Chapter 9
Complex Numbers

Abstract We develop the properties of the number system called the complex numbers. We also describe derivatives and integrals of some basic functions of complex numbers.

9.1 Complex Numbers

Most people first encounter complex numbers as solutions of quadratic equations $x^2+bx+c=0$, that is, zeros z of the function $f(x)=x^2+bx+c$.

Example 9.1. Take as an example the equation $x^2+1=0$. The quadratic formula for the roots gives $z=\pm\sqrt{-1}$. There is no real number $\sqrt{-1}$. We introduce a new number $\mathrm{i}=\sqrt{-1}$.

Definition 9.1. A complex number z is defined as the sum of a real number x and a real multiple y of i,

$$z=x+\mathrm{i}y,$$

where i denotes a square root of minus one, $\mathrm{i}^2=-1$. The number $x=\mathrm{Re}\,(z)$ is called the *real part* of z, and the real number $y=\mathrm{Im}\,(z)$ is called its *imaginary part*. A complex number whose imaginary part is zero is called (naturally enough) real. A complex number whose real part is zero is called *purely imaginary*. The *complex conjugate* of z is $\overline{z}=x-\mathrm{i}y$.

You might be wondering how solving an equation such as $r^2+1=0$ might arise in calculus. Consider the differential equation

$$y''+y=0.$$

We know that $\sin x$ and $\cos x$ solve the equation. But what about $y=\mathrm{e}^{rx}$? The second derivative of $y=\mathrm{e}^{rx}$ is r^2 times y, so $y=\mathrm{e}^{rx}$ solves the equation

P.D. Lax and M.S. Terrell, *Calculus With Applications*, Undergraduate Texts in Mathematics, 347
DOI 10.1007/978-1-4614-7946-8_9, © Springer Science+Business Media New York 2014

$$y'' + y = (r^2 + 1)e^{rx} = 0$$

if r solves $r^2 + 1 = 0$. We will see in this chapter that e^{ix}, $\sin x$, and $\cos x$ are related, and we will see in Chap. 10 that functions of complex numbers help us solve many useful differential equations. Complex numbers also have very practical applications; they are used to analyze alternating current circuits.[1]

9.1a Arithmetic of Complex Numbers

We now describe a natural way of doing arithmetic with complex numbers. To add complex numbers, we add their real and imaginary parts separately:

$$(x + iy) + (u + iv) = x + u + i(y + v).$$

Similarly, for subtraction, $(x + iy) - (u + iv) = x - u + i(y - v)$. To multiply complex numbers, we use the distributive law:

$$(x + iy)(u + iv) = xu + iyu + xiv + iyiv.$$

Rewrite xi as ix and yi as iy, since we are assuming that multiplication of real numbers and i is commutative. Then, since $i^2 = -1$, we can write the product above as

$$(xu - yv) + i(yu + xv).$$

Example 9.2. Multiplication includes squaring:

$$(-i)^2 = -1, \quad (3 - i)^2 = 9 - 6i + i^2 = 8 - 6i, \quad (5i)^2 = -25.$$

It is easy to divide a complex number by a real number r: $\dfrac{x + iy}{r} = \dfrac{x}{r} + i\dfrac{y}{r}$. To express the quotient of two complex numbers $\dfrac{x + iy}{u + iv}$ as a complex number in the form $s + it$, multiply the numerator and denominator by the complex conjugate[2] $\overline{u + iv} = u - iv$. We get

$$\frac{x + iy}{u + iv} = \frac{(x + iy)(u - iv)}{(u + iv)(u - iv)} = \frac{xu + yv + i(yu - xv)}{u^2 + v^2} = \frac{xu + yv}{u^2 + v^2} + i\frac{yu - xv}{u^2 + v^2}.$$

Notice that the indicated division by $u^2 + v^2$ can be carried out unless both u and v are zero. In that case, the divisor $u + iv$ is zero, so we do not expect to be able to carry out the division.

[1] For electrical engineers, the letter i denotes current and nothing but current. So they denote the square root of -1 by the letter j.

[2] For physicists, the conjugate is denoted by an asterisk: $(u + iv)^* = u - iv$.

Example 9.3. One handy example of a quotient is the reciprocal of i,

$$\frac{1}{\mathrm{i}} = \frac{1}{\mathrm{i}}\frac{(-\mathrm{i})}{(-\mathrm{i})} = \frac{-\mathrm{i}}{1} = -\mathrm{i}.$$

Example 9.4. Division is needed almost every time you solve an equation. Let us try to solve for a if

$$\mathrm{i}(\mathrm{i}+a) = 2a+1.$$

Use properties of complex arithmetic and collect like terms to get $\mathrm{i}^2 - 1 = (2 - \mathrm{i})a$. Next divide both sides by $2-\mathrm{i}$:

$$a = \frac{-2}{2-\mathrm{i}} = \frac{-2}{2-\mathrm{i}}\frac{2+\mathrm{i}}{2+\mathrm{i}} = \frac{-4-2\mathrm{i}}{4+1} = -\frac{4}{5} - \frac{2}{5}\mathrm{i}.$$

Addition and multiplication of complex numbers are defined so that the associative, commutative, and distributive rules of arithmetic hold. For complex numbers v, w, and z, we have

- Associativity rules: $v+(w+z) = (v+w)+z$ and $v(wz) = (vw)z$,
- Commutativity rules: $v+w = w+v$ and $vw = wv$,
- Distributivity rule: $v(w+z) = vw+vz$.
- The numbers $0 = 0+0\mathrm{i}$ and $1 = 1+\mathrm{i}0$ are distinguished, inasmuch as adding 0 and multiplying by 1 do not alter a number.

