# Chapter 1
# Vectors and Coordinate Systems

The notion of a *vector*, or more precisely of a *vector applied at a point*, originates in physics when dealing with an observable quantity. By this or simply by *observable*, one means anything that can be measured in the physical space—the space of physical events— via a suitable measuring process. Examples are the velocity of a point particle, or its acceleration, or a force acting on it. These are characterised at the point of application by a *direction*, an *orientation* and a *modulus* (or *magnitude*). In the following pages we describe the physical space in terms of points and applied vectors, and use these to describe the physical observables related to the motion of a point particle with respect to a coordinate system (a reference frame). The geometric structures introduced in this chapter will be more rigorously analysed in the next chapters.

## 1.1 Applied Vectors

We refer to the common intuition of a physical space made of points, where the notions of *straight* line between two points and of the length of a segment (or equivalently of distance of two points) are assumed to be given. Then, a vector $v$ can be denoted as

$$v = B - A \qquad \text{or} \qquad v = AB,$$

where $A$, $B$ are two points of the physical space. Then, $A$ is the point of application of $v$, its direction is the straight line joining $B$ to $A$, its orientation the one of the arrow pointing from $A$ towards $B$, and its modulus the real number $\|B - A\| = \|A - B\|$, that is the length (with respect to a fixed unit) of the segment $AB$.



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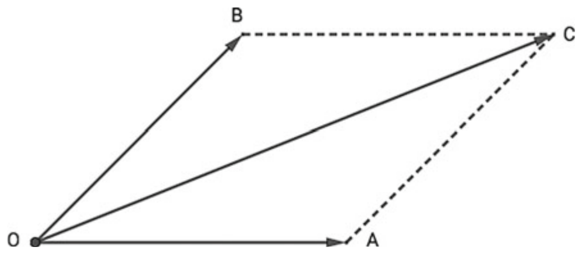


**Fig. 1.1** The parallelogram rule

If $\mathcal{S}$ denotes the usual three dimensional physical space, we denote by

$$\mathcal{W}^3 = \{B - A \mid A, B \in \mathcal{S}\}$$

the collection of all applied vectors at any point of $\mathcal{S}$ and by

$$\mathcal{V}^3_A = \{B - A \mid B \in \mathcal{S}\}$$

the collection of all vectors applied at $A$ in $\mathcal{S}$. Then

$$\mathcal{W}^3 = \bigcup_{A \in \mathcal{S}} \mathcal{V}^3_A.$$

*Remark 1.1.1* Once fixed a point $O$ in $\mathcal{S}$, one sees that there is a bijection between the set $\mathcal{V}^3_O = \{B - O \mid B \in \mathcal{S}\}$ and $\mathcal{S}$ itself. Indeed, each point $B$ in $\mathcal{S}$ uniquely determines the element $B - O$ in $\mathcal{V}^3_O$, and each element $B - O$ in $\mathcal{V}^3_O$ uniquely determines the point $B$ in $\mathcal{S}$.

It is well known that the so called *parallelogram rule* defines in $\mathcal{V}^3_O$ a sum of vectors, where

$$(A - O) + (B - O) = (C - O),$$

with $C$ the fourth vertex of the parallelogram whose other three vertices are $A$, $O$, $B$, as shown in Fig. 1.1.

The vector $\mathbf{0} = O - O$ is called the *zero vector* (or *null* vector); notice that its modulus is zero, while its direction and orientation are undefined.

It is evident that $\mathcal{V}^3_O$ is closed with respect to the notion of sum defined above. That such a sum is associative and abelian is part of the content of the proposition that follows.

**Proposition 1.1.2** *The datum* $(\mathcal{V}^3_O, +, \mathbf{0})$ *is an abelian group.*

*Proof* Clearly the zero vector $\mathbf{0}$ is the neutral (identity) element for the sum in $\mathcal{V}^3_O$, that added to any vector leave the latter unchanged. Any vector $A - O$ has an inverse

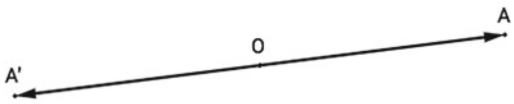


**Fig. 1.2** The opposite of a vector: $A^{'} - O = -(A - O)$

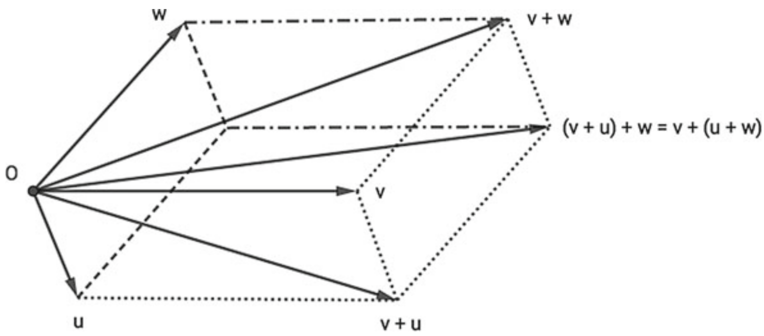


**Fig. 1.3** The associativity of the vector sum

with respect to the sum (that is, any vector has an opposite vector) given by $A' - O$, where $A'$ is the symmetric point to $A$ with respect to $O$ on the straight line joining $A$ to $O$ (see Fig. 1.2).

From its definition the sum of two vectors is a commutative operation. For the associativity we give a pictorial argument in Fig. 1.3. □

There is indeed more structure. The physical intuition allows one to consider multiples of an applied vector. Concerning the collection $\mathcal{V}_O^3$, this amounts to define an operation involving vectors applied in $O$ and real numbers, which, in order not to create confusion with vectors, are called (real) *scalars*.

