Chapter 11
Rotations

The notion of *rotation* appears naturally in physics, and is geometrically formulated in terms of a euclidean structure as a suitable linear map on a real vector space. The aim of this chapter is to analyse the main properties of rotations using the spectral theory previously developed, as well as to recover known results from classical mechanics, using the geometric language we are describing.

11.1 Skew-Adjoint Endomorphisms

In analogy to the Definition 10.2.1 of a self-adjoint endomorphism, we have the following.

Definition 11.1.1 An endomorphism ϕ of the euclidean vector space E^n is called *skew-adjoint* if

$$\phi(v) \cdot w = - v \cdot \phi(w), \qquad \text{for all} \quad v, w \in E^n.$$

From the Definition 4.1.7 we call a matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$ *skew-symmetric* (or *anti-symmetric*) if ${}^tA = -A$, that is if $a_{ij} = -a_{ji}$, for any i, j. Notice that the skew-symmetry condition for A clearly implies for its diagonal elements that $a_{ii} = 0$. The following result is an analogous of the Theorem 10.2.3 and can be established in a similar manner.

Theorem 11.1.2 *Let $\phi \in \mathrm{End}(E^n)$ and $\mathcal{B}$ an orthonormal basis for E^n. The endomorphism ϕ is skew-adjoint if and only if $M_\phi^{\mathcal{B},\mathcal{B}}$ is skew-symmetric.*

G. Landi and A. Zampini, *Linear Algebra and Analytic Geometry for Physical Sciences*, Undergraduate Lecture Notes in Physics,
https://doi.org/10.1007/978-3-319-78361-1_11

Proposition 11.1.3 *Let $\phi \in \mathrm{End}(E^n)$ be skew-adjoint. It holds that*

(a) the euclidean vector space E^n has an orthogonal decomposition

$$E^n = \mathrm{Im}(\phi) \oplus \ker(\phi),$$

(b) the rank of ϕ is even.

Proof (a) Let $u \in E^n$ and $v \in \ker(\phi)$. We can write

$$0 = u \cdot \phi(v) = -\phi(u) \cdot v.$$

Since this is valid for any $u \in E^n$, the element $\phi(u)$ ranges over the whole space $\mathrm{Im}(\phi)$, so we have that $\ker(\phi) = (\mathrm{Im}(\phi)^{\perp}$.

(b) From ${}^tM_{\phi}^{\mathcal{B},\mathcal{B}} = -M_{\phi}^{\mathcal{B},\mathcal{B}}$, it follows $\det(M_{\phi}^{\mathcal{B},\mathcal{B}}) = (-1)^n \det(M_{\phi}^{\mathcal{B},\mathcal{B}})$. Thus a skew-adjoint endomorphism on an odd dimensional euclidean space is singular (that is it is not invertible). From the orthogonal decomposition for E^n of point (a) we conclude that the restriction $\widetilde{\phi}_{\mathrm{Im}(\phi)} : \mathrm{Im}(\phi) \to \mathrm{Im}(\widetilde{\phi})$ is regular (that is it is invertible). Since such a restriction is skew-adjoint, we have that $\dim(\mathrm{Im}(\phi)) = \dim(\mathrm{Im}(\widetilde{\phi})) = \mathrm{rk}(\phi)$ is even. □

A skew-adjoint endomorphism ϕ on E^n can have only the zero as (real) eigenvalue, so it is not diagonalisable. Indeed, if λ is an eigenvalue for ϕ, that is $\phi(v) = \lambda v$ for $v \neq 0_{E^n} \in E^n$, from the skew-symmetry condition we have that $0 = v \cdot \phi(v) = \lambda\, v \cdot v$, which implies $\lambda = 0$. Also, since its characteristic polynomial has non real roots, it does not have a Jordan form (see Theorem 9.5.1).

Although not diagonalisable, a skew-adjoint endomorphism has nonetheless a canonical form.

Proposition 11.1.4 *Given a skew-adjoint invertible endomorphism $\phi : E^{2p} \to E^{2p}$, there exists an orthonormal basis $\mathcal{B}$ for E^{2p} with respect to which the representing matrix for ϕ is of the form,*

$$M_{\phi}^{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 0 & \mu_1 & & & & \\ -\mu_1 & 0 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & \mu_p \\ & & & & -\mu_p & 0 \end{pmatrix}$$

with $\mu_j \in \mathbb{R}$ for $j = 1, \ldots, p$.

Proof The map $S = \phi^2 = \phi \circ \phi$ is a self-adjoint endomorphism on E^{2p}, so there exists an orthonormal basis of eigenvectors for S. Given $S(w_j) = \lambda_j w_j$ with $\lambda_j \in \mathbb{R}$, each eigenvalue λ_j has even multiplicity, since the identity

$$S(\phi(w_i)) = \phi(S(w_i)) = \lambda_i \phi(w_i)$$

shows that w_i and $\phi(w_i)$ are eigenvectors of S with the same eigenvalue λ_i. We label then the spectrum of S by $(\lambda_1, \ldots, \lambda_k)$ and the basis $\mathcal{C} = \big(w_1, \phi(w_1), \ldots, w_k, \phi(w_k)\big)$. We also have

$$\lambda_i = w_i \cdot S(w_i) = w_i \cdot \phi^2(w_i) = -\phi(w_i) \cdot \phi(w_i) = -\|\phi(w_i)\|^2$$

and, since we took ϕ to be invertible, we have $\lambda_i < 0$. Define the set $\mathcal{B} = (e_1, \ldots, e_{2p})$ of vectors as

$$e_{2j-1} = w_j, \qquad e_{2j} = \frac{1}{\sqrt{|\lambda_j|}}\,\phi(w_j)$$

for $j = 1, \ldots, p$. A direct computation shows that $e_j \cdot e_k = \delta_{jk}$ with $j, k = 1, \ldots, 2p$ and

$$\phi(e_{2j-1}) = \sqrt{|\lambda_j|}\, e_{2j}, \qquad \phi(e_{2j}) = -\sqrt{|\lambda_j|}\, e_{2j-1}.$$

Thus $\mathcal{B}$ is an orthonormal basis with respect to which the matrix representing the endomorphism ϕ has the form above, with $\mu_j = \sqrt{|\lambda_j|}$. □

Corollary 11.1.5 *If ϕ is a skew-adjoint endomorphism on E^n, then there exists an orthonormal basis $\mathcal{B}$ for E^n with respect to which the associated matrix of ϕ has the form*

$$M_{\phi}^{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 0 & \mu_1 & & & & & & & \\ -\mu_1 & 0 & & & & & & & \\ & & \ddots & & & & & & \\ & & & \ddots & & & & & \\ & & & & 0 & \mu_p & & & \\ & & & & -\mu_p & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix},$$

with $\mu_j \in \mathbb{R}$, $j = 1, \cdots, p$, and $2p \leq n$.

The study of antisymmetric matrices makes it natural to introduce the notion of Lie algebra.

Definition 11.1.6 Given $A, B \in \mathbb{R}^{n,n}$, one defines the map $[\ ,\ ] : \mathbb{R}^{n,n} \times \mathbb{R}^{n,n} \to \mathbb{R}^{n,n}$,

$$[A, B] = AB - BA$$

as the *commutator* of A and B. Using the properties of the matrix product is it easy to prove that the following hold, for any $A, B, C \in \mathbb{R}^{n,n}$ and any $\alpha \in \mathbb{R}$:

(1) $[A, B] = -[B, A]$, $\quad [\alpha A, B] = \alpha[A, B]$, $\quad [A + B, C] = [A, C] + [B, C]$, that is the commutator is bilinear and antisymmetric,
(2) $[AB, C] = A[B, C] + [A, C]B$,
(3) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$; this is called the *Jacoby identity*.

