Chapter 5
The Determinant

The notion of *determinant* of a matrix plays an important role in linear algebra. While the rank measures the linear independence of the row (or column) vectors of a matrix, the determinant (which is defined only for square matrices) is used to control the invertibility of a matrix and in explicitly constructing the inverse of an invertible matrix.

5.1 A Multilinear Alternating Mapping

The determinant can be defined as an abstract function by using multilinear algebra. We shall define it constructively and using a recursive procedure.

Definition 5.1.1 The *determinant* of a 2×2 matrix is the map

$$\det : \mathbb{R}^{2,2} \to \mathbb{R}, \qquad A \mapsto \det(A) = |A|$$

defined as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \mapsto \quad \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

The above definition shows that the determinant can be though of as a function of the column vectors of $A = (C_1, C_2)$, that is

$$\det : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \qquad (C_1, C_2) \mapsto a_{11}a_{22} - a_{12}a_{21}.$$

It is immediate to see that the map det is bilinear on the column of A, that is

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$$\begin{aligned}\det(\lambda C_1 + \lambda' C_1', C_2) &= \lambda \det(C_1, C_2) + \lambda' \det(C_1', C_2) \\ \det(C_1, \lambda C_2 + \lambda' C_2') &= \lambda \det(C_1, C_2) + \lambda' \det(C_1, C_2') \end{aligned} \tag{5.1}$$

for any $C_1,\ C_1',\ C_2,\ C_2' \in \mathbb{R}^2$ and any $\lambda, \lambda' \in \mathbb{R}$.

The map det is indeed alternating (or skew-symmetric), that is

$$\det(C_2, C_1) = -\det(C_1, C_2). \tag{5.2}$$

From (5.2) the determinant of A vanishes if the columns C_1 and C_2 coincide. More generally, $\det(A) = 0$ if $C_2 = \lambda C_1$ for $\lambda \in \mathbb{R}$, since, from (5.1)

$$\det(C_1, C_2) = \det(C_1, \lambda C_1) = \lambda \det(C_1, C_1) = 0.$$

Since the determinant map is bilinear and alternating, one also has

$$\det(C_1 + \lambda C_2, C_2) = \det(C_1, C_2) + \det(\lambda C_2, C_2) = \det(C_1, C_2).$$

Exercise 5.1.2 Given the canonical basis (e_1, e_2) for $\mathbb{R}^2$, we compute

$$\det(e_1, e_1) = \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 0, \qquad \det(e_1, e_2) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\det(e_2, e_1) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \qquad \det(e_2, e_2) = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0.$$

We generalise the definition of determinant to 3×3 and further to $n \times n$ matrices.

Definition 5.1.3 Given a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

one defines $\det : \mathbb{R}^{3,3} \to \mathbb{R}$ as

$$\begin{aligned} \det(A) = |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \end{aligned} \tag{5.3}$$

Exercise 5.1.4 Let us compute the determinant of the following matrix,

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Using the first row as above one gets:

$$\det(A) \;=\; \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \;-\; \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \;=\; 2.$$

It is evident that the map det can be read, as we showed above, as defined on the column vectors of $A = (C_1, C_2, C_3)$, that is

$$\det : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}\,, \qquad (C_1, C_2, C_3) \mapsto \det(A).$$

Remark 5.1.5 It is easy to see that the map det defined in (5.3) is multilinear, that is it is linear in each column argument. Also, for any swap of the columns of A, $\det(A)$ changes its sign. This means that (5.3) is an alternating map (this property generalises the skew-symmetry of the det map on 2×2 matrices). For example,

$$\det(C_2, C_1, C_3) = -\det(C_1, C_2, C_3),$$

with analogous relations holding for any swap of the columns of A. Then $\det(A) = 0$ if one of the columns of A is a multiple of the others, like in

$$\det(C_1, C_2, \lambda C_2) \;=\; \lambda \det(C_1, C_2, C_2) \;=\; -\lambda \det(C_1, C_2, C_2) \;=\; 0.$$

