdot Q + b|^2}{\|A\|^4}\, \|A\|^2 = \frac{|A \cdot Q + b|^2}{\|A\|^2}, \end{aligned}$$

therefore getting

$$\mathrm{d}(Q, H) = \frac{|A \cdot Q + b|}{\|A\|} = \frac{|a_1 x_1' + \cdots + a_n x_n' + b|}{\sqrt{a_1^2 + \cdots + a_n^2}},$$

as claimed. □

Exercise 15.3.10 Consider the line r with cartesian equation $\Sigma_r \,:\, 2x + y + 4 = 0$ and the point $A = (1, -1)$ in $\mathbb{E}^2$ as in the Exercise 15.3.7 above. From the Theorem 15.3.9 we have

$$\mathrm{d}(A, r) = \frac{|2 - 1 + 4|}{\sqrt{4 + 1}} = \sqrt{5}.$$

Exercise 15.3.11 By making again use of the Theorem 15.3.9 it is easy to compute the distance between the point $A = (1, 2, -1)$ and the plane π in $\mathbb{E}^3$ with $\Sigma_\pi \,:\, x + 2y - 2z + 3 = 0$. We have

$$\mathrm{d}(A, \pi) = \frac{|1 + 4 + 2 + 3|}{\sqrt{1 + 4 + 4}} = \frac{10}{3}.$$

We generalise the analysis above with a natural definition for the distance between any two linear affine varieties.

Definition 15.3.12 Let L and L' two linear affine varieties in $\mathbb{E}^n$. The *distance* between them is defined as the non negative real number

$$\mathrm{d}(L, L') = \min\{\mathrm{d}(A, A') \;:\; A \in L,\; A' \in L'\}.$$

It is evident that $\mathrm{d}(L, L') = 0$ if and only if $L \cap L' \neq \emptyset$. It is indeed possible to show that the previous definition is consistent even when $L \cap L' = \emptyset$. Moreover one can show that there exist a point $\bar{A} \in L$ and a point $\bar{A}' \in L'$, such that the minimum distance is attained for them, that is $\mathrm{d}(\bar{A}, \bar{A}') \le \mathrm{d}(A, A')$ for any $A \in L$ and $A' \in L'$. For such a pair of points it is $\mathrm{d}(L, L') = \mathrm{d}(\bar{A}, \bar{A}')$.

In the following pages we shall study the following cases of linear varieties which do not intersect:

- lines r, r' in $\mathbb{E}^2$ which are parallel,
- planes π, π' in $\mathbb{E}^3$ which are parallel,
- a plane π and a line r in $\mathbb{E}^3$ which are parallel,
- lines r, r' in $\mathbb{E}^3$ which are parallel.

Remark 15.3.13 Consider lines r and r' in $\mathbb{E}^2$ which are parallel and distinct. Their cartesian equations are

$$\Sigma_r : \quad ax + by + c = 0, \qquad \Sigma_{r'} : \quad ax + by + c' = 0,$$

for $c' \neq c$. Let $A = (x_1', x_2') \in r$, that is $ax_1' + bx_2' + c = 0$. From the Theorem 15.3.9 it is

$$\mathrm{d}(A, r') = \frac{|ax_1' + bx_2' + c'|}{\sqrt{a^2 + b^2}} = \frac{|c' - c|}{\sqrt{a^2 + b^2}}.$$

From the Definition 15.3.12 we have $\mathrm{d}(A, A') \ge \mathrm{d}(A, r')$. Since the value $\mathrm{d}(A, r')$ we have computed does not depend on the coordinates of $A \in r$, we have that $\mathrm{d}(A, r')$ is the minimum value for $\mathrm{d}(A, A')$ when A ranges in r and A' in r', so we conclude that

$$\mathrm{d}(r, r') = \frac{|c' - c|}{\sqrt{a^2 + b^2}}.$$

Notice that, with respect to the same lines, we also have

$$\mathrm{d}(A', r) = \frac{|c' - c|}{\sqrt{a^2 + b^2}} = \mathrm{d}(A, r').$$

Exercise 15.3.14 Consider the parallel lines $r, r' \subset \mathbb{E}^2$ with cartesian equations

$$\Sigma_r : \quad 2x + y - 3 = 0, \qquad \Sigma_{r'} : \quad 2x + y + 2 = 0.$$

The distance between them is

$$\mathrm{d}(r, r') = \frac{|2 - (-3)|}{\sqrt{5}} = \sqrt{5}.$$

The distance between two parallel hyperplanes in $\mathbb{E}^n$ is given by generalising the proof of the Theorem 15.3.9.

