Chapter 3
Gravity

3.1 Ceaseless, Repetitive Paths Across the Sky

Look up at the Sun as it glides across the bright blue sky, or watch the Moon's nightly voyage. On dark, moonless nights you also might notice a bright planet traveling against the stars.

Ancient astronomers thought that the Moon, Sun, and planets all moved in circles, forever wheeling around the central, unmoving Earth, and the Moon does indeed revolve about our planet. But the Earth and other planets revolve about the Sun, and the Sun does not revolve around the Earth.

So, motion is a matter of perspective. It is always relative, perceived only in relation to something else, by comparison with another object that is either at rest or moving in a different way.

The earliest Sun-centered theories of planetary motion had one fatal flaw; they also initially assumed that the planets move in circular orbits. This explanation couldn't be reconciled with careful observations of the changing positions of the planets in the sky that were meticulously carried out by the Danish astronomer, Tycho Brahe (1546–1601). Johannes Kepler (1571–1630), Brahe's assistant and eventual successor, found that the architecture of the solar system had to be described by noncircular shapes.

After 8 years of computations, Kepler found in 1609 that the observed planetary orbits could be described by ellipses with the Sun at one focus (Kepler 1609). This ultimately became known as *Kepler's first law* of planetary motion. Although the planetary orbits are nearly circular, they are slightly elliptical in shape.

At about the same time, Kepler described how a planet moves at different speeds as it travels along its elliptical orbit. He was able to state the relationship in a precise mathematical form now called *Kepler's second law*, which can be explained with the help of Fig. 3.1. Imagine a line drawn from the Sun to a planet. As the planet swings about its elliptical path, the line (which will increase and decrease in length) sweeps out a surface at a constant rate. This also is known as the *law of equal areas*. During the three equal time intervals shown in Fig. 3.1, the

K. R. Lang, *Essential Astrophysics*, Undergraduate Lecture Notes in Physics,
DOI: 10.1007/978-3-642-35963-7_3,

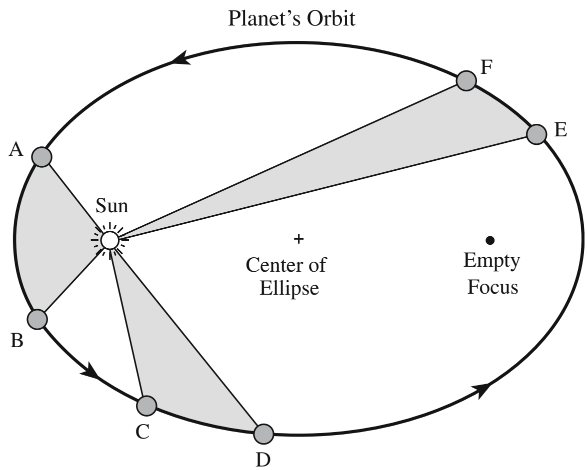

Fig. 3.1 Kepler's first and second laws The German astronomer Johannes Kepler (1571–1630) published his first two laws of planetary orbital motion in 1609. His first law states that the orbit of a planet about the Sun is an ellipse with the Sun at one focus. The other focus of the ellipse is empty. According to Kepler's second law, the line joining a planet to the Sun sweeps out equal areas in equal times. This is also known as the law of equal areas, and is represented by the equality of the *three shaded areas ABS*, *CDS*, and *EFS*. It takes as long to travel from *A* to *B* as from *C* to *D* and from *E* to *F*. A planet moves most rapidly when it is nearest the Sun, at perihelion; a planet's slowest motion occurs when it is farthest from the Sun, at aphelion

planet moves through different arcs because its orbital speed changes, but the areas swept out are identical.

So, a planet moves faster when it is closer to the Sun, and the modern explanation for this involves one of the fundamental concepts of physics, known as the *conservation of angular momentum* (Focus 3.1).

Focus 3.1 Moving along an elliptical trajectory

According to *Kepler's first law*, the planets move in elliptical orbits (Fig. 3.2). A planet's closest point to the Sun, when the planet moves most rapidly, is called the perihelion; and its most distant point is the aphelion, where the planet moves most slowly. The distance between the perihelion and aphelion is the major axis of the orbital ellipse. Half that distance is called the semi-major axis, designated by the symbol, a. The semi-major axis of the Earth's elliptical orbit about the Sun is called the astronomical unit, abbreviated AU. It sets the scale of the solar system, and when combined with the Earth's yearlong orbital period permits the determination of the Sun's mass and the Earth's orbital velocity, but only after astronomers had found out how large the AU is.

The distances to the solar focus and the shape of an ellipse are determined by its eccentricity, e. At perihelion the distance between the planet and the Sun is $[a\ (1 - e)]$ and at aphelion that distance is $[a\ (1 + e)]$. If $e = 0$ its shape is a circle. The ellipse becomes more elongated and squashed as its eccentricity increases toward $e = 1.0$. The eccentricity of the planetary ellipse has been greatly exaggerated in Fig. 3.2, with an eccentricity of about $e = 0.5$.

With the exception of Mercury, all of the major planets have orbits that are nearly circular, with eccentricities of less than $e = 0.1$. This means that the Sun is very near the center of each orbital ellipse. For Mercury, $a = 0.387$ AU and $e = 0.206$, so its distance from the Sun is just 0.307 AU at perihelion and quite a lot greater at aphelion, at 0.467 AU.

Conservation of angular momentum explains why a planet keeps on whirling around the Sun, and why its speed is fastest at perihelion. For a planet of mass, M, orbiting the Sun at speed or velocity, V, and a distance, D,

$$\text{Angular momentum} = M \times V \times D. \tag{3.1}$$

By the way, in physics velocity has an amount, its magnitude, and a direction. Speed is the magnitude of the velocity. In astronomy the velocity is often just given by its observed magnitude in a given direction, the speed, so the orbital velocity is given as its speed along the orbit.

The conservation law states that as long as no outside force is acting on a planet, its angular momentum cannot change. This means that a planet continues moving along without anything pushing or pulling it. The mass does not change, so when the distance from the Sun decreases, at perihelion, the velocity increases to compensate and keep the angular momentum unchanged; at aphelion the distance from the Sun increases so the speed must decrease.

Kepler's third law took another 10 years of work to discover (Kepler 1619). In this musical pattern, each planet produces its own unique "note" as it moves around the Sun, with an orbital period that increases with a planet's distance from the Sun. Kepler's harmonic relationship states that the squares of the planetary periods are in proportion to the cubes of their average distances from the Sun. If P_P denotes the orbital period of a planet measured in years and a_P describes its semi-major axis measured in AU, then Kepler's third law states that $P_P^2 = a_P^3$. This expression is illustrated in Fig. 3.3, for the major planets and for the brighter moons of Jupiter. It also implies that a more distant planet moves with a slower speed. For a circular orbit, the planet's uniform velocity $V_P = 2\pi a_P / P_P =$ constant $\times\ a_P^{-1/2}$, which falls off as the inverse square root of a_P.

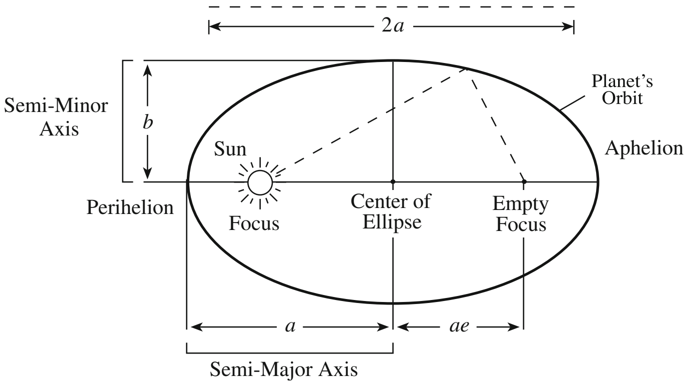

Fig. 3.2 **Ellipse** Each planet moves in an ellipse with the Sun at one focus. The length of a line drawn from the Sun, to a planet and then to the empty focus, denoted by the *dashed line*, is always $2a$, or twice the semi-major axis, a. The eccentricity, or elongation, of the planetary ellipse has been greatly overdone in this figure; planetary orbits look much more like a circle

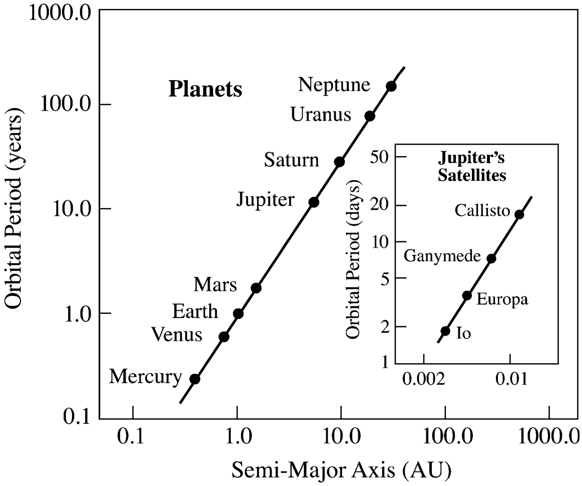

Fig. 3.3 **Kepler's third law** The orbital periods of the major planets in years are plotted against the semi-major axes of their elliptical orbits in astronomical units (AU), using a logarithmic scale. The straight line that connects the points has a slope of $3/2$, thereby verifying Kepler's third law that states that the square of the orbital periods increase with the cubes of the planetary distances. The German astronomer Johannes Kepler (1571–1630) published this third law in 1619. This type of relation applies to any set of bodies in elliptical orbits, including Jupiter's four largest satellites shown in the inset, with a vertical axis in units of days and a horizontal axis that gives the distance from Jupiter in AU units

These were amazing discoveries, but no one yet had explained what holds up the Moon and planets in their orbits. The explanation awaited the discovery of gravity, a principle that rules the universe.