Definition 11.1.7 If $W \subseteq \mathbb{R}^{n,n}$ is a vector subspace such that the commutator maps $W \times W$ into W, we say that W is a *(matrix) Lie algebra*. Its rank is the dimension of W as a vector space.

Excercise 11.1.8 The collection of all antisymmetric matrices $W_A \subset \mathbb{R}^{n,n}$ is a matrix Lie algebra since, if ${}^tA = -A$ and ${}^tB = -B$ it is

$${}^t([A, B]) = {}^tB\,{}^tA - {}^tA\,{}^tB = BA - AB.$$

As a Lie algebra, it is denoted $\mathfrak{so}(n)$ and one easily computed its dimension to be $n(n-1)/2$. As we shall see, this Lie algebra has a deep relation with the orthogonal group $\mathrm{SO}(n)$.

Remark 11.1.9 It is worth noticing that the vector space $W_S \subset \mathbb{R}^{n,n}$ of symmetric matrices is *not* a matrix Lie algebra, since the commutator of two symmetric matrices is an antisymmetric matrix.

Excercise 11.1.10 It is clear that the matrices

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

provide a basis for the three dimensional real vector space of antisymmetric matrices $W_A \subset \mathbb{R}^{3,3}$. As the matrix Lie algebra $\mathfrak{so}(3)$, one computes the commutators:

$$[L_1, L_2] = L_3, \qquad [L_2, L_3] = L_1, \qquad [L_3, L_1] = L_2.$$

Excercise 11.1.11 We consider the most general skew-adjoint endomorphism ϕ on E^3. With respect to the canonical orthonormal basis $\mathcal{E} = (e_1, e_2, e_3)$ it has associated matrix of the form

$$M_\phi^{\mathcal{E},\mathcal{E}} = \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} = \alpha L_1 + \beta L_2 + \gamma L_3.$$

with $\alpha, \beta, \gamma \in \mathbb{R}$. Any vector (x, y, z) in its kernel is a solution of the system

$$\begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is easy to show that the kernel is one-dimensional with $\ker(\phi) = \mathcal{L}((\alpha, \beta, \gamma))$. Since ϕ is defined on a three dimensional space and has a one-dimensional kernel, from the Proposition 11.1.4 the spectrum of the map $S = \phi^2$ is made of the simple eigenvalue $\lambda_0 = 0$ and a multiplicity 2 eigenvalue $\lambda < 0$, which is such that $2\lambda = \mathrm{tr}(M_S^{\mathcal{E},\mathcal{E}})$ with

$$M_S^{\mathcal{E},\mathcal{E}} = \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} = \begin{pmatrix} -\gamma^2 - \beta^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & -\gamma^2 - \alpha^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & -\beta^2 - \alpha^2 \end{pmatrix};$$

thus $\lambda = -(\alpha^2 + \beta^2 + \gamma^2)$. For the corresponding eigenspace $V_\lambda \ni (x, y, x)$ one has

$$\begin{pmatrix} \alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Leftrightarrow \qquad \begin{cases} \alpha(\alpha x + \beta y + \gamma z) = 0 \\ \beta(\alpha x + \beta y + \gamma z) = 0 \\ \gamma(\alpha x + \beta y + \gamma z) = 0 \end{cases}.$$

Such a linear system is equivalent to the single equation $(\alpha x + \beta y + \gamma z) = 0$, which shows that $\ker(S)$ is orthogonal to $\mathrm{Im}(S)$. To be definite, assume $\alpha \neq 0$, and fix as basis for V_λ

$$w_1 = (-\gamma, 0, \alpha),$$
$$w_2 = \phi(w_1) = (-\alpha\beta, \alpha^2 + \gamma^2, -\beta\gamma),$$

with $w_1 \cdot \phi(w_1) = 0$. With the appropriate normalization, we define

$$u_1 = \frac{w_1}{\|w_1\|},$$
$$u_2 = \frac{\phi(w_1)}{\|\phi(w_1)\|},$$
$$u_3 = \frac{1}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}} (\alpha, \beta, \gamma)$$

and verify that $\mathcal{C} = (u_1, u_2, u_3)$ is an orthonormal basis for E^3. With $M^{\mathcal{C},\mathcal{E}}$ the orthogonal matrix of change of bases (see the Theorem 7.9.9), this leads to

$$M^{\mathcal{C},\mathcal{E}} M_\phi^{\mathcal{E},\mathcal{E}} M^{\mathcal{E},\mathcal{C}} = M_\phi^{\mathcal{C},\mathcal{C}} = \begin{pmatrix} 0 & -\rho & 0 \\ \rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho = |\lambda| = \alpha^2 + \beta^2 + \gamma^2,$$

an example indeed of Corollary 11.1.5.

11.2 The Exponential of a Matrix

In Sect. 10.1 we studied the properties of the orthogonal group $\mathrm{O}(n)$ in E^n. Before studying the spectral properties of orthogonal matrices we recall some general results.

Definition 11.2.1 Given a matrix $A \in \mathbb{R}^{n,n}$, its *exponential* is the matrix e^A defined by

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

where the sum is defined component-wise, that is $(e^A)_{jl} = \sum_{k=0}^{\infty} \frac{1}{k!} (A^k)_{jl}$.

We omit the proof that such a limit exists (that is each series converges) for every matrix A, and we omit as well the proof of the following proposition, which lists several properties of the exponential maps on matrices.

Proposition 11.2.2 *Given matrices $A, B \in \mathbb{R}^{n,n}$ and an invertible matrix $P \in \mathrm{GL}(n)$, the following identities hold:*

(a) $e^A \in \mathrm{GL}(n)$*, that is the matrix* e^A *is invertible, with* $(e^A)^{-1} = e^{-A}$ *and* $\det(e^A) = e^{\operatorname{tr} A}$*,*
(b) *if* $A = \mathrm{diag}(a_{11}, \ldots, a_{nn})$*, then* $e^A = \mathrm{diag}(e^{a_{11}}, \ldots, e^{a_{nn}})$*,*
(c) $e^{PAP^{-1}} = Pe^A P^{-1}$*,*
(d) *if* $AB = BA$*, that is* $[A, B] = 0$*, then* $e^A e^B = e^B e^A = e^{A+B}$*,*
(e) *it is* $e^{{}^tA} = {}^t(e^A)$*,*
(f) *if* $W \subset \mathbb{R}^{n,n}$ *is a matrix Lie algebra, the elements* e^M *with* $M \in W$ *form a group with respect to the matrix product.*

Excercise 11.2.3 Let as determine the exponential e^Q of the symmetric matrix

$$Q = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad a \in \mathbb{R}.$$

We can proceed in two ways. On the one hand, it is easy to see that

$$Q^{2k} = \begin{pmatrix} a^{2k} & 0 \\ 0 & a^{2k} \end{pmatrix}, \qquad Q^{2k+1} = \begin{pmatrix} 0 & a^{2k+1} \\ a^{2k+1} & 0 \end{pmatrix}.$$

Thus, by using the definition we compute

$$e^Q = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{a^{2k}}{(2k)!} & \sum_{k=0}^{\infty} \frac{a^{2k+1}}{(2k+1)!} \\ \sum_{k=0}^{\infty} \frac{a^{2k+1}}{(2k+1)!} & \sum_{k=0}^{\infty} \frac{a^{2k}}{(2k)!} \end{pmatrix} = \begin{pmatrix} \cosh a & \sinh a \\ \sinh a & \cosh a \end{pmatrix}.$$

Alternatively, we can use the identities (c) and (b) in the previous proposition, once Q has been diagonalised. It is easy to compute the eigenvalues of Q to be $\lambda_\pm = \pm a$, with diagonalising orthogonal matrix $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. That is, $P\Delta_Q P^{-1} = Q$ with with $\Delta_Q = \mathrm{diag}(-a, a)$,

$$\tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.$$

We then compute

$$\begin{aligned} e^Q &= e^{P\Delta_Q P^{-1}} = Pe^{\Delta_Q}T^{-1} \\ &= \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-a} & 0 \\ 0 & e^a \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cosh a & \sinh a \\ \sinh a & \cosh a \end{pmatrix}. \end{aligned}$$

Notice that $\det(e^Q) = \cosh^2 a - \sinh^2 a = 1 = e^{\mathrm{tr}\, Q}$.

