Chapter 3
Euclidean Vector Spaces

When dealing with vectors of $\mathcal{V}_O^3$ in Chap. 1, we have somehow implicitly used the notions of length for a vector and of orthogonality of vectors as well as amplitude of plane angle between vectors. In order to generalise all of this, in the present chapter we introduce the structure of *scalar product* for any vector space, thus coming to the notion of *euclidean* vector space. A scalar product allows one to speak, among other things, of orthogonality of vectors or of the length of a vector in an arbitrary vector space.

3.1 Scalar Product, Norm

We start by recalling, through an example, how the vector space $\mathbb{R}^3$ can be endowed with a euclidean scalar product using the usual scalar product in the space $\mathcal{V}_O^3$.

Example 3.1.1 The usual scalar product in $\mathcal{V}_O^3$, under the isomorphism $\mathbb{R}^3 \simeq \mathcal{V}_O^3$ (see the Proposition 1.3.9), induces a map

$$\cdot \, : \, \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$$

defined as

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

For vectors (x_1, x_2, x_3), (y_1, y_2, y_3), $(z_1, z_2, z_3) \in \mathbb{R}^3$ and scalars $a, b \in \mathbb{R}$, the following properties are easy to verify.

G. Landi and A. Zampini, *Linear Algebra and Analytic Geometry for Physical Sciences*, Undergraduate Lecture Notes in Physics,
https://doi.org/10.1007/978-3-319-78361-1_3

(i) Symmetry, that is:

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1 y_1 + x_2 y_2 + x_3 y_3$$
$$= y_1 x_1 + y_2 x_2 + y_3 x_3 = (y_1, y_2, y_3) \cdot (x_1, x_2, x_3).$$

(ii) Linearity, that is:

$$\begin{aligned}(a(x_1, x_2, x_3) + b(y_1, y_2, y_3)) \cdot (z_1, z_2, z_3) \\ &= (ax_1 + by_1)z_1 + (ax_2 + by_2)z_2 + (ax_3 + by_3)z_3 \\ &= a(x_1 z_1 + x_2 z_2 + x_3 z_3) + b(y_1 z_1 + y_2 z_2 + by_3 z_3) \\ &= a(x_1, x_2, x_3) \cdot (z_1, z_2, z_3) + b(y_1, y_2, y_3) \cdot (z_1, z_2, z_3).\end{aligned}$$

(iii) Non negativity, that is:

$$(x_1, x_2, x_3) \cdot (x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \geq 0.$$

(iv) Non degeneracy, that is:

$$(x_1, x_2, x_3) \cdot (x_1, x_2, x_3) = 0 \quad \Leftrightarrow \quad (x_1, x_2, x_3) = (0, 0, 0).$$

These last two properties are summarised by saying that the scalar product in $\mathbb{R}^3$ is positive definite.

The above properties suggest the following definition.

Definition 3.1.2 Let V be a finite dimensional real vector space. A *scalar product* on V is a map

$$\cdot\, : V \times V \longrightarrow \mathbb{R} \qquad (v, w) \mapsto v \cdot w$$

that fulfils the following properties. For any $v, w, v_1, v_2 \in V$ and $a_1, a_2 \in \mathbb{R}$ it holds that:

(i) $v \cdot w = w \cdot v$,
(ii) $(a_1 v_1 + a_2 v_2) \cdot w = a_1(v_1 \cdot w) + a_2(v_2 \cdot w)$,
(iii) $v \cdot v \geq 0$,
(iv) $v \cdot v = 0 \quad \Leftrightarrow \quad v = 0_V$.

A finite dimensional real vector space V equipped with a scalar product will be denoted $(V, \cdot)$ and will be referred to as a *euclidean vector space*.

Clearly the properties (i) and (ii) in the previous definition allows one to prove that the scalar product map $\cdot$ is linear also with respect to the second argument. A scalar product is then a suitable *bilinear symmetric map*, also called a bilinear symmetric real form since its range is in $\mathbb{R}$.