Proposition 15.3.15 *If H and H' are parallel hyperplanes in $\mathbb{E}^n$ with cartesian equations*

$$\Sigma_H : \quad a_1x_1 + \cdots + a_nx_n + b = 0, \qquad \Sigma_{H'} : \quad a_1x_1 + \cdots + a_nx_n + b' = 0,$$

then $\mathrm{d}(H, H') = \mathrm{d}(Q, H')$*, where* Q *is an arbitrary point in* H*, and therefore it is*

$$\mathrm{d}(H, H') = \frac{|b - b'|}{\sqrt{a_1^2 + \cdots + a_n^2}}.$$

Proof We proceed as in the Theorem 15.3.9, so we set $X = (x_1, \ldots, x_n)$ and $A = (a_1, \ldots, a_n)$ and write

$$\Sigma_H : \quad A \cdot X + b = 0, \qquad \Sigma_{H'} : \quad A \cdot X + b' = 0.$$

As we argued in the Remark 15.3.13, by setting $Q = \bar{X}$ with $A\bar{X} + b = 0$, as an arbitrary point in H, we have

$$\mathrm{d}(Q, H') = \frac{|A \cdot \bar{X} + b'|}{\|A\|} = \frac{|b' - b|}{\|A\|}$$

and since such a distance does not depend on Q, we conclude that $\mathrm{d}(H, H') = \mathrm{d}(Q, H')$. □

Exercise 15.3.16 The planes

$$\Sigma_\pi : \quad x + 2y - z + 2 = 0, \qquad \Sigma_{\pi'} : \quad x + 2y - z - 4 = 0$$

are parallel and distinct. The distance between them is

$$\mathrm{d}(\pi, \pi') = \frac{|2 + 4|}{\sqrt{1 + 4 + 1}} = \sqrt{6}.$$

It is clear that not all linear affine varieties which are parallel have the same dimension. The next proposition shows a result within this situation.

Proposition 15.3.17 *Let* r *be a line and* H *an hyperplane in* $\mathbb{E}^n$*, with* r *parallel to* H*. It is*

$$\mathrm{d}(r, H) = \mathrm{d}(\bar{P}, H),$$

where $\bar{P}$ *is any point in* r*.*

Proof With the notations previously adopted, we have $A = (a_1, \ldots, a_n)$ and $X = (x_1, \ldots, x_n)$, we represent H by the cartesian equation

$$\Sigma_H : \quad A \cdot X + b = 0$$

and r by the vector equation

$$r: \quad P = \bar{P} + \lambda v$$

where $\bar{P} \in r$ while $v \cdot A = 0$ since r is parallel to H. From the Theorem 15.3.9 we have

$$\mathrm{d}(P, H) = \frac{|A \cdot P + b|}{\|A\|} = \frac{|A \cdot (\bar{P} + \lambda v) + b|}{\|A\|} = \frac{|A \cdot \bar{P} + b|}{\|A\|}.$$

This expression does not depend on λ: this is the reason why $\mathrm{d}(P, H) = \mathrm{d}(\bar{P}, H) = \mathrm{d}(r, H)$. □

Exercise 15.3.18 Consider in $\mathbb{E}^3$ the line r and the plane π given by:

$$\Sigma_r: \quad \begin{cases} 2x - y + z - 2 = 0 \\ y + 2z = 0 \end{cases}, \qquad \Sigma_\pi: \quad 2x - y + z + 3 = 0.$$

Since r is parallel to π, we take the point $P = (1, 0, 0)$ in r and compute the distance between P and π. One gets

$$\mathrm{d}(r, \pi) = \mathrm{d}(P, \pi) = \frac{5}{\sqrt{6}}.$$

Exercise 15.3.19 Consider the lines r and r' in $\mathbb{E}^3$ given by the vector equations

$$r: \quad (x, y, z) = (3, 1, 2) + \lambda(1, 2, 0), \qquad r': \quad (x, y, z) = (-1, -2, 3) + \lambda(1, 2, 0).$$

Since r is parallel to r', the distance between them can be computed by proceeding as in the previous exercises, that is $\mathrm{d}(r, r') = \mathrm{d}(A, r') = \mathrm{d}(B, r)$, where A is an arbitrary point in r and B an arbitrary point in r'.

We illustrate an alternative method. We notice that, if π is a plane orthogonal to both r and r', then the distance $\mathrm{d}(r, r') = \mathrm{d}(P, P')$ where $P = \pi \cap r$ and $P' = \pi \cap r'$. We consider the plane π through the origin which is orthogonal to both r and r', and whose cartesian equation is

$$\Sigma_\pi: \quad x + 2y = 0.$$

Direct calculations show that $P = \pi \cap r = (2, -1, 2)$ and $P' = \pi \cap r' = (0, 0, 3)$, so

$$\mathrm{d}(r, r') = \mathrm{d}(P, P') = \sqrt{6}.$$

We end the section by sketching how to define the distance between skew lines in $\mathbb{E}^3$.

