
CHAPTER XIX

The Alternating Product

The alternating product has applications throughout mathematics. In differential geometry, one takes the maximal alternating product of the tangent space to get a canonical line bundle over a manifold. Intermediate alternating products give rise to differential forms (sections of these products over the manifold). In this chapter, we give the algebraic background for these constructions.

For a reasonably self-contained treatment of the action of various groups of automorphisms of bilinear forms on tensor and alternating algebras, together with numerous classical examples, I refer to:

R. HOWE, Remarks on classical invariant theory, *Trans. AMS* **313** (1989), pp. 539–569

§1 DEFINITION AND BASIC PROPERTIES

Consider the category of modules over a commutative ring R .

We recall that an r -multilinear map $f: E^{(r)} \rightarrow F$ is said to be **alternating** if $f(x_1, \dots, x_r) = 0$ whenever $x_i = x_j$ for some $i \neq j$.

Let \mathfrak{a}_r be the submodule of the tensor product $T^r(E)$ generated by all elements of type

$$x_1 \otimes \cdots \otimes x_r$$

where $x_i = x_j$ for some $i \neq j$. We define

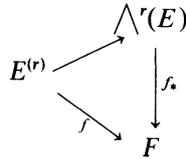
$$\bigwedge^r(E) = T^r(E)/\mathfrak{a}_r.$$

Then we have an r -multilinear map $E^{(r)} \rightarrow \bigwedge^r(E)$ (called canonical) obtained

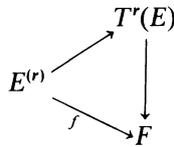
from the composition

$$E^{(r)} \rightarrow T^r(E) \rightarrow T^r(E)/\alpha_r = \bigwedge^r(E).$$

It is clear that our map is alternating. *Furthermore, it is universal with respect to r -multilinear alternating maps on E .* In other words, if $f : E^{(r)} \rightarrow F$ is such a map, there exists a unique linear map $f_* : \bigwedge^r(E) \rightarrow F$ such that the following diagram is commutative:



Our map f_* exists because we can first get an induced map $T^r(E) \rightarrow F$ making the following diagram commutative:



and this induced map vanishes on α_r , hence inducing our f_* .

The image of an element $(x_1, \dots, x_r) \in E^{(r)}$ in the canonical map into $\bigwedge^r(E)$ will be denoted by $x_1 \wedge \dots \wedge x_r$. It is also the image of $x_1 \otimes \dots \otimes x_r$ in the factor homomorphism $T^r(E) \rightarrow \bigwedge^r(E)$.

In this way, \bigwedge^r becomes a functor, from modules to modules. Indeed, let $u : E \rightarrow F$ be a homomorphism. Given elements $x_1, \dots, x_r \in E$, we can map

$$(x_1, \dots, x_r) \mapsto u(x_1) \wedge \dots \wedge u(x_r) \in \bigwedge^r(F).$$

This map is multilinear alternating, and therefore induces a homomorphism

$$\bigwedge^r(u) : \bigwedge^r(E) \rightarrow \bigwedge^r(F).$$

The association $u \mapsto \bigwedge^r(u)$ is obviously functorial.

Example. Open any book on differential geometry (complex or real) and you will see an application of this construction when E is the tangent space of a point on a manifold, or the dual of the tangent space. When taking the dual, the construction gives rise to differential forms.

We let $\bigwedge(E)$ be the direct sum

$$\bigwedge(E) = \bigoplus_{r=0}^{\infty} \bigwedge^r(E).$$

We shall make $\bigwedge(E)$ into a graded R -algebra and call it the **alternating algebra** of E , or also the **exterior algebra**, or the **Grassmann algebra**. We shall first discuss the general situation, with arbitrary graded rings.

Let G be an additive monoid again, and let $A = \bigoplus_{r \in G} A_r$ be a G -graded R -algebra. Suppose given for each A_r a submodule \mathfrak{a}_r , and let $\mathfrak{a} = \bigoplus_{r \in G} \mathfrak{a}_r$. Assume that \mathfrak{a} is an ideal of A . Then \mathfrak{a} is called a **homogeneous ideal**, and we can define a graded structure on A/\mathfrak{a} . Indeed, the bilinear map

$$A_r \times A_s \rightarrow A_{r+s}$$

sends $\mathfrak{a}_r \times A_s$ into \mathfrak{a}_{r+s} and similarly, sends $A_r \times \mathfrak{a}_s$ into \mathfrak{a}_{r+s} . Thus using representatives in A_r, A_s respectively, we can define a bilinear map

$$A_r/\mathfrak{a}_r \times A_s/\mathfrak{a}_s \rightarrow A_{r+s}/\mathfrak{a}_{r+s},$$

and thus a bilinear map $A/\mathfrak{a} \times A/\mathfrak{a} \rightarrow A/\mathfrak{a}$, which obviously makes A/\mathfrak{a} into a graded R -algebra.

We apply this to $T^r(E)$ and the modules \mathfrak{a}_r defined previously. If

$$x_i = x_j \quad (i \neq j)$$

in a product $x_1 \wedge \cdots \wedge x_r$, then for any $y_1, \dots, y_s \in E$ we see that

$$x_1 \wedge \cdots \wedge x_r \wedge y_1 \wedge \cdots \wedge y_s$$

lies in \mathfrak{a}_{r+s} , and similarly for the product on the left. Hence the direct sum $\bigoplus \mathfrak{a}_r$ is an ideal of $T(E)$, and we can define an R -algebra structure on $T(E)/\mathfrak{a}$. The product on homogeneous elements is given by the formula

$$((x_1 \wedge \cdots \wedge x_r), (y_1 \wedge \cdots \wedge y_s)) \mapsto x_1 \wedge \cdots \wedge x_r \wedge y_1 \wedge \cdots \wedge y_s.$$

We use the symbol \wedge also to denote the product in $\bigwedge(E)$. This product is called the **alternating product** or **exterior product**. If $x \in E$ and $y \in E$, then $x \wedge y = -y \wedge x$, as follows from the fact that $(x + y) \wedge (x + y) = 0$.

We observe that \bigwedge is a functor from the category of modules to the category of graded R -algebras. To each linear map $f : E \rightarrow F$ we obtain a map

$$\bigwedge(f) : \bigwedge(E) \rightarrow \bigwedge(F)$$

which is such that for $x_1, \dots, x_r \in E$ we have

$$\bigwedge(f)(x_1 \wedge \cdots \wedge x_r) = f(x_1) \wedge \cdots \wedge f(x_r).$$

Furthermore, $\bigwedge(f)$ is a homomorphism of graded R -algebras.

Proposition 1.1. *Let E be free of dimension n over R . If $r > n$ then $\bigwedge^r(E) = 0$. Let $\{v_1, \dots, v_n\}$ be a basis of E over R . If $1 \leq r \leq n$, then $\bigwedge^r(E)$ is free over R , and the elements*

$$v_{i_1} \wedge \cdots \wedge v_{i_r}, \quad i_1 < \cdots < i_r$$

form a basis of $\bigwedge^r(E)$ over k . We have

$$\dim_R \bigwedge^r(E) = \binom{n}{r}.$$

Proof. We shall first prove our assertion when $r = n$. Every element of E can be written in the form $\sum a_i v_i$, and hence using the formula $x \wedge y = -y \wedge x$ we conclude that $v_1 \wedge \cdots \wedge v_n$ generates $\bigwedge^n(E)$. On the other hand, we know from the theory of determinants that given $a \in R$, there exists a unique multilinear alternating form f_a on E such that

$$f_a(v_1, \dots, v_n) = a.$$

Consequently, there exists a unique linear map

$$\bigwedge^n(E) \rightarrow R$$

taking the value a on $v_1 \wedge \cdots \wedge v_n$. From this it follows at once that $v_1 \wedge \cdots \wedge v_n$ is a basis of $\bigwedge^n(E)$ over R .