Exercise 3.1.3 It is clear that the scalar product considered in $\mathcal{V}_O^3$ satisfies the conditions given in the Definition 3.1.2. The map in the Example 3.1.1 is a scalar product on the vector space $\mathbb{R}^3$. This scalar product is not unique. Indeed, consider for instance $p : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by

$$p((x_1, x_2, x_3), (y_1, y_2, y_3)) = 2x_1y_1 + 3x_2y_2 + x_3y_3.$$

It is easy to verify that such a map p is bilinear and symmetric. With $v = (v_1, v_2, v_3)$, from $p(v, v) = 2v_1^2 + 3v_2^2 + v_3^2$ one has $p(v, v) \geq 0$ and $p(v, v) = 0 \Leftrightarrow v = 0$. We have then that p is a scalar product on $\mathbb{R}^3$.

Definition 3.1.4 On $\mathbb{R}^n$ there is a *canonical scalar product*

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

defined by

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1y_1 + \cdots + x_ny_n = \sum_{j=1}^{n} x_jy_j.$$

The datum $(\mathbb{R}^n, \cdot)$ is referred to as the *canonical euclidean space* and denoted E^n.

The following lines sketch the proof that the above map satisfies the conditions of Definition 3.1.2.

(i) $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = \sum_{j=1}^{n} x_jy_j$
$= \sum_{j=1}^{n} y_jx_j = (y_1, \ldots, y_n) \cdot (x_1, \ldots, x_n)$,

(ii) left to the reader,

(iii) $(x_1, \ldots, x_n) \cdot (x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 \geq 0$,

(iv) $(x_1, \ldots, x_n) \cdot (x_1, \ldots, x_n) = 0 \Leftrightarrow \sum_{i=1}^{n} x_i^2 = 0 \Leftrightarrow x_i = 0, \forall i \Leftrightarrow (x_1, \ldots, x_n) = (0, \ldots, 0)$.

Definition 3.1.5 Let $(V, \cdot)$ be a finite dimensional euclidean vector space. The map

$$\| - \| : V \longrightarrow \mathbb{R}, \qquad v \mapsto \|v\| = \sqrt{v \cdot v}$$

is called *norm*. For any $v \in V$, the real number $\|v\|$ is the norm or the length of the vector v.

Exercise 3.1.6 The norm of a vector $v = (x_1, \ldots, x_n)$ in $E^n = (\mathbb{R}^n, \cdot)$ is

$$\|(x_1, \ldots, x_n)\| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

In particular, for E^3 one has $\|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

The proof of the following proposition is immediate.

Proposition 3.1.7 *Let $(V, \cdot)$ be a finite dimensional euclidean vector space. For any $v \in V$ and any $a \in \mathbb{R}$, the following properties hold:*

(1) $\|v\| \geq 0$,
(2) $\|v\| = 0 \Leftrightarrow v = 0_V$,
(3) $\|av\| = |a| \, \|v\|$.

Proposition 3.1.8 *Let $(V, \cdot)$ be a finite dimensional euclidean vector space. For any $v, w \in V$ the following inequality holds:*

$$|v \cdot w| \;\leq\; \|v\| \, \|w\|.$$

This is called the Schwarz *inequality.*

Proof If either $v = 0_V$ or $w = 0_V$ the claim is obvious, so we may assume that both vectors $v, w \neq 0_V$. Set $a = \|w\|$ and $b = \|v\|$; from (iii) in the Definition 3.1.2, one can write

$$\begin{aligned} 0 \leq \|av \pm bw\|^2 &= (av \pm bw) \cdot (av \pm bw) \\ &= a^2\|v\|^2 \pm \; 2ab(v \cdot w) + b^2\|w\|^2 \\ &= 2ab(\|v\|\|w\| \pm \; v \cdot w). \end{aligned}$$