Rules for the Conjugate of a Complex Number. In the case of complex numbers, we have the additional operation of conjugation. The following rules concerning conjugation are very useful. They can be verified immediately using the rules of arithmetic for complex numbers. We already encountered some of the rules of conjugation when we explained how to divide by a complex number.

- Symmetry: The conjugate of the conjugate is the original number: $\overline{\overline{z}} = z$.
- Additivity: The conjugate of a sum is the sum of the conjugates: $\overline{z+w} = \overline{z}+\overline{w}$.
- The sum of a complex number and its conjugate is real:

$$z+\overline{z} = 2\mathrm{Re}(z). \tag{9.1}$$

- The conjugate of a product is the product of the conjugates: $\overline{zw} = \overline{z}\,\overline{w}$.
- The product of a complex number $z = x+\mathrm{i}y$ and its conjugate is

$$z\overline{z} = x^2+y^2. \tag{9.2}$$

The number $z\overline{z}$ is real and nonnegative.

The Absolute Value of a Complex Number. Since $z\overline{z}$ is real and nonnegative, we can make the following definition.

Definition 9.2. The *absolute value*, $|z|$, of $z = x + \mathrm{i}y$ is the nonnegative square root of $z\bar{z}$:

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}. \tag{9.3}$$

For z real, this definition coincides with the absolute value of a real number. Next, we see that the absolute values of the real and imaginary parts of $z = x + \mathrm{i}y$ are always less than or equal to $|z|$.

$$|\mathrm{Re}\,(z)| = |x| = \sqrt{x^2} \le \sqrt{x^2 + y^2} = |z| \tag{9.4}$$

and

$$|\mathrm{Im}\,(z)| = |y| = \sqrt{y^2} \le \sqrt{x^2 + y^2} = |z|.$$

Example 9.5. To find $|z|$ for $z = 3 + 4\mathrm{i}$, we have

$$|z| = \sqrt{z\bar{z}} = \sqrt{(3 + 4\mathrm{i})(3 - 4\mathrm{i})} = \sqrt{3^2 + 4^2} = 5.$$

Therefore, $|3 + 4\mathrm{i}| = 5$.

The absolute value of z is also called the modulus of z or the magnitude of z.

Example 9.6. There are infinitely many complex numbers with the same absolute value. Let

$$z = \cos\theta + \mathrm{i}\sin\theta,$$

where θ is any real number. Then $z\bar{z} = \cos^2\theta + \sin^2\theta = 1$. So

$$|z| = \sqrt{1} = 1.$$

We show below that this extension of the notion of absolute value to complex numbers has some familiar properties.

Theorem 9.1. Properties of absolute value

(a) Positivity: $|0| = 0$, *and if* $z \neq 0$ *then* $|z| > 0$.
(b) Symmetry: $|\bar{z}| = |z|$.
(c) Multiplicativity: $|wz| = |w||z|$.
(d) Triangle inequality: $|w + z| \le |w| + |z|$.

Proof. Positivity and symmetry follow directly from the definition, as we ask you to check in Problem 9.8.

For multiplicativity we use the properties of complex numbers and conjugation along with the definition of absolute value to get

$$|wz|^2 = wz\overline{wz} = wz\overline{w}\,\overline{z} = w\overline{w}z\overline{z} = |w|^2|z|^2 = (|w||z|)^2,$$

and so $|wz| = |w||z|$.

The proof of the triangle inequality again begins with the definition of absolute value and then uses the additive property and the distributive rule to get

$$|w+z|^2 = (w+z)(\overline{w+z}) = (w+z)(\overline{w}+\overline{z})$$
$$= w\overline{w}+w\overline{z}+z\overline{w}+z\overline{z} = |w|^2+w\overline{z}+z\overline{w}+|z|^2.$$

Observe that $w\overline{z}$ and $z\overline{w}$ are conjugates of each other. According to (9.1), their sum is equal to twice the real part

$$w\overline{z}+z\overline{w} = 2\mathrm{Re}\,(w\overline{z}).$$

As we saw in (9.4), the real part of a complex number does not exceed its absolute value. Therefore,

$$w\overline{z}+z\overline{w} \leq 2|w\overline{z}|.$$

Now by multiplicativity and symmetry, $2|w\overline{z}| = 2|w||\overline{z}| = 2|w||z|$. Therefore,

$$|w+z|^2 = |w|^2+2\mathrm{Re}\,(w\overline{z})+|z|^2 \leq |w|^2+2|w||z|+|z|^2 = (|w|+|z|)^2.$$

Therefore,

$$|w+z| \leq |w|+|z|.$$

□

Example 9.7. We verify the triangle inequality for $w = 1-2\mathrm{i}$ and $z = 3+4\mathrm{i}$. Adding, we get $w+z = 4+2\mathrm{i}$. Taking absolute values, we have

$$|w| = \sqrt{5}, \qquad |z| = \sqrt{25}, \qquad |w+z| = \sqrt{20},$$

and $\sqrt{20} \leq \sqrt{5}+\sqrt{25}$. So we see that $|w+z| \leq |w|+|z|$.