We now prove our statement for $1 \leq r \leq n$. Suppose that we have a relation

$$0 = \sum a_{(i)} v_{i_1} \wedge \cdots \wedge v_{i_r}$$

with $i_1 < \cdots < i_r$ and $a_{(i)} \in R$. Select any r -tuple $(j) = (j_1, \dots, j_r)$ such that $j_1 < \cdots < j_r$ and let j_{r+1}, \dots, j_n be those values of i which do not appear among (j_1, \dots, j_r) . Take the alternating product with $v_{j_{r+1}} \wedge \cdots \wedge v_{j_n}$. Then we shall have alternating products in the sum with repeated components in all the terms except the (j) -term, and thus we obtain

$$0 = a_{(j)} v_{j_1} \wedge \cdots \wedge v_{j_r} \wedge \cdots \wedge v_{j_n}.$$

Reshuffling $v_{j_1} \wedge \cdots \wedge v_{j_n}$ into $v_1 \wedge \cdots \wedge v_n$ simply changes the right-hand side by a sign. From what we proved at the beginning of this proof, it follows that $a_{(j)} = 0$. Hence we have proved our assertion for $1 \leq r \leq n$.

When $r = 0$, we deal with the empty product, and 1 is a basis for $\bigwedge^0(E) = R$ over R . We leave the case $r > n$ as a trivial exercise to the reader.

The assertion concerning the dimension is trivial, considering that there is a bijection between the set of basis elements, and the subsets of the set of integers $(1, \dots, n)$.

Remark. It is possible to give the first part of the proof, for $\bigwedge^n(E)$, without assuming known the existence of determinants. One must then show that \mathfrak{a}_n admits a 1-dimensional complementary submodule in $T^n(E)$. This can be done by simple means, which we leave as an exercise which the reader can look up in the more general situation of §4. When R is a field, this exercise is even more trivial, since one can verify at once that $v_1 \otimes \cdots \otimes v_n$ does not lie in \mathfrak{a}_n . This alternative approach to the theorem then proves the existence of determinants.

Proposition 1.2. *Let*

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

be an exact sequence of free R -modules of finite ranks r , n , and s respectively. Then there is a natural isomorphism

$$\varphi : \bigwedge^r E' \otimes \bigwedge^s E'' \rightarrow \bigwedge^n E.$$

This isomorphism is the unique isomorphism having the following property. For elements $v_1, \dots, v_r \in E'$ and $w_1, \dots, w_s \in E''$, let u_1, \dots, u_s be liftings of w_1, \dots, w_s in E . Then

$$\varphi((v_1 \wedge \cdots \wedge v_r) \otimes (w_1 \wedge \cdots \wedge w_s)) = v_1 \wedge \cdots \wedge v_r \wedge u_1 \wedge \cdots \wedge u_s.$$

Proof. The proof proceeds in the usual two steps. First one shows the existence of a homomorphism φ having the desired effect. The value on the right of the above formula is independent of the choice of u_1, \dots, u_s lifting w_1, \dots, w_s by using the alternating property, so we obtain a homomorphism φ . Selecting in particular $\{v_1, \dots, v_r\}$ and $\{w_1, \dots, w_s\}$ to be bases of E' and E'' respectively, one then sees that φ is both injective and surjective. We leave the details to the reader.

Given a free module E of rank n , we define its **determinant** to be

$$\det E = \bigwedge^{\max} E = \bigwedge^n E.$$

Then Proposition 1.2 may be reformulated by the isomorphism formula

$$\det(E') \otimes \det(E'') \approx \det(E).$$

If $R = k$ is a field, then we may say that \det is an Euler-Poincaré map on the category of finite dimensional vector spaces over k .

Example. Let V be a finite dimensional vector space over \mathbf{R} . By a **volume** on V we mean a norm $\| \cdot \|$ on $\det V$. Since V is finite dimensional, such a norm is equivalent to assigning a positive number c to a given basis of $\det(V)$. Such a basis can be expressed in the form $e_1 \wedge \cdots \wedge e_n$, where $\{e_1, \dots, e_n\}$ is a basis of V . Then for $a \in \mathbf{R}$ we have

$$\|ae_1 \wedge \cdots \wedge e_n\| = |a|c.$$

In analysis, given a volume as above, one then defines a Haar measure μ on V by defining the measure of a set S to be

$$\mu(S) = \int_S \|e_1 \wedge \cdots \wedge e_n\| dx_1 \cdots dx_n,$$

where x_1, \dots, x_n are the coordinates on V with respect to the above basis. As an exercise, show that the expression on the right is independent of the choice of basis.

Proposition 1.2 is a special case of the following more general situation. We consider again an exact sequence of free R -modules of finite rank as above. With respect to the submodule E' of E , we define

$\bigwedge_i^n E =$ submodule of $\bigwedge^n E$ generated by all elements

$$x'_1 \wedge \cdots \wedge x'_i \wedge y_{i+1} \wedge \cdots \wedge y_n$$

with $x'_1, \dots, x'_i \in E'$ viewed as submodule of E .

Then we have a filtration

$$\bigwedge_i^n E \supset \bigwedge_{i+1}^n E.$$

Proposition 1.3. *There is a natural isomorphism*

$$\bigwedge^i E' \otimes \bigwedge^{n-i} E'' \rightarrow \bigwedge_i^n E / \bigwedge_{i+1}^n E.$$

Proof. Let x''_1, \dots, x''_{n-i} be elements of E'' , and lift them to elements y_1, \dots, y_{n-i} of E . We consider the map

$$(x'_1, \dots, x'_i, x''_1, \dots, x''_{n-i}) \mapsto x'_1 \wedge \cdots \wedge x'_i \wedge y_1 \wedge \cdots \wedge y_{n-i}$$

with the right-hand side taken mod $\bigwedge_{i+1}^n E$. Then it is immediate that this map factors through

$$\bigwedge^i E' \otimes \bigwedge^{n-i} E'' \rightarrow \bigwedge_i^n E / \bigwedge_{i+1}^n E,$$

and picking bases shows that one gets an isomorphism as desired.

In a similar vein, we have:

Proposition 1.4. *Let $E = E' \oplus E''$ be a direct sum of finite free modules. Then for every positive integer n , we have a module isomorphism*

$$\bigwedge^n E \approx \bigoplus_{p+q=n} \bigwedge^p E' \otimes \bigwedge^q E''.$$

In terms of the alternating algebras, we have an isomorphism

$$\wedge E \approx \wedge E' \otimes_{su} \wedge E''.$$

where \otimes_{su} is the superproduct of graded algebras.

Proof. Each natural injection of E' and E'' into E induces a natural map on the alternating algebras, and so gives the homomorphism

$$\wedge E' \otimes \wedge E'' \rightarrow \wedge E,$$

which is graded, i.e. for $p = 0, \dots, n$ we have

$$\wedge^p E' \otimes \wedge^{n-p} E'' \rightarrow \wedge^n E.$$

To verify that this yields the desired isomorphism, one can argue by picking bases, which we leave to the reader. The anti-commutation rule of the alternating product immediately shows that the isomorphism is an algebra isomorphism for the super product $\wedge E' \otimes_{su} \wedge E''$.

We end this section with comments on duality. In Exercise 3, you will prove:

Proposition 1.5. *Let E be free of rank n over R . For each positive integer r , we have a natural isomorphism*

$$\wedge^r(E^\vee) \approx \wedge^r(E)^\vee.$$

The isomorphism is explicitly described in that exercise. A more precise property than “natural” would be that the isomorphism is functorial with respect to the category whose objects are finite free modules over R , and whose morphisms are isomorphisms.

Examples. Let L be a free module over R of rank 1. We have the dual module $L^\vee = \text{Hom}_R(L, R)$, which is also free of the same rank. For a positive integer m , we define

$$L^{\otimes -m} = (L^\vee)^{\otimes m} = L^\vee \otimes \cdots \otimes L^\vee \text{ (tensor product taken } m \text{ times).}$$

Thus we have defined the tensor product of a line with itself for negative integers. We define $L^{\otimes 0} = R$. You can easily verify that the rule

$$L^{\otimes p} \otimes L^{\otimes q} \approx L^{\otimes (p+q)}$$

holds for all integers $p, q \in \mathbf{Z}$, with a natural isomorphism. In particular, if $q = -p$ then we get R itself on the right-hand side.

Now let \mathbf{E} be an exact sequence of free modules:

$$\mathbf{E} : 0 \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_m \rightarrow 0.$$

We define the **determinant** of this exact sequence to be

$$\det(\mathbf{E}) = \bigotimes \det(E_i)^{\otimes (-1)^i}.$$

As an exercise, prove that $\det(\mathbf{E})$ has a natural isomorphism with R , functorial with respect to isomorphisms of exact sequences.

