CHAPTER XVI

The Tensor Product

Having considered bilinear maps, we now come to multilinear maps and basic theorems concerning their structure. There is a universal module representing multilinear maps, called the tensor product. We derive its basic properties, and postpone to Chapter XIX the special case of alternating products. The tensor product derives its name from the use made in differential geometry, when this product is applied to the tangent space or cotangent space of a manifold. The tensor product can be viewed also as providing a mechanism for "extending the base"; that is, passing from a module over a ring to a module over some algebra over the ring. This "extension" can also involve reduction modulo an ideal, because what matters is that we are given a ring homomorphism $f: A \to B$, and we pass from modules over A to modules over B. The homomorphism f can be of both types, an inclusion or a canonical map with $B = A/J$ for some ideal J, or a composition of the two.

I have tried to provide the basic material which is immediately used in a variety of applications to many fields (topology, algebra, differential geometry, algebraic geometry, etc.).

§1. TENSOR PRODUCT

Let R be a commutative ring. If $E_1, \dots, E_n, F$ are modules, we denote by

$$L^n(E_1, \dots, E_n; F)$$

the module of n-multilinear maps

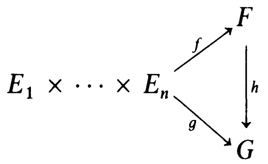
$$f: E_1 \times \cdots \times E_n \to F.$$

We recall that a multilinear map is a map which is linear (i.e., R-linear) in each variable. We use the words *linear* and *homomorphism* interchangeably. *Unless otherwise specified, modules, homomorphisms, linear, multilinear refer to the ring R.*

One may view the multilinear maps of a fixed set of modules $E_1, \dots, E_n$ as the objects of a category. Indeed, if

$$f : E_1 \times \cdots \times E_n \to F \quad \text{and} \quad g : E_1 \times \cdots \times E_n \to G$$

are multilinear, we define a morphism $f \to g$ to be a homomorphism $h : F \to G$ which makes the following diagram commutative:



A universal object in this category is called a **tensor product** of $E_1, \dots, E_n$ (over R).

We shall now prove that a tensor product exists, and in fact construct one in a natural way. By abstract nonsense, we know of course that a tensor product is uniquely determined, up to a unique isomorphism.

Let M be the free module generated by the set of all n-tuples $(x_1, \dots, x_n)$, $(x_i \in E_i)$, i.e. generated by the set $E_1 \times \cdots \times E_n$. Let N be the submodule generated by all the elements of the following type:

$$(x_1, \dots, x_i + x_i', \dots, x_n) - (x_1, \dots, x_i, \dots, x_n) - (x_1, \dots, x_i', \dots, x_n)$$
$$(x_1, \dots, ax_i, \dots, x_n) - a(x_1, \dots, x_n)$$

for all $x_i \in E_i$, $x_i' \in E_i$, $a \in R$. We have the canonical injection

$$E_1 \times \cdots \times E_n \to M$$

of our set into the free module generated by it. We compose this map with the canonical map $M \to M/N$ on the factor module, to get a map

$$\varphi : E_1 \times \cdots \times E_n \to M/N.$$

We contend that φ is multilinear and is a tensor product.

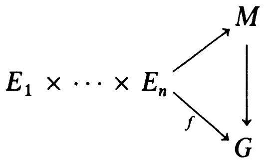
It is obvious that φ is multilinear—our definition was adjusted to this purpose. Let

$$f : E_1 \times \cdots \times E_n \to G$$

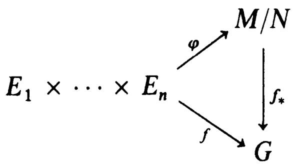
be a multilinear map. By the definition of free module generated by

$$E_1 \times \cdots \times E_n$$

we have an induced linear map $M \to G$ which makes the following diagram commutative:



Since f is multilinear, the induced map $M \to G$ takes on the value 0 on N. Hence by the universal property of factor modules, it can be factored through M/N, and we have a homomorphism $f_*: M/N \to G$ which makes the following diagram commutative:



Since the image of φ generates M/N, it follows that the induced map f_* is uniquely determined. This proves what we wanted.

The module M/N will be denoted by

$$E_1 \otimes \cdots \otimes E_n \quad \text{or also} \quad \bigotimes_{i=1}^{n} E_i.$$

We have constructed a specific tensor product in the isomorphism class of tensor products, and we shall call it **the** tensor product of $E_1, \ldots, E_n$. If $x_i \in E_i$, we write

$$\varphi(x_1, \ldots, x_n) = x_1 \otimes \cdots \otimes x_n = x_1 \otimes_R \cdots \otimes_R x_n.$$

We have for all i,

$$x_1 \otimes \cdots \otimes ax_i \otimes \cdots \otimes x_n = a(x_1 \otimes \cdots \otimes x_n),$$

$$x_1 \otimes \cdots \otimes (x_i + x_i') \otimes \cdots \otimes x_n$$

$$= (x_1 \otimes \cdots \otimes x_n) + (x_1 \otimes \cdots \otimes x_i' \otimes \cdots \otimes x_n)$$

for $x_i, x_i' \in E_i$ and $a \in R$.

If we have two factors, say $E \otimes F$, then every element of $E \otimes F$ can be written as a sum of terms $x \otimes y$ with $x \in E$ and $y \in F$, because such terms generate $E \otimes F$ over R, and $a(x \otimes y) = ax \otimes y$ for $a \in R$.

Remark. If an element of the tensor product is 0, then that element can already be expressed in terms of a finite number of the relations defining the tensor product. Thus if E is a direct limit of submodules E_i then

$$\varinjlim F \otimes E_i = F \otimes \varinjlim E_i = F \otimes E.$$

In particular, every module is a direct limit of finitely generated submodules, and one uses frequently the technique of testing whether an element of $F \otimes E$ is 0 by testing whether the image of this element in $F \otimes E_i$ is 0 when E_i ranges over the finitely generated submodules of E.

Warning. The tensor product can involve a great deal of collapsing between the modules. For instance, take the tensor product over $\mathbf{Z}$ of $\mathbf{Z}/m\mathbf{Z}$ and $\mathbf{Z}/n\mathbf{Z}$ where m, n are integers > 1 and are relatively prime. Then the tensor product

$$\mathbf{Z}/n\mathbf{Z} \otimes \mathbf{Z}/m\mathbf{Z} = 0.$$

Indeed, we have $n(x \otimes y) = (nx) \otimes y = 0$ and $m(x \otimes y) = x \otimes my = 0$. Hence $x \otimes y = 0$ for all $x \in \mathbf{Z}/n\mathbf{Z}$ and $y \in \mathbf{Z}/m\mathbf{Z}$. Elements of type $x \otimes y$ generate the tensor product, which is therefore 0. We shall see later conditions under which there is no collapsing.

In many subsequent results, we shall assert the existence of certain linear maps from a tensor product. This existence is proved by using the universal mapping property of bilinear maps factoring through the tensor product. The uniqueness follows by prescribing the value of the linear maps on elements of type $x \otimes y$ (say for two factors) since such elements generate the tensor product.

We shall prove the associativity of the tensor product.

Proposition 1.1. *Let E_1, E_2, E_3 be modules. Then there exists a unique isomorphism*

$$(E_1 \otimes E_2) \otimes E_3 \to E_1 \otimes (E_2 \otimes E_3)$$

such that

$$(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$$

for $x \in E_1$, $y \in E_2$ and $z \in E_3$.

Proof. Since elements of type $(x \otimes y) \otimes z$ generate the tensor product, the uniqueness of the desired linear map is obvious. To prove its existence, let $x \in E_1$. The map

$$\lambda_x : E_2 \times E_3 \to (E_1 \otimes E_2) \otimes E_3$$

such that $\lambda_x(y, z) = (x \otimes y) \otimes z$ is clearly bilinear, and hence factors through a linear map of the tensor product

$$\bar{\lambda}_x : E_2 \otimes E_3 \to (E_1 \otimes E_2) \otimes E_3.$$

The map

$$E_1 \times (E_2 \otimes E_3) \to (E_1 \otimes E_2) \otimes E_3$$

such that

$$(x, \alpha) \mapsto \bar{\lambda}_x(\alpha)$$

for $x \in E_1$ and $\alpha \in E_2 \otimes E_3$ is then obviously bilinear, and factors through a linear map

$$E_1 \otimes (E_2 \otimes E_3) \to (E_1 \otimes E_2) \otimes E_3,$$

which has the desired property (clear from its construction).

Proposition 1.2. *Let E, F be modules. Then there is a unique isomorphism*

$$E \otimes F \to F \otimes E$$

such that $x \otimes y \mapsto y \otimes x$ for $x \in E$ and $y \in F$.

Proof. The map $E \times F \to F \otimes E$ such that $(x, y) \mapsto y \otimes x$ is bilinear, and factors through the tensor product $E \otimes F$, sending $x \otimes y$ on $y \otimes x$. Since this last map has an inverse (by symmetry) we obtain the desired isomorphism.