Since both a, b are real positive scalars, the above expression reads

$$\mp \, v \cdot w \leq \; \|v\| \, \|w\|$$

which is the claim. □

Definition 3.1.9 The Schwarz inequality can be written in the form

$$\frac{|v \cdot w|}{\|v\| \, \|w\|} \;\leq\; 1, \qquad \text{that is} \qquad -1 \;\leq\; \frac{v \cdot w}{\|v\| \, \|w\|} \;\leq\; 1.$$

Then one can define then *angle* α between the vectors v, w, by requiring that

$$\frac{v \cdot w}{\|v\| \, \|w\|} = \cos \alpha$$

with $0 \leq \alpha \leq \pi$. Notice the analogy between such a definition and the one in Definition (1.3.2) for the geometric vectors in $\mathcal{V}_O^3$.

Proposition 3.1.10 *Let $(V, \cdot)$ be a finite dimensional euclidean vector space. For any $v, w \in V$ the following inequality holds:*

$$\|v + w\| \;\leq\; \|v\| + \|w\|.$$

This is called the triangle*, or* Minkowski *inequality.*

Proof From the definition of the norm and the Schwarz inequality in Proposition 3.1.8, one has $v \cdot w \leq |v \cdot w| \leq \|v\| \, \|w\|$. The following relations are immediate,

$$\begin{aligned} \|v + w\|^2 &= (v + w) \cdot (v + w) \\ &= \|v\|^2 + 2(v \cdot w) + \|w\|^2 \\ &\leq \|v\|^2 + 2\|v\| \, \|w\| + \|w\|^2 \\ &= (\|v\| + \|w\|)^2 \end{aligned}$$

and prove the claim. □

3.2 Orthogonality

As mentioned, with a scalar product one generalises the notion of orthogonality between vectors and then between vector subspaces.

Definition 3.2.1 Let $(V, \cdot)$ be a finite dimensional euclidean vector space. Two vectors $v, w \in V$ are said to be *orthogonal* if $v \cdot w = 0$.

Proposition 3.2.2 *Let $(V, \cdot)$ be a finite dimensional euclidean vector space, and let $w_1, \cdots, w_s$ and v be vectors in V. If v is orthogonal to each w_i, then v is orthogonal to any vector in the linear span $\mathcal{L}(w_1, \ldots, w_s)$.*

Proof From the bilinearity of the scalar product, one has

$$v \cdot (\lambda_1 w_1 + \cdots + \lambda_s w_s) = \lambda_1 (v \cdot w_1) + \cdots + \lambda_s (v \cdot w_s).$$

The right hand side of such expression is obviously zero under the hypothesis of orthogonality, that is $v \cdot w_i = 0$ for any i. □

Proposition 3.2.3 *Let $(V, \cdot)$ be a finite dimensional euclidean vector space. If $v_1, \ldots, v_s$ is a collection of non zero vectors which are mutually orthogonal, that is $v_i \cdot v_j = 0$ for $i \neq j$, then the vectors $v_1, \ldots, v_s$ are linearly independent.*

Proof Let us equate to the zero vector a linear combination of the vectors $v_1, \ldots, v_s$, that is, let

$$\lambda_1 v_1 + \cdots + \lambda_s v_s = 0_V.$$

For $v_i \in \{v_1, \ldots, v_s\}$, we have

$$0 = v_i \cdot (\lambda_1 v_1 + \cdots + \lambda_s v_s) = \lambda_1 (v_i \cdot v_1) + \cdots + \lambda_s (v_i \cdot v_s) = \lambda_i \, \|v_i\|^2.$$

Being $v_i \neq 0_V$ it must be $\lambda_i = 0$. One gets $\lambda_1 = \ldots = \lambda_s = 0$, with the same argument for any vector v_i. □

Definition 3.2.4 Let $(V, \cdot)$ be a finite dimensional euclidean vector space. If $W \subseteq V$ is a vector subspace of V, then the set

$$W^{\perp} = \{v \in V \ : s\, v \cdot w = 0, \forall w \in W\}$$

is called the *orthogonal* complement to W.