Examples. Determinants of vector spaces or free modules occur in several branches of mathematics, e. g. complexes of partial differential operators, homology theories, the theory of determinant line bundles in algebraic geometry, etc. For instance, given a non-singular projective variety V over \mathbf{C} , one defines the **determinant of cohomology** of V to be

$$\det H(V) = \bigotimes \det H^i(V)^{\otimes (-1)^i},$$

where $H^i(V)$ are the cohomology groups. Then $\det H(V)$ is a one-dimensional vector space over \mathbf{C} , but there is no natural identification of this vector space with \mathbf{C} , because *a priori* there is no natural choice of a basis. For a notable application of the determinant of cohomology, following work of Faltings, see Deligne, Le determinant de la cohomologie, in Ribet, K. (ed.), *Current Trends in Arithmetical Algebraic Geometry*, Proc. Arcata 1985. (*Contemporary Math.* vol 67, AMS (1985), pp. 93–178.)

§2. FITTING IDEALS

Certain ideals generated by determinants are coming more and more into use, in several branches of algebra and algebraic geometry. Therefore I include this section which summarizes some of their properties. For a more extensive account, see Northcott's book *Finite Free Resolutions* which I have used, as well as the appendix of the paper by Mazur-Wiles: "Class Fields of abelian extensions of \mathbf{Q} ," which they wrote in a self-contained way. (*Invent. Math.* 76 (1984), pp. 179–330.)

Let R be a commutative ring. Let A be a $p \times q$ matrix and B a $q \times s$ matrix with coefficients in R . Let $r \geq 0$ be an integer. We define the **determinant ideal** $I_r(A)$ to be the ideal generated by all determinants of $r \times r$ submatrices of A . This ideal may also be described as follows. Let S_r^p be the set of sequences

$$J = (j_1, \dots, j_r) \text{ with } 1 \leq j_1 < j_2 < \dots < j_r \leq p.$$

Let $A = (a_{ij})$. Let $1 \leq r \leq \min(p, q)$. Let $K = (k_1, \dots, k_r)$ be another element of S_r^q . We define

$$A_{JK}^{(r)} = \begin{vmatrix} a_{j_1 k_1} & a_{j_1 k_2} & \cdots & a_{j_1 k_r} \\ a_{j_2 k_1} & a_{j_2 k_2} & \cdots & a_{j_2 k_r} \\ \vdots & \vdots & & \vdots \\ a_{j_r k_1} & a_{j_r k_2} & \cdots & a_{j_r k_r} \end{vmatrix}$$

where the vertical bars denote the determinant. With J, K ranging over S_r^p we may view $A_{JK}^{(r)}$ as the JK -component of a matrix $A^{(r)}$ which we call the r -th exterior power of A .

One may also describe the matrix as follows. Let $\{e_1, \dots, e_p\}$ be a basis of R^p and $\{u_1, \dots, u_q\}$ a basis of R^q . Then the elements

$$e_{j_1} \wedge \dots \wedge e_{j_r} \quad (j_1 < j_2 < \dots < j_r)$$

form a basis for $\bigwedge^r R^p$ and similarly for a basis of $\bigwedge^r R^q$. We may view A as a linear map of R^p into R^q , and the matrix $A^{(r)}$ is then the matrix representing the exterior power $\bigwedge^r A$ viewed as a linear map of $\bigwedge^r R^p$ into $\bigwedge^r R^q$. On the whole, this interpretation will not be especially useful for certain computations, but it does give a slightly more conceptual context for the exterior power. Just at the beginning, this interpretation allows for an immediate proof of Proposition 2.1.

For $r = 0$ we define $A^{(0)}$ to be the 1×1 matrix whose single entry is the unit element of R . We also note that $A^{(1)} = A$.

Proposition 2.1. *Let A be a $p \times q$ matrix and B a $q \times s$ matrix. Then*

$$(AB)^{(r)} = A^{(r)}B^{(r)} \quad \text{for } r \geq 0.$$

If one uses the alternating products as mentioned above, the proof simply says that the matrix of the composite of linear maps with respect to fixed bases is the product of the matrices. If one does not use the alternating products, then one can prove the proposition by a direct computation which will be left to the reader.

We have formed a matrix whose entries are indexed by a finite set S_r^p . For any finite set S and doubly indexed family (c_{JK}) with $J, K \in S$ we may also define the **determinant** as

$$\det(c_{JK}) = \sum_{\sigma} \epsilon(\sigma) \left(\prod_{J \in S} c_{J, \sigma(J)} \right)$$

where σ ranges over all permutations of the set.

For $r \geq 0$ we define the **determinant ideal** $I_r(A)$ to be the ideal generated by all the components of $A^{(r)}$, or equivalently by all $r \times r$ subdeterminants of A . We have by definition

$$A^{(0)} = R \quad \text{and} \quad A^{(1)} = \text{ideal generated by the components of } A.$$

Furthermore

$$I_r(A) = 0 \quad \text{for } r > \min(p, q)$$

and the inclusions

$$R = I_0(A) \supset I_1(A) \supset I_2(A) \supset \dots$$

By Proposition 10.1, we also have

$$(1) \quad I_r(AB) \subset I_r(A) \cap I_r(B).$$

Therefore, if $A = UBU'$ where U, U' are square matrices of determinant 1, then

$$(2) \quad I_r(A) = I_r(B).$$

Next, let E be an R -module. Let x_1, \dots, x_q be generators of E . Then we may form the matrix of relations $(a_1, \dots, a_q) \in R^q$ such that

$$\sum_{i=1}^q a_i x_i = 0.$$

Suppose first we take only finitely many relations, thus giving rise to a $p \times q$ matrix A . We form the determinant ideal $I_r(A)$. We let the **determinant ideals** of the family of generators be:

$$I_r(x_1, \dots, x_q) = I_r(x) = \text{ideal generated by } I_r(A) \text{ for all } A.$$

Thus we may in fact take the infinite matrix of relations, and say that $I_r(x)$ is generated by the determinants of all $r \times r$ submatrices. The inclusion relations of (1) show that

$$R = I_0(x) \supset I_1(x) \supset I_2(x) \supset \dots$$

$$I_r(x) = 0 \quad \text{if } r > q.$$

Furthermore, it is easy to see that if we form a submatrix M of the matrix of all relations by taking only a family of relations which generate the ideal of all relations in R^q , then we have

$$I_r(M) = I_r(x).$$

We leave the verification to the reader. We can take M to be a finite matrix when E is finitely presented, which happens if R is Noetherian.

In terms of this representation of a module as a quotient of R^q , we get the following characterization.

Proposition 2.2. *Let $R^q \rightarrow E \rightarrow 0$ be a representation of E as a quotient of R^q , and let x_1, \dots, x_q be the images of the unit vectors in R^q . Then $I_r(x)$ is the ideal generated by all values*

$$\lambda(w_1, \dots, w_r)$$

where $w_1, \dots, w_r \in \text{Ker}(R^q \rightarrow E)$ and $\lambda \in L'_q(R^q, R)$.

Proof. This is immediate from the definition of the determinant ideal.

The above proposition can be useful to replace a matrix computation by a more conceptual argument with fewer indices. The reader can profitably translate some of the following matrix arguments in these more invariant terms.

We now change the numbering, and let the **Fitting ideals** be:

$$F_k(x) = I_{q-k}(x) \quad \text{for } 0 \leq k \leq q$$

$$F_k(x) = R \quad \text{when } k > q.$$

Lemma 2.3. *The Fitting ideal $F_k(x)$ does not depend on the choice of generators (x) .*

Proof. Let y_1, \dots, y_s be elements of E . We shall prove that

$$I_r(x) = I_{r+s}(x, y).$$

The relations of (x, y) constitute a matrix of the form

$$W = \begin{pmatrix} a_{11} & \cdots & a_{1q} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{p1} & \cdots & a_{pq} & 0 & \cdots & 0 \\ b_{11} & \cdots & b_{1q} & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & & \vdots \\ b_{s1} & \cdots & b_{sq} & 0 & \cdots & & 1 \end{pmatrix}$$

By elementary column operations, we can change this to a matrix

$$\begin{pmatrix} A & 0 \\ 0 & 1_s \end{pmatrix}$$

and such operations do not change the determinant ideals by (2). Then we conclude that for all $r \geq 0$ we have

$$I_r(A) = I_{r+s}(W) \subset I_{r+s}(x, y).$$

This proves that $I_r(x) \subset I_{r+s}(x, y)$.