**Definition 1.1.3** Given the scalar $\lambda \in \mathbb{R}$ and the vector $A - O \in \mathcal{V}_O^3$, the *product by a scalar*

$$B - O = \lambda(A - O)$$

is the vector such that:

(i) $A, B, O$ are on the same (straight) line,
(ii) $B - O$ and $A - O$ have the same orientation if $\lambda > 0$, while $A - O$ and $B - O$ have opposite orientations if $\lambda < 0$,
(iii) $\|B - O\| = |\lambda| \, \|A - O\|$.

The main properties of the operation of product by a scalar are given in the following proposition.

**Proposition 1.1.4** *For any pair of scalars* $\lambda, \mu \in \mathbb{R}$ *and any pair of vectors* $A - O,\ B - O \in \mathcal{V}_O^3$, *it holds that:*

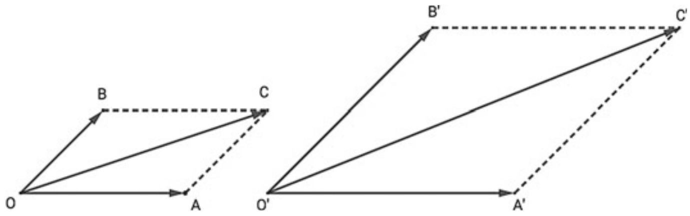


**Fig. 1.4** The scaling $\lambda(C - O) = (C' - O)$ with $\lambda > 1$

1. $\lambda(\mu(A - O)) = (\lambda\mu)(A - O)$,
2. $1(A - O) = A - O$,
3. $\lambda((A - O) + (B - O)) = \lambda(A - O) + \lambda(B - O)$,
4. $(\lambda + \mu)(A - O) = \lambda(A - O) + \mu(A - O)$.

*Proof* 1. Set $C - O = \lambda(\mu(A - O))$ and $D - O = (\lambda\mu)(A - O)$. If one of the scalars $\lambda, \mu$ is zero, one trivially has $C - O = \mathbf{0}$ and $D - O = \mathbf{0}$, so Point 1. is satisfied. Assume now that $\lambda \neq 0$ and $\mu \neq 0$. Since, by definition, both $C$ and $D$ are points on the line determined by $O$ and $A$, the vectors $C - O$ and $D - O$ have the same direction. It is easy to see that $C - O$ and $D - O$ have the same orientation: it will coincide with the orientation of $A - O$ or not, depending on the sign of the product $\lambda\mu \neq 0$. Since $|\lambda\mu| = |\lambda||\mu| \in R$, one has $\|C - O\| = \|D - O\|$.

2. It follows directly from the definition.
3. Set $C - O = (A - O) + (B - O)$ and $C' - O = (A' - O) + (B' - O)$, with $A' - O = \lambda(A - O)$ and $B' - O = \lambda(B - O)$.
   We verify that $\lambda(C - O) = C' - O$ (see Fig. 1.4).
   Since $OA$ is parallel to $OA'$ by definition, then $BC$ is parallel to $B'C'$; $OB$ is indeed parallel to $OB'$, so that the planar angles $\widehat{OBC}$ and $\widehat{OB'C'}$ are equal. Also $\lambda(OB) = OB'$, $\lambda(OA) = OA'$, and $\lambda(BC) = B'C'$. It follows that the triangles $OBC$ and $OB'C'$ are similar: the vector $OC$ is then parallel $OC'$ and they have the same orientation, with $\|OC'\| = \lambda\,\|OC\|$. From this we obtain $OC' = \lambda(OC)$.
4. The proof is analogue to the one in point 3. □

What we have described above shows that the operations of sum and product by a scalar give $\mathcal{V}_O^3$ an algebraic structure which is richer than that of abelian group. Such a structure, that we shall study in detail in Chap. 2, is called in a natural way *vector space*.

## 1.2 Coordinate Systems

The notion of coordinate system is well known. We rephrase its main aspects in terms of vector properties.

**Definition 1.2.1** Given a line $r$, a *coordinate system* $\Lambda$ on it is defined by a point $O \in r$ and a vector $\mathbf{i} = A - O$, where $A \in r$ and $A \neq O$.

The point $O$ is called the *origin* of the coordinate system, the norm $\|A - O\|$ is the *unit of measure* (or *length)* of $\Lambda$, with $\mathbf{i}$ the basis *unit vector*. The orientation of $\mathbf{i}$ is the *orientation* of the coordinate system $\Lambda$.

A coordinate system $\Lambda$ provides a bijection between the points on the line $r$ and $\mathbb{R}$. Any point $P \in r$ singles out the real number $x$ such that $P - O = x\mathbf{i}$; viceversa, for any $x \in \mathbb{R}$ one has the point $P \in r$ defined by $P - O = x\mathbf{i}$. One says that $P$ has coordinate $x$, and we shall denote it by $P = (x)$, with respect to the coordinate system $\Lambda$ that is also denoted as $(O; x)$ or $(O; \mathbf{i})$.

**Definition 1.2.2** Given a plane $\alpha$, a coordinate system $\Pi$ on it is defined by a point $O \in \alpha$ and a pair of non zero distinct (and not having the same direction) vectors $\mathbf{i} = A - O$ and $\mathbf{j} = B - O$ with $A, B \in \alpha$, and $\|A - O\| = \|B - O\|$.

The point $O$ is the origin of the coordinate system, the (common) norm of the vectors $\mathbf{i}, \mathbf{j}$ is the unit length of $\Pi$, with $\mathbf{i}, \mathbf{j}$ the basis *unit vectors*. The system is oriented in such a way that the vector $\mathbf{i}$ coincides with $\mathbf{j}$ after an anticlockwise rotation of angle $\phi$ with $0 < \phi < \pi$. The line defined by $O$ and $\mathbf{i}$, with its given orientation, is usually referred to as a the *abscissa axis*, while the one defined by $O$ and $\mathbf{j}$, again with its given orientation, is called *ordinate axis*.