3.2 Universal Gravitational Attraction

What moves the planets within their well-defined orbits? Kepler supposed that some invisible magnetic force emanated from the rotating Sun, and that this force pushed the planets through space. The farther the planet is from the Sun, the weaker the solar force and the slower a planet's motion – as described by Kepler's harmonic relationship.

Roughly half a century later, the great English scientist Isaac Newton (1643–1727) proposed another unseen agent, the invisible gravitational force of the Sun. Newton showed that the pull of gravity is universal, with an unlimited range and capacity to act on all matter, thereby holding the Moon, comets, and planets in their orbits.

Gravitation is the driving and organizing force of the universe (see Mac Dougal 2013); that is why it is known as *universal gravitation*. It binds stars and galaxies together and is responsible for their formation. The pull of gravity keeps our feet on the ground, so we rotate with the spinning Earth and stay on it. The atmosphere and oceans similarly are held close to the planet by its relentless gravitational pull.

It is gravity that explains why and how things fall. We might suppose, as Aristotle once did, that a heavy object will fall faster than a lighter one, in direct proportion to its weight, but that is not the case. Unless some outside force is involved, such as wind, all objects fall at the same rate, regardless of their weight. Galileo Galilei (1564–1642) stated the idea in his *Discoursi* (Galilei 1638), and apparently the same idea was stated 17 centuries before that by the Roman poet Lucretius (c. 99 BC–c. 55 BC) in *De rerum natura* (Lucretius 55 BC).

Galileo also proposed that any undisturbed body will fall with uniform acceleration, and he showed that the distance traveled by an object falling from rest is proportional to the square of the elapsed time. The distance, d, after time, t, is given by $d = gt^2/2$, where $g \approx 9.8$ m s^{-2} is the local acceleration of gravity on the Earth. In these ways Galileo provided a scientific foundation for Newton's subsequent theory of universal gravitation.

Newton realized that the power of gravity, whose pull influences the motion of falling bodies, seems undiminished even at the top of the highest mountains. He therefore argued that the Earth's gravitational force extends to our Moon, and showed that this force can pull the Moon into its orbit.

Newton showed that motions everywhere, whether in the celestial heavens on the ground, are described by the same concepts and that all material objects are subject to gravitation. Therefore, everything in the observable universe moves in predictable and verifiable ways. The basic ideas are that a moving body will continue to move in a straight line, unless acted on by an outside force, and that every object attracts every other object as the result of universal gravitation. These insights resulted in Sir Isaac Newton becoming the first person in England to be knighted for his scientific work.

It was his friend, the English astronomer Edmond Halley (1656–1742), who persuaded the secretive Newton to write his greatest work, the *Philosophiae naturalis principia mathematica*, or the *Mathematical Principles of Natural Philosophy*, commonly known as the *Principia* (Newton 1687). It was presented to the Royal Society of London in 1686, which withdrew from publishing it due to insufficient funds; Halley, a wealthy man, paid for the publication the following year.

The enormous reach of gravity can be traced to two causes. First, gravitational force decreases relatively slowly with distance, which gives gravity a much greater range than other natural forces, such as the strong force that holds the nucleus of an atom together. Second, gravitation has no positive and negative charge, like electricity, or opposite polarities like magnets. This means that there is no gravitational repulsion between masses. That is, the force of gravity acting between two objects always pulls them together and never pushes them apart. The attractive forces among unlike electrical charges in an atom cancel one another, shielding it from the electrical forces of any other atom.

The gravitational force is mutual, so any two objects attract each other, and every atom in the universe feels the gravitational attraction of every other atom. Their attraction is proportional to the product of their masses, which possess inertia, the tendency to resist any change in motion. Mass is an intrinsic aspect of an object. It is different from weight, which decreases with distance from the main source of gravity. An astronaut, for example, weighs less after leaving the Earth, but his or her mass is just the same.

As expected, the strength of the gravitational force decreases with increasing distance, and Newton used Kepler's relationship between a planet's orbital period and distance to show that the force of gravity falls off as the inverse square of the distance from the center of the main source of gravity, the Sun.

Newton also demonstrated that the force of gravity at the Earth's surface is the same as the force, diminished by distance, which holds our Moon in place during its endless journey around the Earth. In effect our planet's gravity is forever pulling on the Moon, so it is perpetually falling toward the Earth while maintaining the same mean distance from it. Without the Earth's gravitational pull, the Moon would not orbit our planet but instead would travel out into space, never returning to Earth. The Sun's gravity similarly deflects the moving planets into their curved paths, so they forever revolve around the Sun (Newton 1687).

The gravitational power of an individual object depends on its mass and diminishes with distance from it. Expressed mathematically, any mass, M_1, produces a gravitational force, F_G, on another mass, M_2, given by the expression:

$$F_G = \frac{GM_1M_2}{D^2}, \tag{3.2}$$

where the universal gravitational constant denoted G, has the value $G \approx 6.674 \times 10^{-11}$ N m^2 kg^{-2}, and D is the distance between the centers of the two masses. This expression for the force is sometimes called an *inverse square law*, since the force of gravity is inversely proportional to the square of the distance or separation. The SI unit of force is appropriately called the newton, abbreviated N, and it is equal to the amount of net force required to accelerate a mass of 1 kg at a rate of 1 m every second squared, so 1 N = 1 kg m s^{-2}, and we can express the universal constant of gravitation with the units $G \approx 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.

Any two masses attract one another with a gravitational force that varies in proportion to the product of the masses and the inverse square of the separation between their centers. The constant of proportionality – the *universal gravitational constant* G – was not measured until 71 years after Newton's death, and then indirectly by Henry Cavendish (1731–1810). Cavendish's aim was to determine the mass density of the Earth and because the planet's radius was known, he could effectively weigh the world. After nearly a year of meticulous observations, Cavendish (1798) announced that the Earth has a mass density of $\rho_E = 5{,}488 \pm 33$ kg m^{-3} (when corrected for a small arithmetical error in his paper). His result meant that the mass of the Earth is $M_E = 4\pi R_E^3 \rho_E / 3 \approx 6 \times 10^{24}$ kg, where the approximate radius of the Earth, $R_E \approx 6.4 \times 10^6$ m, was known at the time. (Distances part way around the surface of the Earth had been found by the surveying technique of triangulation, and combined to determine the Earth's circumference and a radius of about 6,400 km.) In Cavendish's time, mass and weight were assumed to be equal and, as he stated in his correspondence, he succeeded in weighing the world. It weighed in at a little more than 6 billion trillion metric tons. (A metric ton is 1,000 kg or 2,205 pounds).

Although he didn't specifically determine the gravitational constant, the value implied from Cavendish's work is $G = 6.754 \times 10^{-11}$ m^3 kg^{-1} s^{-2} (Brush and Holton 2001). A very precise value of G, accurate to more than the third decimal place, is still unknown, since gravity is a relatively weak force when compared to other forces that might act on the relevant experimental apparatus (Heyl 1930; Rose et al. 1969; Luther and Towler 1982; Gillies 1997; Fixler et al. 2007). The currently accepted value is:

$$G = 6.67428 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} = 6.67428 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}, \tag{3.3}$$

with an uncertainty of 1 part in 10^4. For computations involving the orbits of either the natural or the artificial satellites around planets, and the trajectories of spacecraft visiting them, astronomers use the product of G and the planet's mass M_P, since GM_P is known more accurately than either term alone. The geocentric gravitational constant, GM_E, is, for example, a primary astronomical constant:

$$GM_E = 3.986004391 \times 10^{14}\,\mathrm{m^3\,s^{-2}}, \tag{3.4}$$

where $M_E = 5.9736 \times 10^{24}$ kg is the mass of the Earth, which is given together with other physical properties of the planet in Table 3.1.

Example: How fast are the Moon and planets moving, and how do we measure the mass of the planets?

Assuming a circular orbit at a distance D with period P, the average orbital speed will be $V = 2\pi D/P$, where $\pi = 3.14159$. The mean distance of our Moon from the Earth is $D = 384{,}400$ km $= 3.844 \times 10^8$ m and its orbital period around the Earth is $P = 27.3$ Earth days, where 1 day = 86,400 s, and its average orbital speed is about 1.02 km s^{-1}. For the Earth's orbit around the Sun, $D = 1$ AU $= 1.496 \times 10^{11}$ m and P = 1 yr = 3.156 $\times 10^7$ s, with an average orbital speed of $V = 2\pi D/P = 29.78$ km s^{-1}; since one hour 3,600 s, the Earth is moving at about 107,200 km/h, a lot faster than a vehicle on the highway. Jupiter is located at a distance of about 5.2 AU from the Sun, so from Kepler's third law, in which the square of the orbital periods scale as the cubes of the planetary distance, the orbital period P_J of Jupiter about the Sun will be $P_J = (5.2 \text{ AU}/1.0 \text{ AU})^{3/2} = 11.86$ years. Its average orbital speed is about 13 km s^{-1}, which is about three times slower than the Earth's orbital speed.