Remark 15.3.20 If r and r' are skew lines in $\mathbb{E}^3$, then there exist a point $P \in r$ and a point $P \in r'$ which are the intersections of the lines r and r' with the unique line s

orthogonally intersecting both r and r'. The line s is the *minimum distance line for r and r'*, and the distance $\mathrm{d}(r, r') = \mathrm{d}(P, P')$.

Exercise 15.3.21 We consider the skew lines in $\mathbb{E}^3$,

$$r: \quad (x, y, z) = \lambda(1, -1, 1), \qquad r': \quad (x, y, z) = (0, 0, 1) + \mu(1, 0, 1).$$

The subspace $N \subset E^3$ which is orthogonal to both the directions S_r and $S_{r'}$ is $N = \mathcal{L}((1, 0, -1))$. The minimum distance line s for the given r and r' has the direction $S_s = N$, and intersects r in a point P and r' in a point P'. Since $P \in r$ and $P \in r'$, there exists a value for λ and a value for μ such that $P = Q(\lambda)$ and $P' = Q'(\mu)$ with

$$Q(\lambda) + t(1, 0, -1) = Q'(\mu),$$

where ν is the parameter for s. The points $P = s \cap r$ and $P' = s \cap r'$ are then those corresponding to the values of the parameters λ and μ solving such a relation, that is

$$s: \quad \begin{cases} \lambda + \nu = \mu \\ -\lambda = 0 \\ \lambda - t = 1 + \mu \end{cases}.$$

One finds $\lambda = 0$, $\mu = \nu = -\frac{1}{2}$, so $P = (0, 0, 0)$, $P' = \frac{1}{2}(-1, 0, 1)$ and

$$\mathrm{d}(r, r') = \mathrm{d}(P, P') = \frac{1}{\sqrt{2}}.$$

15.4 Bundles of Lines and of Planes

A useful notion for several kinds of problems in affine geometry is that of bundle of lines and bundle of planes.

Definition 15.4.1 Given a point A in $\mathbb{E}^2$, the *bundle of concurrent lines* with center (or *point of concurrency*) A is the set of all the lines through A in $\mathbb{E}^2$; we shall denote it by $\mathcal{F}_A$.

The next result is immediate.

Proposition 15.4.2 *With $A = (x_0, y_0) \in \mathbb{E}^2$, the cartesian equation of an arbitrary line in the bundle $\mathcal{F}_A$ through A is given by*

$$\Sigma_{\mathcal{F}_A}: \quad \alpha(x - x_0) + \beta(y - y_0) = 0$$

for any choice of the real parameters α and β such that $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

Notice that the parameters α and β label a line in $\mathcal{F}_A$, but there is not a bijection between pairs (α, β) and lines in $\mathcal{F}_A$: the pairs (α_0, β_0) and $(\rho\alpha_0, \rho\beta_0)$, for $\rho \neq 0$, give the same line in $\mathcal{F}_A$.

Exercise 15.4.3 The cartesian equation for the bundle $\mathcal{F}_A$ of lines through $A = (1, -2)$ in $\mathbb{E}^2$ is

$$\Sigma_{\mathcal{F}_A} : \quad \alpha(x-1) + \beta(y+2) = 0.$$

The result described in the next proposition (whose proof we omit) shows that the bundle $\mathcal{F}_A$ can be *generated* by any pair of distinct lines concurrent in A.

Proposition 15.4.4 *Let $A \in \mathbb{E}^2$ be the unique intersection of the lines*

$$\Sigma_r : \quad ax + by + c = 0, \qquad \Sigma_{r'} : \quad a'x + b'y + c' = 0.$$

Any relation

$$\Sigma_{(\alpha,\beta)} : \quad \alpha(ax + by + c) + \beta(a'x + b'y + c') = 0$$

with $\mathbb{R}^2 \ni (\alpha, \beta) \neq (0, 0)$ is the cartesian equation for a line in the bundle $\mathcal{F}_A$ of lines with center A, and for any element s of $\mathcal{F}_A$ there exists a pair $\mathbb{R}^2 \ni (\alpha, \beta) \neq (0, 0)$ such that the cartesian equation of s can be written as

$$\Sigma_{(\alpha,\beta)} : \quad \alpha(ax + by + c) + \beta(a'x + b'y + c') = 0. \tag{15.1}$$

Definition 15.4.5 If the bundle $\mathcal{F}_A$ is given by (15.1), the distinct lines r, r' are called the *generators* of the bundle. To stress the role of the generating lines, we also write in such a case $\mathcal{F}_A = \mathcal{F}(r, r')$.

Exercise 15.4.6 The line r whose cartesian equation is $\Sigma_r : x + y + 1$ is an element in the bundle $\mathcal{F}_A$ in the Exercise 15.4.3, corresponding to the parameters $(\alpha, \beta) = (1, 1)$ or equivalently $(\alpha, \beta) = (\rho, \rho)$ with $\rho \neq 0$.