Conversely, let C be a matrix of relations between the generators (x, y) . We also have a matrix of relations

$$Z = \begin{pmatrix} & & & C & & \\ b_{11} & \cdots & b_{1q} & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{s1} & \cdots & b_{sq} & 0 & \cdots & 1 \end{pmatrix}$$

By elementary row operations, we can bring this matrix into the same shape

as B above, with some matrix of relations A' for (x) , namely

$$Z' = \begin{pmatrix} A' & 0 \\ B & 1_s \end{pmatrix}$$

Then

$$I_r(A') = I_{r+s}(Z') = I_{r+s}(Z) \supset I_{r+s}(C),$$

whence $I_{r+s}(C) \subset I_r(x)$. Taking all possible matrices of relations C shows that $I_{r+s}(x, y) \subset I_r(x)$, which combined with the previous inequality yields $I_{r+s}(x, y) = I_r(x)$.

Now given two families of generators (x) and (y) , we simply put them side by side (x, y) and use the new numbering for the F_k to conclude the proof of the lemma.

Now let E be a finitely generated R -module with presentation

$$0 \rightarrow K \rightarrow R^q \rightarrow E \rightarrow 0,$$

where the sequence is exact and K is defined as the kernel. Then K is generated by q -vectors, and can be viewed as an infinite matrix. The images of the unit vectors in R^q are generators (x_1, \dots, x_q) . We define the **Fitting ideal** of the module to be

$$F_k(E) = F_k(x).$$

Lemma 2.3 shows that the ideal is independent of the choice of presentation. The inclusion relations of a determinant ideal $I_r(A)$ of a matrix now translate into reverse inclusion relations for the Fitting ideals, namely:

Proposition 2.4.

(i) *We have*

$$F_0(E) \subset F_1(E) \subset F_2(E) \subset \dots$$

(ii) *If E can be generated by q elements, then*

$$F_q(E) = R.$$

(iii) *If E is finitely presented then $F_k(E)$ is finitely generated for all k .*

This last statement merely repeats the property that the determinant ideals of a matrix can be generated by the determinants associated with a finite submatrix if the row space of the matrix is finitely generated.

Example. Let $E = R^q$ be the free module of dimension q . Then:

$$F_k(E) = \begin{cases} 0 & \text{if } 0 \leq k < q \\ R & \text{if } k \geq q. \end{cases}$$

This is immediate from the definitions and the fact that the only relation of a basis for E is the trivial one.

The Fitting ideal $F_0(E)$ is called the **zero-th** or **initial Fitting ideal**. In some applications it is the only one which comes up, in which case it is called “**the**” **Fitting ideal** $F(E)$ of E . It is the ideal generated by all $q \times q$ determinants in the matrix of relations of q generators of the module.

For any module E we let $\text{ann}_R(E)$ be the annihilator of E in R , that is the set of elements $a \in R$ such that $aE = 0$.

Proposition 2.5. *Suppose that E can be generated by q elements. Then*

$$(\text{ann}_R(E))^q \subset F(E) \subset \text{ann}_R(E).$$

In particular, if E can be generated by one element, then

$$F(E) = \text{ann}_R(E).$$

Proof. Let x_1, \dots, x_q be generators of E . Let a_1, \dots, a_q be elements of R annihilating E . Then the diagonal matrix whose diagonal components are a_1, \dots, a_q is a matrix of relations, so the definition of the Fitting ideal shows that the determinant of this matrix, which is the product $a_1 \cdots a_q$ lies in $I_q(E) \subset F_0(E)$. This proves the inclusion

$$\text{ann}_R(E)^q \subset F(E).$$

Conversely, let A be a $q \times q$ matrix of relations between x_1, \dots, x_q . Then $\det(A)x_i = 0$ for all i so $\det(A) \in \text{ann}_R(E)$. Since $F(E)$ is generated by such determinants, we get the reverse inclusion which proves the proposition.

Corollary 2.6. *Let $E = R/\mathfrak{a}$ for some ideal \mathfrak{a} . Then $F(E) = \mathfrak{a}$.*

Proof. The module R/\mathfrak{a} can be generated by one element so the corollary is an immediate consequence of the proposition.

Proposition 2.7. *Let*

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

be an exact sequence of finite R -modules. For integers $m, n \geq 0$ we have

$$F_m(E')F_n(E'') \subset F_{m+n}(E).$$

In particular for $F = F_0$,

$$F(E')F(E'') \subset F(E).$$

Proof. We may assume E' is a submodule of E . We pick generators x_1, \dots, x_p of E' and elements y_1, \dots, y_q in E such that their images y''_1, \dots, y''_q in E'' generate E'' . Then (x, y) is a family of generators for E . Suppose first that $m \leq p$ and $n \leq q$. Let A be a matrix of relations among y''_1, \dots, y''_q with q columns. If (a_1, \dots, a_q) is such a relation, then

$$a_1y_1 + \dots + a_qy_q \in E'$$

so there exist elements $b_1, \dots, b_p \in R$ such that

$$\sum a_iy_i + \sum b_jx_j = 0.$$

Thus we can find a matrix B with p columns and the same number of rows as A such that (B, A) is a matrix of relations of (x, y) . Let C be a matrix of relations of (x_1, \dots, x_p) . Then

$$\begin{pmatrix} B & A \\ C & 0 \end{pmatrix}$$

is a matrix of relations of (x, y) . If D'' is a $(q - n) \times (q - n)$ subdeterminant of A and D' is a $(p - m) \times (p - m)$ subdeterminant of C then $D''D'$ is a

$$(p + q - m - n) \times (p + q - m - n)$$

subdeterminant of the matrix

$$\begin{pmatrix} B & A \\ C & 0 \end{pmatrix}$$

and $D''D' \in F_{m+n}(E)$. Since $F_m(E')$ is generated by determinants like D' and $F_n(E'')$ is generated by determinants like D'' , this proves the proposition in the present case.

If $m > p$ and $n > q$ then $F_{m+n}(E) = F_m(E') = F_n(E'') = R$ so the proposition is trivial in this case.

Say $m \leq p$ and $n > q$. Then $F_n(E'') = R = F_q(E'')$ and hence

$$F_m(E')F_n(E'') = F_q(E'')F_m(E') \subset F_{p+n}(E) \subset F_{m+n}(E)$$

where the inclusion follows from the first case. A similar argument proves the remaining case with $m > p$ and $n \leq q$. This concludes the proof.

Proposition 2.8. *Let E', E'' be finite R -modules. For any integer $n \geq 0$ we have*

$$F_n(E' \oplus E'') = \sum_{r+s=n} F_r(E')F_s(E'').$$

Proof. Let x_1, \dots, x_p generate E^i and y_1, \dots, y_q generate E'' . Then (x, y) generate $E' \oplus E''$. By Proposition 2.6 we know the inclusion

$$\sum F_r(E')F_s(E'') \subset F_n(E' \oplus E''),$$

so we have to prove the converse. If $n \geq p + q$ then we can take $r \geq p$ and $s \geq q$ in which case

$$F_r(E') = F_s(E'') = F_n(E) = R$$

and we are done. So we assume $n < p + q$. A relation between (x, y) in the direct sum splits into a relation for (x) and a relation for (y) . The matrix of relations for (x, y) is therefore of the form

$$C = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix}$$

where A' is the matrix of relations for (x) and A'' the matrix of relations for (y) . Thus

$$F_n(E' \oplus E'') = \sum_C I_{p+q-n}(C)$$

where the sum is taken over all matrices C as above. Let D be a

$$(p + q - n) \times (p + q - n)$$

subdeterminant. Then D has the form

$$D = \begin{vmatrix} B' & 0 \\ 0 & B'' \end{vmatrix}$$

where B' is a $k' \times (p - r)$ matrix, and B'' is a $k'' \times (q - s)$ matrix with some positive integers k', k'', r, s satisfying

$$k' + k'' = p + q - n \quad \text{and} \quad r + s = n.$$

Then $D = 0$ unless $k' = p - r$ and $k'' = q - s$. In that case

$$D = \det(B')\det(B'') \in F_r(E')F_s(E''),$$

which proves the reverse inclusion and concludes the proof of the proposition.