We can estimate the mass of a planet, M, from the motion of one of its moons, or natural satellites, using Kepler's third law, $M = 4\pi^2 D^3/(GP^2)$, where the Newtonian constant of gravitation $G = 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2}. For our Moon, with the distance D and orbital period P given just above, we infer a mass of the Earth $M_E \approx 6.0 \times 10^{24}$ kg. The orbital parameters for Jupiter's natural satellite Io are $D = 421{,}700$ km $= 4.217 \times 10^8$ m and $P = 1.77$ Earth days $= 1.53 \times 10^5$ s, and with these parameters we obtain the mass of Jupiter $M_J \approx 1.9 \times 10^{27}$ kg $\approx 318\, M_E$.

Any object has a gravitational potential stored within it due to its efforts at overcoming relentless gravity. Two separated objects, for example, have worked against the gravitational attraction that pulls them together, achieving a reserve of energy and a potential for future action.

According to the conservation of energy, a fundamental law of physics, energy cannot be created or destroyed, just transformed. So the energy that went into overcoming the pull of gravity is stored in any object, and this stored potential energy can be converted into the kinetic energy of motion.

This *gravitational potential energy* is due to an object's position and is associated with the gravitational force. It depends on the height of the object, its mass, and the strength of the gravitational field it is in. For a very small mass m, as tiny as a point, the gravitational potential energy U when separated by a distance r in the gravitational field of another point mass M is:

$$U = -\frac{GMm}{r}. \tag{3.5}$$

Table 3.1 Earth's orbital and physical properties

Orbital characteristics
Po = orbital period of Earth about Sun = 365.25636 days = 1.000 sidereal year
Vo = average orbital speed of Earth about Sun = 2.9783×10^4 m s^{-1} = 107,200 km h^{-1}
a_E = AU = astronomical unit = mean Earth–Sun distance = $1.495\,9787 \times 10^{11}$ m
$\pi_\odot$ = solar parallax = arcsin $(a_e/\mathrm{AU}) \approx a_e/\mathrm{AU} \approx$ 8.794143 seconds of arc (for Earth's equatorial radius $a_e = 6.3781 \times 10^6$ m)
e = eccentricity = 0.01671
$a_E\,(1 + e)$ = aphelion = $1.52\,098\,232 \times 10^{11}$ m = 1.01671388 AU
$a_E\,(1 - e)$ = perihelion = $1.47\,098\,290 \times 10^{11}$ m = 0.98329134 AU
Physical characteristics
Age = 4.6×10^9 year
M_E = mass = 5.9736×10^{24} kg
$M_\odot/M_E$ = inverse mass = 332 946
a_e = equatorial radius = 6.3781×10^6 m
a_p = polar radius = 6.3568×10^6 m
$f = (a_e - a_p)/a_p$ = flattening = 0.0033528 = 1/298.25642
R_E = mean radius = $(a_e^2 a_p)^{1/3} \approx 6.371 \times 10^6$ m
ρ_E = mean mass density = $3M_E/(4\pi R_E^3)$ = 5515 kg m^{-3}
GM_E = geocentric gravitational attraction = 3.986×10^{14} m^3 s^{-2}
g_E = equatorial gravitational acceleration = $GM_E/a_E^2 = 9.780$ m s^{-2}
V_{escE} = surface escape velocity of Earth = $(2GM_E/R_E)^{1/2} \approx 1.1186 \times 10^4$ m s^{-1} ≈ 11.2 km s^{-1}
B = magnetic field strength of Earth (equator to poles) = 0.3–0.6 G = (3–6) $\times 10^{-5}$ T (magnetic field poles reverse every 250,000 years)
P_r = rotation period = 24 h = 8.64×10^4 s
$\mathrm{d}P_r/\mathrm{d}t$ = slow down of rotation = 0.002 s century^{-1}
ω = angular velocity of rotation = 7.292×10^{-5} radians s^{-1}
V_r = equatorial rotation velocity = 465.12 m s^{-1} = 1,674.4 km h^{-1}
A = albedo (Bond) = 0.306 or albedo (geometric) = 0.367
T = mean surface temperature = 287.2 K
Atmosphere
P = mean surface pressure at sea level = 1 bar = 1.01×10^5 Pa
N_2 = nitrogen molecule = 78.08 % by volume
O_2 = oxygen molecule = 20.95 % by volume
Ar = argon = 0.92 %
CO_2 = carbon dioxide = 0.038 %
H_2O = water vapor ≈ 1 % variable

The negative sign is a convention, not important for most physical purposes where differences in energy are used.

This expression can be used to determine the escape velocity from the gravity of an object of radius R; just equate the kinetic energy $mV^2/2$ to GMm/R to get the velocity V of escape, or V_{esc}, given by:

$$V_{esc} = \left(\frac{2GM}{R}\right)^{1/2}, \tag{3.6}$$

which is independent of the small mass m.

For a self-gravitating sphere of uniform mass density, rather than a point mass, the gravitational potential energy is given by integrating the potential energy over all parts of the sphere, resulting in:

$$U = \frac{-3GM^2}{5R}, \tag{3.7}$$

where R is the radius of the sphere and the mass, M is given by

$$M = \frac{4}{3}\pi R^3 \rho, \tag{3.8}$$

for a mass density ρ.

The *gravitational binding energy* of a sphere held together by its gravity is $3GM^2/(5R)$, without the minus sign; it is the amount of energy required to pull all of the material apart and the amount of energy released, mainly by heat, during its formation.

Because a precise value of the gravitational constant, G, is only known to three significant figures, the orbits of the planets are calculated using the *Gaussian constant of gravitation*, denoted by the symbol k, first proposed by the German mathematician Carl Friedrich Gauss (1777–1855). It is given by (Gauss 1809):

$$k^2 = \frac{4\pi^2 a_E^3}{P_E^2(M_E + M_\odot)}, \tag{3.9}$$

where a_E is the semi-major axis of the Earth's orbit about the Sun, the orbital period of the Earth, P_E, is one year, and M_E and $M_\odot$ respectively denote the mass of the Earth and the Sun. Here $a_E =$ AU is the astronomical unit, while the symbol a_e with a lowercase subscript e, is the equatorial radius of the Earth. The $a_E = 1.496 \times 10^{11}$ m and $M_\odot = 1.989 \times 10^{30}$ kg. By using this constant Gauss was able to simplify the calculation of planetary orbits; he had previously used it in his 1801 prediction of the orbit of the first asteroid, Ceres, which had been lost from view.

The Canadian-American astronomer Simon Newcomb (1835–1909) determined the value of k with such great precision (Newcomb 1895) that it is still used in computing the planetary ephemerides and is one of the defining astronomical constants. This value is:

$$k = 0.01720209895(\mathrm{AU})^{3/2} M_\odot^{-1/2} (D)^{-1}, \tag{3.10}$$

where AU denotes the astronomical unit and the mean solar day $D = 86{,}400$ s. Thus, k^2 is the Newtonian constant of gravitation expressed in units of the astronomical unit, the solar mass, and the day. The derived, heliocentric, or Sun-centered, gravitational constant, $GM_\odot$, is given by $GM_\odot = (\mathrm{AU})^3 k^2/D^2 = 1.327\ 244 \times 10^{20}\ \mathrm{m^3\ s^{-2}}$.

For objects near the Earth, the local acceleration of gravity g can be considered to be approximately constant and the expression for the gravitational potential energy relative to the Earth's surface becomes:

$$U = mgh, \tag{3.11}$$

where h is the height above the Earth's surface and g is the surface value of the acceleration of gravity, or $g = 9.780$ m s^{-2} at the Earth's equator. The local acceleration of gravity, g, determines how things fall. For an object of mass, M, and radius, R, we have:

$$g = \frac{GM}{R^2}, \tag{3.12}$$

where the universal constant of gravitation G = 6.674 × 10^{-11} m^3 kg^{-1} s^{-2}. For the Earth, the local acceleration of gravity at the equator is $g_E = 9.780$ m s^{-2}, increasing to about 9.832 m s^{-2} at the poles. The detailed mathematical expressions for the variation of g with altitude and latitude on the Earth are given in Focus 3.2.

Focus 3.2 The Earth's gravity

The gravitational acceleration of the Earth, g, depends on the distance from the planet's center. The value g_H at altitude H above the Earth's mean radius, R_E, is given by:

$$g_H = \frac{g_0 R_E^2}{(R_E + H)^2}, \tag{3.13}$$

where the standard gravity $g_0 = 9.8331$ ms^{-2} at $R_E = 6.371 \times 10^6$ m.

As we previously discussed in Chap. 1, Focus 1.2, the surface of the Earth is not perfectly round, being extended at the equator and squashed at the poles. The surface radius r at latitude ϕ of the Earth geoid is:

$$r = a_e\left(1 - f\sin^2\phi\right), \tag{3.14}$$

where the equatorial radius $a_e = 6.378\ 140 \times 10^8$ m, and the flattening factor f is given by

$$f = \frac{a_e - a_p}{a_p} = \frac{3}{2}J_2 + \frac{1}{2}m = 0.0033528 = 1/298.25642. \tag{3.15}$$

The polar radius $a_P = 6.356755 \times 10^6$ m; the dynamical form factor J_2 for the Earth is given by

$$J_2 = 0.0010826359. \tag{3.16}$$

The effective gravity of the Earth is reduced by its rotation, and this reduction is greatest at the equator. The ratio m of centrifugal acceleration at the equator to the gravitational acceleration at the equator is given by:

$$m = \frac{\omega^2 a_e^3}{GM_E} = 0.00346 \tag{3.17}$$

where the angular velocity of the Earth's rotation $\omega = 2\pi$ radians/86,400 s = 7.292×10^{-5} rad s^{-1}. Therefore, the effective gravity of the Earth is reduced by rotation, but at most by about 3 % and near the equator.