Proposition 3.2.5 *Let $W \subseteq V$ be a vector subspace of a euclidean vector space $(V, \cdot)$. Then,*

(i) $W^{\perp}$ is a vector subspace of V,
(ii) $W \cap W^{\perp} = \{0_V\}$, and the sum between W and $W^{\perp}$ is direct.

Proof (i) Let $v_1, v_2 \in W^{\perp}$, that is $v_1 \cdot w = 0$ and $v_2 \cdot w = 0$ for any $w \in W$. With arbitrary scalars $\lambda_1, \lambda_2 \in \mathbb{R}$, one has

$$(\lambda_1 v_1 + \lambda_2 v_2) \cdot w = \lambda_1 (v_1 \cdot w) + \lambda_2 (v_2 \cdot w) = 0$$

for any $w \in W$; thus $\lambda_1 v_1 + \lambda_2 v_2 \in W^{\perp}$. The claim follows by recalling the Proposition 2.2.2.

(ii) If $w \in W \cap W^{\perp}$, then $w \cdot w = 0$, which then gives $w = 0_V$. □

Remark 3.2.6 Let $W = \mathcal{L}(w_1, \ldots, w_s) \subset V$. One has

$$W^{\perp} = \{v \in V \mid v \cdot w_i = 0, \forall i = 1, \ldots, s\}.$$

The inclusion $W^{\perp} \subseteq \mathcal{L}(w_1, \ldots, v_s)$ is obvious, while the opposite inclusion $\mathcal{L}(w_1, \ldots, w_s) \subseteq W^{\perp}$ follows from the Proposition 3.2.2.

Exercise 3.2.7 Consider the vector subspace $W = \mathcal{L}((1, 0, 1)) \subset E^3$. From the previous remark we have

$$W^{\perp} = \{(x, y, z) \in E^3 \mid (x, y, z) \cdot (1, 0, 1) = 0\} = \{(x, y, z) \in E^3 \mid x + z = 0\},$$

that is $W^{\perp} = \mathcal{L}((1, 0, -1), (0, 1, 0))$.

Exercise 3.2.8 Let $W \subset E^4$ be defined by

$$W = \mathcal{L}((1, -1, 1, 0), (2, 1, 0, 1)).$$

By recalling the Proposition 3.2.3 and the Corollary 2.5.7 we know that the orthogonal subspace $W^{\perp}$ has dimension 2. From the Remark 3.2.6, it is given by

$$W^{\perp} = \left\{ (x, y, z, t) \in E^4 \ : \ \begin{cases} (x, y, z, t) \cdot (1, -1, 1, 0) = 0 \\ (x, y, z, t) \cdot (2, 1, 0, 1) = 0 \end{cases} \right\},$$

that is by the common solutions of the following linear equations,

$$x - y + z = 0$$
$$2x + y + t = 0 \,.$$

Such solutions can be written as

$$\begin{cases} z = y - x \\ t = -2x - y \end{cases}$$

for arbitrary values of x, y. By choosing, for example, $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$, for the orthogonal subspace $W^{\perp}$ one can show that $W^{\perp} = \mathcal{L}((1, 0, -1, -2), (0, 1, 1, -1))$ (this kind of examples and exercises will be clearer after studying homogeneous linear systems of equations).

3.3 Orthonormal Basis

We have seen in Chap. 2 that the orthogonal cartesian coordinate system $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ for the vector space $\mathcal{V}_O^3$ can be seen as having a basis whose vectors are mutually orthogonal and have norm one.

In this section we analyse how to select in a finite dimensional euclidean vector space $(V, \cdot)$, a basis whose vectors are mutually orthogonal and have norm one.

Definition 3.3.1 Let $I = \{v_1, \ldots, v_r\}$ be a system of vectors of a vector space V. If V is endowed with a scalar product, I is called *orthonormal* if

$$v_i \cdot v_j = \delta_{ij} = \begin{cases} 1 \text{ if } & i = j \\ 0 \text{ if } & i \neq j \end{cases} .$$

Remark 3.3.2 From the Proposition 3.2.3 one has that any orthonormal system of vectors if free, that is its vectors are linearly independent.

