Chapter 6
Systems of Linear Equations

Linear equations and system of them are ubiquitous and an important tool in all of physics. In this chapter we shall present a systematic approach to them and to methods for their solutions.

6.1 Basic Notions

Definition 6.1.1 An equation in *n unknown variables* $x_1, \dots, x_n$ with coefficients in $\mathbb{R}$ is called *linear* if it has the form

$$a_1x_1 + \cdots + a_nx_n = b,$$

with $a_i \in \mathbb{R}$ and $b \in \mathbb{R}$. A *solution* for such a linear equation is an n-tuple of real numbers $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ which, when substituted for the unknowns, yield an 'identity', that is

$$a_1\alpha_1 + \cdots + a_n\alpha_n = b.$$

Exercise 6.1.2 It is easy to see that the element $(2, 6, 1) \in \mathbb{R}^3$ is a solution for the equation with real coefficients given by

$$3x_1 - 2x_2 + 7x_3 = 1.$$

Clearly, this is not the only solution for the equation: the element $(\frac{1}{3}, 0, 0)$ is for instance a solution of the same equation.

Definition 6.1.3 A collection of m linear equations in the n unknown variables $x_1, \dots, x_n$ and with real coefficients is called a *linear system* of m equations in n

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unknowns. We shall adopt the following notation

$$\Sigma : \begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases} .$$

A *solution* for a given linear system is an n-tuple $(\alpha_1, \ldots, \alpha_n)$ in $\mathbb{R}^n$ which simultaneously solves each equation of the system. The collection of the solutions of the system Σ is then a subset of $\mathbb{R}^n$, denoted by S_Σ and called the space of solutions of Σ.

A system Σ is called *compatible* or *solvable* if its space of solutions is non void, $S_\Sigma \neq \emptyset$; it will be said to be *incompatible* if $S_\Sigma = \emptyset$.

Exercise 6.1.4 The element $(1, -1) \in \mathbb{R}^2$ is a solution of the system

$$\begin{cases} x + y = 0 \\ x - y = 2 \end{cases} .$$

The following system

$$\begin{cases} x + y = 0 \\ x + y = 1 \end{cases} .$$

has no solutions.

In the present chapter we study conditions under which a linear system is compatible and in such a case find methods to determine its space of solutions. We shall make a systematic use of the matrix formalism described in the previous Chaps. 4 and 5.

Definition 6.1.5 There are two matrices naturally associated to the linear system Σ as given in the Definition 6.1.3:

1. the *matrix of the coefficients* of Σ, $A = (a_{ij}) \in \mathbb{R}^{m,n}$,
2. the *matrix of the inhomogeneous terms* of Σ, $B = {}^t(b_1, \ldots, b_m) \in \mathbb{R}^{m,1}$.

The *complete* or *augmented* matrix of the linear system Σ is given by

$$(A, B) = (a_{ij} \,|\, b_i) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right).$$

By using these matrices the system Σ can be represented as follows

$$\Sigma : \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or more succinctly as

$$\Sigma : AX = B$$

where the array of unknowns is written as $X = {}^t(x_1, \dots, x_n)$ and (abusing notations) thought to be an element in $\mathbb{R}^{n,1}$.

Definition 6.1.6 Two linear systems $\Sigma : AX = B$ and $\Sigma' : A'X = B'$ are called *equivalent* if their spaces of solutions coincide, that is $\Sigma \sim \Sigma'$ if $S_\Sigma = S_{\Sigma'}$. Notice that the vector of unknowns for the two systems is the same.

Remark 6.1.7 The linear systems $AX = B$ and $A'X = B'$ are trivially equivalent

- if (A', B') results from (A, B) by adding null rows,
- if (A', B') is given by a row permutation of (A, B).

The following linear systems are evidently equivalent:

$$\begin{cases} x + y = 0 \\ x - y = 2 \end{cases}, \qquad \begin{cases} x - y = 2 \\ x + y = 0 \end{cases}.$$

Remark 6.1.8 Notice that for a permutation of the columns of the matrix of its coefficients a linear system Σ changes to a system that is in general not equivalent to the starting one. As an example, consider the compatible linear system $AX = B$ given in Exercise 6.1.4. If the columns of A are swapped one has

$$(A, B) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{C_1 \leftrightarrow C_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 1 & 2 \end{array}\right) = (A', B).$$

One checks that the solution $(1, -1)$ of the starting system is not a solution for the system $A'X = B$.

6.2 The Space of Solutions for Reduced Systems

Definition 6.2.1 A linear system $AX = B$ is called *reduced* if the matrix A of its coefficients is reduced by rows in the sense of Sect. 4.4. Solving a reduced system is quite elementary, as the following exercises show.