As before, it is immediate to see that a coordinate system $\Pi$ on $\alpha$ allows one to define a bijection between points on $\alpha$ and *ordered* pairs of real numbers. Any $P \in \alpha$ uniquely provides, via the parallelogram rule (see Fig. 1.5), the ordered pair $(x, y) \in \mathbb{R}^2$ with $P - O = x\mathbf{i} + y\mathbf{j}$; conversely, for any given ordered pair $(x, y) \in \mathbb{R}^2$, one defines $P \in \alpha$ as given by $P - O = x\mathbf{i} + y\mathbf{j}$.

With respect to $\Pi$, the elements $x \in \mathbb{R}$ and $y \in \mathbb{R}$ are the *coordinates* of $P$, and this will be denoted by $P = (x, y)$. The coordinate system $\Pi$ will be denoted $(O; \mathbf{i}, \mathbf{j})$ or $(O; x, y)$.

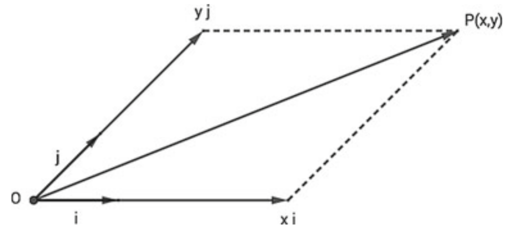


**Fig. 1.5** The bijection $P(x, y) \leftrightarrow P - O = x\mathbf{i} + y\mathbf{j}$ in a plane

**Definition 1.2.3** A coordinate system $\Pi = (O; \mathbf{i}, \mathbf{j})$ on a plane $\alpha$ is called an *orthogonal cartesian coordinate system* if $\phi = \pi/2$, where $\phi$ is as before the width of the anticlockwise rotation under which $\mathbf{i}$ coincides with $\mathbf{j}$.

In order to introduce a coordinate system for the physical three dimensional space, we start by considering three unit-length vectors in $\mathcal{V}_O^3$ given as $\mathbf{u} = U - O$, $\mathbf{v} = V - O$, $\mathbf{w} = W - O$, and we assume the points $O, U, V, W$ not to be on the same plane. This means that any two vectors, $\mathbf{u}$ and $\mathbf{v}$ say, determine a plane which does not contain the third point, say $W$. Seen from $W$, the vector $\mathbf{u}$ will coincide with $\mathbf{v}$ under an anticlockwise rotation by an angle that we denote by $\widehat{\mathbf{uv}}$.

**Definition 1.2.4** An ordered triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of unit vectors in $\mathcal{V}_O^3$ which do not lie on the same plane is called *right-handed* if the three angles $\widehat{\mathbf{uv}}$, $\widehat{\mathbf{vw}}$, $\widehat{\mathbf{wu}}$, defined by the prescription above are smaller than $\pi$. Notice that the order of the vectors matters.

**Definition 1.2.5** A coordinate system $\Sigma$ for the space $\mathcal{S}$ is given by a point $O \in \mathcal{S}$ and three non zero distinct (and not lying on the same plane) vectors $\mathbf{i} = A - O$, $\mathbf{j} = B - O$ and $\mathbf{k} = C - O$, with $A, B, C \in \mathcal{S}$, and $\|A - O\| = \|B - O\| = \|C - O\|$ and $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ giving a right-handed triple.

The point $O$ is the *origin* of the coordinate system, the common length of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is the unit measure in $\Sigma$, with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the basis *unit vectors*. The line defined by $O$ and $\mathbf{i}$, with its orientation, is the abscissa axis, that defined by $O$ and $\mathbf{j}$ is the ordinate axis, while the one defined by $O$ and $\mathbf{k}$ is the quota axis.

With respect to the coordinate system $\Sigma$, one establishes, via $\mathcal{V}_O^3$, a bijection between ordered triples of real numbers and points in $\mathcal{S}$. One has

$$P \quad \leftrightarrow \quad P - O \quad \leftrightarrow \quad (x, y, z)$$

with $P - O = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ as in Fig. 1.6. The real numbers $x, y, z$ are the *components* (or *coordinates*) of the applied vector $P - O$, and this will be denoted by $P = (x, y, z)$. Accordingly, the coordinate system will be denoted by $\Sigma = (O; \mathbf{i}, \mathbf{j}, \mathbf{k}) = (O; x, y, z)$. The coordinate system $\Sigma$ is called cartesian orthogonal if the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are pairwise orthogonal.

By writing $\mathbf{v} = P - O$, it is convenient to denote by $v_x, v_y, v_z$ the components of $\mathbf{v}$ with respect to a cartesian coordinate system $\Sigma$, so to have

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}.$$

In order to simplify the notations, we shall also write this as

$$\mathbf{v} = (v_x, v_y, v_z),$$

implicitly assuming that such components of $\mathbf{v}$ refer to the cartesian coordinate system $(O; \mathbf{i}, \mathbf{j}, \mathbf{k})$. Clearly the components of a given vector $\mathbf{v}$ depend on the particular coordinate system one is using.

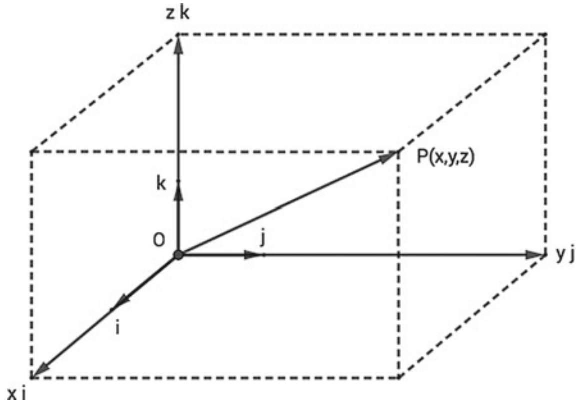


**Fig. 1.6** The bijection $P(x, y, z) \leftrightarrow P - O = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in the space