At sea level, then we can estimate the surface gravitational acceleration, g, at latitude, ϕ, from the formula derived by the French astronomer and mathematician Alexis Claude de Clairault (1713–1765). Known as *Clairault's theorem*, it is (Clairault 1743):

$$g = g_e\left[1 + \left(\frac{5m}{2} - f\right)\sin^2\phi\right] \tag{3.18}$$

where the surface equatorial acceleration of gravity is given by:

$$g_e = \frac{GM_E}{a_e^2}\left(1 + \frac{3}{2}J_2 - m\right) = 9.780\,327\ \mathrm{m\,s^{-2}}. \tag{3.19}$$

Expressed numerically, Clairault's theorem becomes:

$$g = 9.780327\left(1 + 0.0053024\sin^2\phi - 0.0000058\sin^2 2\phi\right)\mathrm{m\,s^{-2}}, \tag{3.20}$$

at latitude ϕ.

3.3 Mass of the Sun

The concept of universal gravitation, and Newton's expression for the gravitational force, can be used to derive *Kepler's third law* in the form (see Chap. 4):

$$P_P^2 = \frac{4\pi^2}{G(M_P + M_\odot)}a_P^3, \tag{3.21}$$

where the constant $\pi = 3.14159$, the universal gravitational constant $G \approx 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, the a_P is the semi-major axis of the planet's orbital ellipse in meters, P_P is the orbital period in seconds, and M_P and $M_\odot$ respectively denote the mass of the planet and the mass of the Sun in kilograms.

Within the solar system, the dominant mass is that of the Sun, which far surpasses the mass of any other object there. That is why we call it a solar system, governed by the central Sun. The sum $(M_P + M_\odot)$ is therefore, to the first approximation, a constant equal to the Sun's mass, $M_\odot$, regardless of the planet under consideration, and Kepler's third law becomes:

$$P_P^2 = \frac{4\pi^2}{GM_\odot} a_P^3. \tag{3.22}$$

Since any planet mass, M_p, is much smaller than the Sun's mass, $M_\odot$, we have:

$$M_\odot = \frac{4\pi^2 a_P^3}{GP_P^2}, \tag{3.23}$$

where and $\pi = 3.141592$.

We can use the Earth's orbital motion at a mean distance of one astronomical unit, or 1 AU $= 1.496 \times 10^{11}$ m and its orbital period of one year, or 1 year $= 3.1556926 \times 10^7$ s to infer the mass of the Sun, $M_\odot$ from:

$$M_\odot = 4\pi^2 (\mathrm{AU})^3 \Big/ \left[G(1\mathrm{yr})^2\right] = 1.989 \times 10^{30}\,\mathrm{kg}. \tag{3.24}$$

It is the benchmark unit for specifying the mass of the stars and galaxies.

The ratio of the mass of the Earth, M_E, to the mass of the Sun, $M_\odot$, which is independent of G, is given by

$$\frac{M_E}{M_\odot} = \frac{1}{332{,}946}. \tag{3.25}$$

The Sun is about 333,000 times more massive than the Earth, and contains more than 99.9 % of the mass of the entire solar system, so our assumption that our planet's motion is controlled by the massive Sun is amply justified.

The first person to estimate the mass of the Sun was Newton, in the *Principia*, where he calculated that the ratio of the mass of the Earth to the mass of the Sun was 1/28,700. After an improved value for the distance to the Sun was available, he revised his result to obtain a ratio of 1/169,282 in the third edition of the *Principia*. The modern value is of 1/332,946 is a result of improved determinations of the AU.

3.4 Tidal Effects

3.4.1 The Ocean Tides

While walking along the beach we might notice that the waves are rising farther and farther up the shore, steadily advancing and enlarging the bounds of the sea. The tide is flooding the beach. But several hours later it retreats and goes down again. We say that the tide is rising and falling, while the sea runs in and out, and Newton showed that the Moon's attraction is the main cause of the *ocean tides*.

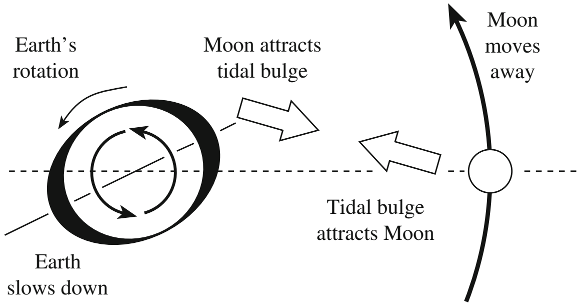

Fig. 3.4 Cause of the Earth's ocean tides The Moon's gravitational attraction causes two tidal bulges in the Earth's ocean water, one on the closest side to the Moon and one on the farthest side. The Earth's rotation twists the closest bulge ahead of the Earth−Moon line (*dashed line*), which produces a lag between the time the Moon is directly overhead and the time of highest tide. The Moon pulls on the nearest tidal bulge, slowing down the Earth's rotation. At the same time, the tidal bulge nearest the Moon produces a force that tends to pull the Moon ahead in its orbit, causing the Moon to spiral slowly outward

Table 3.2 Orbital and physical properties of the Moon

Orbital characteristics
D_M = mean distance of Moon = 3.844×10^8 m = 384,400 km
V_M = mean orbital speed of Moon = 1.022×10^3 m s^{-1} = 1.022 km s^{-1}
V_{escE} = escape velocity of Earth at Moon's distance = $2GM_E/D_{moon})^{1/2}$ = 1.44×10^3 m s^{-1}
dD_M/dt = rate of increase of Moon's distance = 0.0382 ± 0.0007 m year^{-1}
P_M = orbital period of Moon = 27.3216 days = sidereal month = fixed star to fixed star (time from new moon to new moon is 29.530589 days = synodic month)
e = eccentricity = 0.0549
D_M (1 + e) = apogee = 405,410 km
D_M (1 − e) = perigee = 362,570 km
Physical characteristics
Age = 4.6×10^9 year
M_M = mass = 7.348×10^{22} kg
$M_\odot/M_M$ = inverse mass = 27,068,708.7
M_E/M_M = Earth-Moon mass ratio = 81.30056
$\mu = M_M/M_E$ = Moon-Earth mass ratio = 0.0123
R_M = mean radius = 1.737×10^6 m
ρ_M = mean mass density = $3M_M/\left(4\pi R_M^3\right)$ = 3346.4 kg m^{-3}
g_M = equatorial surface gravity = 1.622 m s^{-2}
V_{escM} = escape velocity from surface = $(2GM_M/R_M)^{1/2} \approx 2.38 \times 10^3$ m s^{-1} = 2.38 km s^{-1}
P_r = sidereal rotation period = 27.3216 days (synchronous)
A = albedo = 0.136
T = mean equatorial surface temperature = 220 K (Temperature range at lunar equator 100–390 K)

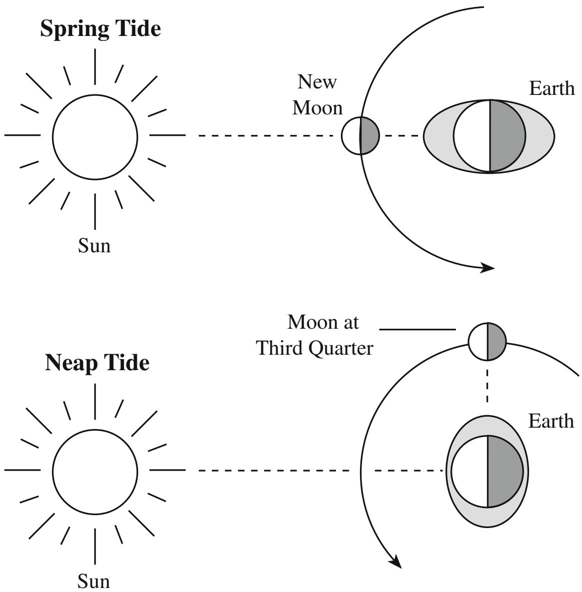

Fig. 3.5 Earth's spring and neap ocean tides The height of the tides and the phase of the Moon depend on the relative positions of the Earth, Moon, and Sun. When the tide-raising forces of the Sun and the Moon are in the same direction, they reinforce one another, making the highest high tides and the lowest low tides. These spring tides (*top*) occur at either new or full Moon. The range of tides is least when the Moon is at first or third quarter and the tide-raising forces of the Sun and the Moon are at right angles to one another. The tidal forces are then in opposition, producing the lowest high tides and the highest low tides, or the neap tides (*bottom*). In this diagram, the height of the tides is greatly exaggerated in comparison to the size of the Earth

Because the Moon's gravitational force decreases with increasing distance, the Moon pulls hardest on the ocean facing it and least on the opposite ocean, whereas the Earth between is pulled with an intermediate force. In this way, the Moon's gravity draws out the ocean into the shape of an egg and creates two high tides. As the Earth's rotation carries the continents past the two tidal humps, we experience the rise and fall of water, the ebb and flow of the tides, twice every day (Fig. 3.4).