Corollary 2.9. *Let*

$$E = \bigoplus_{i=1}^s R/\alpha_i$$

where α_i is an ideal. Then $F(E) = \alpha_1 \cdots \alpha_s$.

Proof. This is really a corollary of Proposition 2.8 and Corollary 2.6.

§3. UNIVERSAL DERIVATIONS AND THE DE RHAM COMPLEX

In this section, all rings R, A , etc. are assumed commutative.

Let A be an R -algebra and M an A -module. By a **derivation** $D: A \rightarrow M$ (over R) we mean an R -linear map satisfying the usual rules

$$D(ab) = aDb + bDa.$$

Note that $D(1) = 2D(1)$ so $D(1) = 0$, whence $D(R) = 0$. Such derivations form an A -module $\text{Der}_R(A, M)$ in a natural way, where aD is defined by $(aD)(b) = aDb$.

By a **universal derivation** for A over R , we mean an A -module Ω , and a derivation

$$d: A \rightarrow \Omega$$

such that, given a derivation $D: A \rightarrow M$ there exists a unique A -homomorphism $f: \Omega \rightarrow M$ making the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{d} & \Omega \\ & \searrow d & \swarrow f \\ & & M \end{array}$$

It is immediate from the definition that a universal derivation (d, Ω) is uniquely determined up to a unique isomorphism. By definition, we have a functorial isomorphism

$$\text{Der}_R(A, M) \approx \text{Hom}_A(\Omega, M).$$

We shall now prove the existence of a universal derivation.

The following general remark will be useful. Let

$$f_1, f_2: A \rightarrow B$$

be two homomorphisms of R -algebras, and let J be an ideal in B such that $J^2 = 0$. Assume that $f_1 \equiv f_2 \pmod{J}$; this means that $f_1(x) \equiv f_2(x) \pmod{J}$ for all x in A . Then

$$D = f_2 - f_1$$

is a derivation. This fact is immediately verified as follows:

$$\begin{aligned} f_2(ab) &= f_2(a)f_2(b) = [f_1(a) + D(a)][f_1(b) + D(b)] \\ &= f_1(ab) + f_1(b)D(a) + f_1(a)D(b). \end{aligned}$$

But the A -module structure of J is given via f_1 or f_2 (which amount to the same thing in light of our assumptions on f_1, f_2), so the fact is proved.

Let the tensor product be taken over R .

Let $\mathbf{m}_A: A \otimes A \rightarrow A$ be the multiplication homomorphism, such that $\mathbf{m}_A(a \otimes b) = ab$. Let $J = \text{Ker } \mathbf{m}_A$. We define the module of **differentials**

$$\Omega_{A/R} = J/J^2,$$

as an ideal in $(A \otimes A)/J^2$. The A -module structure will always be given via the embedding on the first factor:

$$A \rightarrow A \otimes A \quad \text{by } a \mapsto a \otimes 1.$$

Note that we have a direct sum decomposition of A -modules

$$A \otimes A = (A \otimes 1) \oplus J,$$

and therefore

$$(A \otimes A)/J^2 = (A \otimes 1) \oplus J/J^2.$$

Let

$$d: A \rightarrow J/J^2 \text{ be the } R\text{-linear map } a \mapsto 1 \otimes a - a \otimes 1 \text{ mod } J^2.$$

Taking $f_1: a \mapsto a \otimes 1$ and $f_2: a \mapsto 1 \otimes a$, we see that $d = f_2 - f_1$. Hence d is a derivation when viewed as a map into J/J^2 .

We note that J is generated by elements of the form

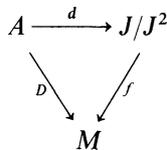
$$\sum x_i dy_i.$$

Indeed, if $\sum x_i \otimes y_i \in J$, then by definition $\sum x_i y_i = 0$, and hence

$$\sum x_i \otimes y_i = \sum x_i(1 \otimes y_i - y_i \otimes 1),$$

according to the A -module structure we have put on $A \otimes A$ (operation of A on the left factor.)

Theorem 3.1. *The pair $(J/J^2, d)$ is universal for derivations of A . This means: Given a derivation $D: A \rightarrow M$ there exists a unique A -linear map $f: J/J^2 \rightarrow M$ making the following diagram commutative.*



Proof. There is a unique R -bilinear map

$$f: A \otimes A \rightarrow M \quad \text{given by} \quad x \otimes y \mapsto xDy,$$

which is A -linear by our definition of the A -module structure on $A \otimes A$. Then by definition, the diagram is commutative on elements of A , when we take f restricted to J , because

$$f(1 \otimes y - y \otimes 1) = Dy.$$

Since J/J^2 is generated by elements of the form $x dy$, the uniqueness of the map in the diagram of the theorem is clear. This proves the desired universal property.

We may write the result expressed in the theorem as a formula

$$\text{Der}_R(A, M) \approx \text{Hom}_A(J/J^2, M).$$

The reader will find exercises on derivations which give an alternative way of constructing the universal derivation, especially useful when dealing with finitely generated algebras, which are factors of polynomial rings.

I insert here without proofs some further fundamental constructions, important in differential and algebraic geometry. The proofs are easy, and provide nice exercises.

Let $R \rightarrow A$ be an R -algebra of commutative rings. For $i \geq 0$ define

$$\Omega_{A/R}^i = \bigwedge^i \Omega_{A/R}^1,$$

where $\Omega_{A/R}^0 = A$.

Theorem 3.2. *There exists a unique sequence of R -homomorphisms*

$$d_i: \Omega_{A/R}^i \rightarrow \Omega_{A/R}^{i+1}$$

such that for $\omega \in \Omega^i$ and $\eta \in \Omega^j$ we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^i \omega \wedge d\eta.$$

Furthermore $d \circ d = 0$.

The proof will be left as an exercise.

Recall that a **complex** of modules is a sequence of homomorphisms

$$\dots \rightarrow E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \rightarrow$$

such that $d^i \circ d^{i-1} = 0$. One usually omits the superscript on the maps d . With this terminology, we see that the $\Omega_{A/R}^i$ form a complex, called the **De Rham complex**.

Theorem 3.3. *Let k be a field of characteristic 0, and let $A = k[X_1, \dots, X_n]$ be the polynomial ring in n variables. Then the De Rham complex*

$$0 \rightarrow k \rightarrow A \rightarrow \Omega_{A/k}^1 \rightarrow \dots \rightarrow \Omega_{A/k}^n \rightarrow 0$$

is exact.

Again the proof will be left as an exercise. *Hint:* Use induction and integrate formally.

Other results concerning connections will be found in the exercises below.

§4. THE CLIFFORD ALGEBRA

Let k be a field. By an **algebra** throughout this section, we mean a k -algebra given by a ring homomorphism $k \rightarrow A$ such that the image of k is in the center of A .

Let E be a finite dimensional vector space over the field k , and let g be a symmetric form on E . We would like to find a universal algebra over k , in which we can embed E , and such that the square in the algebra corresponds to the value of the quadratic form in E . More precisely, by a **Clifford algebra** for g , we shall mean a k -algebra $C(g)$, also denoted by $C_g(E)$, and a linear map $\rho: E \rightarrow C(g)$ having the following property: If $\psi: E \rightarrow L$ is a linear map of E into a k -algebra L such that

$$\psi(x)^2 = g(x, x) \cdot 1 \quad (1 = \text{unit element of } L)$$

for all $x \in E$, then there exists a unique algebra-homomorphism

$$C(\psi) = \psi_* : C(g) \rightarrow L$$

such that the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\rho} & C(g) \\ & \searrow \psi & \swarrow \psi_* \\ & & L \end{array}$$

By abstract nonsense, a Clifford algebra for g is uniquely determined, up to a unique isomorphism. Furthermore, it is clear that if $(C(g), \rho)$ exists, then $C(g)$ is generated by the image of ρ , i.e. by $\rho(E)$, as an algebra over k .

We shall write $\rho = \rho_g$ if it is necessary to specify the reference to g explicitly.

We have trivially

$$\rho(x)^2 = g(x, x) \cdot 1$$

for all $x \in E$, and

$$\rho(x)\rho(y) + \rho(y)\rho(x) = 2g(x, y) \cdot 1$$

as one sees by replacing x by $x + y$ in the preceding relation.