More generally $\det(A) = 0$ if one of the columns of A is a linear combination of the others as in

$$\det(\lambda C_2 + \mu C_3, C_2, C_3) \;=\; \lambda \det(C_2, C_2, C_3) \;+\; \mu \det(C_3, C_2, C_3) \;=\; 0.$$

Exercise 5.1.6 If (e_1, e_2, e_3) is the canonical basis for $\mathbb{R}^3$, generalising Exercise 5.1.2 one finds $\det(e_i, e_i, e_i) = 0$, $\det(e_i, e_i, e_j) = 0$ and $\det(e_1, e_2, e_3) = \det(I_3) = 1$, with I_3 the 3×3 unit matrix.

We have seen that the determinant of a 3×3 matrix A makes use of the determinant of a 2×2 matrix: such a determinant is given as the alternating sum of the elements in the first row of A, times the determinant of suitable 2×2 submatrices in A. This procedure is generalised to define the determinant of $n \times n$ matrices.

Definition 5.1.7 Consider the matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$, or

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

For any pair (i, j) we denote by A_{ij} the $(n-1) \times (n-1)$ submatrix of A obtained by erasing the i-th row and the j-th column of A, Firstly, the number $\det(A_{ij})$ is called the *minor* of the element a_{ij}. Then the *cofactor* α_{ij} of the element a_{ij} (or associated with a_{ij}) is defined as

$$\alpha_{ij} = (-1)^{i+j} \det(A_{ij}).$$

Exercise 5.1.8 With $A \in \mathbb{R}^{3,3}$ given by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & -2 & -1 \\ 2 & 5 & 0 \end{pmatrix},$$

we easily compute for instance,

$$A_{11} = \begin{pmatrix} -2 & -1 \\ 5 & 0 \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$$

and

$$\alpha_{11} = (-1)^{1+1}|A_{11}| = 5, \qquad \alpha_{12} = (-1)^{1+2}|A_{12}| = -2.$$

Definition 5.1.9 Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$. One defines its determinant by the formula

$$\det(A) = a_{11}\alpha_{11} + a_{12}\alpha_{12} + \cdots + a_{1n}\alpha_{1n}. \tag{5.4}$$

Such an expression is also referred to as the expansion of the determinant of the matrix A with respect to its first row.

The above definition is recursive: the determinant of a $n \times n$ matrix involves the determinants of a $(n-1) \times (n-1)$ matrices, starting from the definition of the determinant of a 2×2 matrix. The Definition 5.1.3 is indeed the expansion with respect to the first row as written in (5.4).

That the determinant $\det(A)$ of a matrix A can be equivalently defined in terms of a similar expansion with respect to any row or column of A is the content of the following important theorem, whose proof we omit.

Theorem 5.1.10 (Laplace) *For any $i = 2, \ldots, n$ it holds that*

$$\det(A) = a_{i1}\alpha_{i1} + a_{i2}\alpha_{i2} + \cdots + a_{in}\alpha_{in}. \tag{5.5}$$

This expression is called the expansion of the determinant of A with respect to its i-th row.

For any $j = 1, \ldots, n$, it holds that

$$\det(A) = a_{1j}\alpha_{1j} + a_{2j}\alpha_{2j} + \cdots + a_{nj}\alpha_{nj} \tag{5.6}$$

and this expression is the expansion of the determinant of A with respect to its j-th column.

The expansions (5.5) or (5.6) are called the *cofactor expansion* of the determinant with respect to the corresponding row or column.

Exercise 5.1.11 Let $I_n \in \mathbb{R}^{n,n}$ be the $n \times n$ unit matrix. It is immediate to compute

$$\det(I_n) = 1.$$

From the Laplace theorem the following statement is obvious.

Corollary 5.1.12 *Let* $A \in \mathbb{R}^{n,n}$. *Then* $\det({}^tA) = \det(A)$.