Absolute Value and Sequence Convergence. Just as with sequences of real numbers, absolute values help us define what it means for numbers to be close, and hence what it means for a sequence of complex numbers to converge. A sequence of complex numbers $\{z_n\} = \{z_1, z_2, \ldots, z_n, \ldots\}$ is said to converge to z if for any tolerance $\varepsilon > 0$, $|z_n - z|$ is less than ε for all z_n far enough out in the sequence.

A sequence of complex numbers z_n is a *Cauchy sequence* if $|z_n - z_m|$ can be made less than any prescribed tolerance for all n and m large enough. A sequence $z_n = x_n + \mathrm{i}y_n$ of complex numbers gives rise to two sequences of real numbers: the real parts $x_1, x_2, \ldots$ and the imaginary parts $y_1, y_2, \ldots$. How is the behavior of $\{z_n\}$ related to the behavior of these sequences of real and imaginary parts $\{x_n\}$ and $\{y_n\}$? For example, suppose $\{z_n\}$ is a Cauchy sequence. According to (9.4),

$$|x_n - x_m| = |\mathrm{Re}(z_n - z_m)| \leq |z_n - z_m|$$

and

$$|y_n - y_m| = |\mathrm{Im}(z_n - z_m)| \leq |z_n - z_m|.$$

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of real numbers, and hence converge to limits x and y. The sequence $\{z_n\}$ converges to $z = x + \mathrm{i}y$, because according to the triangle inequality,

$$|z - z_n| = |x - x_n + \mathrm{i}(y - y_n)| \le |x - x_n| + |y - y_n|,$$

and this sum on the right will be as small as desired whenever n is large enough.

On the other hand, suppose x_n and y_n are Cauchy sequences of real numbers. Then the sequence of complex numbers

$$z_n = x_n + \mathrm{i}y_n$$

is a Cauchy sequence of complex numbers, because

$$|z_n - z_m| = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}$$

can be made as small as desired for all n and m large enough. Take n and m so large that *both* of $|x_n - x_m|$ and $|y_n - y_m|$ are less than ε. Then

$$|z_n - z_m| < \sqrt{\varepsilon^2 + \varepsilon^2} = \sqrt{2}\varepsilon.$$

In summary, the convergence of a sequence of complex numbers boils down to the convergence of the sequences of their real and imaginary parts. As a result, much of the work we did in Chap. 1 to prove theorems about convergence of sequences can be extended to complex numbers.

Example 9.8. Suppose $z_n = x_n + \mathrm{i}y_n$ is a Cauchy sequence tending to $z = x + \mathrm{i}y$. Then $x_n \to x$ and $y_n \to y$. Therefore, the sequence

$$z_n^2 = x_n^2 - y_n^2 + 2\mathrm{i}x_n y_n \quad \text{tends to} \quad x^2 - y^2 + 2\mathrm{i}xy.$$

That is, $z_n^2 \to z^2$.

Alternatively, without mentioning x_n or y_n, we have

$$|z^2 - z_n^2| = |z + z_n||z - z_n|.$$

The factor $|z + z_n|$ is nearly $|z + z|$ for n large, and the factor $|z - z_n|$ tends to zero. Therefore, z_n^2 tends to z^2.

9.1b Geometry of Complex Numbers

We now present a *geometric representation* of complex numbers. This turns out to be a very useful way of thinking about complex numbers, just as it is useful to think of the real numbers as points of the number line. The complex numbers are conveniently represented as points in a plane, called the *complex number plane*.

To each point (x,y) *of the Cartesian plane we associate the complex number* $x+\mathrm{i}y$. The horizontal axis consists of real numbers. It is called the *real axis*. The vertical axis consists of purely imaginary numbers and is called the *imaginary axis*. See Fig. 9.1.

The complex conjugate has a simple geometric interpretation in the complex plane. The complex conjugate of $x+\mathrm{i}y$, $x-\mathrm{i}y$, is obtained from $x+\mathrm{i}y$ by *reflection across the real axis*. The geometric interpretation of the absolute value $\sqrt{x^2+y^2}$ of $x+\mathrm{i}y$ is very striking: it is the *distance* of $x+\mathrm{i}y$ *from the origin*. To visualize geometrically the sum of two complex numbers z and w, move the coordinate system rigidly and parallel to itself, i.e., without rotating it, so that the origin ends up where the point z was originally located. Then the point w will end up where the point $z+w$ was located in the original coordinate system. It follows from this geometric description of addition that the four points 0, w, z, $w+z$ are the vertices of a parallelogram; in particular, it follows that the distance of w to $w+z$ equals the distance from 0 to z. See on the left in Fig. 9.2.