Exercise 6.2.2 Let the linear system Σ be given by

$$\Sigma : \begin{cases} x + y + 2z = 4 \\ \quad y - 2z = -3 \\ \qquad z = 2 \end{cases} \quad \text{with} \quad (A, B) = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

It is reduced, and has the only solution $(x, y, z) = (-1, 1, 2)$. This is easily found by noticing that the third equation gives $z = 2$. By inserting this value into the second equation one has $y = 1$, and by inserting both these values into the first equation one eventually gets $x = -1$.

Exercise 6.2.3 To solve the linear system

$$\Sigma : \begin{cases} 2x + y + 2z + t = 1 \\ 2x + 3y - z = 3 \\ x + z = 0 \end{cases} \quad \text{with} \quad (A, B) = \left(\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 1 \\ 2 & 3 & -1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

one proceeds as in the previous exercise. The last equation gives $z = -x$. By setting $x = \tau$, one gets the solutions $(x, y, z, t) = (\tau, -\tau + 1, -\tau, \tau)$ with $\tau \in \mathbb{R}$. Clearly Σ has an infinite number of solutions: the space of solutions for Σ is bijective to elements $\tau \in \mathbb{R}$.

Exercise 6.2.4 The linear system $\Sigma : AX = B$, with

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

is trivially not compatible since the last equation would give $0 = 1$.

Remark 6.2.5 If A is reduced by row, the Exercises 6.2.2 and 6.2.3 show that one first determines the value of the unknown corresponding to the pivot (special) element of the bottom row and then replaces such unknown by its value in the remaining equations. This amounts to *delete*, or *eliminate* one of the unknowns. Upon iterating this procedure one completely solves the system. This procedure is showed in the following displays where the pivot elements are bold typed:

$$(A, B) = \left(\begin{array}{ccc|c} \mathbf{1} & 1 & 2 & 4 \\ 0 & \mathbf{1} & -2 & -3 \\ 0 & 0 & \mathbf{1} & 2 \end{array}\right).$$

Here one determines z at first then y and finally x. As for the Exercise 6.2.3, one writes

$$(A, B) = \left(\begin{array}{cccc|c} 2 & 1 & 2 & \mathbf{1} & 1 \\ 2 & \mathbf{3} & -1 & 0 & 3 \\ 1 & 0 & \mathbf{1} & 0 & 0 \end{array}\right)$$

where one determines z, then y and after those one determines t.

The previous exercises suggest the following method that we describe as a proposition.

Proposition 6.2.6 (The method of eliminations) *Let $\Sigma : AX = B$ be a reduced system.*

(1) From the Remark 6.1.7 we may assume that (A, B) has no null rows.
(2) If A has null rows they correspond to equations like $0 = b_i$ with $b_i \neq 0$ since the augmented matrix (A, B) has no null rows. This means that the system is not compatible, $S_\Sigma = \emptyset$.
(3) If A has no null rows, then $m \leq n$. Since A is reduced, it has m pivot elements, so its rank is m. Starting from the bottom row one can then determine the unknown corresponding to the pivot element and then, by substituting such an unknown in the remaining equations, iterate the procedure thus determining the space of solutions.

We describe the general procedure when A is a complete upper triangular matrix.

$$(A, B) = \left(\begin{array}{cccccccccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1m} & * & \dots & * & a_{1n} & & b_1 \\ 0 & a_{22} & a_{23} & \dots & a_{2m} & * & \dots & * & a_{2n} & & b_2 \\ 0 & 0 & a_{33} & \dots & a_{3m} & * & \dots & * & a_{3n} & & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{mm} & * & \dots & * & a_{mn} & & b_m \end{array}\right)$$

with all diagonal elements $a_{ii} \neq 0$. The equation corresponding to the bottom line of the matrix is

$$a_{mm}x_m + a_{mm+1}x_{m+1} + \cdots + a_{mn}x_n = b_m$$

with $a_{mm} \neq 0$. By dividing both sides of the equation by a_{mm}, one has

$$x_m = a_{mm}^{-1}(b_m - a_{mm+1}x_{m+1} + \cdots - a_{mn}x_n).$$

Then x_m is a function of $x_{m+1}, \dots, x_n$. From the $(m-1)$-th row one analogously obtains

$$x_{m-1} = a_{m-1m-1}^{-1}(b_{m-1} - a_{m-1m}x_m - a_{m-1m+1}x_{m+1} + \cdots - a_{m-1n}x_n).$$

By replacing x_m with its value (as a function of $x_{m+1}, \dots, x_n$) previously determined, one writes x_{m-1} as a function of the last unknowns $x_{m+1}, \dots, x_n$. The natural iterations of this process leads to write the unknowns $x_{m-2}, x_{m-3}, \dots, x_1$ as functions of the remaining ones $x_{m+1}, \dots, x_n$.