Excercise 11.2.4 Let us determine the exponential e^M of the anti-symmetric matrix

$$M = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad a \in \mathbb{R}.$$

Since M is not diagonalisable, we explicitly compute e^M as we did in the previous exercise, finding

$$M^{2k} = (-1)^k \begin{pmatrix} a^{2k} & 0 \\ 0 & a^{2k} \end{pmatrix}, \qquad M^{2k+1} = (-1)^k \begin{pmatrix} 0 & a^{2k+1} \\ -a^{2k+1} & 0 \end{pmatrix}.$$

By putting together all terms, one finds

$$e^M = \begin{pmatrix} \sum_{k=0}^{\infty}(-1)^k \frac{a^{2k}}{(2k)!} & \sum_{k=0}^{\infty}(-1)^k \frac{a^{2k+1}}{(2k+1)!} \\ -\sum_{k=0}^{\infty}(-1)^k \frac{a^{2k+1}}{(2k+1)!} & \sum_{k=0}^{\infty}(-1)^k \frac{a^{2k}}{(2k)!} \end{pmatrix} = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}.$$

We see that if M is a 2×2 anti-symmetric matrix, the matrix e^M is special orthogonal. This is an example for the point (f) in the Proposition 11.2.2.

Excercise 11.2.5 In order to further explore the relations between anti-symmetric matrices and special orthogonal matrices, consider the matrix

$$M = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad a \in \mathbb{R}.$$

In parallel with the computations from the previous exercise, it is immediate to see that

$$e^M = \begin{pmatrix} \cos a & \sin a & 0 \\ -\sin a & \cos a & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This hints to the conclusion that $e^M \in \mathrm{SO}(3)$ if $M \in \mathbb{R}^{3,3}$ is anti-symmetric.

The following proposition generalises the results of the exercises above, and provides a further example for the claim (f) from the Proposition 11.2.2, since the set $W_A \subset \mathbb{R}^{n,n}$ of antisymmetric matrices is the (matrix) Lie algebra $\mathfrak{so}(n)$, as shown in the exercise 11.1.8.

Proposition 11.2.6 *If $M \in \mathbb{R}^{n,n}$ is anti-symmetric, then e^M is special orthogonal. The restriction of the exponential map to the Lie algebra $\mathfrak{so}(n)$ of anti-symmetric matrices is surjective onto* $\mathrm{SO}(n)$.

Proof We focus on the first claim which follows from point (a) of Proposition 11.2.2. If $M \in \mathbb{R}^{n,n}$ is anti-symmetric, ${}^tM = -M$ and $\mathrm{tr}(M) = 0$. Thus ${}^t(e^M) = e^{{}^tM} = e^{-M} = (e^M)^{-1}$ and $\det(e^M) = e^{\mathrm{tr}(M)} = e^0 = 1$. □

Remark 11.2.7 As the Exercise 11.2.5 directly shows, the restriction of the exponential map to the Lie algebra $\mathfrak{so}(n)$ of anti-symmetric matrices is *not* injective into $\mathrm{SO}(n)$.

In the Example 11.3.1 below, we shall sees explicitly that the exponential map, when restricted to 2-dimensional anti-symmetric matrices, is indeed *surjective* onto the group $\mathrm{SO}(2)$.

11.3 Rotations in Two Dimensions

We study now spectral properties of orthogonal matrices. We start with the orthogonal group $\mathrm{O}(2)$.

Example 11.3.1 Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{R}^{2,2}$. The condition ${}^tAA = A\,{}^tA = I_2$ is equivalent to the conditions for its entries given by

$$\begin{aligned} a_{11}^2 + a_{12}^2 &= 1 \\ a_{21}^2 + a_{22}^2 &= 1 \\ a_{11}a_{21} + a_{12}a_{22} &= 0. \end{aligned}$$

To solve these equations, let us assume $a_{11} \neq 0$ (the case $a_{22} \neq 0$ is analogous). We have then $a_{21} = -(a_{12}a_{22})/a_{11}$ from the third equation while, from the others, we have

$$a_{22}^2 \left(\frac{a_{12}^2}{a_{11}^2} + 1 \right) = 1 \qquad \Rightarrow \qquad a_{22}^2 = a_{11}^2.$$

There are two possibilities.

- If $a_{11} = a_{22}$, it follows that $a_{12} + a_{21} = 0$, so the matrix A can be written as

  $$A_+ = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \qquad \text{with } a^2 + b^2 = 1,$$

  and $\det(A_+) = a^2 + b^2 = 1$. One can write $a = \cos\varphi,\ b = \sin\varphi$, for $\varphi \in \mathbb{R}$, so to get

  $$A_+ = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

- If $a_{11} = -a_{22}$, it follows that $a_{12} = a_{21}$, so the matrix A can be written as

  $$A_- = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad \text{with } a^2 + b^2 = 1,$$

  and we can write

  $$A_- = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

  with $\det(A_-) = -a^2 + b^2 = -1$.

Finally, it is easy to see that $a_{11} = 0$ would imply $a_{22} = 0$ and $a_{12}^2 = a_{21}^2 = 1$. These four cases correspond to $\varphi = \pm\frac{\pi}{2}$ for A_+ or A_-, according to wether $a_{12} = -a_{21}$ or $a_{12} = a_{21}$ respectively.

We see that A_+ makes up the special orthogonal group SO(2), while A_- the orthogonal transformations in E^2 which in physics are usually called *improper* rotations.

Given the 2π-periodicity of the trigonometric functions, we see that any element in the special orthogonal group SO(2) corresponds bijectively to an angle $\varphi \in [0, 2\pi)$.

On the other hand, any improper orthogonal transformation can be factorised as the product of a SO(2) matrix times the matrix $Q = \mathrm{diag}(1, -1)$,

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

Thus, an improper orthogonal transformation 'reverses' one of the axis of any given orthogonal basis for E^2 and so changes its orientation.

Remark 11.3.2 Being O(2) a group, the product of two improper orthogonal transformations is a special orthogonal transformation. We indeed compute

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} \cos\varphi' & \sin\varphi' \\ \sin\varphi' & -\cos\varphi' \end{pmatrix} = \begin{pmatrix} \cos(\varphi'-\varphi) & \sin(\varphi'-\varphi) \\ -\sin(\varphi'-\varphi) & \cos(\varphi'-\varphi) \end{pmatrix} \in \mathrm{SO}(2).$$

Proposition 11.3.3 *A matrix* $A \in \mathrm{SO}(2)$ *is diagonalisable if and only if* $A = \pm I_2$. *An orthogonal matrix* A *with* $\det(A) = -1$ *is diagonalisable, with spectrum given by* $\lambda = \pm 1$.