Also, from the Laplace theorem it is immediate to see that $\det(A) = 0$ if A has a null column or a null row. We can still think of the determinant of the matrix A as a function defined on its columns. If $A = (C_1, \cdots, C_n)$, one has $\det(A) = \det(C_1, \dots, C_n)$, that is

$$\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \;\to\; \mathbb{R}, \qquad (C_1, \dots, C_n) \;\mapsto\; \det(A).$$

The following result, that can be proven by using the Definition 5.1.9, generalises properties already seen for the matrices of order two and three.

Proposition 5.1.13 *Let* $A = (C_1, \cdots, C_n) \in \mathbb{R}^{n,n}$. *One has the following properties:*

(i) *For any* $\lambda, \lambda' \in \mathbb{R}$ *and* $C_1' \in \mathbb{R}^n$, *it holds that*

$$\det\left(\lambda C_1 + \lambda' C_1', C_2, \dots, C_n\right) = \lambda \det(C_1, C_2, \dots, C_n) + \lambda' \det(C_1', C_2, \dots, C_n).$$

Analogous properties hold for any other column of A.

(ii) *If* $A' = (C_{\sigma(1)}, \dots, C_{\sigma(n)})$, *where* $\sigma = (\sigma(1), \dots, \sigma(n))$ *is a permutation of the columns transforming* $A \mapsto A'$, *it holds that*

$$\det(A') = (-1)^\sigma \det(A),$$

where $(-1)^\sigma$ *is the parity of the permutation* σ, *that is* $(-1)^\sigma = 1$ *if* σ *is given by an even number of swaps, while* $(-1)^\sigma = -1$ *if* σ *is given by an odd number of swaps.*

Corollary 5.1.14 *Let* $A = (C_1, \cdots, C_n) \in \mathbb{R}^{n,n}$. *Then,*

(i) $\det(\lambda C_1, C_2, \dots, C_n) = \lambda \det(A)$,
(ii) *if* $C_i = C_j$ *for any pair* i, j, *then* $\det(A) = 0$,
(iii) $\det(\alpha_2 C_2 + \cdots + \alpha_n C_n, C_2, \dots, C_n) = 0$; *that is the determinant of a matrix* A *is zero if a column of* A *is a linear combination of its other columns,*
(iv) $\det(C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n, C_2, \dots, C_n) = \det(A)$.

Proof (i) it follows from the Proposition 5.1.13, with $\lambda' = 0$,

(ii) if $C_i = C_j$, the odd permutation σ which swaps C_i with C_j does not change the matrix A; then from the Proposition 5.1.13, $\det(A) = -\det(A) \Rightarrow \det(A) = 0$,

(iii) from 5.1.13 we can write

$$\det(\alpha_2 C_2 + \cdots + \alpha_n C_n, C_2, \ldots, C_n) = \sum_{i=2}^{n} \alpha_i \det(C_i, C_2, \ldots, C_n) = 0$$

since, by point (ii), one has $\det(C_i, C_2, \ldots, C_n) = 0$ for any $i = 2, \ldots, n$,

(iv) from the previous point we have

$$\begin{aligned} &\det(C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n, C_2, \ldots, C_n) \\ &\qquad = \det(C_1, C_2, \ldots, C_n) + \sum_{i=2}^{n} \alpha_i \det(C_i, C_2, \ldots, C_n) = \det(A). \end{aligned}$$

This concludes the proof. □

Remark 5.1.15 From the Laplace theorem it follows that the determinant of A is an alternating and multilinear function even when it is defined via the expansion with respect to the rows of A.

We conclude this section with the next useful theorem, whose proof we omit.

Theorem 5.1.16 (Binet) *Given $A, B \in \mathbb{R}^{n,n}$ it holds that*

$$\det(AB) = \det(A)\det(B). \tag{5.7}$$

5.2 Computing Determinants via a Reduction Procedure

The Definition 5.1.9 and the Laplace theorem allow one to compute the determinant of any square matrix. In this section we illustate how the reduction procedure studied in the previous chapter can be used when computing a determinant. We start by considering upper triangular matrices.