Theorem 4.1. *Let g be a symmetric bilinear form on a finite dimensional vector space E over k . Then the Clifford algebra $(C(g), \rho)$ exists. The map ρ is injective, and $C(g)$ has dimension 2^n over k , if $n = \dim E$.*

Proof. Let $T(E)$ be the tensor algebra as in Chapter XVI, §7. In that algebra, we let I_g be the two-sided ideal generated by all elements

$$x \otimes x - g(x, x) \cdot 1 \text{ for } x \in E.$$

We define $C_g(E) = T(E)/I_g$. Observe that E is naturally embedded in $T(E)$ since

$$T(E) = k \oplus E \oplus (E \otimes E) \oplus \cdots$$

Then the natural embedding of E in TE followed by the canonical homomorphisms of $T(E)$ onto $C_g(E)$ defines our k -linear map $\rho: E \rightarrow C_g(E)$. It is immediate from the universal property of the tensor product that $C_g(E)$ as just defined satisfies the universal property of a Clifford algebra, which therefore exists. The only problem is to prove that it has the stated dimension over k .

We first prove that the dimension is $\leq 2^n$. We give a proof only when the characteristic of k is $\neq 2$ and leave characteristic 2 to the reader. Let $\{v_1, \dots, v_n\}$ be an orthogonal basis of E as given by Theorem 3.1 of Chapter XV. Let $e_i = \psi(v_i)$, where $\psi: E \rightarrow L$ is given as in the beginning of the section. Let $c_i = g(v_i, v_i)$. Then we have the relations

$$e_i^2 = c_i, \quad e_i e_j = -e_j e_i \text{ for all } i \neq j.$$

This immediately implies that the subalgebra of L generated by $\psi(E)$ over k is generated as a vector space over k by all elements

$$e_1^{v_1} \cdots e_n^{v_n} \text{ with } v_i = 0 \text{ or } 1 \text{ for } i = 1, \dots, n.$$

Hence the dimension of this subalgebra is $\leq 2^n$. In particular, $\dim C_g(E) \leq 2^n$ as desired.

There remains to show that there exists at least one $\psi: E \rightarrow L$ such that L is generated by $\psi(E)$ as an algebra over k , and has dimension 2^n ; for in that case, the homomorphism $\psi_*: C_g(E) \rightarrow L$ being surjective, it follows that $\dim C_g(E) \geq 2^n$ and the theorem will be proved. We construct L in the following way. We first need some general notions.

Let M be a module over a commutative ring. Let $i, j \in \mathbf{Z}/2\mathbf{Z}$. Suppose M is a direct sum $M = M_0 \oplus M_1$ where 0, 1 are viewed as the elements of $\mathbf{Z}/2\mathbf{Z}$. We then say that M is $\mathbf{Z}/2\mathbf{Z}$ -graded. If M is an algebra over the ring, we say

it is a **$\mathbf{Z}/2\mathbf{Z}$ -graded algebra** if $M_i M_j \subset M_{i+j}$ for all $i, j \in \mathbf{Z}/2\mathbf{Z}$. We simply say **graded**, omitting the $\mathbf{Z}/2\mathbf{Z}$ prefix when the reference to $\mathbf{Z}/2\mathbf{Z}$ is fixed throughout a discussion, which will be the case in the rest of this section.

Let A, B be graded modules as above, with $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$. Then the tensor product $A \otimes B$ has a direct sum decomposition

$$A \otimes B = \bigoplus_{i,j} A_i \otimes B_j.$$

We define a grading on $A \otimes B$ by letting $(A \otimes B)_0$ consist of the sum over indices i, j such that $i + j = 0$ (in $\mathbf{Z}/2\mathbf{Z}$), and $(A \otimes B)_1$ consist of the sum over the indices i, j such that $i + j = 1$.

Suppose that A, B are graded algebras over the given commutative ring. There is a unique bilinear map of $A \otimes B$ into itself such that

$$(a \otimes b)(a' \otimes b') = (-1)^{ij} aa' \otimes bb'$$

if $a' \in A_i$ and $b \in B_j$. Just as in Chapter XVI, §6, one verifies associativity and the fact that this product gives rise to a graded algebra, whose product is called the **super tensor product**, or **super product**. As a matter of notation, when we take the super tensor product of A and B , we shall denote the resulting algebra by

$$A \otimes_{su} B$$

to distinguish it from the ordinary algebra $A \otimes B$ of Chapter XVI, §6.

Next suppose that E has dimension 1 over k . Then the factor polynomial ring $k[X]/(x^2 - c_1)$ is immediately verified to be the Clifford algebra in this case. We let t_1 be the image of X in the factor ring, so $C_g(E) = k[t_1]$ with $t_1^2 = c_1$. The vector space E is imbedded as kt_1 in the direct sum $k \oplus kt_1$.

In general we now take the super tensor product inductively:

$$C_g(E) = k[t_1] \otimes_{su} k[t_2] \otimes_{su} \cdots \otimes_{su} k[t_n], \text{ with } k[t_i] = k[X]/(X^2 - c_i).$$

Its dimension is 2^n . Then E is embedded in $C_g(E)$ by the map

$$a_1 v_1 + \cdots + a_n v_n \mapsto a_1 t_1 \oplus \cdots \oplus a_n t_n.$$

The desired commutation rules among t_i, t_j are immediately verified from the definition of the super product, thus concluding the proof of the dimension of the Clifford algebra.

Note that the proof gives an explicit representation of the relations of the algebra, which also makes it easy to compute in the algebra. Note further that the alternating algebra of a free module is a special case, taking $c_i = 0$ for all i . Taking the c_i to be algebraically independent shows that the alternating algebra is a specialization of the generic Clifford algebra, or that Clifford algebras are what one calls perturbations of the alternating algebra. Just as for the alternating algebra, we have immediately from the construction:

Theorem 4.2. *Let g, g' be symmetric forms on E, E' respectively. Then we*

have an algebra isomorphism

$$C(g \oplus g') \approx C(g) \otimes_{su} C(g').$$

Examples. Clifford algebras have had increasingly wide applications in physics, differential geometry, topology, group representations (finite groups and Lie groups), and number theory. First, in topology I refer to Adams [Ad 62] and [ABS 64] giving applications of the Clifford algebra to various problems in topology, notably a description of the way Clifford algebras over the reals are related to the existence of vector fields on spheres. The multiplication in the Clifford algebra gives rise to a multiplication on the sphere, whence to vector fields. [ABS 64] also gives a number of computations related to the Clifford algebra and its applications to topology and physics. For instance, let $E = \mathbf{R}^n$ and let g be the negative of the standard dot product. Or more invariantly, take for E an n -dimensional vector space over \mathbf{R} , and let g be a *negative definite* symmetric form on E . Let $C_n = C(g)$.

The operation

$$v_1 \otimes \cdots \otimes v_r \mapsto v_r \otimes \cdots \otimes v_1 = (v_1 \otimes \cdots \otimes v_r)^* \text{ for } v_i \in E$$

induces an endomorphism of $T^r(E)$ for $r \geq 0$. Since $v \otimes v - g(v, v) \cdot 1$ (for $v \in E$) is invariant under this operation, there is an induced endomorphism $*$: $C_n \rightarrow C_n$, which is actually an involution, that is $x^{**} = x$ and $(xy)^* = y^*x^*$ for $x \in C_n$. We let $\text{Spin}(n)$ be the subgroup of units in C_n generated by the unit sphere in E (i.e. the set of elements such that $g(v, v) = -1$), and lying in the even part of C_n . Equivalently, $\text{Spin}(n)$ is the group of elements x such that $xx^* = 1$. The name dates back to Dirac who used this group in his study of electron spin. Topologists and others view that group as being the universal covering group of the special orthogonal group $SO(n) = SU_n(\mathbf{R})$.

An account of some of the results of [Ad 62] and [ABS 64] will also be found in [Hu 75], Chapter 11. Second I refer to two works encompassing two decades, concerning the heat kernel, Dirac operator, index theorem, and number theory, ranging from Atiyah, Bott and Patodi [ABP 73] to Faltings [Fa 91], see especially §4, entitled "The local index theorem for Dirac operators". The vector space to which the general theory is applied is mostly the cotangent space at a point on a manifold. I recommend the book [BGV 92], Chapter 3.