Proof From the previous example we have:

(a) The eigenvalues λ for a special orthogonal matrix are given by the solutions of the equation

$$p_{A_+}(T) = (\cos\varphi - T)^2 + \sin^2\varphi = T^2 - 2(\cos\varphi)\,T + 1 = 0,$$

which are $\lambda_\pm = \cos\varphi \pm \sqrt{\cos^2\varphi - 1}$. This shows that A_+ is diagonalisable if and only if $\cos^2 = 1$, that is $A_+ = \pm I_2$.

(b) Improper orthogonal matrices A_- turn to be diagonalisable since they are symmetric. The eigenvalue equation is

$$p_{A_-} = (T + \cos\varphi)(T - \cos\varphi) - \sin^2\varphi = T^2 - 1 = 0,$$

giving $\lambda_\pm = \pm 1$. □

11.4 Rotations in Three Dimensions

We move to the analysis of rotations in three dimensional spaces.

Excercise 11.4.1 From the Exercise 11.1.11 we know that the anti-symmetric matrices in $\mathbb{R}^{3,3}$ form a three dimensional vector space, thus any anti-symmetric matrix M is labelled by a triple (α, β, γ) of real parameters. The vector $a = (\alpha, \beta, \gamma)$ is the generator, with respect the canonical basis $\mathcal{E}$ of E^3, of the kernel of the endomorphism ϕ associated to M with respect to the basis $\mathcal{E}$, $M = M_\phi^{\mathcal{E},\mathcal{E}}$.

Moreover, from the same exercise we know that there exists an orthogonal matrix P which reduces M to its canonical form (see Corollary 11.1.5), that is $M = P\,\mathfrak{a}_M P^{-1}$ with

$$\mathfrak{a}_M = \begin{pmatrix} 0 & -\rho & 0 \\ \rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{and} \quad \rho^2 = \alpha^2 + \beta^2 + \gamma^2,$$

with respect to an orthonormal basis $\mathcal{C}$ for E^3 such that $P = M^{\mathcal{E},\mathcal{C}}$, the matrix of change of basis. From the Exercise 11.2.5 it is

$$\mathrm{SO}(3) \ni e^{\mathfrak{a}_M} = \begin{pmatrix} \cos\rho & -\sin\rho & 0 \\ \sin\rho & \cos\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{11.1}$$

and, if $R = e^M$, from the Proposition 11.2.2 one has $R = Pe^{\mathfrak{a}_M}P^{-1}$.

The only real eigenvalue of the orthogonal transformation $e^{\mathfrak{a}_M}$ is then $\lambda = 1$, corresponding to the 1-dimensional eigenspace spanned by the vector $a = (\alpha, \beta, \gamma)$. The vector line $\mathcal{L}(a)$ is therefore left unchanged by the isometry ϕ of E^3 corresponding to the matrix R, that is such that $M_\phi^{\mathcal{E},\mathcal{E}} = R$.

From the Proposition 11.2.6 we know that given $R \in \mathrm{SO}(3)$, there exists an anti-symmetric matrix $M \in \mathbb{R}^{3,3}$ such that $R = e^M$. The previous exercise gives then the proof of the following theorem.

Theorem 11.4.2 *For any matrix $R \in \mathrm{SO}(3)$ with $R \neq I_3$ there exists an orthonormal basis $\mathcal{B}$ on E^3 with respect to which the matrix R has the form* (11.1).

This theorem, that is associated with the name of Euler, can also be stated as follow:

Theorem 11.4.3 *Any special orthogonal matrix $R \in \mathrm{SO}(3)$ has the eigenvalue $+1$.*

Those isometries $\phi \in \mathrm{End}(E^3)$ whose representing matrices $M_\phi^{\mathcal{E},\mathcal{E}}$ with respect to an orthonormal basis $\mathcal{E}$ are special orthogonal are also called 3-dimensional *rotation* endomorphisms or rotations tout court. With a language used for the euclidean affine spaces (Chap. 15), we then have:

- For each rotation R of E^3 there exists a unique vector line (a direction) which is left unchanged by the action of the rotation. Such a vector line is called the *rotation axis*.
- The width of the rotation around the rotation axis is given by an angle ρ obtained from (11.1), and implicitly given by

$$1 + 2\cos\rho = \mathrm{tr} \begin{pmatrix} \cos\rho & -\sin\rho & 0 \\ \sin\rho & \cos\rho & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{tr}\, P^{-1}RP = \mathrm{tr}R, \tag{11.2}$$

from the cyclic property of the trace.

Excercise 11.4.4 Consider the rotation of E^3 whose matrix $\mathcal{E} = (e_1, e_2.e_3)$ is

$$R = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix}$$

with respect to the canonical basis. Such a matrix is the product $R = R_1 R_2$ of two special orthogonal matrices. The matrix R_1 is a rotation by an angle α with rotation axis the vector line $\mathcal{L}(e_1)$ and angular width α, while R_2 is a rotation by the angle β with rotation axis $\mathcal{L}(e_3)$. We wish to determine the rotation axis for R with corresponding angle. A direct calculation yields

$$R = \begin{pmatrix} \cos\alpha & \sin\alpha\, \cos\beta & \sin\alpha\, \sin\beta \\ -\sin\alpha & \cos\alpha\, \cos\beta & \cos\alpha\, \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix}.$$

Since $R \neq I_3$ for $\alpha \neq 0$ and $\beta \neq 0$, the rotation axis is given by the eigenspace corresponding to the eigenvalue $\lambda = 1$. This eigenspace is found to be spanned by the vector v with

$$\begin{aligned} v &= \big(\sin\alpha(1-\cos\beta),\ (\cos\alpha-1)(1-\cos\beta),\ \sin\beta(1-\cos\alpha)\big) \qquad \text{if } \alpha \neq 0,\ \beta \neq 0, \\ v &= (1,0,0) \qquad \text{if } \alpha = 0, \\ v &= (0,0,1) \qquad \text{if } \beta = 0. \end{aligned}$$

The rotation angle ρ can be obtained (implicitly) from the Eq. (11.2) as

$$1 + 2\cos\rho = \mathrm{tr}(R) = \cos\alpha + \cos\beta + \cos\alpha\, \cos\beta.$$

Excercise 11.4.5 Since the special orthogonal group $\mathrm{SO}(n)$ is non abelian for $n > 2$, for the special orthogonal matrix given by $R' = R_2 R_1$ one has $R' \neq R$. The matrix R' can be written as

$$R' = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha\, \cos\beta & \cos\alpha\, \cos\beta & \sin\beta \\ \sin\alpha\, \sin\beta & -\sin\beta\, \cos\alpha & \cos\beta \end{pmatrix}.$$

One now computes that while the rotation angle is the same as in the previous exercise, the rotation axis is spanned by the vector v' with

$$\begin{aligned} v' &= \big(\sin\alpha\, \sin\beta,\ (1-\cos\alpha)\sin\beta,\ (1-\cos\alpha)(1+\cos\beta)\big) \qquad \text{if } \alpha \neq 0,\ \beta \neq 0, \\ v' &= (1,0,0) \qquad \text{if } \alpha = 0, \\ v' &= (0,0,1) \qquad \text{if } \beta = 0. \end{aligned}$$

Excercise 11.4.6 Consider the matrix $R'' = Q_1 Q_2$ given by

$$R'' = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta' & \sin\beta' \\ 0 & \sin\beta' & -\cos\beta' \end{pmatrix}.$$

Now neither Q_1 nor Q_2 are (proper) rotation matrix: both Q_1 and Q_2 are in O(3), but $\det(Q_1) = \det(Q_2) = -1$ (see the Example 11.3.1, where O(2) has been described, and the Remark 11.3.2). The matrix R'' is nonetheless special orthogonal since O(3) is a group and $\det(R'') = 1$.