The tensor product has various functorial properties. First, suppose that

$$f_i : E_i' \to E_i \qquad (i = 1, \ldots, n)$$

is a collection of linear maps. We get an induced map on the product,

$$\prod f_i : \prod E_i' \to \prod E_i.$$

If we compose $\prod f_i$ with the canonical map into the tensor product, then we get an induced linear map which we may denote by $T(f_1, \ldots, f_n)$ which makes the following diagram commutative:

$$\begin{array}{ccc} E_1' \times \cdots \times E_n' & \longrightarrow & E_1' \otimes \cdots \otimes E_n' \\ \downarrow {\scriptstyle \prod f_i} & & \downarrow {\scriptstyle T(f_1, \ldots, f_n)} \\ E_1 \times \cdots \times E_n & \longrightarrow & E_1 \otimes \cdots \otimes E_n \end{array}$$

It is immediately verified that T is functorial, namely that if we have a composite of linear maps $f_i \circ g_i$ $(i = 1, \dots, n)$ then

$$T(f_1 \circ g_1, \dots, f_n \circ g_n) = T(f_1, \dots, f_n) \circ T(g_1, \dots, g_n)$$

and

$$T(\mathrm{id}, \dots, \mathrm{id}) = \mathrm{id}.$$

We observe that $T(f_1, \dots, f_n)$ is the unique linear map whose effect on an element $x_1' \otimes \cdots \otimes x_n'$ of $E_1' \otimes \cdots \otimes E_n'$ is

$$x_1' \otimes \cdots \otimes x_n' \mapsto f_1(x_1') \otimes \cdots \otimes f_n(x_n').$$

We may view T as a map

$$\prod_{i=1}^{n} L(E_i', E_i) \to L\left(\bigotimes_{i=1}^{n} E_i', \bigotimes_{i=1}^{n} E_i\right),$$

and the reader will have no difficulty in verifying that this map is multilinear. We shall write out what this means explicitly for two factors, so that our map can be written

$$(f, g) \mapsto T(f, g).$$

Given homomorphisms $f : F' \to F$ and $g_1, g_2 : E' \to E$, then

$$T(f, g_1 + g_2) = T(f, g_1) + T(f, g_2),$$

$$T(f, ag_1) = aT(f, g_1).$$

In particular, select a fixed module F, and consider the functor $\tau = \tau_F$ (from modules to modules) such that

$$\tau(E) = F \otimes E.$$

Then τ gives rise to a linear map

$$\tau : L(E', E) \to L(\tau(E'), \tau(E))$$

for each pair of modules E', E, by the formula

$$\tau(f) = T(\mathrm{id}, f).$$

Remark. By abuse of notation, it is sometimes convenient to write

$$f_1 \otimes \cdots \otimes f_n \quad \text{instead of} \quad T(f_1, \dots, f_n).$$

This should not be confused with the tensor product of elements taken in the tensor product of the modules

$$L(E'_1, E_1) \otimes \cdots \otimes L(E'_n, E_n).$$

The context will always make our meaning clear.

§2. BASIC PROPERTIES

The most basic relation relating linear maps, bilinear maps, and the tensor product is the following: For three modules E, F, G,

$$\boxed{L(E, L(F, G)) \approx L^2(E, F; G) \approx L(E \otimes F, G).}$$

The isomorphisms involved are described in a natural way.

(i) $L^2(E, F; G) \to L(E, L(F, G))$.

If $f : E \times F \to G$ is bilinear, and $x \in E$, then the map

$$f_x : F \to G$$

such that $f_x(y) = f(x, y)$ is linear. Furthermore, the map $x \mapsto f_x$ is linear, and is associated with f to get (i).

(ii) $L(E, L(F, G)) \to L^2(E, F; G)$.

Let $\varphi \in L(E, L(F, G))$. We let $f_\varphi : E \times F \to G$ be the bilinear map such that

$$f_\varphi(x, y) = \varphi(x)(y).$$

Then $\varphi \mapsto f_\varphi$ defines (ii).

It is clear that the homomorphisms of (i) and (ii) are inverse to each other and therefore give isomorphisms of the first two objects in the enclosed box.

(iii) $L^2(E, F; G) \to L(E \otimes F, G)$.

This is the map $f \mapsto f_*$ which associates to each bilinear map f the induced linear map on the tensor product. The association $f \mapsto f_*$ is injective (because f_* is uniquely determined by f), and it is surjective, because any linear map of the tensor product composed with the canonical map $E \times F \to E \otimes F$ gives rise to a bilinear map on $E \times F$.

Proposition 2.1. *Let* $E = \bigoplus_{i=1}^{n} E_i$ *be a direct sum. Then we have an isomorphism*

$$F \otimes E \leftrightarrow \bigoplus_{i=1}^{n} (F \otimes E_i).$$

Proof. The isomorphism is given by abstract nonsense. We keep F fixed, and consider the functor $\tau : X \mapsto F \otimes X$. As we saw above, τ is linear. We have projections $\pi_i : E \to E$ of E on E_i. Then

$$\pi_i \circ \pi_i = \pi_i, \qquad \pi_i \circ \pi_j = 0 \qquad \text{if} \quad i \neq j,$$

$$\sum_{i=1}^{n} \pi_i = \text{id}.$$

We apply the functor τ, and see that $\tau(\pi_i)$ satisfies the same relations, hence gives a direct sum decomposition of $\tau(E) = F \otimes E$. Note that $\tau(\pi_i) = \text{id} \otimes \pi_i$.

Corollary 2.2. *Let* I *be an indexing set, and* $E = \bigoplus_{i \in I} E_i$. *Then we have an isomorphism*

$$\left(\bigoplus_{i \in I} E_i\right) \otimes F \approx \bigoplus_{i \in I} (E_i \otimes F).$$

Proof. Let S be a finite subset of I. We have a sequence of maps

$$\left(\bigoplus_{i \in S} E_i\right) \times F \to \bigoplus_{i \in S} (E_i \otimes F) \to \bigoplus_{i \in I} (E_i \otimes F)$$

the first of which is bilinear, and the second is linear, induced by the inclusion of S in I. The first is the obvious map. If $S \subset S'$, then a trivial commutative diagram shows that the restriction of the map

$$\left(\bigoplus_{i \in S'} E_i\right) \times F \to \bigoplus_{i \in I} (E_i \otimes F)$$

induces our preceding map on the sum for $i \in S$. But we have an *injection*

$$\left(\bigoplus_{i \in S} E_i\right) \times F \to \left(\bigoplus_{i \in S'} E_i\right) \times F.$$

Hence by compatibility, we can define a bilinear map

$$\left(\bigoplus_{i \in I} E_i\right) \times F \to \bigoplus_{i \in I} (E_i \otimes F),$$

and consequently a linear map

$$\left(\bigoplus_{i \in I} E_i\right) \otimes F \to \bigoplus_{i \in I} (E_i \otimes F).$$

In a similar way, one defines a map in the opposite direction, and it is clear that these maps are inverse to each other, hence give an isomorphism.

Suppose now that E is free, of dimension 1 over R. Let $\{v\}$ be a basis, and consider $F \otimes E$. Every element of $F \otimes E$ can be written as a sum of terms $y \otimes av$ with $y \in F$ and $a \in R$. However, $y \otimes av = ay \otimes v$. In a sum of such terms, we can then use linearity on the left,

$$\sum_{i=1}^{n} (y_i \otimes v) = \left(\sum_{i=1}^{n} y_i\right) \otimes v, \qquad y_i \in F.$$

Hence every element is in fact of type $y \otimes v$ with some $y \in F$.

We have a bilinear map

$$F \times E \to F$$

such that $(y, av) \mapsto ay$, inducing a linear map

$$F \otimes E \mapsto F.$$

We also have a linear map $F \to F \otimes E$ given by $y \mapsto y \otimes v$. It is clear that these maps are inverse to each other, and hence that we have an isomorphism

$$F \otimes E \approx F.$$

Thus every element of $F \otimes E$ can be written *uniquely* in the form $y \otimes v$, $y \in F$.

Proposition 2.3. *Let E be free over R, with basis $\{v_i\}_{i \in I}$. Then every element of $F \otimes E$ has a unique expression of the form*

$$\sum_{i \in I} y_i \otimes v_i, \qquad y_i \in F$$

with almost all $y_i = 0$.

Proof. This follows at once from the discussion of the 1-dimensional case, and the corollary of Proposition 2.1.

Corollary 2.4. *Let E, F be free over R, with bases $\{v_i\}_{i \in I}$ and $\{w_j\}_{j \in J}$ respectively. Then $E \otimes F$ is free, with basis $\{v_i \otimes w_j\}$. We have*

$$\dim(E \otimes F) = (\dim E)(\dim F).$$

Proof. Immediate from the proposition.

We see that when E is free over R, then there is no collapsing in the tensor product. Every element of $F \otimes E$ can be viewed as a "formal" linear combination of elements in a basis of E with coefficients in F.

In particular, we see that $R \otimes E$ (or $E \otimes R$) is isomorphic to E, under the correspondence $x \mapsto x \otimes 1$.

Proposition 2.5. *Let E, F be free of finite dimension over R. Then we have an isomorphism*

$$\mathrm{End}_R(E) \otimes \mathrm{End}_R(F) \to \mathrm{End}_R(E \otimes F)$$

which is the unique linear map such that

$$f \otimes g \mapsto T(f, g)$$

for $f \in \mathrm{End}_R(E)$ and $g \in \mathrm{End}_R(F)$.

[We note that the tensor product on the left is here taken in the tensor product of the two modules $\mathrm{End}_R(E)$ and $\mathrm{End}_R(F)$.]

Proof. Let $\{v_i\}$ be a basis of E and let $\{w_j\}$ be a basis of F. Then $\{v_i \otimes w_j\}$ is a basis of $E \otimes F$. For each pair of indices (i', j') there exists a unique endomorphism $f = f_{i,i'}$ of E and $g = g_{j,j'}$ of F such that

$$f(v_i) = v_{i'} \quad \text{and} \quad f(v_\nu) = 0 \quad \text{if } \nu \neq i$$

$$g(w_j) = w_{j'} \quad \text{and} \quad g(w_\mu) = 0 \quad \text{if } \mu \neq j.$$

Furthermore, the families $\{f_{i,i'}\}$ and $\{g_{j,j'}\}$ are bases of $\mathrm{End}_R(E)$ and $\mathrm{End}_R(F)$ respectively. Then

$$T(f, g)(v_\nu \otimes w_\mu) = \begin{cases} v_{i'} \otimes w_{j'} & \text{if } (\nu, \mu) = (i, j) \\ 0 & \text{if } (\nu, \mu) \neq (i, j). \end{cases}$$

Thus the family $\{T(f_{i,i'}, g_{j,j'})\}$ is a basis of $\mathrm{End}_R(E \otimes F)$. Since the family $\{f_{i,i'} \otimes g_{j,j'}\}$ is a basis of $\mathrm{End}_R(E) \otimes \mathrm{End}_R(F)$, the assertion of our proposition is now clear.

In Proposition 2.5, we see that the ambiguity of the tensor sign in $f \otimes g$ is in fact unambiguous in the important special case of free, finite dimensional modules. We shall see later an important application of Proposition 2.5 when we discuss the tensor algebra of a module.

Proposition 2.6. *Let*

$$0 \to E' \xrightarrow{\varphi} E \xrightarrow{\psi} E'' \to 0$$

be an exact sequence, and F any module. Then the sequence

$$F \otimes E' \to F \otimes E \to F \otimes E'' \to 0$$

is exact.