Finally, I refer to Bröcker and Tom Dieck for applications of the Clifford algebra to representation theory, starting with their Chapter I, §6, [BtD 85].

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EXERCISES

1. Let E be a finite dimensional vector space over a field k . Let x_1, \dots, x_p be elements of E such that $x_1 \wedge \dots \wedge x_p \neq 0$, and similarly $y_1 \wedge \dots \wedge y_p \neq 0$. If $c \in k$ and

$$x_1 \wedge \dots \wedge x_p = cy_1 \wedge \dots \wedge y_p$$

show that x_1, \dots, x_p and y_1, \dots, y_p generate the same subspace. Thus non-zero decomposable vectors in $\bigwedge^p E$ up to non-zero scalar multiples correspond to p -dimensional subspaces of E .

2. Let E be a free module of dimension n over the commutative ring R . Let $f: E \rightarrow E$ be a linear map. Let $\alpha_r(f) = \text{tr} \bigwedge^r(f)$, where $\bigwedge^r(f)$ is the endomorphism of $\bigwedge^r(E)$ into itself induced by f . We have

$$\alpha_0(f) = 1, \quad \alpha_1(f) = \text{tr}(f), \quad \alpha_n(f) = \det f,$$

and $\alpha_r(f) = 0$ if $r > n$. Show that

$$\det(1 + f) = \sum_{r \geq 0} \alpha_r(f).$$

[Hint: As usual, prove the statement when f is represented by a matrix with variable coefficients over the integers.] Interpret the $\alpha_r(f)$ in terms of the coefficients of the characteristic polynomial of f .

3. Let E be a finite dimensional free module over the commutative ring R . Let E^\vee be its dual module. For each integer $r \geq 1$ show that $\bigwedge^r E$ and $\bigwedge^r E^\vee$ are dual modules to each other, under the bilinear map such that

$$(v_1 \wedge \dots \wedge v_r, v'_1 \wedge \dots \wedge v'_r) \mapsto \det \langle (v_i, v'_j) \rangle$$

where $\langle v_i, v'_j \rangle$ is the value of v'_j on v_i , as usual, for $v_i \in E$ and $v'_j \in E^\vee$.

4. Notation being as in the preceding exercise, let F be another R -module which is free, finite dimensional. Let $f: E \rightarrow F$ be a linear map. Relative to the bilinear map of the preceding exercise, show that the transpose of $\bigwedge^r f$ is $\bigwedge^r(f)$, i.e. is equal to the r -th alternating product of the transpose of f .
5. Let R be a commutative ring. If E is an R -module, denote by $L'_a(E)$ the module of

r -multilinear alternating maps of E into R itself (i.e. the r -multilinear alternating forms on E). Let $L_a^0(E) = R$, and let

$$\Omega(E) = \bigoplus_{r=0}^{\infty} L_a^r(E).$$

Show that $\Omega(E)$ is a graded R -algebra, the multiplication being defined as follows. If $\omega \in L_a^r(E)$ and $\psi \in L_a^s(E)$, and v_1, \dots, v_{r+s} are elements of E , then

$$(\omega \wedge \psi)(v_1, \dots, v_{r+s}) = \sum \epsilon(\sigma)\omega(v_{\sigma_1}, \dots, v_{\sigma_r})\psi(v_{\sigma(r+1)}, \dots, v_{\sigma s}),$$

the sum being taken over all permutations σ of $(1, \dots, r + s)$ such that $\sigma_1 < \dots < \sigma_r$ and $\sigma(r + 1) < \dots < \sigma s$.

Derivations

In the following exercises on derivations, all rings are assumed commutative. Among other things, the exercises give another proof of the existence of universal derivations.

Let $R \rightarrow A$ be a R -algebra (of commutative rings, according to our convention). We denote the module of universal derivations of A over R by $(d_{A/R}, \Omega_{A/R}^1)$, but we do not assume that it necessarily exists. Sometimes we write d instead of $d_{A/R}$ for simplicity if the reference to A/R is clear.

- Let $A = R[X_\alpha]$ be a polynomial ring in variables X_α , where α ranges over some indexing set, possibly infinite. Let Ω be the free A -module on the symbols dX_α , and let

$$d: A \rightarrow \Omega$$

be the mapping defined by

$$df(X) = \sum_{\alpha} \frac{\partial f}{\partial X_{\alpha}} dX_{\alpha}.$$

Show that the pair (d, Ω) is a universal derivation $(d_{A/R}, \Omega_{A/R}^1)$.

- Let $A \rightarrow B$ be a homomorphism of R -algebras. Assume that the universal derivations for $A/R, B/R$, and B/A exist. Show that one has a natural exact sequence:

$$B \otimes_A \Omega_{A/R}^1 \rightarrow \Omega_{B/R}^1 \rightarrow \Omega_{B/A}^1 \rightarrow 0.$$

[Hint: Consider the sequence

$$0 \rightarrow \text{Der}_A(B, M) \rightarrow \text{Der}_R(B, M) \rightarrow \text{Der}_R(A, M)$$

which you prove is exact. Use the fact that a sequence of B -modules

$$N' \rightarrow N \rightarrow N'' \rightarrow 0$$

is exact if and only if its Hom into M is exact for every B -module M . Apply this to the sequence of derivations.]

- Let $R \rightarrow A$ be an R -algebra, and let I be an ideal of A . Let $B = A/I$. Suppose that the universal derivation of A over R exists. Show that the universal derivation of B over R

also exists, and that there is a natural exact sequence

$$I/I^2 \xrightarrow{d_{A/R}} B \otimes_A \Omega_{A/R}^1 \rightarrow \Omega_{B/R}^1 \rightarrow 0.$$

[Hint: Let M be a B -module. Show that the sequence

$$0 \rightarrow \text{Der}_R(B, M) \rightarrow \text{Der}_R(A, M) \rightarrow \text{Hom}_B(I/I^2, M)$$

is exact.]

9. Let $R \rightarrow B$ be an R -algebra. Show that the universal derivation of B over R exists as follows. Represent B as a quotient of a polynomial ring, possibly in infinitely many variables. Apply Exercises 6 and 7.
10. Let $R \rightarrow A$ be an R -algebra. Let S_0 be a multiplicative subset of R , and S a multiplicative subset of A such that S_0 maps into S . Show that the universal derivation of $S^{-1}A$ over $S_0^{-1}R$ is $(d, S^{-1}\Omega_{A/R}^1)$, where

$$d(a/s) = (sd_{A/R}(a) - ad_{A/R}(s))/s^2.$$

11. Let B be an R -algebra and M a B -module. On $B \oplus M$ define a product

$$(b, x)(b', y) = (bb', by + b'x).$$

Show that $B \oplus M$ is a B -algebra, if we identify an element $b \in B$ with $(b, 0)$. For any R -algebra A , show that the algebra homomorphisms $\text{Hom}_{\text{Alg}/R}(A, B \oplus M)$ consist of pairs (φ, D) , where $\varphi: A \rightarrow B$ is an algebra homomorphism, and $D: A \rightarrow M$ is a derivation for the A -module structure on M induced by φ .

12. Let A be an R -algebra. Let $\varepsilon: A \rightarrow R$ be an algebra homomorphism, which we call an **augmentation**. Let M be an R -module. Define an A -module structure on M via ε , by

$$a \cdot x = \varepsilon(a)x \quad \text{for} \quad a \in A \quad \text{and} \quad x \in M.$$

Write M_ε to denote M with this new module structure. Let:

$$\text{Der}_\varepsilon(A, M) = A\text{-module of derivations for the } \varepsilon\text{-module structure on } M$$

$$I = \text{Ker } \varepsilon.$$

Then $\text{Der}_\varepsilon(A, M)$ is an A/I -module. Note that there is an R -module direct sum decomposition $A = R \oplus I$. Show that there is a natural A -module isomorphism

$$\Omega_{A/R}/I\Omega_{A/R} \approx I/I^2$$

and an R -module isomorphism

$$\text{Der}_\varepsilon(A, M) \approx \text{Hom}_R(I/I^2, M).$$

In particular, let $\eta: A \rightarrow I/I^2$ be the projection of A on I/I^2 relative to the direct sum decomposition $A = R \oplus I$. Then η is the universal ε -derivation.