Exercise 15.4.7 Consider the following cartesian equation,

$$\Sigma_{(\alpha,\beta)} : \quad \alpha(x - y + 3) + \beta(2x + y + 3) = 0,$$

depending on a pair of real parameters $(\alpha, \beta) \neq (0, 0)$. Since the relations $x - y + 3 = 0$ and $2x + y + 3 = 0$ yield the cartesian equations for a pair of non parallel lines in $\mathbb{E}^2$, the equation $\Sigma_{(\alpha,\beta)}$ is the cartesian equation for a bundle $\mathcal{F}$ of lines in $\mathbb{E}^2$. We compute:

(a) the centre A of the bundle $\mathcal{F}$,
(b) the line $s_1 \in \mathcal{F}$ which is orthogonal to the line r_1 whose cartesian equation is $\Sigma_{r_1} : 3x + y - 1 = 0$,

(c) the line $s_2 \in \mathcal{F}$ which is parallel to the line r_2 whose cartesian equation is $\Sigma_{r_2} : x - y = 0$,
(d) the line $s_3 \in \mathcal{F}$ through the point $B = (1, 1)$.

We proceed as follows:

(a) The centre of the bundle is given by the intersection

$$\begin{cases} x - y + 3 = 0 \\ 2x + y + 3 = 0 \end{cases},$$

which is found to be $A = (-2, 1)$.
(b) We write the cartesian equation of the bundle $\mathcal{F}$,

$$\Sigma_{\mathcal{F}} : \quad (\alpha + 2\beta)x + (-\alpha + \beta)y + 3(\alpha + \beta) = 0.$$

As a consequence, the direction of an arbitrary line in the bundle $\mathcal{F}$ is spanned by the vector $v_{(\alpha,\beta)} = (\alpha + 2\beta, \beta - \alpha)$. In order for the line $s_1 \in \mathcal{F}$ to be orthogonal to r_1 we require

$$(\alpha + 2\beta, \beta - \alpha) \cdot (-1, 3) = 0 \qquad \Rightarrow \qquad (\alpha, \beta) = \rho(7, -2)$$

with $\rho \neq 0$. The line s has the cartesian equation $\Sigma_{(7,-2)} : x - 3y + 5 = 0$.
(c) In order for an element $s_2 \in \mathcal{F}$ to be parallel to r_2 we require that its direction coincides with the direction of r_2, which is $\mathcal{L}((1, -1))$. We impose then

$$\alpha + 2\beta = -(\beta - \alpha) \qquad \Rightarrow \qquad (\alpha, \beta) = \rho(1, 0)$$

with $\rho \neq 0$. So we have that s_2 is given by the cartesian equation $\Sigma_{(1,0)} : x - y + 3 = 0$. The line s_2 turns out to be indeed one of the generators of the bundle $\mathcal{F}$.
(d) We have now to require that the coordinates of B solve the equation $\Sigma_{(\alpha,\beta)}$, that is

$$(\alpha + 2\beta) + (\beta - \alpha) + 3(\alpha + \beta) = 0 \qquad \Rightarrow \qquad 3\alpha + 6\beta = 0,$$

giving $(\alpha, \beta) = \rho(2, -1)$ with $\rho \neq 0$. The line s_3 is therefore given by $\Sigma_{(2,-1)} : y - 1 = 0$.

Remark 15.4.8 Notice that the computations in (d) above can be generalised. If $\mathcal{F}_A$ is a bundle of lines through A, for any point $B \neq A$ there always exists a unique line in $\mathcal{F}_A$ which passes through B. We denote it as the line $r_{AB} \in \mathcal{F}_A$.

Definition 15.4.9 Let $\Sigma_r : ax + by + c = 0$ be the cartesian equation of the line r in $\mathbb{E}^2$. The set of all lines which are parallel to r is said to define a *bundle of parallel lines* or an *improper* bundle. The most convenient way to describe an improper bundle of lines is

$$\Sigma_{\mathcal{F}} : \quad ax + by + h = 0, \qquad \text{with} \quad h \in \mathbb{R}.$$

Exercise 15.4.10 We consider the line r in $\mathbb{E}^2$ given by $\Sigma_r : \ 2x - y + 3 = 0$. We wish to determine the lines s which are parallel to r and whose distance from r is $\mathrm{d}(s, r) = \sqrt{5}$.