Proof. Given $x'' \in E''$ and $y \in F$, there exists $x \in E$ such that $x'' = \psi(x)$, and hence $y \otimes x''$ is the image of $y \otimes x$ under the linear map

$$F \otimes E \to F \otimes E''.$$

Since elements of type $y \otimes x''$ generate $F \otimes E''$, we conclude that the preceding linear map is surjective. One also verifies trivially that the image of

$$F \otimes E' \to F \otimes E$$

is contained in the kernel of

$$F \otimes E \to F \otimes E''.$$

Conversely, let I be the image of $F \otimes E' \to F \otimes E$, and let

$$f : (F \otimes E)/I \to F \otimes E''$$

be the canonical map. We shall define a linear map

$$g : F \otimes E'' \to (F \otimes E)/I$$

such that $g \circ f = \text{id}$. This obviously will imply that f is injective, and hence will prove the desired converse.

Let $y \in F$ and $x'' \in E''$. Let $x \in E$ be such that $\psi(x) = x''$. We define a map $F \times E'' \to (F \otimes E)/I$ by letting

$$(y, x'') \mapsto y \otimes x \pmod{I},$$

and contend that this map is well defined, i.e. independent of the choice of x such that $\psi(x) = x''$. If $\psi(x_1) = \psi(x_2) = x''$, then $\psi(x_1 - x_2) = 0$, and by hypothesis, $x_1 - x_2 = \varphi(x')$ for some $x' \in E'$. Then

$$y \otimes x_1 - y \otimes x_2 = y \otimes (x_1 - x_2) = y \otimes \varphi(x').$$

This shows that $y \otimes x_1 \equiv y \otimes x_2 \pmod{I}$, and proves that our map is well defined. It is obviously bilinear, and hence factors through a linear map g, on the tensor product. It is clear that the restriction of $g \circ f$ on elements of type $y \otimes x$ is the identity. Since these elements generate $F \otimes E$, we conclude that f is injective, as was to be shown.

It is not always true that the sequence

$$0 \to F \otimes E' \to F \otimes E \to F \otimes E'' \to 0$$

is exact. It is exact if the first sequence in Proposition 2.6 splits, i.e. if E is essentially the direct sum of E' and E''. This is a trivial consequence of Proposition 2.1, and the reader should carry out the details to get accustomed to the formalism of the tensor product.

Proposition 2.7. *Let $\mathfrak{a}$ be an ideal of R. Let E be a module. Then the map $(R/\mathfrak{a}) \times E \to E/\mathfrak{a}E$ induced by*

$$(a, x) \mapsto ax \pmod{\mathfrak{a}E}, \qquad a \in R, x \in E$$

is bilinear and induces an isomorphism

$$(R/\mathfrak{a}) \otimes E \xrightarrow{\approx} E/\mathfrak{a}E.$$

Proof. Our map $(a, x) \mapsto ax \pmod{\mathfrak{a}E}$ clearly induces a bilinear map of $R/\mathfrak{a} \times E$ onto $E/\mathfrak{a}E$, and hence a linear map of $R/\mathfrak{a} \otimes E$ onto $E/\mathfrak{a}E$. We can construct an inverse, for we have a well-defined linear map

$$E \to R/\mathfrak{a} \otimes E$$

such that $x \mapsto \bar{1} \otimes x$ (where $\bar{1}$ is the residue class of 1 in $R/\mathfrak{a}$). It is clear that $\mathfrak{a}E$ is contained in the kernel of this last linear map, and thus that we obtain a homomorphism

$$E/\mathfrak{a}E \to R/\mathfrak{a} \otimes E,$$

which is immediately verified to be inverse to the homomorphism described in the statement of the proposition.

The association $E \mapsto E/\mathfrak{a}E \approx R/\mathfrak{a} \otimes E$ is often called a **reduction map**. In §4, we shall interpret this reduction map as an extension of the base.

§3. FLAT MODULES

The question under which conditions the left-hand arrow in Proposition 2.6 is an injection gives rise to the theory of those modules for which it is, and we follow Serre in calling them flat. Thus formally, the following conditions are equivalent, and define a **flat** module F, which should be called **tensor exact**.

F 1. For every exact sequence

$$E' \to E \to E''$$

the sequence

$$F \otimes E' \to F \otimes E \to F \otimes E''$$

is exact.

F 2. For every short exact sequence

$$0 \to E' \to E \to E'' \to 0$$

the sequence

$$0 \to F \otimes E' \to F \otimes E \to F \otimes E'' \to 0$$

is exact.

F 3. For every injection $0 \to E' \to E$ the sequence

$$0 \to F \otimes E' \to F \otimes E$$

is exact.

It is immediate that **F 1** implies **F 2** implies **F 3**. Finally, we see that **F 3** implies **F 1** by writing down the kernel and image of the map $E' \to E$ and applying **F 3**. We leave the details to the reader.

The following proposition gives tests for flatness, and also examples.

Proposition 3.1.

(i) *The ground ring is flat as module over itself.*

(ii) *Let* $F = \bigoplus F_i$ *be a direct sum. Then* F *is flat if and only if each* F_i *is flat.*

(iii) *A projective module is flat.*

The properties expressed in this proposition are basically categorical, cf. the comments on abstract nonsense at the end of the section. In another vein, we have the following tests having to do with localization.

Proposition 3.2.

(i) *Let* S *be a multiplicative subset of* R. *Then* $S^{-1}R$ *is flat over* R.

(ii) *A module* M *is flat over* R *if and only if the localization* $M_{\mathfrak{p}}$ *is flat over* $R_{\mathfrak{p}}$ *for each prime ideal* $\mathfrak{p}$ *of* R.

(iii) *Let* R *be a principal ring. A module* F *is flat if and only if* F *is torsion free.*

The proofs are simple, and will be left to the reader. More difficult tests for flatness will be proved below, however.

Examples of non-flatness. If R is an entire ring, and a module M over R has torsion, then M is not flat. (Prove this, which is immediate.)

There is another type of example which illustrates another bad phenomenon. Let R be some ring in a finite extension K of $\mathbf{Q}$, and such that R is a finite module over $\mathbf{Z}$ but not integrally closed. Let R' be its integral closure. Let $\mathfrak{p}$ be a maximal ideal of R and suppose that $\mathfrak{p}R'$ is contained in two distinct maximal ideals $\mathfrak{P}_1$ and $\mathfrak{P}_2$. Then it can be shown that R' is not flat over R, otherwise R' would be free over the local ring $R_\mathfrak{p}$, and the rank would have to be 1, thus precluding the possibility of the two primes $\mathfrak{P}_1$ and $\mathfrak{P}_2$. It is good practice for the reader actually to construct a numerical example of this situation. The same type of example can be constructed with a ring $R = k[x,y]$, where k is an algebraically closed field, even of characteristic 0, and x, y are related by an irreducible polynomial equation $f(x,y) = 0$ over k. We take R not integrally closed, such that its integral closure exhibits the same splitting of a prime $\mathfrak{p}$ of R into two primes. In each one of these similar cases, one says that there is a singularity at $\mathfrak{p}$.

As a third example, let R be the power series ring in more than one variable over a field k. Let $\mathfrak{m}$ be the maximal ideal. Then $\mathfrak{m}$ is not flat, because otherwise, by Theorem 3.8 below, $\mathfrak{m}$ would be free, and if $R = k[[x_1, \ldots, x_n]]$, then $x_1, \ldots, x_n$ would be a basis for $\mathfrak{m}$ over R, which is obviously not the case, since x_1, x_2 are linearly dependent over R when $n \geqq 2$. The same argument, of course, applies to any local ring R such that $\mathfrak{m}/\mathfrak{m}^2$ has dimension $\geqq 2$ over $R/\mathfrak{m}$.

Next we come to further criteria when a module is flat. For the proofs, we shall snake it all over the place. Cf. the remark at the end of the section.

Lemma 3.3. *Let F be flat, and suppose that*

$$0 \to N \to M \to F \to 0$$

is an exact sequence. Then for any E, we have an exact sequence

$$0 \to N \otimes E \to M \otimes E \to F \otimes E \to 0.$$

Proof. Represent E as a quotient of a flat L by an exact sequence

$$0 \to K \to L \to E \to 0.$$

Then we have the following exact and commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & N \otimes K & \longrightarrow & M \otimes K & \longrightarrow & F \otimes K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N \otimes L & \longrightarrow & M \otimes L & \longrightarrow & F \otimes L & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N \otimes E & \longrightarrow & M \otimes E & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The top right 0 comes by hypothesis that F is flat. The 0 on the left comes from the fact that L is flat. The snake lemma yields the exact sequence

$$0 \to N \otimes E \to M \otimes E$$

which proves the lemma.

Proposition 3.4. *Let*

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence, and assume that F'' is flat. Then F is flat if and only if F' is flat. More generally, let

$$0 \to F^0 \to F^1 \to \cdots \to F^n \to 0$$

be an exact sequence such that $F^1, \ldots, F^n$ are flat. Then F^0 is flat.

Proof. Let $0 \to E' \to E$ be an injection. We have an exact and commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & F' \otimes E' & \longrightarrow & F \otimes E' & \longrightarrow & F'' \otimes E' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F' \otimes E & \longrightarrow & F \otimes E & \longrightarrow & F'' \otimes E & & \end{array}$$

The 0 on top is by hypothesis that F'' is flat, and the two zeros on the left are justified by Lemma 3.3. If F' is flat, then the first vertical map is an injection, and the snake lemma shows that F is flat. If F is flat, then the middle column is an injection. Then the two zeros on the left and the commutativity of the left square show that the map $F' \otimes E' \to F' \otimes E$ is an injection, so F' is flat. This proves the first statement.

The proof of the second statement is done by induction, introducing kernels and cokernels at each step as in dimension shifting, and apply the first statement at each step. This proves the proposition

To give flexibility in testing for flatness, the next two lemmas are useful, in relating the notion of flatness to a specific module. Namely, we say that F is ***E*-flat** or **flat for** E, if for every monomorphism

$$0 \to E' \to E$$

the tensored sequence

$$0 \to F \otimes E' \to F \otimes E$$

is also exact.