Derivations and connections

13. Let $R \rightarrow A$ be a homomorphism of commutative rings, so we view A as an R -algebra.

Let E be an A -module. A **connection** on E is a homomorphism of abelian groups

$$\nabla: E \rightarrow \Omega_{A/R}^1 \otimes_A E$$

such that for $a \in A$ and $x \in E$ we have

$$\nabla(ax) = a\nabla(x) + da \otimes x,$$

where the tensor product is taken over A unless otherwise specified. The kernel of ∇ , denoted by E_∇ , is called the **submodule of horizontal elements**, or the **horizontal submodule** of (E, ∇) .

(a) For any integer $i \geq 1$, define

$$\Omega_{A/R}^i = \bigwedge^i \Omega_{A/R}^1.$$

Show that ∇ can be extended to a homomorphism of R -modules

$$\nabla_i: \Omega_{A/R}^i \otimes E \rightarrow \Omega_{A/R}^{i+1} \otimes E$$

by

$$\nabla_i(\omega \otimes x) = d\omega \otimes x + (-1)^i \omega \wedge \nabla(x).$$

(b) Define the **curvature** of the connection to be the map

$$K = \nabla_1 \circ \nabla: E \rightarrow \Omega_{A/R}^2 \otimes_A E.$$

Show that K is an A -homomorphism. Show that

$$\nabla_{i+1} \circ \nabla_i(\omega \otimes x) = \omega \wedge K(x)$$

for $\omega \in \Omega_{A/R}^i$ and $x \in E$.

(c) Let $\text{Der}(A/R)$ denote the A -module of derivations of A into itself, over R . Let ∇ be a connection on E . Show that ∇ induces a unique A -linear map

$$\nabla: \text{Der}(A/R) \rightarrow \text{End}_R(E)$$

such that

$$\nabla(D)(ax) = D(ax) + a\nabla(D)(x).$$

(d) Prove the formula

$$[\nabla(D_1), \nabla(D_2)] - \nabla([D_1, D_2]) = (D_1 \wedge D_2)(K);$$

In this formula, the bracket is defined by $[f, g] = f \circ g - g \circ f$ for two endomorphisms f, g of E . Furthermore, the right-hand side is the composed mapping

$$E \xrightarrow{K} \Omega_{A/R}^2 \otimes E \xrightarrow{D_1 \wedge D_2} A \otimes E \approx E.$$

14. (a) For any derivation D of a ring A into itself, prove **Leibniz's rule**:

$$D^n(xy) = \sum_{i=0}^n \binom{n}{i} D^i(x)D^{n-i}(y).$$

(b) Suppose A has characteristic p . Show that D^p is a derivation.

15. Let A/R be an algebra, and let E be an A -module with a connection ∇ . Assume that R has characteristic p . Define

$$\psi : \text{Der}(A/R) \rightarrow \text{End}_R(E)$$

by

$$\psi(D) = (\nabla(D))^p - \nabla(D^p).$$

Prove that $\psi(D)$ is A -linear. [Hint: Use Leibniz's formula and the definition of a connection.] Thus the image of ψ is actually in $\text{End}_A(E)$.

Some Clifford exercises

16. Let $C_g(E)$ be the Clifford algebra as defined in §4. Define $F_i(C_g) = (k + E)^i$, viewing E as embedded in C_g . Define the similar object $F_i(\wedge E)$ in the alternating algebra. Then $F_{i+1} \supset F_i$ in both cases, and we define the i -th graded module $\text{gr}_i = F_i/F_{i-1}$. Show that there is a natural (functorial) isomorphism

$$\text{gr}_i(C_g(E)) \xrightarrow{\cong} \text{gr}_i(\wedge E).$$

17. Suppose that $k = \mathbf{R}$, so E is a real vector space, which we now assume of even dimension $2m$. We also assume that g is non-degenerate. We omit the index g since the symmetric form is now fixed, and we write C^+ , C^- for the spaces of degree 0 and 1 respectively in the $\mathbf{Z}/2\mathbf{Z}$ -grading. For elements x, y in C^+ or C^- , define their **supercommutator** to be

$$\{x, y\} = xy - (-1)^{(\text{deg } x)(\text{deg } y)}yx.$$

Show that F_{2m-1} is generated by supercommutators.

18. Still assuming g non-degenerate, let J be an automorphism of (E, g) (i.e. $g(Jx, Jy) = g(x, y)$ for all $x, y \in E$) such that $J^2 = -\text{id}$. Let $E_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{R}} E$ be the extension of scalars from \mathbf{R} to \mathbf{C} . Then $E_{\mathbf{C}}$ has a direct sum decomposition

$$E_{\mathbf{C}} = E_{\mathbf{C}}^+ \oplus E_{\mathbf{C}}^-$$

into the eigenspaces of J , with eigenvalues 1 and -1 respectively. (Proof?) There is a representation of $E_{\mathbf{C}}$ on $\wedge E_{\mathbf{C}}^+$, i.e. a homomorphism $E_{\mathbf{C}} \rightarrow \text{End}_{\mathbf{C}}(\wedge E_{\mathbf{C}}^+)$ whereby an element of $E_{\mathbf{C}}^+$ operates by exterior multiplication, and an element of $E_{\mathbf{C}}^-$ operates by inner multiplication, defined as follows.

For $x' \in E_{\mathbf{C}}^-$ there is a unique \mathbf{C} -linear map having the effect

$$x'(x_1 \wedge \cdots \wedge x_r) = -2 \sum_{i=1}^r (-1)^{i-1} \langle x', x_i \rangle x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_r.$$

Prove that under this operation, you get an isomorphism

$$C_g(E)_{\mathbf{C}} \rightarrow \text{End}_{\mathbf{C}}(\bigwedge E_{\mathbf{C}}^+).$$

[Hint: Count dimensions.]

19. Consider the Clifford algebra over \mathbf{R} . The standard notation is C_n if $E = \mathbf{R}^n$ with the negative definite form, and C'_n if $E = \mathbf{R}^n$ with the positive definite form. Thus $\dim C_n = \dim C'_n = 2^n$.

(a) Show that

$$\begin{aligned} C_1 &\approx \mathbf{C} & C_2 &\approx \mathbf{H} \text{ (the division ring of quaternions)} \\ C'_1 &\approx \mathbf{R} \times \mathbf{R} & C'_2 &\approx M_2(\mathbf{R}) \text{ (} 2 \times 2 \text{ matrices over } \mathbf{R}) \end{aligned}$$

20. Establish isomorphisms:

$$\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \approx \mathbf{C} \times \mathbf{C}; \quad \mathbf{C} \otimes_{\mathbf{R}} \mathbf{H} \approx M_2(\mathbf{C}); \quad \mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \approx M_4(\mathbf{R})$$

where $M_d(F) = d \times d$ matrices over F . For the third one, with $\mathbf{H} \otimes \mathbf{H}$, define an isomorphism

$$f: \mathbf{H} \otimes_{\mathbf{R}} \mathbf{H} \rightarrow \text{Hom}_{\mathbf{R}}(\mathbf{H}, \mathbf{H}) \approx M_4(\mathbf{R})$$

by $f(x \otimes y)(z) = xz\bar{y}$, where if $y = y_0 + y_1i + y_2j + y_3k$ then

$$\bar{y} = y_0 - y_1i - y_2j - y_3k.$$

21. (a) Establish isomorphisms

$$C_{n+2} \approx C'_n \otimes C_2 \quad \text{and} \quad C'_{n+2} \approx C_n \otimes C'_2.$$

[Hint: Let $\{e_1, \dots, e_{n+2}\}$ be the orthonormalized basis with $e_i^2 = -1$. Then for the first isomorphism map $e_i \mapsto e'_i \otimes e_1e_2$ for $i = 1, \dots, n$ and map e_{n+1}, e_{n+2} on $1 \otimes e_1$ and $1 \otimes e_2$ respectively.]

(b) Prove that $C_{n+8} \approx C_n \otimes M_{16}(\mathbf{R})$ (which is called the **periodicity property**).

(c) Conclude that C_n is a semi-simple algebra over \mathbf{R} for all n .

From (c) one can tabulate the simple modules over C_n . See [ABS 64], reproduced in Husemoller [Hu 75], Chapter 11, §6.