The parallel lines to r are the elements r_h of the improper bundle $\mathcal{F}$ whose cartesian equation is

$$\Sigma_h : \quad 2x - y + h = 0.$$

From the Proposition 15.3.15 we have

$$\sqrt{5} \ = \ \mathrm{d}(r_h, r) \ = \ \frac{|h-3|}{\sqrt{5}} \quad \Rightarrow \quad |h-3| = 5 \quad \Rightarrow \quad h - 3 = \pm 5.$$

The solutions of the exercise are

$$\Sigma_{r_8} : \quad 2x - y + 8 = 0, \qquad \Sigma_{r_{-2}} : \quad 2x - y - 2 = 0.$$

In a way similar to above, one has the notion of bundle of planes in a three dimensional affine space.

Definition 15.4.11 Let r be a line in $\mathbb{E}^3$. The bundle $\mathcal{F}_r$ of planes through r is the set of all planes π in $\mathbb{E}^3$ which contains r, that is $r \subset \pi$. The line r is called the *carrier* of the bundle $\mathcal{F}_r$.

Moreover, if π is a plane in $\mathbb{E}^3$, the set of all planes in $\mathbb{E}^3$ which are parallel to π gives the *(improper) bundle of parallel planes to* π.

The following proposition is the analogue of the Proposition 15.4.2.

Proposition 15.4.12 *Let r be the line in $\mathbb{E}^3$ with cartesian equation given by*

$$\Sigma_r : \quad \begin{cases} ax + by + cz + d = 0 \\ a'x + b'y + c'z + d' = 0 \end{cases}.$$

For any choice of the parameters $(\alpha, \beta) \neq (0, 0)$ the relation

$$\Sigma_{(\alpha,\beta)} : \quad \alpha(ax + by + cz + d) + \beta(a'x + b'y + c'z + d') = 0 \tag{15.2}$$

yields the cartesian equation for a plane in the bundle $\mathcal{F}_r$ with carrier line r, and for any plane π in such a bundle there is a pair $(\alpha, \beta) \neq (0, 0)$ such that the cartesian equation of π is given by (15.2).

Definition 15.4.13 If the bundle $\mathcal{F}_r$ of planes is given by the cartesian equation (15.2), the planes $\Sigma_\pi : \ ax + by + cz + d = 0$ and $\Sigma_{\pi'} : \ a'x + b'y + c'z + d' = 0$ are called the *generators* of $\mathcal{F}_r$. In such a case the equivalent notation $\mathcal{F}(\pi, \pi')$ will also be used.

Remark 15.4.14 Clearly, the bundle $\mathcal{F}_r$ is generated by any two distinct planes $\pi, \ \pi'$ through r.

Exercise 15.4.15 Given the line r whose vector equation is

$$r: \quad (x, y, z) = (1, 2, -1) + \lambda(2, 3, 1),$$

we determine the bundle $\mathcal{F}_r$ of planes through r. In order to obtain a cartesian equation for r, we eliminate the parameter λ from the vector equation above, as follows

$$\begin{cases} x = 1 + 2\lambda \\ y = 2 + 3\lambda \\ \lambda = z + 1 \end{cases} \quad \Rightarrow \quad \begin{cases} x = 1 + 2(z+1) \\ y = 2 + 3(z+1) \end{cases} \quad \Rightarrow \quad \Sigma_r : \begin{cases} x - 2z - 3 = 0 \\ y - 3z - 5 = 0 \end{cases}.$$

The cartesian equation for the bundle is then given by

$$\Sigma_{\mathcal{F}_r} : \quad \alpha(x - 2z - 3) + \beta(y - 3z - 5) = 0$$

with any $(\alpha, \beta) \neq (0, 0)$.

Let us next find the plane $\pi \in \mathcal{F}_r$ which passes through $A = (1, 2, 3)$. The condition $A \in \pi$ yields

$$\alpha(1 - 6 - 3) + \beta(2 - 9 - 5) = 0 \qquad \Rightarrow \qquad 2\alpha + 3\beta = 0.$$

We can pick $(\lambda, \mu) = (3, -2)$, giving $\Sigma_\pi : \ 3(x - 2z - 3) - 2(y - 3z - 5) = 0$, that is $\Sigma_\pi : \ 3x - 2y + 1 = 0$.

We also find the plane $\sigma \in \mathcal{F}_r$ which is orthogonal to $v = (1, -1, 1)$. We know that a vector orthogonal to a plane $\pi \in \mathcal{F}_r$ with equation

$$\Sigma_{\mathcal{F}_r} : \quad \alpha x + \beta y - (2\alpha + 3\beta)z - 3\alpha - 5\beta = 0,$$

is given by $(\alpha, \beta, -2\alpha - 3\beta)$. The conditions we have to meet are then

$$\begin{cases} \alpha = -\beta \\ \alpha = -2\alpha - 3\beta \end{cases} \quad \Rightarrow \quad \alpha = -\beta.$$

If we fix $(\lambda, \mu) = (1, -1)$, we have $\Sigma_\sigma : \ (x - 2z - 3) - (y - 3z - 5) = 0$, that is

$$\Sigma_\sigma : \ x - y + z + 2 = 0.$$

15.5 Symmetries

We introduce a few notions related to symmetries which are useful to solve problems in several branches of geometry and physics.