Proposition 5.2.1 *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$. If A is diagonal then,*

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

More generally, if A is an upper (respectively a lower) triangular matrix, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

Proof The claim for a diagonal matrix is evident. With A an upper (respectively a lower) triangular matrix, by expanding $\det(A)$ with respect to the first column (respectively row) the submatrix A_{11} is upper (respectively lower) triangular. The result then follows by a recursive argument. □

Remark 5.2.2 In Sect. 4.4 we defined the type (s), (λ) and (D) elementary transformations on the rows of a matrix. If A is a square matrix, transformed under one of these transformations into the matrix A', we have the following results:

- (s) : $\det(A') = -\det(A)$ (Proposition 5.1.13),
- (λ) : $\det(A') = \lambda \det(A)$ (Corollary 5.1.14),
- (D) : $\det(A') = \det(A)$ (Corollary 5.1.14).

It is evident that the above relations are valid when A is mapped into A' with elementary transformations on its columns.

Exercise 5.2.3 Let us use row transformations on the matrix A:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow[R_3 \mapsto R_3 - R_1]{R_2 \mapsto R_2 - 2R_1} A'$$

$$A' = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[R'_3 \mapsto R'_3 + R'_2]{} A''$$

$$A'' = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{pmatrix}.$$

Since we have used only type (D) transformations, from the Remark 5.2.2 $\det(A) = \det(A'')$ and from Proposition 5.2.1 we have $\det(A'') = 1 \cdot (-1) \cdot 5 = -5$.

Exercise 5.2.4 Via a sequence of elementary transformations,

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} A'$$

$$A' = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R'_3 \mapsto R'_3 - R'_1]{R'_2 \mapsto R'_2 - 2R'_1} A''$$

$$A'' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R''_3 \mapsto R''_3 - R''_2} A'''$$

$$A''' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{pmatrix}.$$

Since we used once a type (s) transformation $\det(A) = -\det(A''') = -3$.

Remark 5.2.5 The sequence of transformations defined in the Exercise 5.2.3 does not alter the space of rows of the matrix A, that is $R(A) = R(A'')$. The sequence of transformations defined in the Exercise 5.2.4 does alter both the spaces of rows and of columns of the matrix A.

Proposition 5.2.6 *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be reduced by rows and without null rows. It holds that*

$$\det(A) = (-1)^{\sigma} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$$

where $a_{i,\sigma(i)}$ is the pivot element of the i-th row and σ is the permutation of the columns mapping A into the corresponding (complete) upper triangular matrix.

Proof Let $B = (b_{ij}) \in \mathbb{R}^{n,n}$ be the complete upper triangular matrix obtained from A with the permutation σ. From the Proposition 5.1.13 we have $\det(A) = (-1)^{\sigma}\det(B)$, with $(-1)^{\sigma}$ the parity of σ. From the Proposition 5.2.1 we have $\det(B) = b_{11}b_{22}\cdots b_{nn}$, with $b_{11} = a_{1,\sigma(1)}, \ \ldots\ , b_{nn} = a_{n,\sigma(n)}$ by construction, thus obtaining the claim. □

The above proposition suggests that a sequence of type (D) transformations on the rows of a square matrix simplifies the computation of its determinant. We summarise this suggestion as a remark.

Remark 5.2.7 In order to compute the determinant of the matrix $A \in \mathbb{R}^{n,n}$:

- riduce A by row with only type (D) transformations to a matrix A'; this is always possible from the Proposition 4.4.3. Then $\det(A) = \det(A')$ from the Remark 5.2.2;
- compute the determinant of A'. Then,
  - if A' has a null row, from the Corollary 5.1.14 one has $\det(A') = 0$;
  - if A' has no null rows, from the Proposition 5.2.6 one has

$$\det(A') = (-1)^{\sigma} a'_{1,\sigma(1)} \cdots a'_{n,\sigma(n)}$$

  with $\sigma = (\sigma(1), \ldots, \sigma(n))$.