Lemma 3.5. *Assume that F is E-flat. Then F is also flat for every submodule and every quotient module of E.*

Proof. The submodule part is immediate because if $E'_1 \subset E'_2 \subset E$ are submodules, and $F \otimes E'_1 \to F \otimes E$ is a monomorphism so is $F \otimes E'_1 \to F \otimes E'_2$ since the composite map with $F \otimes E'_2 \to F \otimes E$ is a monomorphism. The only question lies with a factor module. Suppose we have an exact sequence

$$0 \to N \to E \to M \to 0.$$

Let M' be a submodule of M and E' its inverse image in E. Then we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

We tensor with F to get the exact and commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & K & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & F \otimes N & \longrightarrow & F \otimes E' & \longrightarrow & F \otimes M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F \otimes N & \longrightarrow & F \otimes E & \longrightarrow & F \otimes M & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where K is the questionable kernel which we want to prove is 0. But the snake lemma yields the exact sequence

$$0 \to K \to 0$$

which concludes the proof.

Lemma 3.6. *Let $\{E_i\}$ be a family of modules, and suppose that F is flat for each E_i. Then F is flat for their direct sum.*

Proof. Let $E = \bigoplus E_i$ be their direct sum. We have to prove that given any submodule E' of E, the sequence

$$0 \to F \otimes E' \to F \otimes E = \bigoplus F \otimes E_i$$

is exact. Note that if an element of $F \otimes E'$ becomes 0 when mapped into the direct sum, then it becomes 0 already in a finite subsum, so without loss of generality we may assume that the set of indices is finite. Then by induction, we can assume that the set of indices consists of two elements, so we have two modules E_1 and E_2, and $E = E_1 \oplus E_2$. Let N be a submodule of E. Let $N_1 = N \cap E_1$ and let N_2 be the image of N under the projection on E_2. Then we have the following commutative and exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & N_1 & \longrightarrow & N & \longrightarrow & N_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_1 & \longrightarrow & E & \longrightarrow & E_2 & &
\end{array}$$

Tensoring with F we get the exact and commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & F \otimes N_1 & \longrightarrow & F \otimes N & \longrightarrow & F \otimes N_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F \otimes E_1 & \longrightarrow & F \otimes E & \longrightarrow & F \otimes E_2 & &
\end{array}$$

The lower left exactness is due to the fact that $E = E_1 \oplus E_2$. Then the snake lemma shows that the kernel of the middle vertical map is 0. This proves the lemma.

The next proposition shows that to test for flatness, it suffices to do so only for a special class of exact sequences arising from ideals.

Proposition 3.7. *F is flat if and only if for every ideal $\mathfrak{a}$ of R the natural map*

$$\mathfrak{a} \otimes F \to \mathfrak{a}F$$

is an isomorphism. In fact, F is flat if and only for every ideal $\mathfrak{a}$ of R tensoring the sequence

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$$

with F yields an exact sequence.

Proof. If F is flat, then tensoring with F and using Proposition 2.7 shows that the natural map is an isomorphism, because $\mathfrak{a}M$ is the kernel of $M \to M/\mathfrak{a}M$. Conversely, assume that this map is an isomorphism for all ideals $\mathfrak{a}$. This means that F is R-flat. By Lemma 3.6 it follows that F is flat for an arbitrary direct sum of R with itself, and since any module M is a quotient of such a direct sum, Lemma 3.5 implies that F is M-flat, thus concluding the proof.

Remark on abstract nonsense. The proofs of Proposition 3.1(i), (ii), (iii), and Propositions 3.3 through 3.7 are basically rooted in abstract nonsense, and depend only on arrow theoretic arguments. Specifically, as in Chapter XX, §8, suppose that we have a bifunctor T on two distinct abelian categories $\mathfrak{A}$ and $\mathfrak{B}$ such that for each A, the functor $B \mapsto T(A, B)$ is right exact and for each B the functor $A \mapsto T(A, B)$ is right exact. Instead of "flat" we call an object A of $\mathfrak{A}$ **T-exact** if $B \mapsto T(A, B)$ is an exact functor; and we call an object L of $\mathfrak{B}$ **${}^tT$-exact** if $A \mapsto T(A, L)$ is exact. Then the references to the base ring and free modules can be replaced by abstract nonsense conditions as follows.

In the use of L in Lemma 3.3, we need to assume that for every object E of $\mathfrak{B}$ there is a ${}^tT$-exact L and an epimorphism

$$L \to E \to 0.$$

For the analog of Proposition 3.7, we need to assume that there is some object R in $\mathfrak{B}$ for which F is R-exact, that is given an exact sequence

$$0 \to \mathfrak{a} \to R$$

then $0 \to T(F, \mathfrak{a}) \to T(F, R)$ is exact; and we also need to assume that R is a generator in the sense that every object B is the quotient of a direct sum of R with itself, taken over some family of indices, and T respects direct sums.

The snake lemma is valid in arbitrary abelian categories, either because its proof is "functorial," or by using a representation functor to reduce it to the category of abelian groups. Take your pick.

In particular, we really don't need to have a commutative ring as base ring, this was done only for simplicity of language.

We now pass to somewhat different considerations.

Theorem 3.8. *Let R be a commutative local ring, and let M be a finite flat module over R. Then M is free. In fact, if $x_1, \dots, x_n \in M$ are elements of M whose residue classes are a basis of $M/\mathfrak{m}M$ over $R/\mathfrak{m}$, then $x_1, \dots, x_n$ form a basis of M over R.*

Proof. Let $R^{(n)} \to M$ be the map which sends the unit vectors of $R^{(n)}$ on $x_1, \dots, x_n$ respectively, and let N be its kernel. We get an exact sequence

$$0 \to N \to R^{(n)} \to M,$$

whence a commutative diagram

$$\begin{array}{ccccccc} & & \mathfrak{m} \otimes N & \longrightarrow & \mathfrak{m} \otimes R^{(n)} & \longrightarrow & \mathfrak{m} \otimes M \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & N & \longrightarrow & R^{(n)} & \longrightarrow & M \end{array}$$

in which the rows are exact. Since M is assumed flat, the map h is an injection. By the snake lemma one gets an exact sequence

$$0 \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h,$$

and the arrow on the right is merely

$$R^{(n)}/\mathfrak{m}R^{(n)} \to M/\mathfrak{m}M,$$

which is an isomorphism by the assumption on $x_1, \ldots, x_n$. It follows that $\operatorname{coker} f = 0$, whence $\mathfrak{m}N = N$, whence $N = 0$ by Nakayama if R is Noetherian, so N is finitely generated. If R is not assumed Noetherian, then one has to add a slight argument as follows in case M is finitely presented.

Lemma 3.9. *Assume that M is finitely presented, and let*

$$0 \to N \to E \to M \to 0$$

be exact, with E finite free. Then N is finitely generated.

Proof. Let

$$L_1 \to L_2 \to M \to 0$$

be a finite presentation of M, that is an exact sequence with L_1, L_2 finite free. Using the freeness, there exists a commutative diagram

$$\begin{array}{ccccccccc} & & L_1 & \longrightarrow & L_2 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

such that $L_2 \to E$ is surjective. Then the snake lemma gives at once the exact sequence

$$0 \to \operatorname{coker}(L_1 \to N) \to 0,$$

so $\operatorname{coker}(L_1 \to N) = 0$, whence N is an image of L_1 and is therefore finitely generated, thereby proving the lemma, and also completing the proof of Theorem 3.8 when M is finitely presented.

We still have not proved Theorem 3.8 in the fully general case. For this we use Matsumura's proof (see his *Commutative Algebra*, Chapter 2), based on the following lemma.

Lemma 3.10. *Assume that M is flat over R. Let $a_i \in A$, $x_i \in M$ for $i = 1, \dots, n$, and suppose that we have the relation*

$$\sum_{i=1}^{n} a_i x_i = 0.$$

Then there exists an integer s and elements $b_{ij} \in A$ and $y_j \in M$ $(j = 1, \dots, s)$ such that

$$\sum_i a_i b_{ij} = 0 \quad \text{for all } j \quad \text{and} \quad x_i = \sum_j b_{ij} y_j \quad \text{for all } i.$$

Proof. We consider the exact sequence

$$0 \to K \to R^{(n)} \to R$$

where the map $R^{(n)} \to R$ is given by

$$(b_1, \dots, b_n) \mapsto \sum_{i=1}^{n} a_i b_i,$$

and K is its kernel. Since M is flat it follows that

$$K \otimes M \to M^{(n)} \xrightarrow{f_M} M$$

is exact, where f_M is given by

$$f_M(z_1, \dots, z_n) = \sum_{i=1}^{n} a_i z_i.$$

Therefore there exist elements $\beta_j \in K$ and $y_j \in M$ such that

$$(x_1, \dots, x_n) = \sum_{j=1}^{s} \beta_j \, y_j.$$

Write $\beta_j = (b_{1j}, \dots, b_{nj})$ with $b_{ij} \in R$. This proves the lemma.

We may now apply the lemma to prove the theorem in exactly the same way we proved that a finite projective module over a local ring is free in Chapter X, Theorem 4.4, by induction. This concludes the proof.

Remark. In the applications I know of, the base ring is Noetherian, and so one gets away with the very simple proof given at first. I did not want to obstruct the simplicity of this proof, and that is the reason I gave the additional technicalities in increasing order of generality.

Applications of homology. We end this section by pointing out a connection between the tensor product and the homological considerations of Chapter XX, §8 for those readers who want to pursue this trend of thoughts. The tensor product is a bifunctor to which we can apply the considerations of Chapter XX, §8. Let M, N be modules. Let

$$\cdots \to E_i \to E_{i-1} \to E_0 \to M \to 0$$

be a free or projective resolution of M, i.e. an exact sequence where E_i is free or projective for all $i \geqq 0$. We write this sequence as

$$E_M \to M \to 0.$$

Then by definition,

$$\mathrm{Tor}_i(M, N) = i\text{-th homology of the complex } E \otimes N, \text{ that is of}$$

$$\cdots \to E_i \otimes N \to E_{i-1} \otimes N \to \cdots \to E_0 \otimes N \to 0.$$

This homology is determined up to a unique isomorphism. I leave to the reader to pick whatever convention is agreeable to fix one resolution to determine a fixed representation of $\mathrm{Tor}_i(M, N)$, to which all others are isomorphic by a unique isomorphism.

Since we have a bifunctorial isomorphism $M \otimes N \approx N \otimes M$, we also get a bifunctorial isomorphism

$$\mathrm{Tor}_i(M, N) \approx \mathrm{Tor}_i(N, M)$$

for all i. See Propositions 8.2 and 8.2′ of Chapter XX.