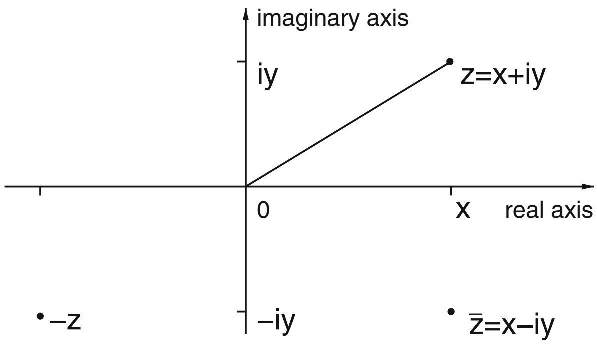

Fig. 9.1 Some complex numbers

Now consider the triangle whose vertices are 0, w, and $w+z$, as on the right in Fig. 9.2. The length of the side 0 to $w+z$ is $|w+z|$, the length of the side 0 to w is $|w|$, and using the parallelogram interpretation, the length of the side w to $w+z$ is $|z|$. According to a famous inequality of geometry, the length of any one side of a triangle does not exceed the sum of the lengths of the other two; therefore,

$$|w+z| \le |w|+|z|.$$

This is precisely the triangle inequality and is the reason for its name. Our earlier proof of this inequality made no reference to geometry, so it may be regarded as a proof of a theorem about triangles with the aid of complex numbers! In Problems 9.16 and 9.17 we shall give further examples of how to prove geometric results with the aid of complex numbers.

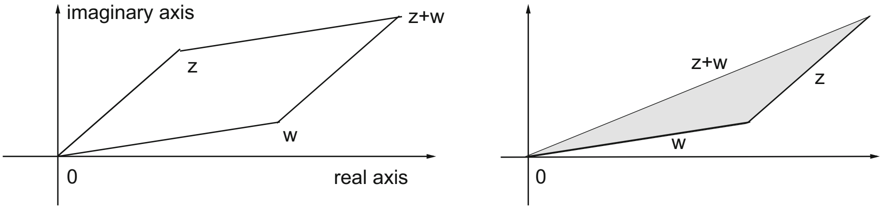

Fig. 9.2 *Left*: addition of complex numbers. *Right*: the triangle inequality illustrated

We have seen that addition of complex numbers can be visualized using Cartesian coordinates. Next, we see how multiplication of complex numbers can be visualized using *polar coordinates*.

Suppose p is a complex number with absolute value 1. Such a point lies on the unit circle. The Cartesian coordinates of p are $(\cos\theta,\ \sin\theta)$, where θ is the radian measure of the angle made by the real axis and the ray from 0 through p. So the complex number p with $|p| = 1$ is

$$p = \cos\theta + \mathrm{i}\sin\theta. \tag{9.5}$$

Let z be any complex number other than 0, and denote its absolute value by $r = |z|$. Define $p = \frac{z}{r}$. Then p has absolute value 1, so p can be represented in the form (9.5). Therefore,

$$z = r(\cos\theta + \mathrm{i}\sin\theta), \quad \text{where} \quad r = |z|. \tag{9.6}$$

See Fig. 9.3. The numbers (r,θ) are called the *polar coordinates* of the point (x,y), and (9.6) is called the *polar form* of the complex number. The angle θ is called the *argument* of z, denoted by $\arg z$; it is the angle between the positive real axis and the ray connecting the origin to z. In (9.6), we may replace θ by θ plus any integer multiple of 2π.

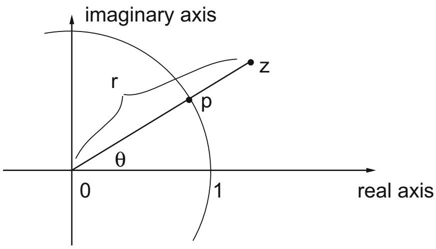

Fig. 9.3 Polar coordinates (r,θ) for z. The case $|z| > 1$ is drawn

Let z and w be a pair of complex numbers. Represent each in polar form,

$$z = r(\cos\theta + \mathrm{i}\sin\theta), \qquad w = s(\cos\phi + \mathrm{i}\sin\phi).$$

Multiplying these together, we get that

$$zw = rs(\cos\theta + \mathrm{i}\sin\theta)(\cos\phi + \mathrm{i}\sin\phi) \tag{9.7}$$

$$= rs\big((\cos\theta\cos\phi - \sin\theta\sin\phi) + \mathrm{i}(\cos\theta\sin\phi + \sin\theta\cos\phi)\big).$$

We use the addition laws for the cosine and sine, Eq. (3.17), to rewrite the product formula (9.7) in a particularly simple form. Recalling that r denotes $|z|$ and s denotes $|w|$, we find that

$$zw = |z||w|[\cos(\theta+\phi) + \mathrm{i}\sin(\theta+\phi)]. \tag{9.8}$$

This formula gives the polar form of the product zw. It is a symbolic statement of the following theorem.

Theorem 9.2. Multiplication rule for complex numbers in polar form

(a) The absolute value of the product zw is the product of the absolute values of its factors.
(b) The argument of the product zw is the sum of the arguments of its factors z and w.

In symbols:

$$|zw| = |z||w|, \tag{9.9}$$

$$\arg(zw) = \arg z + \arg w + 2\pi n, \tag{9.10}$$

where n is 0 or 1.

Example 9.9. Let $z = w = -i$, where $\arg z = \arg w = \dfrac{3\pi}{2}$. Then $zw = -1$, $\arg(zw) = \arg(-1) = \pi$, and $\arg z + \arg w = 3\pi$. Therefore, in this case,

$$\arg z + \arg w = \arg(zw) + 2\pi.$$

Square Root of a Complex Number. The complex number z^2 has twice the argument of z, and its absolute value is $|z|^2$. This suggests that to find square roots, we must halve the argument and form the square root of the absolute value.