Definition 3.3.3 A basis $\mathcal{B}$ for $(V, \cdot)$ is called *orthonormal* if it is an orthonormal system.

By such a definition, the basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ of $\mathcal{V}_O^3$ as well as the canonical basis for E^n are orthonormal.

Remark 3.3.4 Let $\mathcal{B} = (e_1, \ldots, e_n)$ be an orthonormal basis for $(V, \cdot)$ and let $v \in V$. The vector v can be written with respect to $\mathcal{B}$ as

$$v = (v \cdot e_1)e_1 + \cdots + (v \cdot e_n)e_n.$$

Indeed, from

$$v = a_1 e_1 + \cdots + a_n e_n$$

one can consider the scalar products of v with each e_i, and the orthogonality of these yields

$$a_1 = v \cdot e_1, \quad \dots \quad , a_n = v \cdot e_n.$$

Thus the components of a vector with respect to an orthonormal basis are given by the scalar products of the vector with the corresponding basis elements.

Definition 3.3.5 Let $\mathcal{B} = (e_1, \dots, e_n)$ be an orthonormal basis for $(V, \cdot)$. With $v \in V$, the vectors

$$(v \cdot e_1)e_1, \; \dots \; , (v \cdot e_n)e_n,$$

which give a linear decomposition of v, are called the *orthogonal projections* of v along $e_1, \dots, e_n$.

The next proposition shows that in an any finite dimensional real vector space $(V, \cdot)$, with respect to an orthonormal basis for V the scalar product has the same form than the canonical scalar product in E^n.

Proposition 3.3.6 *Let $\mathcal{B} = (e_1, \dots, e_n)$ be an orthonormal basis for $(V, \cdot)$. With $v, w \in V$, let it be $v = (a_1, \dots, a_n)_{\mathcal{B}}$ and $w = (b_1, \dots, b_n)_{\mathcal{B}}$. Then one has*

$$v \cdot w = a_1 b_1 + \dots + a_n b_n.$$

Proof This follows by using the bilinearity of the scalar product and the relations $e_i \cdot e_j = \delta_{ij}$. □

Any finite dimensional real vector space can be shown to admit an orthonormal basis. This is done via the so called *Gram-Schmidt orthonormalisation method*. Its proof is constructive since, out of any given basis, the method provides an explicit orthonormal basis via linear algebra computations.

Proposition 3.3.7 (Gram-Schmidt method) *Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis for the finite dimensional euclidean space $(V, \cdot)$. The vectors*

$$\begin{aligned} e_1 &= \frac{v_1}{\|v_1\|}, \\ e_2 &= \frac{v_2 - (v_2 \cdot e_1)e_1}{\|v_2 - (v_2 \cdot e_1)e_1\|}, \\ &\vdots \\ e_n &= \frac{v_n - \sum_{i=1}^{n-1}(v_n \cdot e_i)e_i}{\|v_n - \sum_{i=1}^{n-1}(v_n \cdot e_i)e_i\|} \end{aligned}$$

form an orthonormal basis $(e_1, \dots, e_n)$ for V.

Proof We start by noticing that $\|e_j\| = 1$, for $j = 1, \dots, n$, from the way these vectors are defined. The proof of orthogonality is done by induction. As induction basis we prove explicitly that $e_1 \cdot e_2 = 0$. Being $e_1 \cdot e_1 = 1$, one has

$$e_1 \cdot e_2 = \frac{e_1 \cdot v_2 \;-\; (v_2 \cdot e_1)e_1 \cdot e_1}{\|v_1\| \, \|v_2 - (v_2 \cdot e_1)e_1\|} = 0 \, .$$