Following general principles, we say that M has **Tor-dimension** $\leqq d$ if $\mathrm{Tor}_i(M, N) = 0$ for all $i > d$ and all N. From Chapter XX, §8 we get the following result, which merely replaces T-exact by flat.

Theorem 3.11. *The following three conditions are equivalent concerning a module M.*

(i) *M is flat.*

(ii) $\mathrm{Tor}_1(M, N) = 0$ *for all N.*

(iii) $\mathrm{Tor}_i(M, N) = 0$ *for all $i \geqq 1$ and all N, in other words, M has* Tor-*dimension* 0.

Remark. Readers willing to use this characterization can replace some of the preceding proofs from 3.3 to 3.6 by a Tor-dimension argument, which is more formal, or at least formal in a different way, and may seem more rapid. The snake lemma was used ad hoc in each case to prove the desired result. The general homology theory simply replaces this use by the corresponding formal homological step, once the general theory of the derived functor has been carried out.

§4. EXTENSION OF THE BASE

Let R be a commutative ring and let E be a R-module. We specify R since we are going to work with several rings in a moment. Let $R \to R'$ be a homomorphism of commutative rings, so that R' is an R-algebra, and may be viewed as an R-module also. We have a 3-multilinear map

$$R' \times R' \times E \to R' \otimes E$$

defined by the rule

$$(a, b, x) \mapsto ab \otimes x.$$

This induces therefore a R-linear map

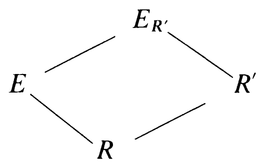
$$R' \otimes (R' \otimes E) \to R' \otimes E$$

and hence a R-bilinear map $R' \times (R' \otimes E) \to R' \otimes E$. It is immediately verified that our last map makes $R' \otimes E$ into a R'-module, which we shall call the **extension of E over R'**, and denote by $E_{R'}$. We also say that $E_{R'}$ is obtained by **extension of the base** ring from R to R'.

Example 1. Let $\mathfrak{a}$ be an ideal of R and let $R \to R/\mathfrak{a}$ be the canonical homomorphism. Then the extension of E to $R/\mathfrak{a}$ is also called the **reduction** of E modulo $\mathfrak{a}$. This happens often over the integers, when we reduce modulo a prime p (i.e. modulo the prime ideal (p)).

Example 2. Let R be a field and R' an extension field. Then E is a vector space over R, and $E_{R'}$ is a vector space over R'. In terms of a basis, we see that our extension gives what was alluded to in the preceding chapter. This example will be expanded in the exercises.

We draw the same diagrams as in field theory:



to visualize an extension of the base. From Proposition 2.3, we conclude:

Proposition 4.1. *Let E be a free module over R, with basis $\{v_i\}_{i \in I}$. Let $v'_i = 1 \otimes v_i$. Then $E_{R'}$ is a free module over R', with basis $\{v'_i\}_{i \in I}$.*

We had already used a special case of this proposition when we observed that the dimension of a free module is defined, i.e. that two bases have the same cardinality. Indeed, in that case, we reduced modulo a maximal ideal of R to reduce the question to a vector space over a field.

When we start changing rings, it is desirable to indicate R in the notation for the tensor product. Thus we write

$$E_{R'} = R' \otimes E = R' \otimes_R E.$$

Then we have transitivity of the extension of the base, namely, if $R \to R' \to R''$ is a succession of homomorphisms of commutative rings, then we have an isomorphism

$$R'' \otimes_R E \approx R'' \otimes_{R'} (R' \otimes_R E)$$

and this isomorphism is one of R''-modules. The proof is trivial and will be left to the reader.

If E has a multiplicative structure, we can extend the base also for this multiplication. Let $R \to A$ be a ring-homomorphism such that every element in the image of R in A commutes with every element in A (i.e. an R-algebra). Let $R \to R'$ be a homomorphism of commutative rings. We have a 4-multilinear map

$$R' \times A \times R' \times A \to R' \otimes A$$

defined by

$$(a, x, b, y) \mapsto ab \otimes xy.$$

We get an induced R-linear map

$$R' \otimes A \otimes R' \otimes A \to R' \otimes A$$

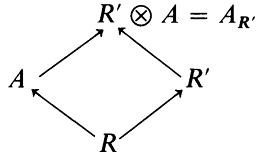
and hence an induced R-bilinear map

$$(R' \otimes A) \times (R' \otimes A) \to R' \otimes A.$$

It is trivially verified that the law of composition on $R' \otimes A$ we have just defined is associative. There is a unit element in $R' \otimes A$, namely, $1 \otimes 1$. We have a ring-homomorphism of R' into $R' \otimes A$, given by $a \mapsto a \otimes 1$. In this way one sees at once that $R' \otimes A = A_{R'}$ is an R'-algebra. We note that the map

$$x \mapsto 1 \otimes x$$

is a ring-homomorphism of A into $R' \otimes A$, and that we get a commutative diagram of ring homomorphisms,



For the record, we give some routine tests for flatness in the context of base extension.

Proposition 4.2. *Let $R \to A$ be an R-algebra, and assume A commutative.*

(i) **Base change.** *If F is a flat R-module, then $A \otimes_R F$ is a flat A-module.*

(ii) **Transitivity.** *If A is a flat commutative R-algebra and M is a flat A-module, then M is flat as R-module.*

The proofs are immediate, and will be left to the reader.

§5. SOME FUNCTORIAL ISOMORPHISMS

We recall an abstract definition. Let $\mathfrak{A}$, $\mathfrak{B}$ be two categories. The functors of $\mathfrak{A}$ into $\mathfrak{B}$ (say covariant, and in one variable) can be viewed as the objects of a category, whose morphisms are defined as follows. If L, M are two such functors, a morphism $H : L \to M$ is a rule which to each object X of $\mathfrak{A}$ associates a morphism $H_X : L(X) \to M(X)$ in $\mathfrak{B}$, such that for any morphism $f : X \to Y$ in $\mathfrak{A}$, the following diagram is commutative:

$$\begin{array}{ccc} L(X) & \xrightarrow{H_X} & M(X) \\ {\scriptstyle L(f)}\downarrow & & \downarrow{\scriptstyle M(f)} \\ L(Y) & \xrightarrow[H_Y]{} & M(Y) \end{array}$$

We can therefore speak of isomorphisms of functors. We shall see examples of these in the theory of tensor products below. In our applications, our categories are additive, that is, the set of morphisms is an additive group, and the composition law is $\mathbf{Z}$-bilinear. In that case, a functor L is called **additive** if

$$L(f + g) = L(f) + L(g).$$

We let R be a commutative ring, and we shall consider additive functors from the category of R-modules into itself. For instance we may view the dual module as a functor,

$$E \mapsto E^{\vee} = L(E, R) = \mathrm{Hom}_R(E, R).$$

Similarly, we have a functor in two variables,

$$(E, F) \mapsto L(E, F) = \mathrm{Hom}_R(E, F),$$

contravariant in the first, covariant in the second, and bi-additive.

We shall give several examples of functorial isomorphisms connected with the tensor product, and for this it is most convenient to state a general theorem, giving us a criterion when a morphism of functors is in fact an isomorphism.

Proposition 5.1. *Let L, M be two functors (both covariant or both contravariant) from the category of R-modules into itself. Assume that both functors are additive. Let $H : L \to M$ be a morphism of functors. If $H_E : L(E) \to M(E)$ is an isomorphism for every 1-dimensional free module E over R, then H_E is an isomorphism for every finite-dimensional free module over R.*

Proof. We begin with a lemma.

Lemma 5.2. *Let E and E_i $(i = 1, \dots, m)$ be modules over a ring. Let $\varphi_i : E_i \to E$ and $\psi_i : E \to E_i$ be homomorphisms having the following properties:*

$$\psi_i \circ \varphi_i = \text{id}, \qquad \psi_i \circ \varphi_j = 0 \quad \text{if} \quad i \neq j$$

$$\sum_{i=1}^{m} \varphi_i \circ \psi_i = \text{id},$$

Then the map

$$x \mapsto (\psi_1 x, \dots, \psi_m x)$$

is an isomorphism of E onto the direct product $\prod_{i=1}^{m} E_i$, and the map

$$(x_1, \dots, x_m) \mapsto \varphi_1 x_1 + \cdots + \varphi_m x_m$$

is an isomorphism of the product onto E. Conversely, if E is equal to the direct sum of submodules E_i $(i = 1, \dots, m)$, if we let ψ_i be the inclusion of E_i in E, and φ_i the projection of E on E_i, then these maps satisfy the above-mentioned properties.

Proof. The proof is routine, and is essentially the same as that of Proposition 3.1 of Chapter III. We shall leave it as an exercise to the reader.

We observe that the families $\{\varphi_i\}$ and $\{\psi_i\}$ satisfying the properties of the lemma behave functorially: If T is an additive contravariant functor, say, then the families $\{T(\psi_i)\}$ and $\{T(\varphi_i)\}$ also satisfy the properties of the lemma. Similarly if T is a covariant functor.

To apply the lemma, we take the modules E_i to be the 1-dimensional components occurring in a decomposition of E in terms of a basis. Let us assume for instance that L, M are both covariant. We have for each module E a commutative diagram

$$\begin{array}{ccc} L(E) & \xrightarrow{H_E} & M(E) \\ {\scriptstyle L(\varphi_i)}\uparrow & & \uparrow{\scriptstyle M(\varphi_i)} \\ L(E_i) & \xrightarrow[H_{E_i}]{} & M(E_i) \end{array}$$

and a similar diagram replacing φ_i by ψ_i, reversing the two vertical arrows. Hence we get a direct sum decomposition of $L(E)$ in terms of $L(\psi_i)$ and $L(\varphi_i)$, and similarly for $M(E)$, in terms of $M(\psi_i)$ and $M(\varphi_i)$. By hypothesis, H_{E_i} is an isomorphism. It then follows trivially that H_E is an isomorphism. For instance, to prove injectivity, we write an element $v \in L(E)$ in the form

$$v = \sum L(\varphi_i)v_i,$$

with $v_i \in L(E_i)$. If $H_E v = 0$, then

$$0 = \sum H_E L(\varphi_i)v_i = \sum M(\varphi_i)H_{E_i}v_i,$$

and since the maps $M(\varphi_i)$ $(i = 1, \ldots, m)$ give a direct sum decomposition of $M(E)$, we conclude that $H_{E_i}v_i = 0$ for all i, whence $v_i = 0$, and $v = 0$. The surjectivity is equally trivial.