Let us use these properties to locate a square root of i. The square root of 1 is 1, and half of $\frac{\pi}{2}$ is $\frac{\pi}{4}$. This describes the number

$$z = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}} = \frac{1+\mathrm{i}}{\sqrt{2}}.$$

Let us verify that $z^2 = \mathrm{i}$:

$$\left(\frac{1+\mathrm{i}}{\sqrt{2}}\right)^2 = \frac{1+2\mathrm{i}+\mathrm{i}^2}{2} = \mathrm{i}.$$

It works. Next we show that when we have z in polar form, it is possible to take powers and roots of z easily.

De Moivre's Theorem. By the multiplication rule, if

$$z = r(\cos\theta + \mathrm{i}\sin\theta),$$

then

$$z^2 = r^2(\cos 2\theta + \mathrm{i}\sin 2\theta),$$
$$z^3 = r^3(\cos 3\theta + \mathrm{i}\sin 3\theta),$$

and for every positive integer n,

$$\big(r(\cos\theta + \mathrm{i}\sin\theta)\big)^n = r^n\big(\cos(n\theta) + \mathrm{i}\sin(n\theta)\big). \tag{9.11}$$

This result is known as de Moivre's theorem. The reciprocal is

$$\frac{1}{r(\cos\theta + \mathrm{i}\sin\theta)} = r^{-1}(\cos\theta - \mathrm{i}\sin\theta),$$

because $r(\cos\theta + \mathrm{i}\sin\theta)\frac{1}{r}(\cos\theta - \mathrm{i}\sin\theta) = \frac{r}{r}(\cos^2\theta + \sin^2\theta) = 1$. Similarly, we ask you to verify in Problem 9.10 that for $n = 1, 2, 3, \ldots$,

$$\text{if } z = r(\cos\theta + \mathrm{i}\sin\theta) \qquad \text{then } z^{-n} = r^{-n}\big(\cos(n\theta) - \mathrm{i}\sin(n\theta)\big).$$

After our successful polar representation of positive and negative integer powers of z, we tackle the problem of rational powers, $z^{p/q}$. Since $z^{p/q} = (z^{1/q})^p$, we first settle the problem of finding the qth roots of z, for $q = 2, 3, 4, \ldots$.

By analogy to (9.11), we shall tentatively represent a qth root of z by

$$w_1 = r^{1/q}\left(\cos\left(\frac{\theta}{q}\right) + \mathrm{i}\sin\left(\frac{\theta}{q}\right)\right).$$

Indeed, this is a qth root of z, for its qth power is

$$(w_1)^q = (r^{1/q})^q\left(\cos\left(q\frac{\theta}{q}\right) + \mathrm{i}\sin\left(q\frac{\theta}{q}\right)\right) = r(\cos\theta + \mathrm{i}\sin\theta) = z,$$

but it is not the only qth root. Another root, different from w_1, is the number

$$w_2 = r^{1/q}\left(\cos\left(\frac{\theta + 2\pi}{q}\right) + \mathrm{i}\sin\left(\frac{\theta + 2\pi}{q}\right)\right).$$

You may also check, using the periodicity of the sine and cosine functions, that the q numbers

$$w_{k+1} = r^{1/q}\left(\cos\left(\frac{\theta + 2k\pi}{q}\right) + \mathrm{i}\sin\left(\frac{\theta + 2k\pi}{q}\right)\right), \qquad k = 0, 1, \ldots, q-1, \tag{9.12}$$

yield the q distinct qth roots of z, if $z \neq 0$. If $z = 0$, they are all equal to zero.

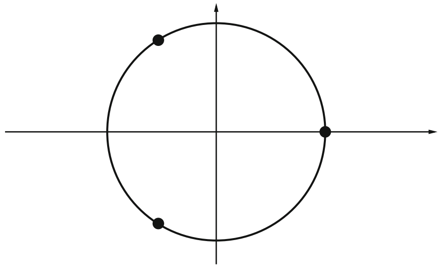

Fig. 9.4 The cube roots of 1, given in Example 9.10. All three lie on the unit circle

Example 9.10. The three cube roots of $1 = \cos 0 + \mathrm{i}\sin 0$ are

$$\cos\frac{0}{3} + \mathrm{i}\sin\frac{0}{3} = 1, \quad \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3} = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2},$$

and

$$\cos\frac{4\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3} = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2},$$

which correspond to taking $k = 0$, 1, and 2 in (9.12). See Fig. 9.4.

We end this section on the geometry of complex numbers by showing how to use products of complex numbers to find the area A of a triangle with vertices at three complex numbers in the plane. We take the case in which one of the vertices of the triangle is the origin. We lose no generality by this assumption, for the area of the triangle with vertices p, q, r is the same as the area of the translated triangle $0, q-p, r-p$.