To understand the Moon's tidal producing force, we need to know its mass and distance, and they are given with other physical information about the Moon in Table 3.2.

The Moon creates most of the ocean waves, but the Sun also contributes to the size and rhythm of the waves. Although more massive than the Moon, our Sun also is much farther away; as a result the tide-producing force of the Moon is about

2.2 times that of the Sun. Near both full and new Moons, the tide-raising forces of the Sun and the Moon are in the same direction, producing the spring tide (Fig. 3.5). They reinforce one another's tides and produce high tides that can be a few times higher than normal. Two weeks later, the two tidal forces are in opposition and interfere with one another, and the range of these neap tides is then lower than any others.

Example: Moon tides and Sun tides

We can estimate the tide-producing capability of the Moon from the difference ΔF_M of its gravitational force, F_M, on the near and far sides of the Earth from:

$$\Delta F_M = \frac{GM_M M_E}{(D_M - R_E)^2} - \frac{GM_M M_E}{(D_M + R_E)^2} = \frac{4GM_M M_E R_E}{D_M^3}, \tag{3.26}$$

assuming $D_M \gg R_E$, where the Newtonian gravitational constant $G = 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, the mass of the Moon $M_M = 7.348 \times 10^{22}$ kg, the mass of the Earth $M_E = 5.974 \times 10^{24}$ kg, the mean radius of the Earth $R_E = 6.371 \times 10^6$ m, and the average distance of the Moon from the center of the Earth is $D_M = 3.844 \times 10^8$ m.

The ratio of the Moon's tide-producing force ΔF_M to the Sun's tide-producing force $\Delta F_\odot$ is:

$$\frac{\Delta F_M}{\Delta F_\odot} = \frac{M_M}{M_\odot}\left(\frac{D_\odot}{D_M}\right)^3 = 2.18, \tag{3.27}$$

where the mass of the moon is $M_M = 7.348 \times 10^{22}$ kg, the Sun's mass $M_\odot = 1.989 \times 10^{30}$ kg, the mean distance of the moon is $D_M = 3.844 \times 10^8$ m, and the mean Sun-Earth distance is $D_\odot = 1$ AU $= 1.496 \times 10^{11}$ m. In other words, the Moon tides are about twice the Sun tides when they are in the same direction in the sky.

The Moon is now moving away from the Earth at a rate of 0.038 m year^{-1}, so its tide-producing force is gradually weakening, while the Sun remains at the same mean distance from the Earth with an unchanging tidal effect. The Moon tides will be equal to the Sun tides when the Moon has moved out to a distance of $D_M \approx 4.98 \times 10^8$ m, or an additional 1.14×10^8 m beyond its current distance. At the current rate of recession, that will happen about 3 billion years from now.

3.4.2 Tidal Locking into Synchronous Rotation

A planet's gravitational force pulls any natural satellite, or moon, into a slightly elongated shape along an axis pointing toward the planet. That is, a planet's gravitation produces two tidal bulges in the solid body of the satellite; one on the closest side to the planet and one on the satellite's farthest side. If the satellite's rotation twists the closest bulge ahead of the planet-satellite line, the planet pulls back on it. As a result, one hemisphere of the satellite always faces the planet, and the satellite takes as long to rotate as it does to orbit the planet. Then we say that the satellite has been tidally locked into synchronous rotation with the planet.

The Moon is in *synchronous rotation* about the Earth, so the Moon's rotation period is the same as the time it takes for the Moon to orbit the Earth, which is 27.32 Earth days. As a result, from Earth we always see the Moon's near side and never its far side. Only when a spacecraft passes beyond the Moon and looks back at it can we see the far side of the Moon. Most of the major moons, or large natural satellites, in the solar system have synchronous rotation with their planet.

If the mass of two orbiting bodies is comparable and their physical separation is relatively small, they both may be tidally locked to one another. This is the case for Pluto and its nearby large moon, Charon. Mutual tidal locking also occurs for close binary stars.

3.4.3 The Days are Getting Longer

As the Earth rotates, the bulge raised on its surface by the Moon's gravity is always a little ahead of the Moon rather than directly under it. The Moon pulls back on the bulge and, in the process, slows down the planet.

When the ocean tides flood and ebb, they create eddies in the water, producing friction on the ocean floor, which heats the water ever so slightly and dissipates energy at the expense of the Earth's rotation. The tides therefore act as brakes on the spinning Earth, slowing it by friction. As a result of this *tidal friction*, the rotation of the Earth is slowing down and the day is becoming longer at a rate of 2 ms, or 0.002 s, per century (Focus 3.3). In other words, the days are getting longer at the rate of 1 s every 50,000 years, and tomorrow will be 60 billionths of a second longer than today.

Focus 3.3 Tidal friction slows the rotation of the Earth

In most of the ocean, the tidal currents are confined to the top of the deep sea, never reaching its bottom. Most of the tidal energy is therefore dissipated in shallow seas near land, where the turbulent tidal water reaches the ocean bottom, at depths of 100 m or less.

When the tide moves toward a beach at velocity, V, the frictional energy, ΔE, dissipated by tidal currents on the sea bottom per unit time, Δt, per unit area, ΔA, is

$$\frac{\Delta E}{\Delta t \Delta A} = \gamma \rho V^3 \approx 2\,\mathrm{J\,s^{-1}\,m^{-2}}, \tag{3.28}$$

where the density of sea water is $\rho \approx 1{,}000\,\mathrm{kg\,m^{-3}}$, a typical velocity $V \approx 1\,\mathrm{m\,s^{-1}}$, and the stress on the sea bottom is $\gamma \rho V^2$ with an empirical drag coefficient $\gamma \approx 0.002$ for wind stress on the ground and a river's stress on its bed as well as tidal currents in the bottom of the sea.

In 1919, Sir G. I. Taylor (1886–1975), a British expert on turbulence in air and water, used this equation to obtain $\Delta E/\Delta t \approx 10^{11}\mathrm{J\ s^{-1}}$ for the Irish Sea alone, and in the following year Sir Harold Jeffreys (1891–1989) estimated that the total rate of energy loss by tidal friction in the shallow seas surrounding Europe, Asia, and North and South America is $\Delta E/\Delta t \approx 10^{13}\ \mathrm{J\ s^{-1}}$ (Taylor 1919; Jeffreys 1920). This is comparable to the estimate obtained by considering the flux of energy convected into the shallow seas by tidal currents. Jeffreys (1975), Munk and MacDonald (1960, 1975), and Lambeck (1978, 1980) have discussed both the flux and bottom friction methods.

The lost energy comes from the Earth's rotational energy, which is equal to

$$\frac{1}{2} M_E V_{rot}^2 = \frac{2\pi^2 M_E R_E^2}{P_E^2} \tag{3.29}$$

where the mass of the Earth $M_E = 5.9736 \times 10^{24}$ kg, $V_{rot} = 2\pi R_E/P_E$, the constant $\pi \approx 3.14159$, the mean radius of the Earth $R_E = 6.371 \times 10^6$ m, and the rotation period of the Earth $P_E = 24$ h $= 8.640 \times 10^4$ s. For a period change ΔP in time interval Δt, the loss in rotational energy is:

$$2\pi^2 M_E R_E^2 \left[\frac{1}{P_E^2} - \frac{1}{(P_E + \Delta P_E)^2} \right] \approx \frac{4\pi^2 M_E R_E^2 \Delta P_E}{P_E^3}. \tag{3.30}$$

Setting this equal to the ΔE and collecting terms, we obtain:

$$\frac{\Delta P_E}{\Delta t} = \frac{P_E^3}{4\pi^2 M_E R_E^2} \frac{\Delta E}{\Delta t} \approx 6.7 \times 10^{-13}\,\mathrm{s\,s^{-1}} \approx 0.002 \text{ s per century}, \tag{3.31}$$

for $\Delta E/\Delta t = 10^{13}\ \mathrm{J\ s^{-1}}$ and one century 100 years $= 3.156 \times 10^9$ s.

The long term increase in the length of the day of roughly 2 ms per century has been documented over the past 2,700 years from historical records of occultations of stars by the Moon and solar and lunar eclipses (Stephenson and Morrison 1984).

Paleontologists have made indirect historical measurements of the Earth's rotation through studies of fossil corals. The growth patterns of these corals consist of annual bands and fine daily ridges, produced by the effects of seasonal and daily changes of water temperature on the growth rate. The days were shorter in the past, but the year was the same, so the number of days per year increases as we go back in time. Ancient corals confirm this, and they show a greater number of daily ridges per annual band than modern corals. Careful counting reveals that the day was only 22 h long when we look back 400 million years. Studies of daily grown increments have been extended to fossilized algae called stromatolites, which indicate that the day may have been only 10 h long about 2 billion years ago (Wells 1963; Mazzullo 1971).

3.4.4 The Moon is Moving Away from the Earth

The Moon pulls the Earth's oceans, and the oceans pull back, in accord with Newton's third law that every action has an equal and opposite reaction. The net effect is to swing the Moon outward into a more distant orbit. This is because the tidal bulge on the side facing the Moon is displaced ahead of the Moon and this bulge pulls the Moon forward.