**Exercise 1.2.6** One has

1. The zero (null) vector $\mathbf{0} = O - O$ has components $(0, 0, 0)$ with respect to any coordinate system whose origin is $O$, and it is the only vector with this property.
2. Given a coordinate system $\Sigma = (O; \mathbf{i}, \mathbf{j}, \mathbf{k})$, the basis unit vectors have components
$$\mathbf{i} = (1, 0, 0)\ , \quad \mathbf{j} = (0, 1, 0)\ , \quad \mathbf{k} = (0, 0, 1).$$
3. Given a coordinate system $\Sigma = (O; \mathbf{i}, \mathbf{j}, \mathbf{k})$ for the space $\mathcal{S}$, we call *coordinate plane* each plane determined by a pair of axes of $\Sigma$. We have $\mathbf{v} = (a, b, 0)$, with $a, b \in \mathbb{R}$, if $\mathbf{v}$ is on the plane $xy$, $\mathbf{v}' = (0, b', c')$ if $\mathbf{v}'$ is on the plane $yz$, and $\mathbf{v}'' = (a'', 0, c'')$ if $\mathbf{v}''$ is on the plane $xz$.

*Example 1.2.7* The motion of a point mass in three dimensional space is described by a map $t \in \mathbb{R} \mapsto \mathbf{x}(t) \in \mathcal{V}^3_O$ where $t$ represents the *time* variable and $\mathbf{x}$(t) is the position of the point mass at time $t$. With respect to a coordinate system $\Sigma = (O; x, y, z)$ we then write

$$\mathbf{x}(t) = (x(t), y(t), z(t)) \qquad \text{or equivalently} \qquad \mathbf{x}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

The corresponding *velocity* is a vector applied in $\mathbf{x}(t)$, that is $\mathbf{v}(t) \in \mathcal{V}^3_{\mathbf{x}(t)}$, with components

$$\mathbf{v}(t) = (v_x(t), v_y(t), v_z(t)) = \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = (\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}),$$

while the acceleration is the vector $\mathbf{a}(t) \in \mathcal{V}^3_{\mathbf{x}(t)}$ with components

$$\mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = (\frac{\mathrm{d}x^2}{\mathrm{d}t^2}, \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}, \frac{\mathrm{d}^2 z}{\mathrm{d}t^2}).$$

One also uses the notations

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \dot{\mathbf{x}} \qquad \text{and} \qquad \mathbf{a} = \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \dot{\mathbf{v}} = \ddot{\mathbf{x}}.$$

In the newtonian formalism for the dynamics, a *force* acting on the given point mass is a vector applied in $\mathbf{x}(t)$, that is $\mathbf{F} \in \mathcal{V}^3_{\mathbf{x}(t)}$ with components $\mathbf{F} = (F_x, F_y, F_z)$, and the *second law of dynamics* is written as

$$m\,\mathbf{a} = \mathbf{F}$$

where $m > 0$ is the value of the *inertial mass* of the moving point mass. Such a relation can be written component-wise as

$$m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F_x, \qquad m\,\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F_y, \qquad m\,\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = F_z.$$

A coordinate system for $\mathcal{S}$ allows one to express the operations of sum and product by a scalar in $\mathcal{V}^3_O$ in terms of elementary algebraic expressions.

**Proposition 1.2.8** *With respect to the coordinate system* $\Sigma = (O; \mathbf{i}, \mathbf{j}, \mathbf{k})$*, let us consider the vectors* $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ *and* $\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$*, and the scalar* $\lambda \in \mathbb{R}$*. One has:*

*(1)* $\mathbf{v} + \mathbf{w} = (v_x + w_x)\mathbf{i} + (v_y + w_y)\mathbf{j} + (v_z + w_z)\mathbf{k}$,
*(2)* $\lambda\mathbf{v} = \lambda v_x\mathbf{i} + \lambda v_y\mathbf{j} + \lambda v_z\mathbf{k}$.

*Proof* (1) Since $\mathbf{v} + \mathbf{w} = (v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) + (w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k})$, by using the commutativity and the associativity of the sum of vectors applied at a point, one has

$$\mathbf{v} + \mathbf{w} = (v_x\mathbf{i} + w_x\mathbf{i}) + (v_y\mathbf{j} + w_y\mathbf{j}) + (v_z\mathbf{k} + w_z\mathbf{k}).$$

Being the product distributive over the sum, this can be regrouped as in the claimed identity.

(2) Along the same lines as (1). □

*Remark 1.2.9* By denoting $\mathbf{v} = (v_x, v_y, v_z)$ and $\mathbf{w} = (w_x, w_y, w_z)$, the identities proven in the proposition above are written as

$$(v_x, v_y, v_z) + (w_x, w_y, w_z) = (v_x + w_x, v_y + w_y, v_z + w_z),$$
$$\lambda(v_x, v_y, v_z) = (\lambda v_x, \lambda v_y, \lambda v_z).$$

This suggests a generalisation we shall study in detail in the next chapter. If we denote by $\mathbb{R}^3$ the set of ordered triples of real numbers, and we consider a pair of elements $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ in $\mathbb{R}^3$, with $\lambda \in \mathbb{R}$, one can introduce a sum of triples and a product by a scalar:

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3),$$
$$\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3).$$

## 1.3 More Vector Operations

In this section we recall the notions—originating in physics—of scalar product, vector product and mixed products.

Before we do this, as an elementary consequence of the Pythagora's theorem, one has the following (see Fig. 1.6)

**Proposition 1.3.1** *Let* $\mathbf{v} = (v_x, v_y, v_z)$ *be an arbitrary vector in* $\mathcal{V}_O^3$ *with respect to the cartesian orthogonal coordinate system* $(O; \mathbf{i}, \mathbf{j}, \mathbf{z})$. *One has*

$$\|\mathbf{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

**Definition 1.3.2** Let us consider a pair of vectors $\mathbf{v}, \mathbf{w} \in \mathcal{V}_O^3$. The *scalar product* of $\mathbf{v}$ and $\mathbf{w}$, denoted by $\mathbf{v} \cdot \mathbf{w}$, is the real number

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \; \|\mathbf{w}\| \cos\alpha$$

with $\alpha = \widehat{\mathbf{vw}}$ the plane angle defined by $\mathbf{v}$ and $\mathbf{w}$. Since $\cos\alpha = \cos(-\alpha)$, for this definition one has $\cos\widehat{\mathbf{vw}} = \cos\widehat{\mathbf{wv}}$.