Remark 6.2.7 Since the m unknowns $x_1, \dots, x_m$ can be expressed as functions of the remaining ones, the $n - m$ unknowns $x_{m+1}, \dots, x_n$, the latter are said to be *free* unknowns. By choosing an arbitrary numerical value for them, $x_{m+1} = \lambda_1, \dots, x_n = \lambda_{n-m}$, with $\lambda_i \in \mathbb{R}$, one obtains a solution, since the matrix A is reduced, of the linear system. This allows one to define a bijection

$$\mathbb{R}^{n-m} \quad \Leftrightarrow \quad S_\Sigma$$

where n is the number of unknowns of Σ and $m = \mathrm{rk}(A)$. One usually labels this result by saying that the linear system has ∞^{n-m} solutions.

6.3 The Space of Solutions for a General Linear System

One of the possible methods to solve a general linear system $AX = B$ uses the notions of row reduction for a matrix as described in Sect. 4.4. From the definition at the beginning of that section one has the following proposition.

Theorem 6.3.1 *Let $\Sigma : AX = B$ be a linear system, and let (A', B') be a transformed by row matrix of (A, B). The linear systems Σ and the transformed one $\Sigma' : A'X = B'$ are equivalent.*

Proof We denote as usual $A = (a_{ij})$ and $B = {}^t(b_1, \ldots, b_m)$. If (A', B') is obtained from (A, B) under a type (e) elementary transformation, the claim is obvious as seen in Remark 6.1.7. If (A', B') is obtained from (A, B) under a type (λ) transformation by the row R_i the claim follows by noticing that, for any $\lambda \neq 0$, the linear equation

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i$$

is equivalent to the equation

$$\lambda a_{i1}x_1 + \cdots + \lambda a_{in}x_n = \lambda b_i .$$

Let now (A', B') be obtained from (A, B) via a type (D) elementary transformation,

$$R_i \mapsto R_i + \lambda R_j$$

with $j \neq i$. To be definite we take $i = 2$ and $j = 1$. We then have

$$(A', B') = \begin{pmatrix} R_1 \\ R_2 + \lambda R_1 \\ \vdots \\ R_m \end{pmatrix} .$$

Let us assume that $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a solution for Σ, that is

$$a_{i1}\alpha_1 + \cdots + a_{in}\alpha_n = b_i$$

for any $i = 1, \ldots, m$. That all but the second equation of Σ' are solved by α is obvious; it remains to verify whether α solves also the second equation in it that is, to show that

$$(a_{21} + \lambda a_{11})x_1 + \cdots + (a_{2n} + \lambda a_{1n})x_n = b_2 + \lambda b_1.$$

If we add the equation for $i = 2$ to λ times the equation for $i = 1$, we obtain

$$(a_{21} + \lambda a_{11})\alpha_1 + \cdots + (a_{2n} + \lambda a_{1n})\alpha_n = b_2 + \lambda b_1$$

thus $(\alpha_1, \ldots, \alpha_n)$ is a solution for Σ' and $S_\Sigma \subseteq S_{\Sigma'}$. The inclusion $S_{\Sigma'} \subseteq S_\Sigma$ is proven in an analogous way. □

By using the above theorem one proves a general method to solve linear systems known as *Gauss' elimination method* or *Gauss' algorithm*.

Theorem 6.3.2 *The space S_Σ of the solutions of the linear system $\Sigma : AX = B$ is determined via the following steps.*

(1) Reduce by rows the matrix (A, B) to (A', B') with A' reduced by row.
(2) Using the method given in the Proposition 6.2.6 determine the space $S_{\Sigma'}$ of the solutions for the system $\Sigma' : A'X = B'$.
(3) From the Theorem 6.3.1 it is $\Sigma \sim \Sigma'$ that is $S_\Sigma = S_{\Sigma'}$.

Exercise 6.3.3 Let us solve the following linear system

$$\Sigma = \begin{cases} 2x + y + z = 1 \\ x - y - z = 0 \\ x + 2y + 2z = 1 \end{cases}$$

whose complete matrix is

$$(A, B) = \left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & 2 & 2 & 1 \end{array}\right).$$

By reducing such a matrix by rows, we have

$$(A, B) \xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \mapsto R_2 + R_1} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ -3 & 0 & 0 & -1 \end{array}\right)$$
$$\xrightarrow[R_3 \mapsto R_3 + R_2]{} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) = (A', B').$$

Since A' is reduced the linear system $\Sigma' : A'X = B'$ is reduced and then solvable by the Gauss' method. We have

$$\Sigma' : \begin{cases} 2x + y + z = 1 \\ 3x = 1 \end{cases} \implies \begin{cases} y + z = \frac{1}{3} \\ x = \frac{1}{3} \end{cases}.$$

It is now clear that one unknown is free so the linear system has ∞^1 solutions. By choosing $z = \lambda$ the space S_Σ of solutions for Σ is

$$S_\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y, z) = (\tfrac{1}{3}, \tfrac{1}{3} - \lambda, \lambda),\ \lambda \in \mathbb{R}\}.$$

On the other end, by choosing $y = \alpha$ the space S_Σ can be written as

$$S_\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y, z) = (\tfrac{1}{3}, \alpha, \tfrac{1}{3} - \alpha),\ \alpha \in \mathbb{R}\}.$$

It is obvious that we are representing the same subset $S_\Sigma \subset \mathbb{R}^3$ in two different ways.