One finds that the rotation axis is the vector line spanned by the vector v'' with

$$
\begin{aligned}
v'' &= \big(\sin\alpha\,\sin\beta',\ (1-\cos\alpha)\sin\beta',\ (1-\cos\alpha)(1-\cos\beta')\big) \qquad \text{if } \alpha \neq 0,\ \beta' \neq 0,\\
v'' &= (1,0,0) \qquad \text{if } \alpha = 0,\\
v'' &= (0,0,1) \qquad \text{if } \beta' = 0.
\end{aligned}
$$

One way to establish this result without doing explicit computation, is to observe that R'' is obtained from R' in Exercise 11.4.5 under a transposition and the identification $\beta' = \pi - \beta$.

Excercise 11.4.7 As an easy application of the Theorem 10.1.13 we know that, if $\mathcal{B} = (u_1, u_2, u_3)$ and $\mathcal{C} = (v_1, v_2, v_3)$ are orthonormal bases in E^3, then the orthogonal endomorphism ϕ mapping $v_k \mapsto u_k$ is represented by a matrix whose entry Φ_{ab} is given by the scalar product $u_b \cdot v_a$

$$M_\phi^{\mathcal{C},\mathcal{C}} = \Phi = \Big(\Phi_{ab} = u_b \cdot v_a\Big)_{a,b=1,2,3}.$$

It is easy indeed to see that the matrix element $({}^t\Phi\,\Phi)_{ks}$ is given by

$$\sum_{a=1}^{3} \Phi_{ak}\Phi_{as} = \sum_{a=1}^{3}(u_a \cdot v_k)(u_a \cdot v_s) = v_k \cdot v_s = \delta_{ks}$$

thus proving that Φ is orthogonal. Notice that $M_\phi^{\mathcal{B},\mathcal{B}} = {}^t\Phi = \Phi^{-1}$.

Excercise 11.4.8 Let $\mathcal{E} = (e_1, e_2, e_3)$ be an orthonormal basis for E^3. We compute the rotation matrix corresponding to the change of basis $\mathcal{E} \to \mathcal{B}$ with $\mathcal{B} = (u_1, u_2, u_3)$ for any given basis $\mathcal{B}$ with the same orientation (see the Definition 10.1.15) of $\mathcal{E}$.

Firstly, consider a vector u of norm 1. Since such a vector defines a point on a sphere of radius 1 in the three dimensional physical space $\mathcal{S}$, which can be identified by a *latitude* and a *longitude*, its components with respect to $\mathcal{E}$ are determined by two angles. With respect to Figure 11.1 we write them as

$$u = (\sin\varphi\,\sin\theta, -\cos\varphi\,\sin\theta, \cos\theta)$$

with $\theta \in (0, \pi)$ and $\varphi \in [0, 2\pi)$. Then, to complete u to an orthonormal basis for E^3 with $u'_3 = u$, one finds,

$$
\begin{aligned}
u'_1 &= u_N = (\cos\varphi, \sin\varphi, 0),\\
u'_2 &= (-\sin\varphi\,\cos\theta, \cos\varphi\,\cos\theta, \sin\theta).
\end{aligned}
$$

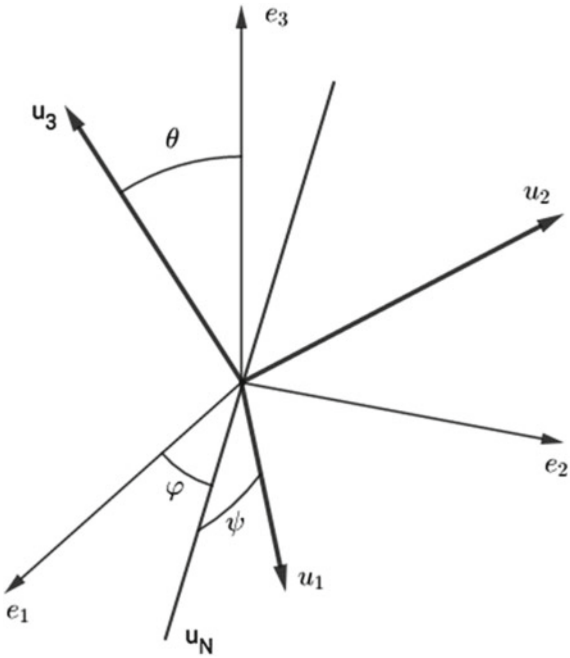

Fig. 11.1 The Euler angles

The rotation matrix (with respect to the basis $\mathcal{E}$) of the transformation $\mathcal{E} \to (u_1', u_2', u_3')$ is given by

$$R'(\theta, \varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi\,\cos\theta & \sin\varphi\,\sin\theta \\ \sin\varphi & \cos\varphi\,\cos\theta & -\cos\varphi\,\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Since the choice of u_1', u_2' is unique up to a rotation around the orthogonal vector u, we see that the most general SO(3) rotation matrix mapping $e_3 \to u$ is given by

$$\begin{aligned} R(\theta, \varphi, \psi) = \; & R(\theta, \varphi) \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ = & \begin{pmatrix} \cos\varphi\,\cos\psi - \sin\varphi\,\cos\theta\,\sin\psi & -\cos\varphi\,\sin\psi - \sin\varphi\,\cos\theta\,\cos\psi & \sin\varphi\,\cos\theta \\ \sin\varphi\,\cos\psi + \cos\varphi\,\cos\theta\,\sin\psi & -\sin\varphi\,\sin\psi + \cos\varphi\,\cos\theta\,\cos\psi & -\cos\varphi\,\sin\theta \\ \sin\theta\,\sin\psi & \sin\theta\,\cos\psi & \cos\theta \end{pmatrix} \end{aligned}$$

with $\psi \in [0, 2\pi)$. This shows that the proper 3-dimensional rotations, that is the group SO(3), can be parametrised by 3 angles. Such angles are usually called *Euler angles*, and clearly there exist several (consistent and equivalent) different choices for them.

Our result depends on the assumption that $\sin\theta \neq 0$, which means that $u_1 \neq \pm e_3$ (this corresponds to the case when u_1 is the north-south pole direction). The most general rotation matrix representing an orthogonal transformation with$e_1 \to u_1 = \pm e_3$ is given by

$$R(\psi) = \begin{pmatrix} 0 & \cos\psi & \mp\sin\psi \\ 0 & \sin\psi & \pm\cos\psi \\ \pm 1 & 0 & 0 \end{pmatrix}.$$

We finally remark that the rotation matrix $R(\theta, \varphi, \psi)$ can be written as the product

$$R(\theta, \varphi, \psi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This identity shows that we can write

$$R(\theta, \beta, \psi) = e^{\varphi L_3}\, e^{\theta L_1}\, e^{\psi L_3}.$$

where L_1 and L_3 are the matrices in Exercise 11.1.10. These matrices are the 'generators' of the rotations around the first and third axis, respectively.

In applications to the dynamics of a rigid body, with reference to the Figure 11.1, the angle φ parametrises the motion of *precession* of the axis u_3 around the axis e_3, the angle θ the motion of *nutation* of the axis u_3 and the angle φ the *intrinsic rotation* around the axis u_3. The unit vector u_N indicates the *line of nodes*, the intersection of the plane $(e_1 e_2)$ with the plane $(u_1 u_2)$.