Definition 15.5.1 Consider a point $C \in \mathbb{E}^n$.

(a) Let $P \in \mathbb{E}^n$ be an arbitrary point in $\mathbb{E}^n$. The *symmetric point to P with respect to C* is the element $P' \in \mathbb{E}^n$ that belongs to the line r_{CP} passing through C and P, and such that $d(P', C) = d(P, C)$ with $P' \neq P$.
(b) Let $X \subset \mathbb{E}^n$ be a set of points. The *symmetric points to X with respect to C* is the set $X' \subset \mathbb{E}^n$ given by every point P' which is symmetric to any P in X with respect to C.
(c) Let $X \subset \mathbb{E}^n$. We say that X *is symmetric with respect to C* if $X = X'$, that is if X contains the symmetric point (with respect to C) to any of its points. In such a case, C is called a *symmetry centre* for X.

Exercise 15.5.2 In the euclidean affine plane $\mathbb{E}^2$ consider the point $C = (2, 3)$. Given the point $P = (1, -1)$, we determine its symmetric P' with respect to C. And with the line $\Sigma_r : \ 2x - y - 3 = 0$, we determine its symmetric r' with respect to C.

We consider the line r_{CP} through P and C, which has the vector equation

$$r_{CP} : \quad (x, y) = (1, -1) + \lambda(1, 4).$$

The distance between P and C is given by $\|P - C\| = \sqrt{17}$, so the point P' can be obtained by finding the value for the parameter λ such that the distance

$$\|P_\lambda - C\| = \|(-1 + \lambda, -4 + 4\lambda)\| = \sqrt{(-1 + \lambda)^2 + (-4 + 4\lambda)^2}$$

be equal to $\|P - C\|$. We have then

$$\sqrt{(-1 + \lambda)^2 + 16(-1 + \lambda)^2} = \sqrt{17} \quad \Rightarrow \quad \sqrt{17(-1 + \lambda)^2} = \sqrt{17} \quad \Rightarrow \quad \sqrt{(-1 + \lambda)^2} = 1$$

that is $\| -1 + \lambda\| = 1$, giving $\lambda = 2$, $\lambda = 0$. For $\lambda = 0$ we have $P_{\lambda=0} = P$, so $P' = P_{\lambda=2} = (3, 7)$.

In order to determine r' we observe that $P \in r$ and we claim that, since r is a line, the set r' symmetric to r with respect to C is a line as well. It is then sufficient to write the line through P' and another point Q' which is symmetric to $Q \in r$ with respect to C. By choosing $Q = (0, -3) \in r$, it is immediate to compute, with the same steps as above, that $Q' = (4, 9)$. We conclude that $r' = r_{CQ'}$, with vector equation

$$r' : \quad (x = 3 + \lambda, \ y = 7 + 2\lambda).$$

Definition 15.5.3 Let A, B be points in $\mathbb{E}^n$. The *midpoint M_{AB}* of the line segment $\overline{AB}$ is the (unique) point of the line r_{AB} with $\|M_{AB} - A\| = \|M_{AB} - B\|$.

Notice that A is the symmetric point to B with respect to M_{AB}, and clearly B is the symmetric point to A with respect to M_{AB} with $M_{AB} = M_{BA}$. One indeed has the vector equality $A - M_{AB} = M_{AB} - B$, giving

$$M_{AB} = \frac{A + B}{2}.$$

The set H_{AB} given by the points

$$H_{AB} = \{P \in \mathbb{E}^n : \|P - A\| = \|P - B\|\}$$

can be shown to be the hyperplane passing through M_{AB} and orthogonal to the line segment $\overline{AB}$. The set H_{AB} is called the *bisecting hyperplane* of the line segment $\overline{AB}$. In $\mathbb{E}^2$ is the bisecting line of $\overline{AB}$, while in $\mathbb{E}^3$ is the *bisecting plane* of $\overline{AB}$.

Exercise 15.5.4 Consider the line segment in $\mathbb{E}^2$ whose endpoints are $A = (1, 2)$ and $B = (3, 4)$. Its midpoint is given by

$$M_{AB} = \frac{A + B}{2} = \frac{(1, 2) + (3, 4)}{2} = (2, 3).$$

A point $P = (x, y)$ belongs to the bisecting line if $\|P - A\|^2 = (x - 1)^2 + (y - 2)^2$ equates $\|P - B\|^2 = (x - 3)^2 + (y - 4)^2 = \|P_B\|^2$, which gives

$$(x-1)^2 + (y-2)^2 = (x-3)^2 + (y-4)^2 \quad \Rightarrow \quad -2x + 1 - 4y + 4 = -6x + 9 - 8y + 16,$$

that is $\Sigma_{H_{AB}} : x + y - 5 = 0$. It is immediate to check that $M \in H_{AB}$. The direction of the bisecting line is spanned by $(1, -1)$, which is orthogonal to the direction vector $B - A = (2, 2)$ spanning the direction of the line r_{AB}.