Notice that the number of free unknowns is the difference between the total number of unknowns and the rank of the matrix A.

Exercise 6.3.4 Let us solve the following linear system,

$$\Sigma : \begin{cases} x + y - z = 0 \\ 2x - y = 1 \\ y + 2z = 2 \end{cases}$$

whose complete matrix is

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right).$$

The reduction procedure gives

$$(A, B) \xrightarrow{R_2 \mapsto R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right)$$

$$\xrightarrow[R_3 \mapsto R_3 - R_2]{} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 2 & 1 \\ 0 & 4 & 0 & 1 \end{array}\right) = (A', B').$$

Since A' is reduced the linear system $\Sigma' : A'X = B'$ is reduced with no free unknowns. This means that $S_{\Sigma'}$ (and then S_Σ) has $\infty^0 = 1$ solution. The Gauss' method provides us a way to find such a solution, namely

$$\Sigma' : \begin{cases} x - y + z = 0 \\ -3y + 2z = 1 \\ 4y = 1 \end{cases} \implies \begin{cases} x - z = -\frac{1}{4} \\ 2z = \frac{7}{4} \\ y = \frac{1}{4} \end{cases}.$$

This gives $S_\Sigma = \{(x, y, z) = (\frac{5}{8}, \frac{1}{4}, \frac{7}{8})\}$. Once more the number of free unknowns is the difference between the total number of unknowns and the rank of the matrix A.

The following exercise shows how to solve a linear system with one coefficient given by a real parameter instead of a fixed real number. By solving such a system we mean to analyse the conditions on the parameter under which the system is solvable and to provide its space of solutions as depending on the possible values of the parameter.

Exercise 6.3.5 Let us study the following linear system,

$$\Sigma_\lambda : \begin{cases} x + 2y + z + t = -1 \\ x + y - z + 2t = 1 \\ 2x + \lambda y + \lambda t = 0 \\ -\lambda y - 2z + \lambda t = 2 \end{cases}$$

with $\lambda \in \mathbb{R}$. When the complete matrix for such a system is reduced, particular care must be taken for some critical values of λ. We have

$$(A, B) = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & -1 \\ 1 & 1 & -1 & 2 & 1 \\ 2 & \lambda & 0 & \lambda & 0 \\ 0 & -\lambda & -2 & \lambda & 2 \end{array}\right)$$

$$\xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \mapsto R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & -1 \\ 0 & -1 & -2 & 1 & 2 \\ 0 & \lambda - 4 & -2 & \lambda - 2 & 2 \\ 0 & -\lambda & -2 & \lambda & 2 \end{array}\right)$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \mapsto R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & -1 \\ 0 & -1 & -2 & 1 & 2 \\ 0 & \lambda - 3 & 0 & \lambda - 3 & 0 \\ 0 & -\lambda + 1 & 0 & \lambda - 1 & 0 \end{array}\right) = (A', B').$$

The transformations $R_3 \mapsto R_3 + R_4$, then $R_3 \mapsto \frac{1}{2} R_3$ and finally $R_4 \mapsto R_4 + (1 - \lambda) R_3$ give a further reduction of (A', B') as

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & -1 \\ 0 & -1 & -2 & 1 & 2 \\ 0 & -1 & 0 & \lambda - 2 & 0 \\ 0 & -\lambda + 1 & 0 & \lambda - 1 & 0 \end{array}\right) \mapsto \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & -1 \\ 0 & -1 & -2 & 1 & 2 \\ 0 & -1 & 0 & \lambda - 2 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \end{array}\right) = (A'', B'')$$

with $a_{44} = (1 - \lambda)(\lambda - 3)$. Notice that the last transformation is meaningful for any $\lambda \in \mathbb{R}$. In the reduced form (A'', B'') we have that R_4 is null if and only if either $\lambda = 3$ or $\lambda = 1$. For such values of the parameter λ either R_3 or R_4 in A' is indeed null. We can now conclude that Σ_λ is solvable for any value of $\lambda \in \mathbb{R}$ and we have

- If $\lambda \in \{1, 3\}$ then $a_{44} = 0$, so $\mathrm{rk}(A) = 3$ and Σ_λ has ∞^1 solutions,
- If $\lambda \notin \{1, 3\}$ then $a_{44} \neq 0$, so $\mathrm{rk}(A) = 4$ and Σ_λ has a unique solution.