We close this section by listing the most interesting properties of orthogonal endomorphisms in E^n with $n > 0$. Endomorphisms ϕ whose representing matrix $M_\phi^{\mathcal{E},\mathcal{E}}$ are special orthogonal, with respect to an orthonormal basis $\mathcal{E}$ for E^n, are called *rotations*. From the Proposition 11.2.6 we know that there exists an anti-symmetric matrix M such that $M_\phi^{\mathcal{E},\mathcal{E}} = e^M$. When $\mathrm{rk}(M) = 2k$, the matrix e^M depends on k angular variables.

From the Corollary 11.1.5 and a direct generalisation of the computations above, one can conclude that for each n-dimensional rotation:

- There exists a vector subspace $V \subset E^n$ which is left unchanged by the action of the rotation, with $\dim(V) = n - \mathrm{rk}(M)$.
- Since $\mathrm{rk}(M)$ is even, we have that, if n is odd, then V is odd dimensional as well, and at least one dimensional. If n is even and the matrix M is invertible, then V is the null space.

11.5 The Lie Algebra $\mathfrak{so}(3)$

We have a closer look at the Lie algebra $\mathfrak{so}(3)$ introduced in the Exercise 11.1.10. As mentioned, it is three dimensional and generated by the three matrices

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which are closed under matrix commutator.

Consider the three dimensional euclidean *totally antisymmetric* Levi-Civita symbol $\varepsilon_{a_1a_2a_3}$ with indices $a_j = 1, 2, 3$ and defined by

$$\varepsilon_{a_1a_2a_3} = \begin{cases} +1 & \text{if } (a_1, a_2, a_3) \text{ is an even permutation of } (1, 2, 3) \\ -1 & \text{if } (a_1, a_2, a_3) \text{ is an odd permutation of } (1, 2, 3) \\ 0 & \text{if any two indices are equal} \end{cases}.$$

One has the identity $\sum_{a=1}^{3} \varepsilon_{abc}\epsilon_{aks} = (\delta_{bk}\delta_{cs} - \delta_{bs}\delta_{ck})$.

Excercise 11.5.1 Using the Levi-Civita symbol, it is easy to see that the generators L_a have components given by

$$(L_a)_{mn} = \varepsilon_{man},$$

while their commutators are written as

$$[L_m, L_n] = \sum_{a=1}^{3} \varepsilon_{mna} L_a.$$

There is an important subtlety when identifying 3×3 antisymmetric matrices with three dimensional vectors. The most general antisymmetric matrix in indeed characterised by three scalars,

$$A = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} = \sum_{a=1}^{3} v_a L_a$$

For the time being, this only defines a triple of numbers (v_1, sv_2, v_3) in E^3. Whether this triple provides the components of a vector in the three dimensional euclidean space, will depend on how it transforms under an orthonormal transformation. Now, we may think of A as the matrix, with respect to the canonical orthonormal basis $\mathcal{E}$ of a skew-adjoint endomorphism ϕ on E^3: $A = M_{\phi}^{\mathcal{E},\mathcal{E}}$. When changing basis to an orthonormal basis $\mathcal{B}$ with matrix of change of basis $R = M^{\mathcal{E},\mathcal{B}} \in \mathrm{O}(3)$, the matrix A is transformed to

$$A' = RAR^{-1} = RA\,{}^tR,$$

since R is orthogonal and thus $R^{-1} = {}^tR$. Since A' is antisymmetric as well, it can be written as $A' = \sum_{a,b=1}^{3} v'_a L_a$ for some (v'_1, v'_2, v'_3). In order to establish the transformation rule from (v_1, v_2, v_3) to (v'_1, v'_2, v'_3), we need an additional result on orthogonal matrices.

Excercise 11.5.2 Using the expression in Sect. 5.3 for the inverse of an invertible matrix, the orthogonality condition for a matrix $R \in \mathrm{O}(3)$, that is $R_{ab} = ({}^tR)_{ba} = (R^{-1})_{ba}$, can be written as

$$R_{ab} = \frac{1}{\det R}(-1)^{a+b}\det(\widehat{R}_{ab}),$$

where $\widehat{R}_{ab}$ is the 2 dimensional matrix obtained by deleting the row a and the column b in the 3 dimensional matrix R. (Then $\det(\widehat{R}_{ab}$ is the minor of the element R_{ab}, see the Definition 5.1.7.) In terms of the Levi-Civita symbol this identity transform to

$$\sum_{j=1}^{3} \varepsilon_{mjn} R_{jq} = \frac{1}{\det R}\sum_{a,b=1}^{3} R_{ma}\varepsilon_{aqb} R_{nb}, \tag{11.3}$$

or, being ${}^tR$ orthogonal as well, with $\det R = \det {}^tR$,

$$\sum_{j=1}^{3} \varepsilon_{mjn} R_{qj} = \frac{1}{\det R}\sum_{a,b=1}^{3} R_{am}\varepsilon_{aqb} R_{bn}. \tag{11.4}$$

Going back to $A = \sum_{a=1}^{3} v_a L_a$ and $A' = \sum_{a,b=1}^{3} v'_a L_a$, we have for their components:

$$A_{mn} = \sum_{j=1}^{3} v_j\,\varepsilon_{mjn} \qquad \text{and} \qquad A'_{mn} = \sum_{j=1}^{3} v'_j\,\varepsilon_{mjn}.$$

We then compute, using the relation (11.3),

$$\begin{aligned} A'_{mn} = (RA\,{}^tR)_{mn} &= \sum_{a,b=1}^{3} R_{ma} A_{ab} R_{nb} \\ &= \sum_{j=1}^{3}\sum_{a,b=1}^{3} v_j\,\varepsilon_{ajb}\,R_{ma} R_{nb} \\ &= (\det R)\sum_{j=1}^{3}\sum_{c=1}^{3} R_{cj} v_j\,\varepsilon_{mcn} = (\det R)\sum_{c=1}^{3}(Rv)_c\,\varepsilon_{mcn}, \end{aligned}$$

that is,

$$v'_j = (\det R)\,(Rv)_j = (\det R)\sum_{c=1}^{3} R_{cj} v_j .$$

This shows that, under an orthogonal transformation between different bases of E^3, the components of an antisymmetric matrix transforms as the components of a vector only if the orientation is preserved, that is only if the transformation is special orthogonal.

Using a terminology from physics, elements in E^3 whose components with respect to orthonormal basis transform as the general theory (see the Proposition 7.9.2) prescribes are called *polar* vectors (or vectors tout court), while elements in E^3 whose components transform as the components of an antisymmetric matrix are called *axial* (or *pseudo)* vectors.

An example of an axial vector is given by the vector product in E^3 of two (polar) vector, that we recall from the Chap. 1. To be definite, let us start with the canonical orthonormal basis $\mathcal{E}$. If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$, the Proposition 1.3.15 define the vector product of v and w as,

$$\tau(v, w) = v \wedge w = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1).$$

Using the Levi-Civita symbol, the components are written as

$$(v \wedge w)_a = \sum_{b,c=1}^{3} \varepsilon_{abc}\, v_b\, w_c .$$

If $R = M^{\mathcal{E},\mathcal{B}} \in \mathrm{O}(3)$ is the change of basis to a new orthonormal basis $\mathcal{B}$ for E^3, on one hand we have $(v \wedge w)'_q = (R(v \wedge w))_q$ while the relation (11.4) yields,

$$\begin{aligned}(v' \wedge w')_q = \sum_{k,j=1}^{3} \varepsilon_{qkj} v'_k w'_j &= \sum_{k,j,b,s=1}^{3} \varepsilon_{qkj} R_{kb} R_{js} v_b w_s \\ &= (\det R) \sum_{a,b,s=1}^{3} R_{qa} \varepsilon_{abs}\, v_b w_s = (\det R)(v \wedge w)'_q .\end{aligned}$$

This shows that the components of a vector product transforms as an axial vector under an orthogonal transformation between different bases of E^3. In a similar manner one shows that the vector product of an axial vector with a polar vector, is a polar vector.