When dealing with a functor of several variables, additive in each variable, one can keep all but one of the variables fixed, and then apply the proposition. We shall do this in the following corollaries.

Corollary 5.3. *Let E', E, F', F be free and finite dimensional over R. Then we have a functorial isomorphism*

$$L(E', E) \otimes L(F', F) \to L(E' \otimes F', E \otimes F)$$

such that

$$f \otimes g \mapsto T(f, g).$$

Proof. Keep E, F', F fixed, and view $L(E', E) \otimes L(F', F)$ as a functor in the variable E'. Similarly, view

$$L(E' \otimes F', E \otimes F)$$

as a functor in E'. The map $f \otimes g \mapsto T(f, g)$ is functorial, and thus by the lemma, it suffices to prove that it yields an isomorphism when E' has dimension 1. Assume now that this is the case; fix E' of dimension 1, and view the two expressions in the corollary as functors of the variable E. Applying the lemma again, it suffices to prove that our arrow is an isomorphism when E has dimension 1. Similarly, we may assume that F, F' have dimension 1. In that case the verification that the arrow is an isomorphism is a triviality, as desired.

Corollary 5.4. *Let E, F be free and finite dimensional. Then we have a natural isomorphism*

$$\mathrm{End}_R(E) \otimes \mathrm{End}_R(F) \to \mathrm{End}_R(E \otimes F).$$

Proof. Special case of Corollary 5.3.

Note that Corollary 5.4 had already been proved before, and that we mention it here only to see how it fits with the present point of view.

Corollary 5.5. *Let E, F be free finite dimensional over R. There is a functorial isomorphism*

$$E^\vee \otimes F \to L(E, F)$$

given for $\lambda \in E^\vee$ and $y \in F$ by the map

$$\lambda \otimes y \mapsto A_{\lambda,y}$$

where $A_{\lambda,y}$ is such that for all $x \in E$, we have $A_{\lambda,y}(x) = \lambda(x)y$.

The inverse isomorphism of Corollary 5.5 can be described as follows. Let $\{v_1, \ldots, v_n\}$ be a basis of E, and let $\{v'_1, \ldots, v'_n\}$ be the dual basis. If $A \in L(E, F)$, then the element

$$\sum_{i=1}^{n} v'_i \otimes A(v_i) \in E^\vee \otimes F$$

maps to A. In particular, if $E = F$, then the element mapping to the identity id_E is called the **Casimir element**

$$\sum_{i=1}^{n} v'_i \otimes v_i,$$

independent of the choice of basis. Cf. Exercise 14.

To prove Corollary 5.5, justify that there is a well-defined homomorphism of $E^\vee \otimes F$ to $L(E, F)$, by the formula written down. Verify that this homomorphism is both injective and surjective. We leave the details as exercises.

Differential geometers are very fond of the isomorphism

$$L(E, E) \to E^\vee \otimes E,$$

and often use $E^\vee \otimes E$ when they think geometrically of $L(E, E)$, thereby emphasizing an unnecessary dualization, and an irrelevant formalism, when it is easier to deal directly with $L(E, E)$. In differential geometry, one applies various functors L to the tangent space at a point on a manifold, and elements of the spaces thus obtained are called **tensors** (of type L).

Corollary 5.6. *Let E, F be free and finite dimensional over R. There is a functorial isomorphism*

$$E^\vee \otimes F^\vee \to (E \otimes F)^\vee.$$

given for $\lambda \in E^\vee$ and $\mu \in F^\vee$ by the map

$$\lambda \otimes \mu \mapsto \Lambda,$$

where Λ is such that, for all $x \in E$ and $y \in F$,

$$\Lambda(x \otimes y) = \lambda(x)\mu(y)$$

Proof. As before.

Finally, we leave the following results as an exercise.

Proposition 5.7. *Let E be free and finite dimensional over R. The trace function on $L(E, E)$ is equal to the composite of the two maps*

$$L(E, E) \to E^\vee \otimes E \to R,$$

where the first map is the inverse of the isomorphism described in Corollary 5.5, and the second map is induced by the bilinear map

$$(\lambda, x) \mapsto \lambda(x).$$

Of course, it is precisely in a situation involving the trace that the isomorphism of Corollary 5.5 becomes important, and that the finite dimensionality of E is used. In many applications, this finite dimensionality plays no role, and it is better to deal with $L(E, E)$ directly.

§6. TENSOR PRODUCT OF ALGEBRAS

In this section, we again let R be a commutative ring. By an **R-algebra** we mean a ring homomorphism $R \to A$ into a ring A such that the image of R is contained in the center of A.

Let A, B be R-algebras. We shall make $A \otimes B$ into an R-algebra. Given $(a, b) \in A \times B$, we have an R-bilinear map

$$M_{a,b}\colon A \times B \to A \otimes B \text{ such that } M_{a,b}(a', b') = aa' \otimes bb'.$$

Hence $M_{a,b}$ induces an R-linear map $m_{a,b}\colon A \otimes B \to A \otimes B$ such that $m_{a,b}(a', b') = aa' \otimes bb'$. But $m_{a,b}$ depends bilinearly on a and b, so we obtain finally a unique R-bilinear map

$$A \otimes B \times A \otimes B \to A \otimes B$$

such that $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. This map is obviously associative, and we have a natural ring homomorphism

$$R \to A \otimes B \quad \text{given by} \quad c \mapsto 1 \otimes c = c \otimes 1.$$

Thus $A \otimes B$ is an R-algebra, called the **ordinary tensor product**.

Application: commutative rings

We shall now see the implication of the above for commutative rings.

Proposition 6.1. *Finite coproducts exist in the category of commutative rings, and in the category of commutative algebras over a commutative ring. If $R \to A$ and $R \to B$ are two homomorphisms of commutative rings, then their coproduct over R is the homomorphism $R \to A \otimes B$ given by*

$$a \mapsto a \otimes 1 = 1 \otimes a.$$

Proof. We shall limit our proof to the case of the coproduct of two ring homomorphisms $R \to A$ and $R \to B$. One can use induction.

Let A, B be commutative rings, and assume given ring-homomorphisms into a commutative ring C,

$$\varphi : A \to C \quad \text{and} \quad \psi : B \to C.$$

Then we can define a $\mathbf{Z}$-bilinear map

$$A \times B \to C$$

by $(x, y) \mapsto \varphi(x)\psi(y)$. From this we get a unique additive homomorphism

$$A \otimes B \to C$$

such that $x \otimes y \mapsto \varphi(x)\psi(y)$. We have seen above that we can define a ring structure on $A \otimes B$, such that

$$(a \otimes b)(c \otimes d) = ac \otimes bd.$$

It is then clear that our map $A \otimes B \to C$ is a ring-homomorphism. We also have two ring-homomorphisms

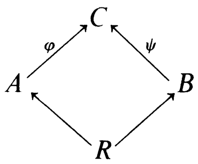
$$A \xrightarrow{f} A \otimes B \quad \text{and} \quad B \xrightarrow{g} A \otimes B$$

given by

$$x \mapsto x \otimes 1 \quad \text{and} \quad y \mapsto 1 \otimes y.$$

The universal property of the tensor product shows that $(A \otimes B, f, g)$ is a coproduct of our rings A and B.

If A, B, C are R-algebras, and if φ, ψ make the following diagram commutative,



then $A \otimes B$ is also an R-algebra (it is in fact an algebra over R, or A, or B, depending on what one wants to use), and the map $A \otimes B \to C$ obtained above gives a homomorphism of R-algebras.

A commutative ring can always be viewed as a $\mathbf{Z}$-algebra (i.e. as an algebra over the integers). Thus one sees the coproduct of commutative rings as a special case of the coproduct of R-algebras.

Graded Algebras. Let G be a commutative monoid, written additively. By a **G-graded ring**, we shall mean a ring A, which as an additive group can be expressed as a direct sum.

$$A = \bigoplus_{r \in G} A_r,$$

and such that the ring multiplication maps $A_r \times A_s$ into A_{r+s}, for all $r, s \in G$.

In particular, we see that A_0 is a subring.

The elements of A_r are called the **homogeneous elements of degree r**.

We shall construct several examples of graded rings, according to the following pattern. Suppose given for each $r \in G$ an abelian group A_r (written additively), and for each pair $r, s \in G$ a map $A_r \times A_s \to A_{r+s}$. Assume that A_0 is a commutative ring, and that composition under these maps is associative and A_0-bilinear. Then the direct sum $A = \bigoplus_{r \in G} A_r$ is a ring: We can define multiplication in the obvious way, namely

$$\left(\sum_{r \in G} x_r\right)\left(\sum_{s \in G} y_s\right) = \sum_{t \in G} \left(\sum_{r+s=t} x_r y_s\right).$$

The above product is called the **ordinary product**. However, there is another way. Suppose the grading is in $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$. We define the **super product** of $x \in A_r$ and $y \in A_s$ to be $(-1)^{rs}xy$, where xy is the given product. It is easily verified that this product is associative, and extends to what is called the **super product** $A \otimes A \to A$ associated with the bilinear maps. If R is a commutative ring such that A is a graded R-algebra, i.e. $RA_r \subset A_r$ for all r (in addition to the condition that A is a graded ring), then with the super product, A is also an R-algebra, which will be denoted by A_{su}, and will be called the **super algebra** associated with A.

Example. In the next section, we shall meet the tensor algebra $T(E)$, which will be graded as the direct sum of $T^r(E)$, and so we get the associated super tensor algebra $T_{\text{su}}(E)$ according to the above recipe.

Similarly, let A, B be graded algebras (graded by the natural numbers as above). We define their **super tensor product**

$$A \otimes_{\text{su}} B$$

to be the ordinary tensor product as graded module, but with the **super product**

$$(a \otimes b)(a' \otimes b') = (-1)^{(\deg b)(\deg a')} aa' \otimes bb'$$

if b, a' are homogeneous elements of B and A respectively. It is routinely verified that $A \otimes_{\text{su}} B$ is then a ring which is also a graded algebra. Except for the sign, the product is the same as the ordinary one, but it is necessary to verify associativity explicitly. Suppose $a' \in A_i$, $b \in B_j$, $a'' \in A_s$, and $b' \in B_r$. Then the reader will find at once that the sign which comes out by computing

$$(a \otimes_{\text{su}} b)(a' \otimes_{\text{su}} b')(a'' \otimes_{\text{su}} b'')$$

in two ways turns out to be the same, namely $(-1)^{ij+js+sr}$. Since bilinearity is trivially satisfied, it follows that $A \otimes_{\text{su}} B$ is indeed an algebra.