We then assume that $e_1, \dots, e_h$ are pairwise orthogonal (this is the inductive hypothesis) and show that $e_1, \dots, e_{h+1}$ are pairwise orthogonal. Consider an integer k such that $1 \le k \le h$. Then,

$$\begin{aligned} e_{h+1} \cdot e_k &= \frac{v_{h+1} - \sum_{i=1}^{h} (v_{h+1} \cdot e_i) e_i}{\|v_{h+1} - \sum_{i=1}^{h} (v_{h+1} \cdot e_i) e_i\|} \cdot e_k \\ &= \frac{v_{h+1} \cdot e_k - \sum_{i=1}^{h} \left((v_{h+1} \cdot e_i)(e_i \cdot e_k)\right)}{\|v_{h+1} - \sum_{i=1}^{h} (v_{h+1} \cdot e_i) e_i\|} \\ &= \frac{v_{h+1} \cdot e_k - v_{h+1} \cdot e_k}{\|v_{h+1} - \sum_{i=1}^{h} (v_{h+1} \cdot e_i) e_i\|} = 0 \end{aligned}$$

where the last equality follows from the inductive hypothesis $e_i \cdot e_k = 0$. The system $(e_1, \dots, e_n)$ is free by Remark 3.3.2, thus giving an orthonormal basis for V. □

Exercise 3.3.8 Let $V = \mathcal{L}(v_1, v_2) \subset E^4$, with $v_1 = (1, 1, 0, 0)$, and $v_2 = (0, 2, 1, 1)$. With the Gram-Schmidt orthogonalization method, we obtain an orthonormal basis for V. Firstly, we have

$$e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \, (1, 1, 0, 0) \, .$$

Set $f_2 = v_2 - (v_2 \cdot e_1)e_1$. We have then

$$\begin{aligned} f_2 &= (0, 2, 1, 1) - \left((0, 2, 1, 1) \cdot \frac{1}{\sqrt{2}} \, (1, 1, 0, 0)\right) \frac{1}{\sqrt{2}} \, (1, 1, 0, 0) \\ &= (0, 2, 1, 1) - (1, 1, 0, 0) \\ &= (-1, 1, 1, 1). \end{aligned}$$

Then, the second vector $e_2 = \frac{f_2}{\|f_2\|}$ is

$$e_2 = \frac{1}{2} \, (-1, 1, 1, 1) \, .$$

Theorem 3.3.9 *Any finite dimensional euclidean vector space* $(V, \cdot)$ *admits an orthonormal basis.*

Proof Since V is finite dimensional, by the Corollary 2.4.4 it has a basis, which can be orthonormalised using the Gram-Schmidt method. □

Theorem 3.3.10 *Let $(V, \cdot)$ be finite dimensional with $\{e_1, \dots, e_r\}$ an orthonormal system of vectors of V. The system can be completed to an orthonormal basis $(e_1, \dots, e_r, e_{r+1}, \dots, e_n)$ for V.*

Proof From the Theorem 2.4.19 the free system $\{e_1, \dots, e_r\}$ can be completed to a basis for V, say

$$\mathcal{B} = (e_1, \dots, e_r, v_{r+1}, \dots, v_n).$$

The Gram-Schmidt method for the system $\mathcal{B}$ does not alter the first r vectors, and provides an orthonormal basis for V. □

Corollary 3.3.11 *Let $(V, \cdot)$ have finite dimension n and let W be a vector subspace of V. Then,*

(1) $\dim(W) + \dim(W^{\perp}) = n$,
(2) $V = W \oplus W^{\perp}$,
(3) $(W^{\perp})^{\perp} = W$.

Proof

(1) Let $(e_1, \dots, e_r)$ be an orthonormal basis for W completed (by the theorem above) to an orthonormal basis $(e_1, \dots, e_r, e_{r+1}, \dots, e_n)$ for V. Since the vectors $e_{r+1}, \dots, e_n$ are then orthogonal to the vectors $e_1, \dots, e_r$, they are (see the Definition 3.2.1) orthogonal to any vector in W, so $e_{r+1}, \dots, e_n \in W^{\perp}$. This gives $\dim(W^{\perp}) \geq n - r$, that is $\dim(W) + \dim(W^{\perp}) \geq n$. From the Definition 3.2.4 the sum of W and $W^{\perp}$ is direct, so, recalling the Corollary 2.5.7, one has $\dim(W) + \dim(W^{\perp}) = \dim(W \oplus W^{\perp}) \leq n$, thus proving the claim.