The definition of a scalar product for vectors in $\mathcal{V}_O^2$ is completely analogue.

*Remark 1.3.3* The following properties follow directly from the definition.

(1) If $\mathbf{v} = \mathbf{0}$, then $\mathbf{v} \cdot \mathbf{w} = 0$.
(2) If $\mathbf{v}, \ \mathbf{w}$ are both non zero vectors, then

$$\mathbf{v} \cdot \mathbf{w} = 0 \quad \Longleftrightarrow \quad \cos\alpha = 0 \quad \Longleftrightarrow \quad \mathbf{v} \perp \mathbf{w}.$$

(3) For any $\mathbf{v} \in \mathcal{V}_O^3$, it holds that:

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$$

and moreover

$$\mathbf{v} \cdot \mathbf{v} = 0 \quad \Longleftrightarrow \quad \mathbf{v} = 0.$$

(4) From (2), (3), if $(O; \mathbf{i}, \mathbf{j}, \mathbf{k})$ is an orthogonal cartesian coordinate system, then

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \qquad \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0.$$

**Proposition 1.3.4** *For any choice of* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}_O^3$ *and* $\lambda \in \mathbb{R}$, *the following identities hold.*

*(i)* $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$,
*(ii)* $(\lambda \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (\lambda \mathbf{w}) = \lambda(\mathbf{v} \cdot \mathbf{w})$,
*(iii)* $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

*Proof* (i) From the definition one has

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\ \|\mathbf{w}\| \cos \widehat{\mathbf{v}\mathbf{w}} = \|\mathbf{w}\|\ \|\mathbf{v}\| \cos \widehat{\mathbf{w}\mathbf{v}} = \mathbf{w} \cdot \mathbf{v}.$$

(ii) Setting $a = (\lambda \mathbf{v}) \cdot \mathbf{w},\ b = \mathbf{v} \cdot (\lambda \mathbf{w})$ and $c = \lambda(\mathbf{v} \cdot \mathbf{w})$, from the Definition 1.3.2 and the properties of the norm of a vector, one has

$$\begin{aligned}
a &= (\lambda \mathbf{v}) \cdot \mathbf{w} = \|\lambda \mathbf{v}\|\ \|\mathbf{w}\| \cos \alpha' = |\lambda| \|\mathbf{v}\|\ \|\mathbf{w}\| \cos \alpha' \\
b &= \mathbf{v} \cdot (\lambda \mathbf{w}) = \|\mathbf{v}\|\ \|\lambda \mathbf{w}\| \cos \alpha'' = \|\mathbf{v}\|\ |\lambda| \|\mathbf{w}\| \cos \alpha'' \\
c &= \lambda(\mathbf{v} \cdot \mathbf{w}) = \lambda(\|\mathbf{v}\|\ \|\mathbf{w}\| \cos \alpha) = \lambda \|\mathbf{v}\|\ \|\mathbf{w}\| \cos \alpha
\end{aligned}$$

where $\alpha' = \widehat{(\lambda \mathbf{v})\mathbf{w}}$, $\alpha'' = \widehat{\mathbf{v}(\lambda \mathbf{w})}$ and $\alpha = \widehat{\mathbf{v}\mathbf{w}}$. If $\lambda = 0$, then $a = b = c = 0$. If $\lambda > 0$, then $|\lambda| = \lambda$ and $\alpha = \alpha' = \alpha''$; from the commutativity and the associativity of the product in $\mathbb{R}$, this gives that $a = b = c$. If $\lambda < 0$, then $|\lambda| = -\lambda$ and $\alpha' = \alpha'' = \pi - \alpha$, thus giving $\cos \alpha' = \cos \alpha'' = -\cos \alpha$. These read $a = b = c$.

(iii) We sketch the proof for parallel $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Under this condition, the result depends on the relative orientations of the vectors. If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ have the same orientation, one has

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \|\mathbf{u}\|\ \|\mathbf{v} + \mathbf{w}\| \\
&= \|\mathbf{u}\|(\|\mathbf{v}\| + \|\mathbf{w}\|) \\
&= \|\mathbf{u}\|\ \|\mathbf{v}\| + \|\mathbf{u}\|\ \|\mathbf{w}\| \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.
\end{aligned}$$

If $\mathbf{v}$ and $\mathbf{w}$ have the same orientation, which is not the orientation of $\mathbf{u}$, one has

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= -\|\mathbf{u}\|\ \|\mathbf{v} + \mathbf{w}\| \\
&= -\|\mathbf{u}\|(\|\mathbf{v}\| + \|\mathbf{w}\|) \\
&= -\|\mathbf{u}\|\ \|\mathbf{v}\| - \|\mathbf{u}\|\ \|\mathbf{w}\| \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.
\end{aligned}$$

We leave the reader to explicitly prove the other cases. □

By expressing vectors in $\mathcal{V}_O^3$ in terms of an orthogonal cartesian coordinate system, the scalar product has an expression that will allow us to define the scalar product of vectors in the more general situation of euclidean spaces.