As the Earth slows down, the angular momentum it loses is transferred to the Moon, which speeds up in its orbit around us. It is not hard to see that this will swing the Moon away from the Earth if we look at the key equations (Focus 3.4). When we do the arithmetic, we find that the change of 0.002 s per century in the length of a day implies an outward motion of the Moon, amounting to about 0.04 m yr^{-1}. Small as it is, this value is just measurable with the laser light sent to corner reflectors, called corner cubes, placed on the Moon by the *Apollo* astronauts. Pulses of light are sent from the Earth to the tiny reflecting mirrors on the Moon, and the time for the light to travel to the Moon and return to Earth is measured. The distance to the Moon is the product of this round-trip light travel time and the speed of light. The lunar laser ranging data indicate that the Moon is moving away from the Earth at a rate of 0.0382 ± 0.0007 m year^{-1} (Dickey et al. 1994).

Focus 3.4 Conservation of angular momentum in the Earth-Moon system

According to one of the unbreakable conservation laws, the angular momentum, or the product of mass, M, velocity, V, and radius, R, is unchanged in a closed system, which is not subject to an outside force. Thus:

$$\text{Conservation of Angular Momentum} = M \times V \times R = \text{constant}.$$

This means that the angular momentum that the Earth loses in slowing down will be transferred to the Moon. For the Earth, the angular momentum is rotational, with $V = 2\pi\ R_E/P_E$, where P_E is the Earth's rotation period of one day and the subscript E denotes the Earth. So, we have:

$$\text{Earth's Rotational Angular Momentum} = 2\pi M_E R_E^2/P_E.$$

Since the length of the Earth's day is increasing by the amount ΔP_E as time goes on, the Earth's rotational angular momentum is decreasing by the amount:

$$\begin{aligned}\text{Decrease in Rotational Angular Momentum} &= 2\pi M_E R_E^2 \left[\frac{1}{P_E} - \frac{1}{(P_E + \Delta P_E)}\right] \\ &= \frac{2\pi M_E R_E^2\, \Delta P_E}{P_E^2}.\end{aligned} \tag{3.32}$$

The loss has to be made up by an equivalent gain somewhere else in order to conserve angular momentum. This is done by an increase in the Moon's orbital angular momentum, which is given by:

$$\text{Moon's Orbital Angular Momentum} = M_M \times V_M \times D_M, \tag{3.33}$$

where M_M is the mass of the Moon, D_M is the distance between the Earth and the Moon, and the orbital velocity of the Moon can be estimated from Kepler's third law with the mass of the Earth and the Moon's distance from it, or by:

$$V_M = 2\pi D_M/P_M \approx (GM_E/D_M)^{1/2}, \tag{3.34}$$

where $G = 6.674 \times 10^{-11}\ \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2}$ is the universal gravitational constant. Substituting this velocity expression into the angular momentum relation, we obtain:

$$\text{Moon's Orbital Angular Momentum} = M_M D_M (GM_E/D_M)^{1/2}. \tag{3.35}$$

Since the mass of the Moon and the mass of the Earth do not change, the Moon's distance has to increase by an amount ΔD_M to provide an increase in the angular momentum.

$$\text{Increase in Orbital Angular Momentum} = M_M\, \Delta D_M \left(\frac{GM_E}{D_M}\right)^{1/2}. \tag{3.36}$$

Setting the loss in rotational angular momentum equal to the gain in orbital angular momentum and collecting terms we obtain

$$\Delta D_M = \frac{2\pi M_E\, R_E^2\, \Delta P_E}{M_M\, P_E^2 \left(\frac{GM_E}{D_M}\right)^{1/2}} \approx 1.8 \times 10^{-9}\,\text{m}\,\text{s}^{-1} \approx 0.057\ \text{m yr}^{-1}, \tag{3.37}$$

where $M_E = 5.974 \times 10^{24}$ kg, $R_E = 6.371 \times 10^6$ m, $\Delta P_E \approx 6.7 \times 10^{-13}$ s s^{-1}, $M_M = 7.348 \times 10^{22}$ kg, $P_E = 24$ h $= 8.640 \times 10^4$ s, $G = 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, $D_M = 3.844 \times 10^8$ m and 1 yr $= 3.156 \times 10^7$ s. This is the approximate amount, of 0.04 m yr^{-1}, measured by sending laser pulses from the Earth to the corner reflectors left on the Moon by astronauts (Dickey et al. 1994).

Example: How close was the Moon to the Earth in their youth?

The mean distance of the Moon is now $D_{moon} = 3.844 \times 10^8$ m, and laser signals to the Moon's corner reflector mirrors indicate that the Moon is moving away from the Earth at the rate of $dD_{moon}/dt = 0.0382\,\mathrm{m\,yr^{-1}}$. The age of the Earth is 4.6 billion year. Assuming that the Moon was formed when the Earth was in its youth, and that the Moon has always been moving away from the Earth at the presently observed rate, then the Moon has moved over a distance of $\Delta D_{moon} = 0.0382 \times 4.6 \times 10^9 = 1.76 \times 10^8$ m, or about half its present distance. If the Moon moved away from the Earth at a faster rate when it was young, then it could have been formed by a collision with the newly formed Earth.

Will the Moon's outward motion carry it away from the Earth altogether? Only the intrusion of a massive third body could achieve that. What will ultimately happen is the following. The combination of the slowing Earth and the receding Moon means that the Earth's day will eventually catch up with the length of the month. When the day and the month are equal, the Moon-induced tides will cease moving; from then on the oceans will rise and fall much more gently under the influence of the Sun. The Moon will hang motionless in the sky, and will be visible from only one hemisphere. At that stage the recession of the Moon will stop.

Then, billions of years from now, the Sun's tidal action will take over; slowing the Earth's rotation even further, until the day becomes longer than the month. At this point, angular momentum will be drawn from the Moon, and it will begin approaching the Earth, heading on a course of self-destruction until it is finally torn apart by the tidal action of the Earth. Perhaps it will form a ring around our planet. In any case, it will probably end its years where it apparently began – close to the Earth. By this time, however, the brighter Sun will have boiled the oceans away, and the Earth will have become a dry and barren place.

3.4.5 A Planet's Differential Gravitational Attraction Accounts for Planetary Rings

One might expect the numerous particles of a planetary ring to have accumulated long ago into larger satellites. But the interesting feature of these rings – and a clue to their origin – is that they do not coexist with large moons. Planetary rings are also usually closer to the planets than their large satellites.

The rings normally are confined to an inner zone where the planet's tidal forces would stretch a large moon until it fractured and split apart, while also preventing small bodies from coalescing to form a larger moon (Fig. 3.6). The outer radius of this zone in which rings are found is called the *Roche limit* after the French mathematician Édouard A. Roche (1820–1883), who described it (Roche 1849, 1850, 1851).

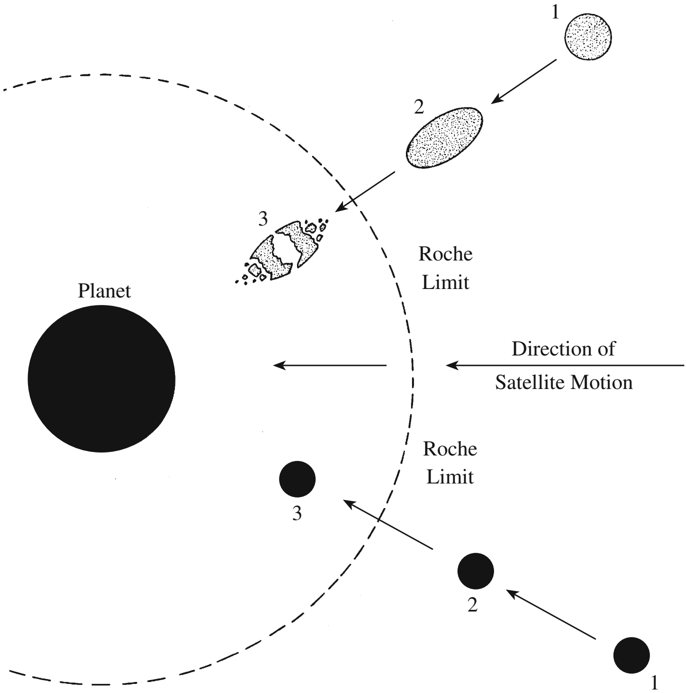

Fig. 3.6 Roche limit A large satellite (*top*) that moves well within a planet's Roche limit (*dashed curve*) will be torn apart by the tidal force of the planet's gravity. This was first investigated in 1849 by the French mathematician Édouard A. Roche (1820–1883). The side of the satellite closer to the planet feels a stronger gravitational pull than the side farther away, and this difference works against the self-gravitation that holds the body together. A small solid satellite (*bottom*) can resist tidal disruption because it has significant internal cohesion in addition to self-gravitation

For a rigid satellite with the same mass density as its planet, the Roche limiting distance from the center of the planet is 1.26 times the planet's radius (Focus 3.5). The Roche limit for a solid body is 1.38 times that radius (Aggarwal and Oberbeck 1974), and Roche's initial calculation, for fluid objects, was 2.446 times the planetary radius. Anywhere inside this distance, a large moon can no longer remain intact, but instead gets torn apart by planetary tides. Nevertheless, because of their material strength and cohesion, small moons less than 100 km across can exist inside the Roche limit without being tidally disrupted, just as the ring particles can.

The sharp edges of planetary rings can be formed by small satellites, which can pass within the Roche limit with enough internal cohesion to withstand the planet's differential gravitational forces. These smaller bodies can also confine ring particles within narrow boundaries. Goldreich and Tremaine (1982) have reviewed the dynamics of planetary rings.