(2) From (1) we have $\dim(W \oplus W^{\perp}) = \dim(W) + \dim(W^{\perp}) = n = \dim(V)$; thus $W \oplus W^{\perp} = V$.

(3) We start by proving the inclusion $(W^{\perp})^{\perp} \supseteq W$.
By definition, it is $(W^{\perp})^{\perp} = \{v \in V \mid v \cdot w = 0,\ \forall w \in W^{\perp}\}$. If $v \in W$, then $v \cdot w = 0$ for any $w \in W^{\perp}$, thus $W \subseteq (W^{\perp})^{\perp}$. Apply now the result in point 1) to $W^{\perp}$: one has

$$\dim(W^{\perp}) + \dim((W^{\perp})^{\perp}) = n.$$

This inequality, together with the point 1) gives $\dim((W^{\perp})^{\perp}) = \dim(W)$; the spaces W and $(W^{\perp})^{\perp}$ are each other subspace with the same dimension, thus they coincide. □

It is worth stressing that for the identity $(W^{\perp})^{\perp} = W$ it is crucial that the vector space V be finite dimensional. For infinite dimensional vector spaces in general only the inclusion $(W^{\perp})^{\perp} \supseteq W$ holds.

Exercise 3.3.12 In Exercise 3.2.7 we considered the subspace of E^3 given by $W = \mathcal{L}((1, 0, 1))$, and computed $W^{\perp} = \mathcal{L}((1, 0, -1), (0, 1, 0))$. It is immediate to verify that

$$\dim(W) + \dim(W^{\perp}) = 1 + 2 = 3 = \dim(E^3).$$

3.4 Hermitian Products

The canonical scalar product in $\mathbb{R}^n$ can be naturally extended to the complex vector space $\mathbb{C}^n$ with a minor modification.

Definition 3.4.1 The *canonical hermitian product* on $\mathbb{C}^n$ is the map

$$\cdot : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

defined by

$$(z_1, \dots, z_n) \cdot (w_1, \dots, w_n) = \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n$$

where $\bar{z}$ denotes the complex conjugate of z (see the Sect. A.5). The datum $(\mathbb{C}^n, \cdot)$ is called the *canonical hermitian vector space* of dimension n.

The following proposition—whose straightforward proof we omit—generalises to the complex case the properties of the canonical scalar product on $\mathbb{R}^n$ shown after Definition 3.1.4. For easy of notation, we shall denote the vectors in $\mathbb{C}^n$ by $z = (z_1, \dots, z_n)$.

Proposition 3.4.2 *For any $z, w, v \in \mathbb{C}^n$ and $a, b \in \mathbb{C}$, the following properties hold:*

(i) $w \cdot z = \overline{z \cdot w}$,
(ii) $(az + bw) \cdot v = \bar{a}(z \cdot v) + \bar{b}(w \cdot v)$
while $v \cdot (az + bw) = a(v \cdot z) + b(v \cdot w)$,
(iii) $z \cdot z = \sum_{i=1}^{n} |z_i|^2 \geq 0$,
(iv) $z \cdot z = 0 \quad \Leftrightarrow \quad z = (0, \dots, 0) \in \mathbb{C}^n$.

Notice that the complex conjugation for the first entry of the hermitian scalar product implies that the hermitian product of a vector with itself is a real positive number. It is this number that gives the *real* norm of a *complex* vector $z = (z_1, \dots, z_n)$, defined as

$$\|z\| = \sqrt{(z_1, \dots, z_n) \cdot (z_1, \dots, z_n)} = \sqrt{\sum_{i=1}^{n} |z_i|^2} \,.$$