Excercise 11.5.3 For example, the change of basis from $\mathcal{B}$ to $\mathcal{B}' = (b'_1 = -b_1, b'_2 = -b_2, b'_3 = -b_3)$ is clearly represented by the matrix $M^{\mathcal{B},\mathcal{B}'} = M^{\mathcal{B}',\mathcal{B}} = -I_3$ which is orthogonal but not special orthogonal. It is immediate to see that we have

$v = (-v_1, -v_2, -v_3)_{\mathcal{B}'}$ and $w = (-w_1, -w_2, -w_3)_{\mathcal{B}'}$, but $v \wedge w = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)_{\mathcal{B}'}$.

From the Example 1.3.17 we see that, since the physical observables position, velocity, acceleration and force are described by polar vectors, both momenta and angular momenta for the dynamics of a point mass are axial vectors.

Excercise 11.5.4 We recall from Sect. 1.4 the action of the operator rot on a vector field $\mathbf{A}(\mathbf{x})$,

$$\mathrm{rot}\,\mathbf{A} = \nabla \wedge \mathbf{A} = \sum_{i,j,k=1}^{3} (\varepsilon_{ijk} \partial_j A_k)\, e_i$$

with respect to an orthonormal basis $\mathcal{E} = (e_1, e_2, e_3)$ of E^3 which represents the physical space $\mathcal{S}$. This identity shows that, if $\mathbf{A}$ is a polar vector (field) then rot $\mathbf{A}$ is an axial vector (field).

Example 11.5.5 The (Lorentz) force $\mathbf{F}$ acting on a point electric charge q whose motion is given by $\mathbf{x}(t)$, in the presence of an electric field $\mathbf{E}(\mathbf{x})$ and a magnetic field $\mathbf{B}(\mathbf{x})$ is written as

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{x}} \wedge \mathbf{B}).$$

We conclude that $\mathbf{E}$ is a polar vector field, while $\mathbf{B}$ is an axial vector field. Indeed, the correct way to describe $\mathbf{B}$ is with an antisymmetric 3×3 matrix.

11.6 The Angular Velocity

When dealing with rotations in physics, an important notion is that of *angular velocity*. This and several related notions can be analysed in terms of the spectral properties of orthogonal matrices that we have illustrated above. It is worth recalling from Chap. 1 that euclidean vector spaces with orthonormal bases are the natural framework for the notion of cartesian orthogonal coordinate systems for the physical space $\mathcal{S}$ (*inertial reference frames*).

Example 11.6.1 Consider the motion $\mathbf{x}(t)$ in E^3 of a point mass such that its distance $\|\mathbf{x}(t)\|$ from the origin of the coordinate system is fixed. We then consider a fixed orthonormal basis $\mathcal{E} = (e_1, e_2, e_3)$, and a orthonormal basis $\mathcal{E}' = (e'_1(t), e'_2(t), e'_3(t))$ which *rotates* with respect to $\mathcal{E}$ in such a way that the components of $\mathbf{x}(t)$ along $\mathcal{E}'$ do not depend on time — the point mass is at *rest* with respect to $\mathcal{E}'$. We can write the position vector $\mathbf{x}(t)$ as

$$\mathbf{x}(t) = \sum_{a=1}^{3} x_a(t) e_a = \sum_{k=1}^{3} x'_k e'_k(t).$$

Since $\mathcal{E}'$ depends on time, the change of the basis is given by a time-dependent orthogonal matrix $M^{\mathcal{E},\mathcal{E}'(t)} = R(t) \in \mathrm{SO}(3)$ as

$$x_k(t) = \sum_{j=1}^{3} R_{kj}(t)x'_j.$$

By differentiating with respect to time t (recall that the dot means time derivative), with $\dot{x}'_j = 0$, the above relation gives,

$$\dot{x}_k = \sum_{a=1}^{3} \dot{R}_a x'_a = \sum_{a,b=1}^{3} \dot{R}_{ka}(R^{-1})_{ab}x_b = \sum_{a,b=1}^{3} \dot{R}_{ka}({}^tR)_{ab}x_b.$$

From the relation $R(t)\,{}^tR(t) = I_3$ it follows that, by differentiating with respect to t,

$$\begin{aligned} &\dot{R}\,{}^tR + R\,{}^t\dot{R} = 0 \qquad \Rightarrow \\ &\qquad \dot{R}\,{}^tR = -R\,({}^t\dot{R}) = -\,{}^t(\dot{R}\,{}^tR) \end{aligned}$$

We see that the matrix $\dot{R}\,{}^tR$ is antisymmetric, so from the Exercise 11.1.11 there exist real scalars $(\omega_1(t), \omega_2(t), \omega_3(t))$ such that

$$\dot{R}\,{}^tR = \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix}. \tag{11.5}$$

A comparison with the Example 1.3.17 than shows that the expression for the velocity,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \dot{R}\,{}^tR \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

can be written as

$$\dot{\mathbf{x}}(t) = \omega(t) \wedge \mathbf{x}(t). \tag{11.6}$$

The triple $\omega(t) = \big(\omega_1(t), \omega_2(t), \omega_3(t)\big)$ is the *angular velocity* vector of the motion described by the rotation $R(t)$.

As we shall see in the Exercise 11.7.1, this relation also describes the rotation of a rigid body with a fixed point.

Excercise 11.6.2 The velocity corresponding to the motion in E^3 given by (here $r > 0$)

$$x(t) = \big(r\cos\alpha(t),\; r\sin\alpha(t), 0\big)$$

with respect to an orthonormal basis $\mathcal{E}$ is

$$\dot{x}(t) = \dot{\alpha}\big(-r \sin\alpha(t),\ r\cos\alpha(t), 0\big) = \omega(t) \wedge x(t)$$

with $\omega(t) = (0, 0, \dot{\alpha})$.

From the Sect. 11.5, we know that the angular velocity is an axial vector, so we write

$$\omega_a(t) \mapsto \omega'_b(t) = (\det P)\ \sum_{a=1}^{3} P_{ab}\omega_a(t).$$

for the transformation of the components under a change of basis in E^3 given by an orthogonal matrix $P \in \mathrm{O}(3)$. Notice that the relation (11.6) shows that the vector $\dot{\mathbf{x}}(t)$, although expressed via an axial vector, is a polar vector, since the vector product between an axial vector and a polar vector yields a polar vector. This is consistent with the formulation of $\dot{\mathbf{x}}(t)$ as the physical velocity of a point mass.

A different perspective on these notions and examples, allows one to study how the dynamics of a point mass is described with respect to different reference systems, in physicists' parlance.

Example 11.6.3 We describe the motion of a point mass with respect to an orthonormal basis $\mathcal{E} = (e_1, e_2, e_3)$ *and* with respect to an orthonormal basis $\mathcal{E}'(t) = (e'_1(t), e'_2(t), e'_3(t))$ that *rotates* with respect to $\mathcal{E}$. So we write

$$\mathbf{x}(t) = \sum_{a=1}^{3} x_a(t)\, e_a = \sum_{k=1}^{3} x'_k(t)\, e'_k(t).$$

Considering the time derivative of both sides, we have

$$\dot{\mathbf{x}}(t) = \sum_{a=1}^{3} \dot{x}_a(t)\, e_a = \sum_{k=1}^{3} \dot{x}'_k(t)\, e'_k(t)\ +\ \sum_{k=1}^{3} x'_k\, \dot{e}'_k(t).$$

Using the results of the Example 11.6.1, the second term can be written by means of an angular velocity $\omega(t)$ and thus we have

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}'(t)\ +\ \omega(t) \wedge \mathbf{x}'(t),$$

where $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity of the point mass with respect to $\mathcal{E}$, while $\mathbf{v}' = \dot{\mathbf{x}}'$ is the velocity of the point mass with respect to $\mathcal{E}'(t)$.