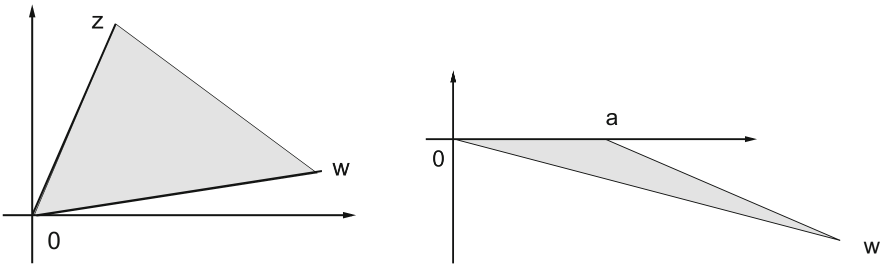

Fig. 9.5 *Left*: we prove that the area of the triangle is $\frac{1}{2}|\mathrm{Im}\,(\bar{z}w)|$. *Right*: the height of this triangle is $|\mathrm{Im}\,(w)|$

Denote by a a positive real number, and by w a complex number. The triangle whose vertices are 0, a, and w has base whose length is a and whose altitude is the absolute value of the imaginary part of w. See Fig. 9.5. Therefore, the area of the triangle with vertices 0, a, w is

$$A(0,a,w) = \frac{1}{2}a|\mathrm{Im}\,(w)| = \frac{1}{2}|\mathrm{Im}\,(aw)|.$$

Now let z and w denote any complex numbers. We claim that the area of the triangle with vertices 0, z, w is

$$A(0,z,w) = \frac{1}{2}\left|\mathrm{Im}\,(\overline{z}w)\right|. \tag{9.13}$$

In the case that z is real and positive, this agrees with the first case. Now consider arbitrary $z \neq 0$. Let p denote any complex number of absolute value 1. Multiplication by p is equivalent to rotation around the origin; therefore, the triangle with vertices 0, pz, pw has the same area as the triangle with vertices 0, z, w:

$$A(0,z,w) = A(0,pz,pw).$$

Choose $p = \dfrac{\overline{z}}{|z|}$. Then $pz = |z|$ is real, so the area is given by the formula

$$A(0,pz,pw) = \frac{1}{2}\left|\mathrm{Im}\,(\overline{p}\,\overline{z}\,pw)\right| = \frac{1}{2}\left|\mathrm{Im}\,(\overline{z}w)\right|.$$

Example 9.11. We find the area of the triangle with vertices $(0,1)$, $(2,3)$, and $(-5,7)$. Translate so that the first vertex is at the origin, giving $(0,0)$, $(2,2)$, and $(-5,6)$, or the complex numbers 0, $z = 2+2\mathrm{i}$, and $w = -5+6\mathrm{i}$. Then

$$A(0,z,w) = \frac{1}{2}|\mathrm{Im}(\overline{z}w)| = \frac{1}{2}\left|\mathrm{Im}\big((2-2\mathrm{i})(-5+6\mathrm{i})\big)\right| = 11.$$

Problems

9.1. For each of the numbers $z = 2+3\mathrm{i}$ and $z = 4-\mathrm{i}$, calculate the following:

(a) $|z|$
(b) $\overline{z}$
(c) Express $\dfrac{1}{z}$ and $\dfrac{1}{\overline{z}}$ in the form $a+b\mathrm{i}$.
(d) Verify that $z+\overline{z} = 2\mathrm{Re}\,(z)$.
(e) Verify that $z\overline{z} = |z|^2$.

9.2. Carry out the following operations with complex numbers:

(a) $(2+3\mathrm{i})+(5-4\mathrm{i})$
(b) $(3-2\mathrm{i})-(8-7\mathrm{i})$
(c) $(3-2\mathrm{i})(4+5\mathrm{i})$
(d) $\dfrac{3-2\mathrm{i}}{4+5\mathrm{i}}$
(e) Solve $2\mathrm{i}z = \mathrm{i}-4z$

9.3. Express $z = 4+(2+\mathrm{i})\mathrm{i}$ in the form $x+\mathrm{i}y$, where x and y are real, and then find the conjugate $\overline{z}$.

9.4. Give an example to show that the ray from 0 to $\mathrm{i}z$ is 90 degrees counterclockwise from the ray from 0 to z, unless z is 0.

9.5. Find the absolute values of the following complex numbers:

(a) $3+4\mathrm{i}$
(b) $5+6\mathrm{i}$
(c) $\dfrac{3+4\mathrm{i}}{5+6\mathrm{i}}$
(d) $\dfrac{1+\mathrm{i}}{1-\mathrm{i}}$

9.6. For each item, describe geometrically the set of complex numbers that satisfy the given condition.

(a) $\mathrm{Im}(z)=3$
(b) $\mathrm{Re}(z)=2$
(c) $2<\mathrm{Im}(z)\le 3$
(d) $|z|=1$
(e) $|z|=0$
(f) $1<|z|<2$

9.7. Show that $\mathrm{Im}\,z=\dfrac{z-\overline{z}}{2\mathrm{i}}$ and $\mathrm{Re}\,z=\dfrac{z+\overline{z}}{2}$.

9.8. Verify the positivity and symmetry properties of absolute value $|z|$, as listed in Theorem 9.1.

9.9. Verify the identity $(a^2+b^2)(c^2+d^2)=(ac-bd)^2+(ad+bc)^2$. Then explain how this proves the property $|z||w|=|zw|$ of absolute value.

9.10. Verify that the reciprocal of $z^n=r^n\big(\cos\theta+\mathrm{i}\sin\theta\big)^n$ is

$$z^{-n}=r^{-n}\big(\cos(n\theta)-\mathrm{i}\sin(n\theta)\big).$$

9.11. Prove that for complex numbers z_1 and z_2,

(a) $|z_1-z_2|^2=|z_1|^2+|z_2|^2-2\mathrm{Re}\,(z_1\overline{z_2})$,
(b) $\big||z_1|-|z_2|\big|\le|z_1-z_2|$.