**Proposition 1.3.5** *Given* $(O; \mathbf{i}, \mathbf{j}, \mathbf{k})$, *an orthogonal cartesian coordinate system for* $\mathcal{S}$; *with vectors* $\mathbf{v} = (v_x, v_y, v_z)$ *and* $\mathbf{w} = (w_x, w_y, w_z)$ *in* $\mathcal{V}_O^3$, *one has*

$$\mathbf{v} \cdot \mathbf{w} = v_x w_x + v_y w_y + v_z w_z.$$

*Proof* With $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ and $\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$, from Proposition 1.3.4, one has

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} = &\ (v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) \cdot (w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}) \\ = &\ v_x w_x \mathbf{i} \cdot \mathbf{i} + v_y w_x \mathbf{j} \cdot \mathbf{i} + v_z w_x \mathbf{k} \cdot \mathbf{i} \\ &+ v_x w_y \mathbf{i} \cdot \mathbf{j} + v_y w_y \mathbf{j} \cdot \mathbf{j} + v_z w_y \mathbf{k} \cdot \mathbf{j} + v_x w_z \mathbf{i} \cdot \mathbf{k} + v_y w_z \mathbf{j} \cdot \mathbf{k} + v_z w_z \mathbf{k} \cdot \mathbf{k}. \end{aligned}$$

The result follows directly from (4) in Remark 1.3.3, that is $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$ as well as $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$. □

**Exercise 1.3.6** With respect to a given cartesian orthogonal coordinate system, consider the vectors $\mathbf{v} = (2, 3, 1)$ and $\mathbf{w} = (1, -1, 1)$. We verify they are orthogonal. From (2) in Remark 1.3.3 this is equivalent to show that $\mathbf{v} \cdot \mathbf{w} = 0$. From Proposition 1.3.5, one has $\mathbf{v} \cdot \mathbf{w} = 2 \cdot 1 + 3 \cdot (-1) + 1 \cdot 1 = 0$.

*Example 1.3.7* If the map $\mathbf{x}(t) : \mathbb{R} \ni t \mapsto \mathbf{x}(t) \in \mathcal{V}_O^3$ describes the motion (notice that the range of the map gives the trajectory) of a point mass (with mass $m$), its *kinetic energy* is defined by

$$T = \frac{1}{2} m \, \|\mathbf{v}(t)\|^2.$$

With respect to an orthogonal coordinate system $\Sigma = (O; \mathbf{i}, \mathbf{j}, \mathbf{k})$, given $\mathbf{v}(t) = (v_x(t), v_y(t), v_z(t))$ as in the Example 1.2.7, we have from the Proposition 1.3.5 that

$$T = \frac{1}{2} m \, (v_x^2 + v_y^2 + v_z^2).$$

Also the following notion will be generalised in the context of euclidean spaces.

**Definition 1.3.8** Given two non zero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{V}_O^3$, the *orthogonal projection* of $\mathbf{v}$ along $\mathbf{w}$ is defined as the vector $\mathbf{v}_\mathbf{w}$ in $\mathcal{V}_O^3$ given by

$$\mathbf{v}_\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \, \mathbf{w}.$$

As the first part of Fig. 1.7 displays, $\mathbf{v}_\mathbf{w}$ is parallel to $\mathbf{w}$.

From the identities proven in Proposition 1.3.4 one easily has

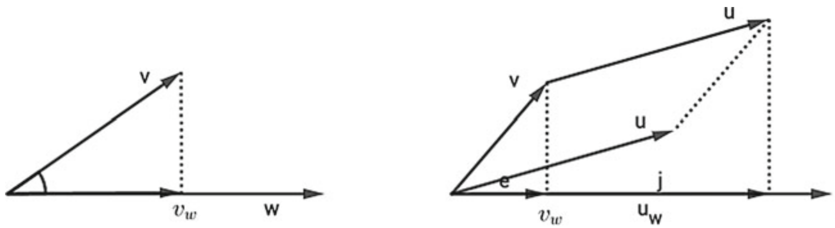


**Fig. 1.7** Orthogonal projections

**Proposition 1.3.9** *For any* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}_O^3$*, the following identities hold:*

*(a)* $(\mathbf{u} + \mathbf{v})_{\mathbf{w}} = \mathbf{u}_{\mathbf{w}} + \mathbf{v}_{\mathbf{w}}$,
*(b)* $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}_{\mathbf{w}} \cdot \mathbf{w} = \mathbf{w}_{\mathbf{v}} \cdot \mathbf{v}$ .

The point (a) is illustrated by the second part of the Fig. 1.7.

*Remark 1.3.10* The scalar product we have defined is a map

$$\sigma : \mathcal{V}_O^3 \times \mathcal{V}_O^3 \longrightarrow \mathbb{R}, \qquad \sigma(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}.$$

Also, the scalar product of vectors on a plane is a map $\sigma : \mathcal{V}_O^2 \times \mathcal{V}_O^2 \longrightarrow \mathbb{R}$.

**Definition 1.3.11** Let $\mathbf{v}, \mathbf{w} \in \mathcal{V}_O^3$. The *vector product* between $\mathbf{v}$ and $\mathbf{w}$, denoted by $\mathbf{v} \wedge \mathbf{w}$, is defined as the vector in $\mathcal{V}_O^3$ whose modulus is

$$\|\mathbf{v} \wedge \mathbf{w}\| = \|\mathbf{v}\| \; \|\mathbf{w}\| \sin \alpha,$$

where $\alpha = \widehat{\mathbf{v}\mathbf{w}}$, with $0 < \alpha < \pi$ is the angle defined by $\mathbf{v}$ e $\mathbf{w}$; the direction of $\mathbf{v} \wedge \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$; and its orientation is such that $(\mathbf{v}, \mathbf{w}, \mathbf{v} \wedge \mathbf{w})$ is a right-handed triple as in Definition 1.2.4.

*Remark 1.3.12* The following properties follow directly from the definition.