The super product in many ways is more natural than what we called the ordinary product. For instance, it is the natural product of cohomology in topology. Cf. Greenberg-Harper, *Algebraic Topology*, Chapter 29. For a similar construction with $\mathbf{Z}/2\mathbf{Z}$-grading, see Chapter XIX, §4.

§7. THE TENSOR ALGEBRA OF A MODULE

Let R be a commutative ring as before, and let E be a module (i.e. an R-module). For each integer $r \geqq 0$, we let

$$T^r(E) = \bigotimes_{i=1}^{r} E \quad \text{and} \quad T^0(E) = R.$$

Thus $T^r(E) = E \otimes \cdots \otimes E$ (tensor product taken r times). Then T^r is a functor, whose effect on linear maps is given as follows. If $f : E \to F$ is a linear map, then

$$T^r(f) = T(f, \ldots, f)$$

in the sense of §1.

From the associativity of the tensor product, we obtain a bilinear map

$$T^r(E) \times T^s(E) \to T^{r+s}(E),$$

which is associative. Consequently, by means of this bilinear map, we can define a ring structure on the direct sum

$$T(E) = \bigoplus_{r=0}^{\infty} T^r(E),$$

and in fact an algebra structure (mapping R on $T^0(E) = R$). We shall call $T(E)$ the **tensor algebra** of E, over R. It is in general *not* commutative. If $x, y \in T(E)$, we shall again write $x \otimes y$ for the ring operation in $T(E)$.

Let $f : E \to F$ be a linear map. Then f induces a linear map

$$T^r(f) : T^r(E) \to T^r(F)$$

for each $r \geqq 0$, and in this way induces a map which we shall denote by $T(f)$ on $T(E)$. (There can be no ambiguity with the map of §1, which should now be written $T^1(f)$, and is in fact equal to f since $T^1(E) = E$.) It is clear that $T(f)$ is the unique linear map such that for $x_1, \ldots, x_r \in E$ we have

$$T(f)(x_1 \otimes \cdots \otimes x_r) = f(x_1) \otimes \cdots \otimes f(x_r).$$

Indeed, the elements of $T^1(E) = E$ are algebra-generators of $T(E)$ over R. We see that $T(f)$ is an algebra-homomorphism. *Thus T may be viewed as a functor from the category of modules to the category of graded algebras, $T(f)$ being a homomorphism of degree* 0.

When E is free and finite dimensional over R, we can determine the structure of $T(E)$ completely, using Proposition 2.3. Let P be an algebra over k. We shall say that P is a **non-commutative polynomial algebra** if there exist elements $t_1, \ldots, t_n \in P$ such that the elements

$$M_{(i)}(t) = t_{i_1} \cdots t_{i_s}$$

with $1 \leqq i_\nu \leqq n$ form a basis of P over R. We may call these elements non-commutative monomials in (t). As usual, by convention, when $r = 0$, the corresponding monomial is the unit element of P. We see that $t_1, \ldots, t_n$ generate P as an algebra over k, and that P is in fact a graded algebra, where P_r consists of linear combinations of monomials $t_{i_1} \cdots t_{i_r}$ with coefficients in R. It is natural to say that $t_1, \ldots, t_n$ are **independent non-commutative variables** over R.

Proposition 7.1. *Let E be free of dimension n over R. Then $T(E)$ is isomorphic to the non-commutative polynomial algebra on n variables over R. In other words, if $\{v_1, \ldots, v_n\}$ is a basis of E over R, then the elements*

$$M_{(i)}(v) = v_{i_1} \otimes \cdots \otimes v_{i_r}, \qquad 1 \leqq i_\nu \leqq n$$

form a basis of $T^r(E)$, and every element of $T(E)$ has a unique expression as a finite sum

$$\sum_{(i)} a_{(i)} M_{(i)}(v), \qquad a_{(i)} \in R$$

with almost all $a_{(i)}$ equal to 0.

Proof. This follows at once from Proposition 2.3.

The tensor product of linear maps will now be interpreted in the context of the tensor algebra.

For convenience, we shall denote the module of endomorphisms $\mathrm{End}_R(E)$ *by* $L(E)$ *for the rest of this section.*

We form the direct sum

$$(LT)(E) = \bigoplus_{r=0}^{\infty} L(T^r(E)),$$

which we shall also write $LT(E)$ for simplicity. (Of course, $LT(E)$ is not equal to $\mathrm{End}_R(T(E))$, so we must view LT as a single symbol.) We shall see that LT is a functor from modules to graded algebras, by defining a suitable multiplication on $LT(E)$. Let $f \in L(T^r(E))$, $g \in L(T^s(E))$, $h \in L(T^m(E))$. We define the product $fg \in L(T^{r+s}(E))$ to be $T(f, g)$, in the notation of §1, in other words to be the unique linear map whose effect on an element $x \otimes y$ with $x \in T^r(E)$ and $y \in T^s(E)$ is

$$x \otimes y \mapsto f(x) \otimes g(y).$$

In view of the associativity of the tensor product, we obtain at once the associativity $(fg)h = f(gh)$, and we also see that our product is bilinear. Hence $LT(E)$ is a k-algebra.

We have an algebra-homomorphism

$$T(L(E)) \to LT(E)$$

given in each dimension r by the linear map

$$f_1 \otimes \cdots \otimes f_r \mapsto T(f_1, \ldots, f_r) = f_1 \cdots f_r.$$

We specify here that the tensor product on the left is taken in

$$L(E) \otimes \cdots \otimes L(E).$$

We also note that the homomorphism is in general neither surjective nor injective. When E is free finite dimensional over R, the homomorphism turns out to be both, and thus we have a clear picture of $LT(E)$ as a non-commutative polynomial algebra, generated by $L(E)$. Namely, from Proposition 2.5, we obtain:

Proposition 7.2. *Let E be free, finite dimensional over R. Then we have an algebra-isomorphism*

$$T(L(E)) = T(\mathrm{End}_R(E)) \to LT(E) = \bigoplus_{r=0}^{\infty} \mathrm{End}_R(T^r(E))$$

given by

$$f \otimes g \mapsto T(f, g).$$

Proof. By Proposition 2.5, we have a linear isomorphism in each dimension, and it is clear that the map preserves multiplication.

In particular, we see that $LT(E)$ is a noncommutative polynomial algebra.

§8. SYMMETRIC PRODUCTS

Let $\mathfrak{S}_n$ denote the symmetric group on n letters, say operating on the integers $(1, \ldots, n)$. An r-multilinear map

$$f : E^{(r)} \to F$$

is said to be **symmetric** if $f(x_1, \ldots, x_r) = f(x_{\sigma(1)}, \ldots, x_{\sigma(r)})$ for all $\sigma \in \mathfrak{S}_r$.

In $T^r(E)$, we let $\mathfrak{b}_r$ be the submodule generated by all elements of type

$$x_1 \otimes \cdots \otimes x_r - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}$$

for all $x_i \in E$ and $\sigma \in \mathfrak{S}_r$. We define the factor module

$$S^r(E) = T^r(E)/\mathfrak{b}_r,$$

and let

$$S(E) = \bigoplus_{r=0}^{\infty} S^r(E)$$

be the direct sum. It is immediately obvious that the direct sum

$$\mathfrak{b} = \bigoplus_{r=0}^{\infty} \mathfrak{b}_r$$

is an ideal in $T(E)$, and hence that $S(E)$ is a graded R-algebra, which is called the **symmetric algebra** of E.

Furthermore, the canonical map

$$E^{(r)} \to S^r(E)$$

obtained by composing the maps

$$E^{(r)} \to T^r(E) \to T^r(E)/\mathfrak{b}_r = S^r(E)$$

is universal for r-multilinear symmetric maps.

We observe that S is a functor, from the category of modules to the category of graded R-algebras. The image of $(x_1, \dots, x_r)$ under the canonical map

$$E^{(r)} \to S^r(E)$$

will be denoted simply by $x_1 \cdots x_r$.

Proposition 8.1. *Let E be free of dimension n over R. Let $\{v_1, \dots, v_n\}$ be a basis of E over k. Viewed as elements of $S^1(E)$ in $S(E)$, these basis elements are algebraically independent over R, and $S(E)$ is therefore isomorphic to the polynomial algebra in n variables over R.*

Proof. Let $t_1, \dots, t_n$ be algebraically independent variables over R, and form the polynomial algebra $R[t_1, \dots, t_n]$. Let P_r be the R-module of homogeneous polynomials of degree r. We define a map of $E^{(r)} \to P_r$ as follows. If $w_1, \dots, w_r$ are elements of E which can be written

$$w_i = \sum_{\nu=1}^{n} a_{i\nu} v_\nu, \qquad i = 1, \dots, r,$$

then our map is given by

$$(w_1, \dots, w_r) \mapsto (a_{11}t_1 + \cdots + a_{1n}t_n) \cdots (a_{r1}t_1 + \cdots + a_{rn}t_n).$$

It is obvious that this map is multilinear and symmetric. Hence it factors through a linear map of $S^r(E)$ into P_r:

$$\begin{array}{ccc} E^{(r)} & \longrightarrow & S^r(E) \\ & \searrow \quad \swarrow & \\ & P_r & \end{array}$$

From the commutativity of our diagram, it is clear that the element $v_{i_1} \cdots v_{i_s}$ in $S^r(E)$ maps on $t_{i_1} \cdots t_{i_s}$ in P_r for each r-tuple of integers $(i) = (i_1, \dots, i_r)$. Since the monomials $M_{(i)}(t)$ of degree r are linearly independent over k, it follows that the monomials $M_{(i)}(v)$ in $S^r(E)$ are also linearly independent over R, and that our map $S^r(E) \to P_r$ is an isomorphism. One verifies at once that the multiplication in $S(E)$ corresponds to the multiplication of polynomials in $R[t]$, and thus that the map of $S(E)$ into the polynomial algebra described as above for each component $S^r(E)$ induces an algebra-isomorphism of $S(E)$ onto $R[t]$, as desired.