With one step further along the same line, by taking a second time derivative results in

$$\begin{aligned}\ddot{\mathbf{x}}(t) &= \ddot{\mathbf{x}}'(t)\ +\ \omega(t) \wedge \dot{\mathbf{x}}'(t)\ +\ \omega(t) \wedge \big(\dot{\mathbf{x}}'(t)\ +\ \omega(t) \wedge \mathbf{x}'(t)\big)\ +\ \dot{\omega}(t) \wedge \mathbf{x}'(t)\\ &= \ddot{\mathbf{x}}'(t)\ +\ 2\,\omega(t) \wedge \dot{\mathbf{x}}'(t)\ +\ \omega(t) \wedge \big(\omega(t) \wedge \mathbf{x}'(t)\big)\ +\ \dot{\omega}(t) \wedge \mathbf{x}'(t).\end{aligned}$$

Using the language of physics, the term $\dot{\mathbf{x}}'(t)$ is the acceleration of the point mass with respect to the 'observer' at rest $\mathcal{E}$, while $\dot{\mathbf{x}}'(t)$ gives its acceleration with respect to the moving 'observer' $\mathcal{E}'(t)$.

With the rotation of $\mathcal{E}'(t)$ with respect to $\mathcal{E}$ given in terms of the angular velocity $\omega(t)$, the term

$$\mathbf{a_C} = 2\,\omega(t) \wedge \dot{\mathbf{x}}'(t)$$

is called the *Coriolis* acceleration, the term

$$\mathbf{a_R} = \omega(t) \wedge \big(\omega(t) \wedge \mathbf{x}'(t)\big)$$

is the *radial* (that is parallel to $\mathbf{x}'(t)$) acceleration, while the term

$$\mathbf{a_T} = \dot{\omega}(t) \wedge \mathbf{x}'(t)$$

is the *tangential* (that is orthogonal to $\mathbf{x}'(t)$) one, and depending on the variation of the angular velocity.

11.7 Rigid Bodies and Inertia Matrix

Example 11.7.1 Consider a system of point masses $\{m_{(j)}\}_{j=1,\dots,N}$ whose mutual distances in E^3 is constant, so that it can be considered as an example of a *rigid body*. The dynamics of each point mass is described by vectors $\mathbf{x}_{(j)}(t)$.

If we do not consider rigid translations, each motion $\mathbf{x}_{(j)}(t)$ is a rotation with the same angular velocity $\omega(t)$ around a fixed point. If we assume, with no loss of generality, that the fixed point coincides with the centre of mass of the system, and we set it to be the origin of E^3, then the total angular momentum of the system (the natural generalization of the angular momentum defined for a single point mass in the Example 1.3.17) is given by (using (11.6))

$$\mathbf{L}(t) = \sum_{j=1}^{N} m_{(j)}\mathbf{x}_{(j)}(t) \wedge \dot{\mathbf{x}}_{(j)}(t) = \sum_{j=1}^{N} m_{(j)}\mathbf{x}_{(j)}(t) \wedge \big(\omega(t) \wedge \mathbf{x}_{(j)}(t)\big).$$

With an orthonormal basis $\mathcal{E} = (e_1, e_2, e_3)$ for E^3, so that $\mathbf{x}_{(j)} = (x_{(j)1}, x_{(j)2}, x_{(j)3})$ and using the definition of vector product in terms of the Levi-Civita symbol, it is straightforward to compute that $\mathbf{L} = (L_1, L_2, L_3)$ is given by

$$L_k = \sum_{s=1}^{3} \Big\{ \sum_{i=1}^{N} m_{(j)}(\|\mathbf{x}_{(j)}\|^2 \delta_{ks} - x_{(j)k}x_{(j)s}) \Big\}\, \omega_s .$$

(In order to lighten notations, we drop for this example the explicit t dependence on the maps.) This expression can be written as

$$L_k = \sum_{s=1}^{3} I_{ks}\,\omega_s$$

where the quantities

$$I_{ks} = \sum_{j=1}^{N} m_{(j)}\big(\|\mathbf{x}_{(j)}\|^2\delta_{ks} - x_{(j)k}x_{(j)s}\big)$$

are the entries of the so called *inertia matrix* $\mathcal{I}$ (or inertia tensor) of the rigid body under analysis.

It is evident that the inertia matrix is symmetric, so from the Proposition 10.5.1, there exists an orthonormal basis for E^3 of eigenvectors for it. Moreover, if λ is an eigenvalue with eigenvector u, we have

$$\lambda\|u\|^2 = \sum_{k,s=1}^{3} I_{ks}u_k u_s = \sum_{j=1}^{N} m_{(j)}\big(\|u\|^2\|\mathbf{x}_{(j)}\|^2 - (u\cdot\mathbf{x}_{(j)})^2\big) \geq 0$$

where the last relation comes from the Schwarz inequality of Proposition 3.1.8. This means that $\mathcal{I}$ has no negative eigenvalues. If (u_1, u_2, u_3) is the orthonormal basis for which the inertia matrix is diagonal, and $(\lambda_1, \lambda_2, \lambda_3)$ are the corresponding eigenvalues, the vector lines $\mathcal{L}(u_a)$ are the so called *principal axes of inertia* for the rigid body, while the eigenvalues are the *moments of inertia*.

We give some basic examples for the inertia matrix of a rigid body.

Excercise 11.7.2 Consider a rigid body given by two point masses with $m_{(1)} = \alpha m_{(2)} = \alpha m$ with $\alpha > 0$, whose position is given in E^3 by the vectors $\mathbf{x}_{(1)} = (0, 0, r)$ and $\mathbf{x}_{(2)} = (0, 0, -\alpha r)$ with $r > 0$. The corresponding inertia matrix is found to be

$$\mathcal{I} = \alpha(1+\alpha)mr^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The principal axes of inertia coincide with the vector lines spanned by the orthonormal basis $\mathcal{E}$. The rigid body has two non zero momenta of inertia; the third momentum of inertia is zero since the rigid body is one dimensional.

Consider a rigid body given by three equal masses $m_{(j)} = m$ and

$$\mathbf{x}_{(1)} = (r, 0, 0), \qquad \mathbf{x}_{(2)} = \frac{1}{2}(-r, \sqrt{3}r, 0), \qquad \mathbf{x}_{(3)} = \frac{1}{2}(-r, -\sqrt{3}r, 0)$$

with $r > 0$, with respect to an orthonormal basis $\mathcal{E}$ in E^3. The inertia matrix is computed to be

$$\mathcal{I} = \frac{3mr^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

so the basis elements $\mathcal{E}$ provide the inertia principal axes.

Finally, consider a rigid body in E^3 consisting of four point masses with $m_{(j)} = m$ and

$$\mathbf{x}_{(1)} = (r, 0, 0), \qquad \mathbf{x}_{(2)} = (-r, 0, 0), \qquad \mathbf{x}_{(3)} = (0, r, 0), \qquad \mathbf{x}_{(4)} = (0, -r, 0)$$

with $r > 0$. The inertia matrix is already diagonal with respect to $\mathcal{E}$ whose basis elements give the principal axes of inertia for the rigid body, while the momenta of inertia is

$$\mathcal{I} = 2mr^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$