We can now study the following three cases:

(a) $\lambda \notin \{1, 3\}$, that is

$$\Sigma_\lambda : \begin{cases} x + 2y + z + t = -1 \\ -y - 2z + t = 2 \\ -y + (\lambda - 2)t = 0 \\ (\lambda - 3)(\lambda - 1)t = 0 \end{cases} .$$

From our assumption, we have that $a_{44} = (\lambda - 3)(\lambda - 1) \neq 0$ so we get $t = 0$. By using the Gauss' method we then write

$$\begin{cases} x = 0 \\ z = -1 \\ y = 0 \\ t = 0 \end{cases} .$$

This shows that for $\lambda \neq 1, 3$ the space S_{Σ_λ} does not depend on λ.

(b) If $\lambda = 1$ we can delete the fourth equation since it is a trivial identity. We have then

$$\Sigma_{\lambda=1} : \begin{cases} x + 2y + z + t = -1 \\ -y - 2z + t = 2 \\ y + t = 0 \end{cases} .$$

The Gauss' method gives us

$$\begin{cases} x = 0 \\ z = t - 1 \\ y = -t \end{cases}$$

and this set of solutions can be written as

$$\{(x, y, z, t) \in \mathbb{R}^4 \mid (x, y, z, t) = (0, -\alpha, \alpha - 1, \alpha),\ \alpha \in \mathbb{R}\}.$$

(c) If $\lambda = 3$ the non trivial part of the system turns out to be

$$\Sigma_{\lambda=3} : \begin{cases} x + 2y + z + t = -1 \\ -y - 2z + t = 2 \\ -y + t = 0 \end{cases}$$

and we write the solutions as

$$\begin{cases} x = -3t \\ z = -1 \\ y = t \end{cases}$$

or equivalently $S_{\Sigma_{\lambda=3}} = \{(x, y, z, t) \in \mathbb{R}^4 \mid (x, y, z, t) = (-3\alpha, \alpha, -1, \alpha),\ \alpha \in \mathbb{R}\}$.

What we have discussed can be given in the form of the following theorem which provides general conditions under which a linear system is solvable.

Theorem 6.3.6 (Rouché–Capelli). *The linear system* $\Sigma : AX = B$ *is solvable if and only if* $\text{rk}(A) = \text{rk}(A, B)$. *In such a case, denoting* $\text{rk}(A) = \text{rk}(A, B) = \rho$ *and with* n *the number of unknowns in* Σ*, the following holds true:*

(a) the number of free unknowns is $n - \rho$*,*
(b) the $n - \rho$ *free unknowns have to be selected in such a way that the remaining* ρ *unknowns correspond to linearly independent columns of* A.

Proof By noticing that the linear system Σ can be written as

$$x_1 C_1 + \cdots + x_n C_n = B$$

with $C_1, \ldots, C_n$ the columns of A, we see that Σ is solvable if and only if B is a linear combination of these columns that is if and only if the linear span of the columns of A coincides with the linear span of the columns of (A, B). This condition is fulfilled if and only if $\text{rk}(A) = \text{rk}(A, B)$.

Suppose then that the system is solvable.

(a) Let $\Sigma' : A'X = B'$ be the system obtained from (A, B) by reduction by rows. From the Remark 6.2.7 the system Σ' has $n - \text{rk}(A')$ free unknowns. Since $\Sigma \sim \Sigma'$ and $\text{rk}(A) = \text{rk}(A')$ the claim follows.
(b) Possibly with a swap of the columns in $A = (C_1, \ldots, C_n)$ (which amounts to renaming the unknown), the result that we aim to prove is the following:

$$x_{\rho+1}, \ldots, x_n \text{ are free } \Leftrightarrow \ C_1, \ldots, C_\rho \text{ are linearly independent.}$$

Let us at first suppose that $C_1, \ldots, C_\rho$ are linearly independent, and set $\overline{A} = (C_1, \ldots, C_\rho)$. By a possible reduction and a swapping of some equations, with $\text{rk}(\overline{A}) = \text{rk}(A, B) = \rho$, the matrix for the system can be written as

$$(A', B') = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1\rho} & * & \ldots & * & b_1 \\ 0 & a_{22} & a_{23} & \ldots & a_{2\rho} & * & \ldots & * & b_2 \\ 0 & 0 & a_{33} & \ldots & a_{3\rho} & * & \ldots & * & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_{\rho\rho} & * & \ldots & * & b_\rho \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

The claim—that $x_{\rho+1}, \ldots, x_n$ can be taken to be free—follows easily from the Gauss' method.