Proposition 8.2. *Let $E = E' \oplus E''$ be a direct sum of finite free modules. Then there is a natural isomorphism*

$$S^n(E' \oplus E'') \approx \bigoplus_{p+q=n} S^pE' \otimes S^qE''.$$

In fact, this is but the n-part of a graded isomorphism

$$S(E' \oplus E'') \approx SE' \otimes SE''.$$

Proof. The isomorphism comes from the following maps. The inclusions of E' and E'' into their direct sum give rise to the functorial maps

$$SE' \otimes SE'' \to SE,$$

and the claim is that this is a graded isomorphism. Note that SE' and SE'' are commutative rings, and so their tensor product is just the tensor product of commutative rings discussed in §6. The reader can either give a functorial map backward to prove the desired isomorphism, or more concretely, SE' is the polynomial ring on a finite family of variables, SE'' is the polynomial ring in another family of variables, and their tensor product is just the polynomial ring in the two families of variables. The matter is easy no matter what, and the formal proof is left to the reader.

EXERCISES

1. Let k be a field and $k(\alpha)$ a finite extension. Let $f(X) = \mathrm{Irr}(\alpha, k, X)$, and suppose that f is separable. Let k' be any extension of k. Show that $k(\alpha) \otimes k'$ is a direct sum of fields. If k' is algebraically closed, show that these fields correspond to the embeddings of $k(\alpha)$ in k'.
2. Let k be a field, $f(X)$ an irreducible polynomial over k, and α a root of f. Show that $k(\alpha) \otimes k'$ is isomorphic, as a k'-algebra, to $k'[X]/(f(X))$.
3. Let E be a finite extension of a field k. Show that E is separable over k if and only if $E \otimes_k L$ has no nilpotent elements for all extensions L of k, and also when $L = k^{\mathrm{a}}$.
4. Let $\varphi : A \to B$ be a commutative ring homomorphism. Let E be an A-module and F a B-module. Let F_A be the A-module obtained from F via the operation of A on F through φ, that is for $y \in F_A$ and $a \in A$ this operation is given by

$$(a, y) \mapsto \varphi(a)y.$$

Show that there is a natural isomorphism

$$\mathrm{Hom}_B(B \otimes_A E, F) \approx \mathrm{Hom}_A(E, F_A).$$

5. **The norm.** Let B be a commutative algebra over the commutative ring R and assume that B is free of rank r. Let A be any commutative R-algebra. Then $A \otimes B$ is both an A-algebra and a B-algebra. We view $A \otimes B$ as an A-algebra, which is also free of rank r. If $\{e_1, \dots, e_r\}$ is a basis of B over R, then

$$1_A \otimes e_1, \dots, 1_A \otimes e_r$$

is a basis of $A \otimes B$ over A. We may then define the **norm**

$$N = N_{A \otimes B, A} : A \otimes B \to A$$

as the unique map which coincides with the determinant of the regular representation.

In other words, if $b \in B$ and b_B denotes multiplication by b, then

$$N_{B,R}(b) = \det(b_B);$$

and similarly after extension of the base. Prove:

(a) Let $\varphi : A \to C$ be a homomorphism of R-algebras. Then the following diagram is commutative:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\varphi \otimes \mathrm{id}} & C \otimes B \\ \downarrow N & & \downarrow N \\ A & \xrightarrow[\varphi]{} & C \end{array}$$

(b) Let $x, y \in A \otimes B$. Then $N(x \otimes_B y) = N(x) \otimes N(y)$. [*Hint*: Use the commutativity relations $e_i e_j = e_j e_i$ and the associativity.]

A little flatness

6. Let M, N be flat. Show that $M \otimes N$ is flat.
7. Let F be a flat R-module, and let $a \in R$ be an element which is not a zero-divisor. Show that if $ax = 0$ for some $x \in F$ then $x = 0$.
8. Prove Proposition 3.2.

Faithfully flat

9. We continue to assume that rings are commutative. Let M be an A-module. We say that M is **faithfully flat** if M is flat, and if the functor

$$T_M : E \mapsto M \otimes_A E.$$

is faithful, that is $E \neq 0$ implies $M \otimes_A E \neq 0$. Prove that the following conditions are equivalent.

(i) M is faithfully flat.

(ii) M is flat, and if $u : F \to E$ is a homomorphism of A-modules, $u \neq 0$, then $T_M(u) : M \otimes_A F \to M \otimes_A E$ is also $\neq 0$.

(iii) M is flat, and for all maximal ideals $\mathfrak{m}$ of A, we have $\mathfrak{m}M \neq M$.

(iv) A sequence of A-modules $N' \to N \to N''$ is exact if and only if the sequence tensored with M is exact.

10. (a) Let $A \to B$ be a ring-homomorphism. If M is faithfully flat over A, then $B \otimes_A M$ is faithfully flat over B.

(b) Let M be faithfully flat over B. Then M viewed as A-module via the homomorphism $A \to B$ is faithfully flat over A if B is faithfully flat over A.

11. Let P, M, E be modules over the commutative ring A. If P is finitely generated (resp. finitely presented) and E is flat, show that the natural homomorphism

$$\mathrm{Hom}_A(P, M) \otimes_A E \to \mathrm{Hom}_A(P, M \otimes_A E)$$

is a monomorphism (resp. an isomorphism).

[*Hint*: Let $F_1 \to F_0 \to P \to 0$ be a finite presentation, say. Consider the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}_A(P, M) \otimes_A E & \longrightarrow & \mathrm{Hom}_A(F_0, M) \otimes_A E & \longrightarrow & \mathrm{Hom}_A(F_1, M) \otimes_A E \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}_A(P, M \otimes_A E) & \longrightarrow & \mathrm{Hom}_A(F_0, M \otimes_A E) & \longrightarrow & \mathrm{Hom}_A(F_1, M \otimes_A E)]. \end{array}$$

Tensor products and direct limits

12. Show that the tensor product commutes with direct limits. In other words, if $\{E_i\}$ is a directed family of modules, and M is any module, then there is a natural isomorphism

$$\varinjlim (E_i \otimes_A M) \approx (\varinjlim E_i) \otimes_A M.$$

13. (D. Lazard) Let E be a module over a commutative ring A. Tensor products are all taken over that ring. Show that the following conditions are equivalent:
    (i) There exists a direct family $\{F_i\}$ of free modules of finite type such that

$$E \approx \varinjlim F_i.$$

    (ii) E is flat.

    (iii) For every finitely presented module P the natural homomorphism

$$\mathrm{Hom}_A(P, A) \otimes_A E \to \mathrm{Hom}_A(P, E)$$

    is surjective.

    (iv) For every finitely presented module P and homomorphism $f: P \to E$ there exists a free module F, finitely generated, and homomorphisms

$$g: P \to F \quad \text{and} \quad h: F \to E$$

    such that $f = h \circ g$.

Remark. The point of Lazard's theorem lies in the first two conditions: *E is flat if and only if E is a direct limit of free modules of finite type.*

[*Hint*: Since the tensor product commutes with direct limits, that (i) implies (ii) comes from the preceding exercise and the definition of flat.

To show that (ii) implies (iii), use Exercise 11.

To show that (iii) implies (iv) is easy from the hypothesis.

To show that (iv) implies (i), use the fact that a module is a direct limit of finitely presented modules (an exercise in Chapter III), and (iv) to get the free modules instead. For complete details, see for instance Bourbaki, *Algèbre*, Chapter X, §1, Theorem 1, p. 14.]

The Casimir element

14. Let k be a commutative field and let E be a vector space over k, of finite dimension n. Let B be a nondegenerate symmetric bilinear form on E, inducing an iso-

morphism $E \to E^\vee$ of E with its dual space. Let $\{v_1, \ldots, v_n\}$ be a basis of E. The B-dual basis $\{v_1', \ldots, v_n'\}$ consists of the elements of E such that $B(v_i, v_j') = \delta_{ij}$.

(a) Show that the element $\sum v_i \otimes v_i'$ in $E \otimes E$ is independent of the choice of basis. We call this element the **Casimir element** (see below).

(b) In the symmetric algebra $S(E)$, let $Q_B = \sum v_i v_i'$. Show that Q_B is independent of the choice of basis. We call Q_B the **Casimir polynomial**. It depends on B, of course.

(c) More generally, let $\mathbf{D}$ be an (associative) algebra over k, let $\mathcal{D}: E \to \mathbf{D}$ be an injective linear map of E into $\mathbf{D}$. Show that the element $\sum \mathcal{D}(v_i)\mathcal{D}(v_i') = \omega_{B,\mathcal{D}}$ is independent of the choice of basis. We call it the **Casimir element** in $\mathbf{D}$, determined by $\mathcal{D}$ and B.

Remark. The terminology of the Casimir element is determined by the classical case, when G is a Lie group, $E = \mathfrak{g} = \text{Lie}(G)$ is the Lie algebra of G (tangent space at the origin with the Lie algebra product determined by the Lie derivative), and $\mathcal{D}(v)$ is the differential operator associated with v (Lie derivative in the direction of v). The Casimir element is then a partial differential operator in the algebra of all differential operators on G. Cf. basic books on manifolds and Lie theory, for instance [JoL 01], Chapter II, §1 and Chapter VII, §2.

15. Let $E = \mathfrak{sl}_n(k)$ = subspace of $\text{Mat}_n(k)$ consisting of matrices with trace 0. Let B be the bilinear form defined by $B(X, Y) = \text{tr}(XY)$. Let $G = SL_n(k)$. Prove:

(a) B is $\mathbf{c}(G)$-invariant, where $\mathbf{c}(g)$ is conjugation by an element $g \in G$.

(b) B is invariant under the transpose $(X, Y) \mapsto ({}^tX, {}^tY)$.

(c) Let $k = \mathbf{R}$. Then B is positive definite on the symmetric matrices and negative definite on the skew-symmetric matrices.

(d) Suppose G is given with an action on the algebra $\mathbf{D}$ of Exercise 14, and that the linear map $\mathcal{D}: E \to \mathbf{D}$ is G-linear. Show that the Casimir element is G-invariant (for the conjugation action on $S(E)$, and the given action on $\mathbf{D}$).