Focus 3.5 The Roche limit

To visualize the significance of the Roche limit, consider two particles of equal mass, m, separated by a distance, R, and located at a distance, D, from a planet of mass, M_P. The gravitational pull of the planet on the particle closest to it will be greater than the pull on the more distant particle. If the difference in pull on the near and far particles, the tidal force, exceeds the mutual gravitational attraction between the two particles, they cannot remain close to each other and will disperse. The outcome of the tug-of-war between the tidal force and the mutual attraction is primarily decided by the particles' distance from the planet. At distances less than the Roche limit, D_{Roche}, particles are pulled apart, and this prevents the accumulation of larger moons. The tidal force will also tear apart any large moon-like object that ventures within the Roche limit.

The gravitational force, F_P, of a planet of mass, M_P, on a smaller mass, m, whose center is located at a distance, D, from the center of the planet is:

$$F_P = \frac{GM_P m}{D^2}, \tag{3.38}$$

where G is the universal constant of gravitation. The planet will pull harder on the side of the object that is closer to it and less hard on the side that is further away. The difference, ΔF, between the force felt by one side and the center of the mass, m, is

$$\Delta F = \frac{GM_P m}{2}\left[\frac{1}{(D-R_m)^2} - \frac{1}{(D+R_m)^2}\right] \approx \frac{2GM_P m}{D^3} R_m, \tag{3.39}$$

where the factor of ½ arises because the center of mass is located midway between the two forces, and R_m is the radius of the smaller object. If it approaches the planet, D becomes smaller and this tidal disruptive force will increase, eventually pulling the object apart at a critical distance D_{Roche} from the center of the planet.

The gravitational binding force, F_B, which attracts the opposite sides of the object and holds it together, is Gm/R_m^2 per unit mass, or for the total mass

$$F_B = \frac{Gm^2}{R_m^2}. \tag{3.40}$$

The Roche limit is reached when the tidal disruptive force, ΔF, equals the binding force, F_B, and when we set these two expression equal and collect terms we obtain:

$$D_{Roche} = \left(\frac{2M_P}{m}\right)^{1/3} R_m. \tag{3.41}$$

This result is expressed in terms of R_m, the radius of the small object, but by using the mass densities ρ_P and ρ_{m} for the planet and small mass respectively, with $M_P = 4\pi\rho_P R_P^3/3$ and $m = 4\pi\rho_m R_m^3/3$, we obtain the Roche limit in terms of the planet radius, R_P:

$$D_{Roche} = \left(\frac{2\rho_P}{\rho_m}\right)^{1/3} R_P, \tag{3.42}$$

which for a planet and smaller object of the same mass density becomes $D_{Roche} = 1.26\ R_P$ (Jeans 1917).

The calculation by Roche used liquid objects whose shapes can distort continuously, and his result is (Roche 1849, 1850, 1851):

$$D_{Roche} = 2.446\left(\frac{\rho_P}{\rho_m}\right)^{1/3} R_P \approx 2.446\, R_P. \tag{3.43}$$

For a satellite with no internal strength and whose density is the same as the planet, the Roche limit is 2.446 times the planetary radius, or about 175,000 km for Jupiter, 147,000 km for Saturn, 62,000 km for Uranus, and 59,000 km for Neptune. Jupiter's insubstantial dusty ring, the magnificent ice particles of Saturn's rings, and the dark boulders in the narrow rings encircling Uranus and Neptune all lie within the Roche limit for the relevant planet. The Earth's Roche limit is 15,584 km, and if our Moon ever ventured within this distance from the Earth's center, it would be pulled apart by tidal forces and our planet would have rings. Nevertheless, the Moon is much farther away from the Earth, at a mean distance of approximately 384,400 km.

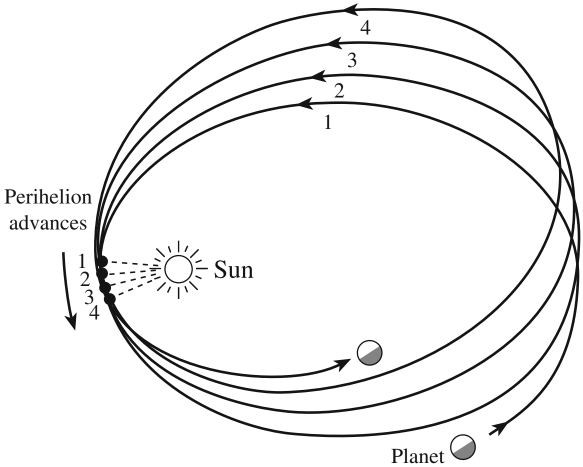

Fig. 3.7 Precession of Mercury's perihelion Instead of always tracing out the same ellipse, the orbit of Mercury pivots around the focus occupied by the Sun. The point of closest approach to the Sun, the perihelion, is slowly rotating ahead of the point predicted by Newton's theory of gravitation. This at first was explained by the gravitational tug of an unknown planet called Vulcan that was supposed to revolve about the Sun inside Mercury's orbit, but we now know that Vulcan does not exist. Albert Einstein (1879–1955) explained Mercury's anomalous motion in 1915 by inventing a new theory of gravity in which the Sun's curvature of nearby space makes the planet move in a slowly revolving ellipse

3.5 What Causes Gravity?

We cannot see the force of gravity and Newton did not know how it was exerted. Albert Einstein (1879–1955) subsequently explained it by supposing that a massive body like a star bends nearby space. This bending of space is the cause of the star's gravity. However, such effects are noticeable only in extreme conditions near a very massive cosmic object like a star, and the differences between Newton's and Einstein's theories of gravity are indistinguishable in everyday life.

One result of the Sun's curvature of nearby space is that planetary orbits are not exactly elliptical. This solved a perplexing problem with the motion of Mercury, the nearest planet to the Sun. Instead of returning to its starting point to form a closed ellipse in one orbital period, Mercury moves slightly ahead in a winding path that can be described as a rotating ellipse (Fig. 3.7). As a result, the point of Mercury's closest approach to the Sun, the perihelion, advances by a small amount – only 43 s of arc per century, beyond the location predicted using Newton's theory.

This anomalous twist in Mercury's motion was discovered in 1854 and recognized as an unexplained problem in 1859 by the French astronomer Urbain Jean Joseph LeVerrier (1811–1877). It was not explained for more than a half-century, when Einstein (1915) proposed that the planet is directed along a path in curved

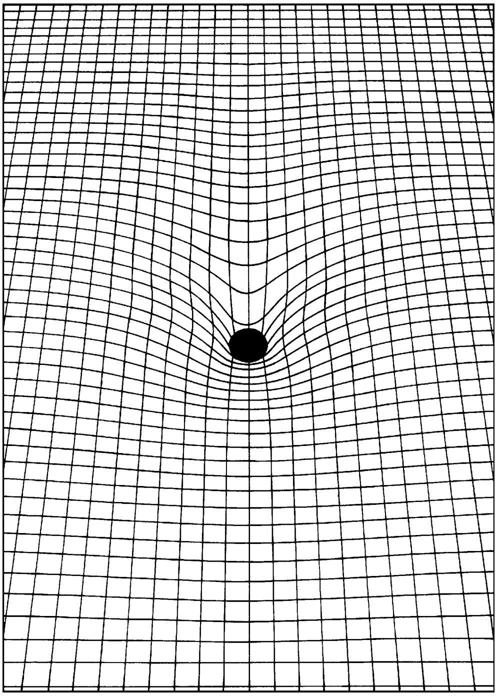

Fig. 3.8 Space curvature A massive object creates a curved indentation on the "flat" space described by Euclidean geometry, which applies in our everyday life on the Earth, where we do not directly encounter astronomical amounts of matter. Notice that the amount of space curvature is greatest in the regions near a cosmic object like a star, whereas farther away, the effect is lessened

space (Fig. 3.8), making the planet overshoot its expected location. Observations of Mercury trace out the invisible curvature. Roseveare (1982) has discussed Mercury's perihelion from Le Verrier to Einstein; also see Nobili and Will (1986).

In the very paper that explained Mercury's anomalous motion, Einstein showed that the curvature of space near the Sun also deflects the path of light from other stars (Fig. 3.9). The otherwise straight trajectory of starlight is bent by the Sun's gravity. The effect can be measured during a solar eclipse when stars pass behind the darkened Sun.

Newton previously had speculated that massive bodies might bend nearby light rays under the assumption that light has mass, and the German astronomer Johann George von Soldner (1726–1833) estimated the amount of light bending produced by the Sun using Newtonian gravity (Soldner 1801). In 1911, Einstein confirmed Soldner's result; however, when he took space curvature into account, the expected deflection was doubled (Einstein 1915).