Again, the result continues to hold by exchanging rows with columns.

Exercise 5.2.8 With the above method we have the following equalities,

$$\begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1.$$

5.3 Invertible Matrices

We now illustrate some use of the determinant in the study of invertible matrices.

Proposition 5.3.1 *Given* $A \in \mathbb{R}^{n,n}$*, it holds that*

$$\det(A) = 0 \quad \Leftrightarrow \quad \mathrm{rk}(A) < n.$$

Proof

($\Leftarrow$) : By hypothesis, the system of the n columns $C_1, \ldots, C_n$ of A is not free, so there is least a column of A which is a linear combination of the other columns. From the Corollary 5.1.14 it is then $\det(A) = 0$.

($\Rightarrow$) : Suppose $\mathrm{rk}(A) = n$. With this assumption A could be reduced by row to a matrix A' having no null rows since $\mathrm{rk}(A) = \mathrm{rk}(A') = n$. From the Proposition 5.2.6, $\det(A')$ is the product of the pivot elements in A' and since by hypothesis they would be non zero, we would have $\det(A') \neq 0$ and from the Remark 5.2.2 $\det(A) = \det(A') \neq 0$ thus contradicting the hypothesis. □

Remark 5.3.2 The equivalence in the above proposition can be stated as

$$\det(A) \neq 0 \quad \Leftrightarrow \quad \mathrm{rk}(A) = n.$$

Proposition 5.3.3 *A matrix* $A = (a_{ij}) \in \mathbb{R}^{n,n}$ *is invertible (or non-singular) if and only if*

$$\det(A) \neq 0.$$

Proof If A is invertible, the matrix inverse A^{-1} exists with $AA^{-1} = I_n$. From the Binet theorem, this yields $\det(A)\det(A^{-1}) = \det(I_n) = 1$ or $\det(A^{-1}) = (\det(A))^{-1} \neq 0$.

If $\det(A) \neq 0$, the inverse of A is the matrix $B = (b_{ij})$ with elements

$$b_{ij} = \frac{1}{\det(A)}\,\alpha_{ji}$$

and α_{ji} the cofactor of a_{ji} as in the Definition 5.1.7. Indeed, an explicit computation shows that

$$(AB)_{rs} = \sum_{k=1}^{n} a_{rk}b_{ks} = \frac{1}{\det(A)}\sum_{k=1}^{n} a_{rk}\alpha_{sk} = \begin{cases} \frac{\det(A)}{\det(A)} = 1 & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases}.$$

The result for $r = s$ is just the cofactor expansion of the determinant given by the Laplace theorem in Theorem 5.1.10, while the result for $r \neq s$ is known as the second Laplace theorem (whose discussion we omit). The above amounts to $AB = I_n$ so that A is invertible with $B = A^{-1}$. □

Notice that in the inverse matrix B there is an index transposition, that is up to the determinant factor, the element b_{ij} of B is given by the cofactor α_{ji} of A.

Exercise 5.3.4 Let us compute the inverse of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is possible if and only if $|A| = ad - bc \neq 0$. In such a case,

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{pmatrix},$$

with $\alpha_{11} = d$, $\alpha_{21} = -b$, $\alpha_{12} = -c$, $\alpha_{22} = a$, so that we get the final result,

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Exercise 5.3.5 Let us compute the inverse of the matrix A from the Exercise 5.1.4,

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix}.$$

From the computation there $\det(A) = 2$, explicit computations show that

$$\begin{array}{lll} \alpha_{11} = (+)\,1 & \alpha_{12} = (-)\,2 & \alpha_{13} = (+)\,(-1) \\ \alpha_{21} = (-)\,1 & \alpha_{22} = (+)\,2 & \alpha_{23} = (-)\,1 \\ \alpha_{31} = (+)\,1 & \alpha_{32} = (-)\,0 & \alpha_{33} = (+)\,1 \end{array}.$$

It is then easy to find that

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ -2 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$