On the other hand, let us assume that $x_{\rho+1}, \ldots, x_n$ are free unknowns for the linear system and let us also suppose that $C_1, \ldots, C_\rho$ are linearly dependent. This would result in the rank of $\overline{A}$ be less that ρ and there would exist a reduction of (A, B) for which the matrix of the linear system turns out to be

$$(A', B') = \begin{pmatrix} a_{11} & \dots & a_{1\rho} & * \dots * \\ \vdots & & \vdots & \vdots \quad \vdots \\ a_{\rho-1\,1} & \dots & a_{\rho-1\,\rho} & * \dots * \\ 0 & \dots & 0 & * \dots * \\ \vdots & & \vdots & \vdots \quad \vdots \\ 0 & \dots & 0 & * \dots * \end{pmatrix}.$$

Since $\mathrm{rk}(A', B') = \mathrm{rk}(A, B) = \rho$ there would then be a non zero row R_i in (A', B') with $i \geq \rho$. The equation corresponding to such an R_i, not depending on the first ρ unknowns, would provide a relation among the $x_{\rho+1}, \dots, x_n$, which would then be not free. □

Remark 6.3.7 If the linear system $\Sigma : AX = B$, with n unknowns and m equations is solvable with $\mathrm{rk}(A) = \rho$, then

(i) Σ is equivalent to a linear system Σ' with ρ equations arbitrarily chosen among the m equations in Σ, provided they are linearly independent.
(ii) there is a bijection between the space S_Σ and $\mathbb{R}^{n-\rho}$.

Exercise 6.3.8 Let us solve the following linear system depending on a parameter $\lambda \in \mathbb{R}$,

$$\Sigma : \begin{cases} \lambda x + z = -1 \\ x + (\lambda - 1)y + 2z = 1 \\ x + (\lambda - 1)y + 3z = 0 \end{cases}.$$

We reduce by rows the complete matrix corresponding to Σ as

$$(A, B) = \left(\begin{array}{ccc|c} \lambda & 0 & 1 & -1 \\ 1 & \lambda - 1 & 2 & 1 \\ 1 & \lambda - 1 & 3 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \mapsto R_3 - 3R_1]{R_2 \mapsto R_2 - 2R_1} \left(\begin{array}{ccc|c} \lambda & 0 & 1 & -1 \\ 1 - 2\lambda & \lambda - 1 & 0 & 3 \\ 1 - 3\lambda & \lambda - 1 & 0 & 3 \end{array}\right)$$

$$\xrightarrow[R_3 \mapsto R_3 - R_2]{} \left(\begin{array}{ccc|c} \lambda & 0 & 1 & -1 \\ 1 - 2\lambda & \lambda - 1 & 0 & 3 \\ -\lambda & 0 & 0 & 0 \end{array}\right) = (A', B').$$

Depending on the values of the parameter λ we have the following cases.

(a) If $\lambda = 1$, the matrix A' is not reduced. We then write

$$(A', B') = \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 3 \\ -1 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_3 \mapsto R_3 - R_2]{} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & -3 \end{array}\right).$$

The last row gives the equation $0 = -3$ and in this case the system has no solution.

(b) If $\lambda \neq 1$ the matrix A' is reduced, so we have:

- If $\lambda \neq 0$, then $\mathrm{rk}(A) = 3 = \mathrm{rk}(A, B)$, so the linear system $\Sigma_{\lambda=0}$ has a unique solution. With $\lambda \notin \{0, 1\}$ the reduced system is

$$\Sigma' : \begin{cases} \lambda x + z = -1 \\ (1 - 2\lambda)x + (\lambda - 1)y = 3 \\ -\lambda x = 0 \end{cases}$$

and the Gauss' method gives $S_{\Sigma_\lambda} = (x, y, z) = (0, 3/(\lambda - 1), -1)$.

- If $\lambda = 0$ the system we have to solve is

$$\Sigma' : \begin{cases} z = -1 \\ x - y = 3 \end{cases}$$

whose solutions are given as

$$S_{\Sigma_{\lambda=0}} = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y, z) = (\alpha + 3, \alpha, -1)\ \alpha \in \mathbb{R}\}.$$

Exercise 6.3.9 Let us show that the following system of vectors,

$$v_1 = (1, 1, 0), \quad v_2 = (0, 1, 1), \quad v_3 = (1, 0, 1),$$

is free and then write $v = (1, 1, 1)$ as a linear combination of v_1, v_2, v_3.