Exercise 15.5.5 Consider the points $A = (1, 2, -1)$ and $B = (3, 0, 1)$ in $\mathbb{E}^3$. The corresponding midpoint is

$$M_{AB} = \frac{A + B}{2} = \frac{(1, 2, -1) + (3, 0, 1)}{2} = (2, 1, 0).$$

The bisecting plane H_{AB} is given by the points $P = (x, y, z)$ fulfilling the condition

$$(x-1)^2 + (y-2)^2 + (z+1)^2 = \|P - A\|^2 = \|P - B\|^2 = (x-3)^2 + y^2 + (z-1)^2$$

which gives

$$\Sigma_\pi : \quad x - y + z - 1 = 0.$$

The bisecting plane is then orthogonal to $(1, -1, 1)$, with r_{AB} having a direction vector given by $B - A = (2, -2, 2)$.

Having defined the notion of symmetry of a set in $\mathbb{E}^n$ with respect to a point, we might wonder about a meaningful definition of symmetry of a set *with respect to an arbitrary linear affine variety* in $\mathbb{E}^n$. Such a task turns out to be quite hard in general, so we focus on the easy case of defining only the notion of symmetry with respect to a hyperplane.

Firstly, a general definition.

Definition 15.5.6 Let $H \subset \mathbb{E}^n$ be a hyperplane.

(a) Let $P \in \mathbb{E}^n$ be an arbitrary point in $\mathbb{E}^n$. The *symmetric point to P with respect to H* is the element $P' \in \mathbb{E}^n$ such that H is the bisecting hyperplane of the line segment $\overline{PP'}$.
(b) Let $X \subset \mathbb{E}^n$ be a set of points. The *symmetric points to X with respect to H* is the set $X' \subset \mathbb{E}^n$ given by every point P' which is symmetric to any P in X with respect to H.
(c) Let $X \subset \mathbb{E}^n$. We say that X *is symmetric with respect to H* if $X = X'$, that is if X contains the symmetric point (with respect to H) to any of its points. In such a case, H is called a *symmetry hyperplane* for X.

Remark 15.5.7 Notice that if P' is the symmetric point to P with respect to the hyperplane H, then the line $r_{PP'}$ is orthogonal to H and $\mathrm{d}(P', H) = \mathrm{d}(P, H)$.

We finish with some examples on the simplest cases in $\mathbb{E}^2$ and $\mathbb{E}^3$.

Exercise 15.5.8 A line is a hyperplane in $\mathbb{E}^2$. Given the point $P = (1, 2)$ we determine its symmetric P' with respect to the line whose equation is $\Sigma_r : 2x + y - 2$.

We observe that if t is the line through P which is orthogonal to P, then P' is the point in t fixed by the condition $\mathrm{d}(P, r) = \mathrm{d}(P', r)$. The direction of t is clearly spanned by the vector (2, 1), so

$$t : \begin{cases} x = 1 + 2\lambda \\ y = 2 + \lambda \end{cases}$$

and the points in t can be written as $Q_\lambda = (1 + 2\lambda, 2 + \lambda)$. By setting

$$\mathrm{d}(Q_\lambda, r) = \mathrm{d}(P, r) \quad \Rightarrow \quad \frac{|2(1 + 2\lambda) + (2 + \lambda) - 2|}{\sqrt{4+1}} = \frac{|2 + 2 - 2|}{\sqrt{4+1}} \quad \Rightarrow \quad |5\lambda + 2| = 2$$

we see that $Q_{\lambda=0} = P$, while $Q_{\lambda=-4/5} = P' = \frac{1}{5}(-3, 6)$.

Exercise 15.5.9 Given $P = (0, 1, -2) \in \mathbb{E}^3$, we determine its symmetric P' with respect to the hyperplane π (which is indeed a plane, since we are in $\mathbb{E}^3$) whose equation is $\Sigma_\pi : 2x + 4y + 4z - 5 = 0$.