9.12. (a) Show that for any pair of complex numbers z and w,

$$|z+w|^2+|z-w|^2=2|z|^2+2|w|^2.$$

(b) Using the parallelogram interpretation of the addition of complex numbers, deduce from (a) that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of its four sides.

9.13. Find all three cube roots of -1 and represent each in the complex plane.

9.14. (a) Show that the two square roots of i are

$$\frac{1}{\sqrt{2}}(1+\mathrm{i}),\ -\frac{1}{\sqrt{2}}(1+\mathrm{i}),$$

and sketch each in the complex plane.

(b) Verify that the two square roots of i are fourth roots of -1. Then find the other two fourth roots of -1. Sketch all four.

9.15. Suppose a triangle has vertices 0, $a = a_1 + \mathrm{i}a_2$, and $b = b_1 + \mathrm{i}b_2$. Derive from the area formula (9.13) that the area of the triangle is

$$\frac{1}{2}|a_1b_2 - a_2b_1|.$$

9.16. Verify the following.

(a) The argument of $\overline{w}$ is the negative of the argument of w.
(b) The argument of $z\overline{w}$ is the difference of the arguments of z and w.
(c) The argument of $\dfrac{z}{w}$ is the difference of the arguments of z and w.
(d) The ray from 0 to z is perpendicular to the ray from 0 to w if and only if the number $z\overline{w}$ is purely imaginary.
(e) Let p be a point on the unit circle. Prove that the ray connecting p to the point 1 on the real axis is perpendicular to the ray connecting p to the point -1.

9.17. Let p and q be complex numbers of absolute value 1. Verify the following.

(a) $(p-1)^2\overline{p}$ is real.
(b) $\big((p-1)(\overline{q}-1)\big)^2\overline{p}q$ is real.
(c) Prove the angle-doubling theorem: If p and q lie on the unit circle, then the angle β between the rays from the origin to p and q is twice the angle α between the rays connecting the point $z = 1$ to p and q. In Fig. 9.6, $\beta = 2\alpha$.

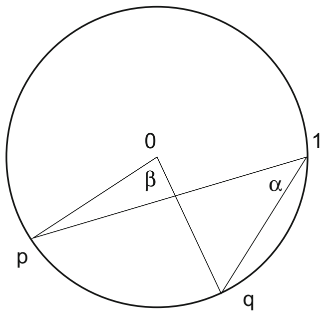

Fig. 9.6 The angle at 0 is twice the angle at 1, in Problem 9.17

9.18. Show that the sequence of complex numbers

$$1,\ 1+z,\ 1+z+z^2,\ 1+z+z^2+z^3,\ldots$$

is a Cauchy sequence, provided that $|z|$ is less than 1. What is the limit of the sequence?

9.19. We explore roots of 1.

(a) Find all zeros of the function $w(x) = x^4 + x^3 + x^2 + x + 1$ by first observing that the function $(x-1)w(x) = x^5 - 1$ is 0 whenever $w(x)$ is 0.
(b) Sketch all n of the nth roots of 1 in the complex plane.
(c) Let r be the nth root of 1 with smallest nonzero argument. Verify that the remaining nth roots are $r^2, r^3, \ldots, r^n$. Explain why the $n-1$ numbers

$$r,\ r^2,\ r^3,\ \ldots,\ r^{n-1}$$

are the roots of $x^{n-1} + x^{n-2} + \cdots + x + 1 = 0$.

9.20. Generalize the geometric series argument used in Problem 1.56 to show that for each complex number z, the partial sums

$$s_n = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$

are a Cauchy sequence.

9.2 Complex-Valued Functions

In this section we discuss the relationship of complex numbers to the concept of function. So far in this text, numbers have entered the concept of function in two places: as *input* and *output*. In the first part of this section, we show how simple it is to replace real numbers by complex ones as output if the input is kept real; such functions are called complex-valued functions with real input. In the second part, we discuss some very special but important functions whose input is complex, and whose output is also complex, i.e., complex-valued functions with complex input.

Definition 9.3. A *complex-valued function of a real variable* means a function

$$f(t) = p(t) + \mathrm{i}q(t),$$

where p and q are real-valued functions. The variable t is real, in some interval that is the domain of p, q and f.

The theory of complex-valued functions is simple, because it can be reduced at one stroke to the theory of real-valued functions. There are two ways of going about it. The first is to observe that everything (well, almost everything) that has been said about real-valued functions makes sense when carried over to complex-valued functions. "Everything" includes the following notions:

(a) The concept of function itself.
(b) The operations of adding and multiplying functions and forming the reciprocal of a function (with the usual proviso that the function should not be zero).

(c) The concept of a continuous function.
(d) The concept of a differentiable function and its derivative.
(e) Higher derivatives.
(f) The integral of a function over an interval.

9.2a Continuity

Let us review the concept of continuity. The intuitive meaning of the continuity of a function f is that to determine $f(t)$ approximately, approximate knowledge of t is sufficient. The precise version was discussed in Definition 2.3 back in Chapter 2. Both the intuitive and the precise definitions make sense for complex-valued functions:

- Intuitive: A complex-valued function $f(t)$ is continuous at t if approximate knowledge of t suffices to determine approximate knowledge of $f(t)$.
- Precise: A complex-valued function f is uniformly continuous on an interval I if given any tolerance $\varepsilon > 0$, there is a precision $\delta > 0$ such that for any pair of numbers t and s in I that differ by less than δ, $f(t)$ and $f(s)$ differ by less than ε.