The successful measurement of this deflection of starlight during the total solar eclipse on 29 May 1919 (Dyson et al. 1920), made Einstein famous, practically overnight. The initial measurements were not exact, and amounted to just a factor of two; nevertheless, the Sun's curvature of nearby space has now been measured with increasingly greater precision for nearly a century, confirming Einstein's prediction to two parts in a hundred thousand, or to the fifth decimal place. His *General Theory of Relativity* (Einstein 1916), which replaces the Sun's gravity

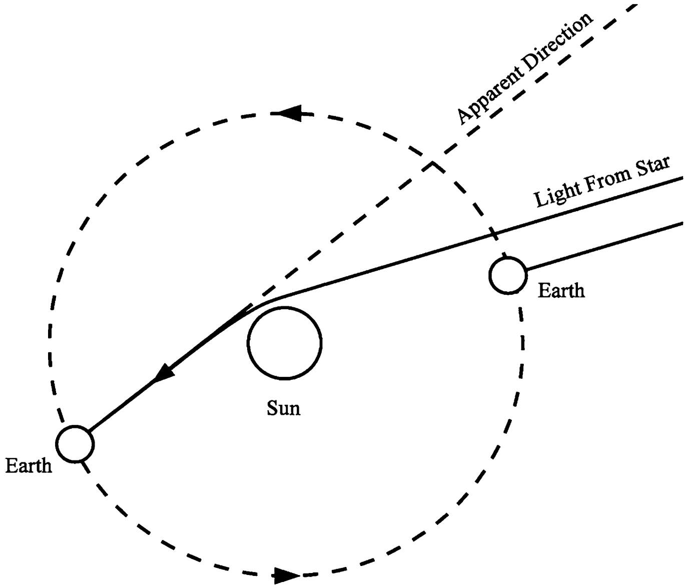

Fig. 3.9 Sun bending starlight As the Earth orbits the Sun, an observer's line of sight to a star or other cosmic object can pass near to the Sun or far from it. The massive Sun curves nearby space, bending the trajectory of starlight passing near to it, and this produces an apparent change in a star's position. The amount of bending and change in stellar position that were predicted by Einstein's *General Theory of Relativity* was first confirmed in 1929 during a total eclipse of the Sun

with geometry, has been verified by so many solar experiments that it now is widely accepted.

Natário (2011) provides a good description of the *General Theory of Relativity* without calculus, whereas Will (1993) has reviewed observational verification of the theory.

The solar curvature of space has been measured with increasingly greater precision for nearly a century, confirming Einstein's prediction. According to Einstein's theory, a light ray passing a minimum distance R_0 from the center of a star of mass M will be deflected by the angle

$$\theta = 2(1+\gamma)\frac{GM}{R_0 c^2} \text{ radians,} \tag{3.44}$$

where $G = 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2} is the universal gravitational constant, the speed of light is $c = 2.9979 \times 10^8$ m s^{-1}, and $\gamma = 1.000000000$, or exactly one, in Einstein's theory of gravitation. Newton's theory of gravitation implies $\gamma = 0$. For the Sun, with the mass $M_\odot = 1.989 \times 10^{30}$ kg, the *light bending* is:

$$\theta = 1.75\left(\frac{R_\odot}{R_0}\right)\left(\frac{1+\gamma}{2}\right) \text{ seconds of arc,} \tag{3.45}$$

where the Sun's radius is $R_{\odot} = 6.955 \times 10^8$ m and we have used 1 radian $= 2.06265 \times 10^5$ $''$, and $''$ denotes second of arc.

In one test, radio astronomers have combined observations of distant quasars from telescopes located on opposite sides of the world, turning the Earth into a gigantic interferometer that accurately measures the positions of radio sources as our line of sight to them nears the Sun (Fomalont and Sramek 1976). The change in position determines the amount of space curvature, as with the solar eclipse results for stars, but with much greater accuracy and without a total solar eclipse, since the Sun is a relatively weak interferometric radio source. Such Very Long Baseline Interferometry, abbreviated VLBI, has confirmed the predicted deflection of radio waves by the Sun to the 0.0003, or 0.03 %, level (Robertson et al. 1991; Shapiro et al. 2004; Fomalont et al. 2009).

Another test of Einstein's theory of gravitation measures the time required for a radar signal to make a round trip between the Earth and a planet (Shapiro et al. 1971), or the time for a radio signal to travel from a spacecraft home (Shapiro et al. 1971; Reasenberg et al. 1979). When the line of sight passes near the Sun, the radio waves travel along a curved path and take slightly longer to return to Earth. The measurements require extremely precise clocks, for the extra time delay caused by the Sun's curvature of nearby space amounts to only one ten-thousandth of a second. Radio links with the *Cassini* spacecraft, for example, indicate that $\gamma = 1.00000 \pm 2.1 \times 10^{-5}$, or precisely unity with an accuracy of 2 parts in 100,000 (Bertotti et al. 2003).

A modern, extended test of Einstein's theory involves the measurement of the periastron advance and light bending of a binary pulsar that has much stronger gravitational fields than found in our solar system, with additional indication of gravitational radiation from observations of its orbital decay (Focus 3.6).

Focus 3.6 Testing relativity with the binary pulsar

The American radio astronomers Russell A. Hulse (1950–) and Joseph H. Taylor, Jr. (1941–) discovered the now famous *binary pulsar* PSR B1913 + 16 in 1974, during a deliberate search for new pulsars using the latest computer technology with the 305 m radio antenna at Arecibo, Puerto Rico (Hulse and Taylor 1975; Hulse 1994). Their mini-computer was programed to scan a range of possible pulsar periodicities, pulse durations, and frequency dispersions, registering a signal whenever a pulsar passed through the telescope beam. After 14 months at Arecibo, and the discovery of 40 pulsars, Hulse, a graduate student at the University of Massachusetts at Amherst, found the enigmatic PSR B1913 + 16, a pulsar in a binary system. PSR denotes pulsar, B designates a binary companion, and 1913 + 16 describes the position of the pulsar in the sky.

The pulsar rotates on its axis 17 times a second, so the pulse repetition period is about 0.059 s. Moreover, the period changes, by about 0.00003 s, and the period change is itself cyclical, increasing and decreasing, rising and falling every 7.75 h. This meant that the pulsar was in orbital motion at this longer period, with the pulses being compressed together when the pulsar approached the Earth and pulled apart when moving away. A pulsar is also a neutron star, which has a mass of about the mass of the Sun collapsed to a radius of about 10 km, and as it turned out, the radio pulsar PSR B1913 + 16 was in rapid orbit with another neutron star that did not emit detectable radio pulses, perhaps because its radiation beam is not aimed at the Earth.

Hulse completed his degree, and left the field of radio astronomy just a few years later. So, precise timing of the radio pulses from PSR B1913 + 16 were continued by Hulse's advisor, Joe Taylor, and his other graduate students, permitting a determination of the orbital parameters of the system, as well as measurements of the mass of the pulsar and its silent companion. The minimum separation of the two neutron stars at periastron is about 1.1 $R_{\odot}$ and the maximum separation at apastron is 4.8 $R_{\odot}$ where the Sun's radius $R_{\odot} = 6.955 \times 10^8$ m. The periastron shift was enormous, at 4.2266 degrees per year, compared with Mercury's 43 s of arc per century, and this permitted the astronomers to infer a mass of $M = 2.828\ M_{\odot}$ solar masses for the binary system, where the Sun's mass $M_{\odot} = 1.989 \times 10^{30}$ kg. The individual masses could be determined from another relativistic effect, and they weighed in at 1.441 $M_{\odot}$ and 1.387 $M_{\odot}$, as would be expected for two neutron stars (Taylor and Weisberg 1989).

More importantly, after four years of measurements and the analysis of about 5 million pulses, Taylor and his colleagues found that the orbital period was slowly becoming shorter, implying a slow shrinking of the average orbit size. The rate of decrease of the orbital period was 76.5 millionths of a second per year or 7.65×10^{-5} s year^{-1}, indicating that the two stars are drawing closer and closer to each other, approaching at about 2.5 m per year. This rate of orbital decay is just the change expected if their orbital energy is being radiated away in the form of gravitational waves, which had never been seen before (Taylor 1992, 1994; Taylor and Weisberg 1982; Weisberg and Taylor 2005).

Einstein (1916) predicted such *gravitational radiation*, showing that any accelerating mass would emit it – as ripples in the curvature of space-time. Gravity waves travel at the speed of light, as electromagnetic radiation does. But while electromagnetic waves move through space, gravity waves are an undulation of space itself.

Gravity waves are produced whenever a mass moves, but they are exceedingly faint when generated and become diluted as they propagate into the increasing volume of space. They are so weak, and their interaction with matter so feeble, that Einstein himself questioned whether they would ever be detected. The gravitational radiation loss of the orbital energy of PSR B1913 + 16 nevertheless exactly matches the amount predicted by Einstein's theory, providing clear and strong evidence for the existence of gravitational radiation – for which Hulse and Taylor received the 1993 Nobel Prize in Physics.

Neither Einstein nor anyone else ever predicted that two neutron stars would be found that emit gravitational waves detected by timing the pulsar emission of one of them, and Taylor, Hulse, and their colleagues did not set out to find a binary neutron star, let alone detect gravitational waves. It was another one of those serendipitous discoveries that make astronomy so wonderfully unexpected and surprising.

A double radio pulsar has also been used to test special and general relativity. Designated PSR B0737–3039, the system also has strong gravitational fields, a rapid perihelion precession of 16.90 degrees per year, and an orbital decay attributed to gravitational radiation (Lyne 2004; Kramer et al. 2006; Breton et al. 2008). Unlike the binary pulsar PSR B1913 + 16, this new system contains two pulsars, attributed to two rotating neutron stars that emit radio pulses. They orbit each other at a speed of 300 km s^{-1} and complete one orbit every 2.4 h.

Joss and Rappaport (1984) have reviewed neutron stars in interacting binary systems. Backer and Hellings (1986) and Taylor (1994) have provided reviews of pulsar timing and general relativity. Kramer and Stairs (2008) have summarized knowledge of the double pulsar, and Hughes (2009) has discussed gravitational waves from merging compact binaries.