We start by recalling that v_1, v_2, v_3 are linearly independent if and only if the rank of the matrix whose columns are the vectors themselves is 3. We have the following reduction,

$$(v_1\ v_2\ v_3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The number of non zero rows of the reduced matrix is 3 so the vectors v_1, v_2, v_3 are linearly independent. Then they are a basis for $\mathbb{R}^3$, so the following relation,

$$xv_1 + yv_2 + zv_3 = v$$

is fullfilled by a unique triple (x, y, z) of coefficients for any $v \in \mathbb{R}^3$. Such a triple is the unique solution of the linear system whose complete matrix is $(A, B) = (v_1\, v_2\, v_3\, v)$. For the case we are considering in this exercise we have

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right).$$

Using for (A, B) the same reduction we used above for A we have

$$(A, B) \quad \mapsto \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \quad \mapsto \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right).$$

The linear system we have then to solve is

$$\begin{cases} x + z = 1 \\ y - z = 0 \\ 2z = 1 \end{cases}$$

giving $(x, y, x) = \frac{1}{2}(1, 1, 1)$. One can indeed directly compute that

$$\tfrac{1}{2}(1, 1, 0) + \tfrac{1}{2}(0, 1, 1) + \tfrac{1}{2}(1, 0, 1) = (1, 1, 1).$$

Exercise 6.3.10 Let us consider the matrix

$$M_\lambda = \begin{pmatrix} \lambda & 1 \\ 1 & \lambda \end{pmatrix}$$

with $\lambda \in \mathbb{R}$. We compute its inverse using the theory of linear systems.

We can indeed write the problem in terms of the linear system

$$\begin{pmatrix} \lambda & 1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

that is

$$\Sigma : \begin{cases} \lambda x + z = 1 \\ x + \lambda z = 0 \\ \lambda y + t = 0 \\ y + \lambda t = 1 \end{cases}.$$

We reduce the complete matrix of the linear system as follows:

$$(A, B) = \left(\begin{array}{cccc|c} \lambda & 0 & 1 & 0 & 1 \\ 1 & 0 & \lambda & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 1 & 0 & \lambda & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \mapsto R_2 - \lambda R_1} \left(\begin{array}{cccc|c} \lambda & 0 & 1 & 0 & 1 \\ 1-\lambda^2 & 0 & 0 & 0 & -\lambda \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 1 & 0 & \lambda & 1 \end{array}\right)$$

$$\xrightarrow[R_4 \mapsto R_4 - \lambda R_3]{} \left(\begin{array}{cccc|c} \lambda & 0 & 1 & 0 & 1 \\ 1-\lambda^2 & 0 & 0 & 0 & -\lambda \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 1-\lambda^2 & 0 & 0 & 1 \end{array}\right) = (A', B').$$

The elementary transformations we used are well defined for any real value of λ. We start by noticing that if $1 - \lambda^2 = 0$ that is $\lambda = \pm 1$, we have

$$(A', B') = \left(\begin{array}{cccc|c} \pm 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \mp 1 \\ 0 & \pm 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

The second and the fourth rows of this matrix show that the corresponding linear system is incompatible. This means that when $\lambda = \pm 1$ the matrix M_λ is not invertible (as we would immediately see by computing its determinant).

We assume next that $1 - \lambda^2 \neq 0$. In such a case we have $\mathrm{rk}(A) = \mathrm{rk}(A, B) = 4$, so there exists a unique solution for the linear system. We write it in the reduced form as

$$\Sigma' : \begin{cases} \lambda x + z = 1 \\ (1 - \lambda^2)x = -\lambda \\ \lambda y + t = 0 \\ (1 - \lambda^2)y = 1 \end{cases}.$$

Its solution is then

$$\begin{cases} z = 1/(1 - \lambda^2) \\ x = -\lambda/(1 - \lambda^2) \\ t = -\lambda/(1 - \lambda^2) \\ y = 1/(1 - \lambda^2) \end{cases},$$

that we write in matrix form as

$$M_\lambda^{-1} = \tfrac{1}{(1-\lambda^2)} \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}.$$

6.4 Homogeneous Linear Systems

We analyse now an interesting class of linear systems (for easy of notation we write $0 = 0_{\mathbb{R}^m}$).

Definition 6.4.1 A linear system $\Sigma : AX = B$ is called *homogeneous* if $B = 0$.

Remark 6.4.2 A linear system $\Sigma : AX = 0$ with $A \in \mathbb{R}^{m,n}$ is always solvable since the null n-tuple (the null vector in $\mathbb{R}^n$) gives a solution for Σ, albeit a trivial one. This also follows form the Rouché-Capelli theorem since one obviously has $\mathrm{rk}(A) = \mathrm{rk}(A, 0)$. The same theorem allows one to conclude that such a trivial solution is indeed the only solution for Σ if and only if $n = \rho = \mathrm{rk}(A)$.

Theorem 6.4.3 *Let* $\Sigma : AX = 0$ *be a homogeneous linear system with* $A \in \mathbb{R}^{m,n}$*. Then* S_Σ *is a vector subspace of* $\mathbb{R}^n$ *with* $\dim S_\Sigma = n - \mathrm{rk}(A)$.