(i) if $\mathbf{v} = \mathbf{0}$ then $\mathbf{v} \wedge \mathbf{w} = \mathbf{0}$,
(ii) if $\mathbf{v}$ and $\mathbf{w}$ are both non zero then

$$\mathbf{v} \wedge \mathbf{w} = 0 \quad \Longleftrightarrow \quad \sin \alpha = 0 \quad \Longleftrightarrow \quad \mathbf{v} \parallel \mathbf{w},$$

(one trivially has $\mathbf{v} \wedge \mathbf{v} = 0$),
(iii) if $(O; \mathbf{i}, \mathbf{j}, \mathbf{k})$ is an orthogonal cartesian coordinate system, then

$$\mathbf{i} \wedge \mathbf{j} = \mathbf{k} = -\mathbf{j} \wedge \mathbf{i}, \qquad \mathbf{j} \wedge \mathbf{k} = \mathbf{i} = -\mathbf{k} \wedge \mathbf{j}, \qquad \mathbf{k} \wedge \mathbf{i} = \mathbf{j} = -\mathbf{i} \wedge \mathbf{k}.$$

We omit to prove the following proposition.

**Proposition 1.3.13** *For any* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}^3_O$ *and* $\lambda \in \mathbb{R}$*, the following identities holds*:

*(i)* $\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v}$,
*(ii)* $(\lambda\mathbf{v}) \wedge \mathbf{w} = \mathbf{v} \wedge (\lambda\mathbf{w}) = \lambda(\mathbf{v} \wedge \mathbf{w})$
*(iii)* $\mathbf{u} \wedge (\mathbf{v} + \mathbf{w}) = \mathbf{u} \wedge \mathbf{v} + \mathbf{u} \wedge \mathbf{w}$,

**Exercise 1.3.14** With respect to a given cartesian orthogonal coordinate system, consider in $\mathcal{V}^3_O$ the vectors $\mathbf{v} = (1, 0, -1)$ e $\mathbf{w} = (-2, 0, 2)$. To verify that they are parallel, we recall the abov e result (ii) in the Remark 1.3.12 and compute, using the Proposition 1.3.15, that $\mathbf{v} \wedge \mathbf{w} = 0$.

**Proposition 1.3.15** *Let* $\mathbf{v} = (v_x, v_y, v_z)$ *and* $\mathbf{w} = (w_x, w_y, w_z)$ *be elements in* $\mathcal{V}^3_O$ *with respect to a given cartesian orthogonal coordinate system. It is*

$$\mathbf{v} \wedge \mathbf{w} = (v_y w_z - v_z w_y, v_z w_x - v_x w_z, v_x w_y - v_y w_x).$$

*Proof* Given the Remark 1.3.12 and the Proposition 1.3.13, this comes as an easy computation. □

*Remark 1.3.16* The vector product defines a map

$$\tau : \mathcal{V}^3_O \times \mathcal{V}^3_O \longrightarrow \mathcal{V}^3_O, \qquad \tau(\mathbf{v}, \mathbf{w}) = \mathbf{v} \wedge \mathbf{w}.$$

Clearly, such a map has no meaning on a plane.

*Example 1.3.17* By slightly extending the Definition 1.3.11, one can use the vector product for additional notions coming from physics. Following Sect. 1.1, we consider vectors $\mathbf{u}, \mathbf{w}$ as elements in $\mathcal{W}^3$, that is vectors applied at arbitrary points in the physical three dimensional space $\mathcal{S}$, with components $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{w} = (w_x, w_y, w_z)$ with respect to a cartesian orthogonal coordinate system $\Sigma = (O; \mathbf{i}, \mathbf{j}, \mathbf{k})$. In parallel with Proposition 1.3.15, we define $\tau : \mathcal{W}^3 \times \mathcal{W}^3 \to \mathcal{W}^3$ as

$$\mathbf{u} \wedge \mathbf{w} = (u_y w_z - u_z w_y, \ u_z w_x - u_x w_z, \ u_x w_y - u_y w_x).$$

If $\mathbf{u} \in \mathcal{V}^3_{\mathbf{x}}$ is a vector applied at $\mathbf{x}$, its *momentum* with respect to a point $\mathbf{x}' \in \mathcal{S}$ is the vector in $\mathcal{W}^3$ defined by

$$\mathbf{M} = (\mathbf{x} - \mathbf{x}') \wedge \mathbf{u}.$$

In particular, if $\mathbf{u} = \mathbf{F}$ is a force acting on a point mass in $\mathbf{x}$, its momentum is $\mathbf{M} = (\mathbf{x} - \mathbf{x}') \wedge \mathbf{F}$.

If $\mathbf{x}(t) \in \mathcal{V}^3_O$ describes the motion of a point mass (with mass $m > 0$), whose velocity is $\mathbf{v}(t)$, then its corresponding *angular momentum* with respect to a point $\mathbf{x}'$ is defined by

$$\mathbf{L}_{\mathbf{x}'}(t) = (\mathbf{x}(t) - \mathbf{x}') \wedge m\mathbf{v}(t).$$

**Exercise 1.3.18** The angular momentum is usually defined with respect to the origin of the coordinate system $\Sigma$, giving $\mathbf{L_O}(t) = \mathbf{x}(t) \wedge m\mathbf{v}(t)$. If we consider a circular uniform motion

$$\mathbf{x}(t) = \Big(x(t) = r\cos(\omega t),\ \ y(t) = r\sin(\omega t),\ \ z(t) = 0\Big),$$

with $r > 0$ the radius of the trajectory and $\omega \in \mathbb{R}$ the angular velocity, then

$$\mathbf{v}(t) = \Big(v_x(t) = -r\omega\sin(\omega t),\ \ y(t) = r\omega\cos(\omega t),\ \ v_z(t) = 0\Big)$$

so that

$$\mathbf{L_O}(t) = (0, 0, mr\omega).$$

Thus, a circular motion on the $xy$ plane has angular momentum along the $z$ axis.