We firstly find the line t through P which is orthogonal to π. The orthogonal subspace to π is spanned by the vector (2, 4, 4) or equivalently $(1, 2, 2)$, so the line t has parametric equation

$$t : \begin{cases} x = \lambda \\ y = 1 + 2\lambda \\ z = -2 + 2\lambda \end{cases} .$$

Since for the symmetric point P' it is $\mathrm{d}(P, \pi) = \mathrm{d}(P', \pi)$, we label a point Q in t by the parameter λ as $Q_\lambda = (\lambda, 1 + 2\lambda, -2 + 2\lambda)$ and impose

$$\mathrm{d}(Q_\lambda, \pi) = \mathrm{d}(P, \pi)$$
$$\Rightarrow \quad \frac{|2\lambda + 4(1+2\lambda) + 4(-2+2\lambda) - 5|}{\sqrt{36}} = \frac{|4-8-5|}{\sqrt{36}} \quad \Rightarrow \quad |18\lambda - 9| = 9.$$

We see that $P = Q_{\lambda=0}$ and $P' = Q_{\lambda=1} = P' = (1, 3, 0)$.

Exercise 15.5.10 In $\mathbb{E}^3$ let us determine the line r' which is symmetric to the line with equation $r : (x, y, z) = (0, 1, -2) + \mu(1, 0, 0)$ with respect to the plane π with equation $\pi : 2x + 4y + 4z - 5 = 0$.

The plane π is the same plane we considered in the previous exercise. Its orthogonal space is spanned by the vector $(1, 2, 2)$. By labelling a point of the line r as $P_\mu = (\mu, 1, -2)$, we find the line t_μ which passes through P_μ and is orthogonal to π. A parametric equation for t_μ is given by

$$t_\mu : \quad \begin{cases} x = \mu + \lambda \\ y = 1 + 2\lambda \\ z = -2 + 2\lambda \end{cases} .$$

We label then points Q in t_μ by writing $Q_{\lambda,\mu} = (\mu + \lambda, 1 + 2\lambda, -2 + 2\lambda)$. We require

$$\mathrm{d}(Q_{\lambda,\mu}, \pi) = \mathrm{d}(P_\mu, \pi)$$

as a condition to determine λ, since μ will yield a parameter for the line r'. We have

$$\mathrm{d}(Q_{\lambda,\mu}, \pi) = \frac{|2(\mu+\lambda) + 4(1+2\lambda) + 4(-2+2\lambda) - 5|}{\sqrt{36}}$$
$$\mathrm{d}(P_\mu, \pi) = \frac{|2\mu + 4 - 8 - 5|}{\sqrt{36}}.$$

From $\mathrm{d}(Q_{\lambda,\mu}, \pi) = \mathrm{d}(P_\mu, \pi)$ we have

$$|2\mu + 18\lambda - 9| = |2\mu - 9| \quad \Rightarrow \quad 2\mu + 18\lambda - 9 = \pm(2\mu - 9).$$

For $\lambda = 0$ we recover $Q_{\lambda=0,\mu} = P_\mu$. The other solution is $\lambda = -\frac{2}{9}\mu + 1$, giving

$$Q_{\lambda=-(2/9)\mu+1,\mu} = P'_\mu = \left(\frac{7}{9}\mu + 1, -\frac{4}{9}\mu + 3, -\frac{4}{9}\mu\right).$$

By a rescaling of the parameter μ, a vector equation for the line r' can be written as

$$r' : (x, y, z) = (1, 3, 0) + \mu(7, -4, -4).$$

Exercise 15.5.11 Consider the set $X \subset \mathbb{E}^2$ given by

$$X = \{(x, y) \in \mathbb{E}^2 \; : \; y = 5x^2\}$$

and the line r whose cartesian equation is $\Sigma_r : x = 0$. We wish to show that r is a symmetry *axis* for X, that is X is symmetric with respect to r. We have then to prove that each point P', symmetric to any point $P \in X$ with respect to r, is an element in X.

Let us consider a generic $P = (x_0, y_0) \in X$ and determine its symmetric with respect to r. The line t through P which is orthogonal to r has the following parametric equation

$$t : \quad \begin{cases} x = x_0 + \lambda \\ y = y_0 \end{cases} .$$

A point in t is then labelled $P_\lambda = (x_0 + \lambda, y_0)$. For its distance from r we compute $\mathrm{d}(P_\lambda, r) = |x_0 + \lambda|$, while $\mathrm{d}(P, r) = |x_0|$. By imposing that these two distances coincide, we have

$$\begin{aligned} \mathrm{d}(P_\lambda, r) = \mathrm{d}(P, r) \quad &\Leftrightarrow \quad |x_0 + \lambda| = |x_0| \\ &\Leftrightarrow \quad (x_0 + \lambda)^2 = x_0^2 \\ &\Leftrightarrow \quad \lambda(2x_0 + \lambda) = 0. \end{aligned}$$

The solution $\lambda = 0$ corresponds to P, the solution $\lambda = -2x_0$ yields $P' = (-x_0, y_0)$. Such calculations do not depend on the fact that P is an element in X. If we consider only points P in X, we have to require that $y_0 = 5x_0^2$. It follows that $y_0 = 5(-x_0)^2$, that is $P' \in X$.