Proof From the Proposition 2.2.2 we have to show that if $X_1, X_2 \in S_\Sigma$ with $\lambda_1, \lambda_2 \in \mathbb{R}$, then $\lambda_1 X_1 + \lambda_2 X_2$ is in S_Σ. Since by hypothesis we have $AX_1 = 0$ and $AX_2 = 0$ we have also $\lambda_1(AX_1) + \lambda_2(AX_2) = 0$. From the properties of the matrix calculus we have in turn $\lambda_1(AX_1) + \lambda_2(AX_2) = A(\lambda_1 X_1 + \lambda_2 X_2)$, thus giving $\lambda_1 X_1 + \lambda_2 X_2$ in S_Σ. We conclude that S_Σ is a vector subspace of $\mathbb{R}^n$.

With $\rho = \mathrm{rk}(A)$, from the Rouché-Capelli theorem we know that Σ has $n - \rho$ free unknowns. This number coincides with the dimension of S_Σ. To show this fact we determine a basis made up of $n - \rho$ elements. Let us assume for simplicity that the free unknowns are the last ones $x_{\rho+1}, \dots, x_n$. Any solution of Σ can then be written as

$$(*, \dots, *, \ x_{\rho+1}, \dots, x_n)$$

where the ρ symbols $*$ stand for the values of $x_1, \dots, x_\rho$ corresponding to each possible value of $x_{\rho+1}, \dots, x_n$. We let now the $(n - \rho)$-dimensional 'vector' $x_{\rho+1}, \dots, x_n$ range over all elements of the canonical basis of $\mathbb{R}^{n-\rho}$ and write the corresponding elements in S_Σ as

$$\begin{aligned} v_1 &= (*, \dots, *, 1, 0, \dots, 0) \\ v_2 &= (*, \dots, *, 0, 1, \dots, 0) \\ &\vdots \\ v_{n-\rho} &= (*, \dots, *, 0, 0, \dots, 1). \end{aligned}$$

The rank of the matrix $(v_1, \dots, v_{n-\rho})$ (that is the matrix whose rows are these vectors) is clearly equal to $n - \rho$, since its last $n - \rho$ columns are linearly independent. This means that its rows, the vectors $v_1, \dots, v_{n-\rho}$, are linearly independent. It is easy to see that such rows generate S_Σ so they are a basis for it and $\dim(S_\Sigma) = n - \rho$. □

It is clear that the general reduction procedure allows one to solve any homogeneous linear system Σ. Since the space S_Σ is in this case a linear space, one can determine a basis for it. The proof of the previous theorem provides indeed an easy method to get such a basis for S_Σ. Once the elements in S_Σ are written in terms of the $n-\rho$ free unknowns a basis for S_Σ is given by fixing for these unknowns the values corresponding to the elements of the canonical basis in $\mathbb{R}^{n-\rho}$.

Exercise 6.4.4 Let us solve the following homogeneous linear system,

$$\Sigma : \begin{cases} x_1 - 2x_3 + x_5 + x_6 = 0 \\ x_1 - x_2 - x_3 + x_4 - x_5 + x_6 = 0 \\ x_1 - x_2 + 2x_4 - 2x_5 + 2x_6 = 0 \end{cases}$$

and let us determine a basis for its space of solutions. The corresponding A matrix is

$$A = \begin{pmatrix} 1 & 0 & -2 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 0 & 2 & -2 & 2 \end{pmatrix}.$$

We reduce it as follows

$$A \mapsto \begin{pmatrix} 1 & 0 & -2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & -1 & 2 & 2 & -3 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix} = A'.$$

Thus $\mathrm{rk}(A) = \mathrm{rk}(A') = 3$. Since the first three rows in A' (and then in A) are linearly independent we choose x_4, x_5, x_6 to be the free unknowns. One clearly has $\Sigma \sim \Sigma' : A'X = 0$ so we can solve

$$\Sigma' : \begin{cases} x_1 - 2x_3 + x_5 + x_6 = 0 \\ x_2 - x_3 - x_4 + 2x_5 = 0 \\ x_3 + x_4 - x_5 + x_6 = 0 \end{cases}.$$

By setting $x_4 = a$, $x_5 = b$ and $x_6 = c$ we have

$$S_\Sigma = \{(x_1, ..., x_6) = (-2a + b - 3c, -b - c, -a + b - c, a, b, c) \mid a, b, c \in \mathbb{R}\}.$$

To determine a basis for S_Σ we let (a, b, c) be the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ of the canonical basis in $\mathbb{R}^3$ since $n - \rho = 6 - 3 = 3$. With this choice we get the following basis

$$\begin{aligned} v_1 &= (-2, 0, -1, 1, 0, 0) \\ v_2 &= (1, -1, 1, 0, 1, 0) \\ v_3 &= (-3, -1, -1, 0, 0, 1). \end{aligned}$$