**Definition 1.3.19** Given an *ordered* triple $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}_O^3$, their *mixed product* is the real number

$$\mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}).$$

**Proposition 1.3.20** *Given a cartesian orthogonal coordinate system in* $\mathcal{S}$ *with* $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$ *and* $\mathbf{w} = (w_x, w_y, w_z)$ *in* $\mathcal{V}_O^3$, *one has*

$$\mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}) = u_x(v_y w_z - v_z w_y) + u_y(v_z w_x - v_x w_z) + u_z(v_x w_y - v_y w_x).$$

*Proof* It follows immediately by Propositions 1.3.5 and 1.3.15. □

In the space $\mathcal{S}$, the vector product between $\mathbf{u} \wedge \mathbf{w}$ is the area of the parallelogram defined by $\mathbf{u}$ and $\mathbf{v}$, while the mixed product $\mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w})$ give the volume of the parallelepiped defined by $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

**Proposition 1.3.21** *Given* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}_O^3$.

1. *Denote* $\alpha = \widehat{\mathbf{v}\mathbf{w}}$ *the angle defined by* $\mathbf{v}$ *and* $\mathbf{w}$. *Then, the area A of the parallelogram whose edges are* $\mathbf{u}$ *and* $\mathbf{v}$, *is given by*

$$A = \|\mathbf{v}\|\ \|\mathbf{w}\| \sin\alpha = \|\mathbf{v} \wedge \mathbf{w}\|.$$

2. *Denote* $\theta = \mathbf{u}\widehat{(\mathbf{v} \wedge \mathbf{w})}$ *the angle defined by* $\mathbf{u}$ *and* $\mathbf{v} \wedge \mathbf{w}$. *Then the volume V of the parallelepiped whose edges are* $\mathbf{u}, \mathbf{v}, \mathbf{w}$, *is given by*

$$V = A\|\mathbf{u}\| \cos\theta = \|\mathbf{u} \cdot \mathbf{v} \wedge \mathbf{w}\|.$$

*Proof* The claim is evident, as shown in the Figs. 1.8 and 1.9. □

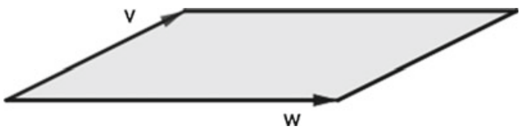


**Fig. 1.8** The area of the parallelogramm with edges **v** and **w**

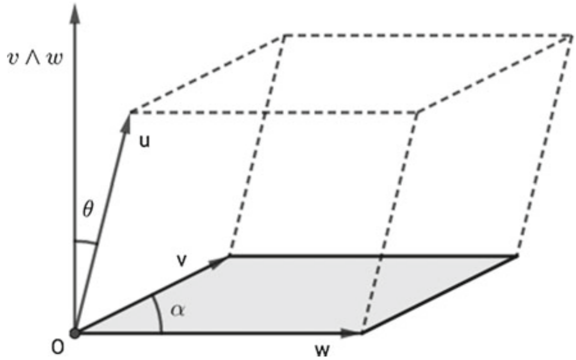


**Fig. 1.9** The volume of the parallelogramm with edges **v**, **w**, **u**

## 1.4 Divergence, Rotor, Gradient and Laplacian

We close this chapter by describing how the notion of vector applied at a point also allows one to introduce a definition of a *vector field*.

The intuition coming from physics requires to consider, for each point $\mathbf{x}$ in the physical space $\mathcal{S}$, a vector applied at $\mathbf{x}$. We describe it as a map

$$\mathcal{S} \ni \mathbf{x} \quad \mapsto \quad \mathbf{A}(\mathbf{x}) \in \mathcal{V}^3_{\mathbf{x}}.$$

With respect to a given cartesian orthogonal reference system for $\mathcal{S}$ we can write this in components as $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{A}(\mathbf{x}) = (A_1(\mathbf{x}), A_2(\mathbf{x}), A_3(\mathbf{x}))$ and one can act on a vector field with partial derivatives (first order differential operators), $\partial_a = (\partial/\partial x_a)$ with $a = 1, 2, 3$, defined as usual by

$$\partial_a(x_b) = (\delta_{ab}), \qquad \text{with} \quad \delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}.$$

Then, (omitting the explicit dependence of $\mathbf{A}$ on $\mathbf{x}$) one defines

$$\begin{aligned} \operatorname{div} \mathbf{A} &= \sum_{k=1}^{3} (\partial_k A_k) \in \mathbb{R} \\ \operatorname{rot} \mathbf{A} &= (\partial_2 A_3 - \partial_3 A_2)\mathbf{i} + (\partial_3 A_1 - \partial_1 A_3)\mathbf{j} + (\partial_1 A_2 - \partial_2 A_1)\mathbf{k} \in \mathcal{V}^3_{\mathbf{x}}. \end{aligned}$$

By introducing the triple $\nabla = (\partial_1, \partial_2, \partial_3)$, such actions can be formally written as a scalar product and a vector product, that is

$$\begin{aligned} \operatorname{div} \mathbf{A} &= \nabla \cdot \mathbf{A} \\ \operatorname{rot} \mathbf{A} &= \nabla \wedge \mathbf{A}\,. \end{aligned}$$

Furthermore, if $f : \mathcal{S} \to \mathbb{R}$ is a real valued function defined on $\mathcal{S}$, that is a (real) *scalar field* on $\mathcal{S}$, one has the grad operator

$$\operatorname{grad} f = \nabla f = (\partial_1 f,\ \partial_2 f,\ \partial_3 f)$$

as well as the Laplacian operator

$$\nabla^2 f = \operatorname{div}(\nabla f) = \Big(\sum_{k=1}^{3} \partial_k \partial_k\Big) f = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f\,.$$

**Exercise 1.4.1** The properties of the mixed products yields a straightforward proof of the identity

$$\operatorname{div}(\operatorname{rot} \mathbf{A}) = \nabla \cdot (\nabla \wedge \mathbf{A}) = 0\,,$$

for any vector field **A**. On the other hand, a direct computation shows also the identity

$$\operatorname{rot}(\operatorname{grad} f) = \nabla \wedge (\operatorname{grad} f) = 0\,,$$

for any scalar field $f$.