Sums and products of uniformly continuous complex-valued functions are uniformly continuous, and so is the reciprocal of a uniformly continuous function that is never zero.

Example 9.12. Let $f(t) = 3 + \mathrm{i}t$. To check uniform continuity of f we may write

$$f(t) - f(s) = (3 + \mathrm{i}t) - (3 + \mathrm{i}s) = \mathrm{i}(t - s).$$

Then

$$|f(t) - f(s)| = |\mathrm{i}(t - s)| = |t - s|.$$

If t and s are within δ, then the complex numbers $f(t)$ and $f(s)$ are as well. Therefore, f is uniformly continuous on every interval.

9.2b Derivative

We turn to the concept of differentiation. The function $f(t)$ is differentiable at t if

$$f_h(t) = \frac{f(t+h) - f(t)}{h}$$

tends to a limit as h tends to zero. This limit is called the derivative of f at t, and is denoted by $f'(t)$. We can separate the difference quotient into its real and imaginary parts:

$$\frac{p(t+h) + \mathrm{i}q(t+h) - p(t) - \mathrm{i}q(h)}{h} = p_h(t) + \mathrm{i}q_h(t).$$

It follows that $f = p + \mathrm{i}q$ is differentiable at t if and only if its real and imaginary parts are differentiable at t.

The usual rules for differentiating sums, products, and reciprocals of differentiable functions hold, i.e.,

$$(f+g)' = f' + g',$$
$$(fg)' = fg' + f'g,$$
$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}.$$

You are urged to consult Chap. 3 to convince yourself that the proofs offered there retain their validity for complex-valued functions as well.

Example 9.13. Suppose $f(t) = z_1 t + z_2$, where z_1 and z_2 are arbitrary complex numbers. Then $f'(t) = z_1$.

Example 9.14. If $f(x) = \dfrac{1}{x+\mathrm{i}}$, then the rule for differentiating the reciprocal of a function yields

$$f'(x) = -\frac{1}{(x+\mathrm{i})^2}.$$

Example 9.15. Again let $f(x) = \dfrac{1}{x+\mathrm{i}}$. Let us find $f'(x)$ a different way. Separate f into its real and imaginary parts:

$$f(x) = \frac{1}{x+\mathrm{i}} = \frac{1}{x+\mathrm{i}}\frac{x-\mathrm{i}}{x-\mathrm{i}} = \frac{x-\mathrm{i}}{x^2+1} = \frac{x}{x^2+1} - \frac{\mathrm{i}}{x^2+1} = a(x) + \mathrm{i}b(x).$$

Differentiating, we get

$$a'(x) = \frac{(x^2+1)1 - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}, \qquad b'(x) = \left(-\frac{1}{x^2+1}\right)' = \frac{2x}{(x^2+1)^2}.$$

To see that the real and imaginary parts of $f'(x)$, as computed in Example 9.14, are $a'(x)$ and $b'(x)$ of Example 9.15, we write

$$f'(x) = -\frac{1}{(x+\mathrm{i})^2} = \frac{-1}{(x+\mathrm{i})^2}\frac{(x-\mathrm{i})^2}{(x-\mathrm{i})^2} = \frac{-x^2+2x\mathrm{i}+1}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} + \mathrm{i}\frac{2x}{(x^2+1)^2}.$$

Example 9.16. We use the quotient rule for differentiating $f(x) = \dfrac{x}{x^2+\mathrm{i}}$, and obtain

$$f'(x) = \frac{(x^2+\mathrm{i})x' - x(x^2+\mathrm{i})'}{(x^2+\mathrm{i})^2} = \frac{\mathrm{i}-x^2}{(x^2+\mathrm{i})^2}.$$

Example 9.17. Now consider $f(x) = (x+\mathrm{i})^2$. If we carry out the indicated squaring, we can split f into its real and imaginary parts:

$$f(x) = x^2 + 2\mathrm{i}x - 1 = x^2 - 1 + 2\mathrm{i}x = a(x) + \mathrm{i}b(x).$$

Differentiating $a(x) = x^2 - 1$ and $b(x) = 2x$, we get $a'(x) = 2x$ and $b'(x) = 2$, so that

$$f'(x) = a'(x) + \mathrm{i}b'(x) = 2x + 2\mathrm{i}.$$

The function $f(x) = (x+\mathrm{i})^2 = (x+\mathrm{i})(x+\mathrm{i})$, when differentiated using the product rule, yields

$$f'(x) = 1(x+\mathrm{i}) + (x+\mathrm{i})1 = 2(x+\mathrm{i}),$$

the same answer we got before.

Next we consider the chain rule. Suppose $g(t)$ is a real-valued function and f is a complex-valued function defined at all values taken on by g. Then we can form their composition $f \circ g$, defined as $f(g(t))$. If f and g are both differentiable, so is the composite, and the derivative of the composite is given by the usual chain rule. The proof is similar to the real-valued case.

Example 9.18. Let $f(x) = \dfrac{1}{x+\mathrm{i}}$ and $g(t) = t^2$. By Example 9.14, $f'(x) = -(x+\mathrm{i})^{-2}$. The derivative of $f \circ g$ can be calculated by the chain rule:

$$\left(\frac{1}{t^2+\mathrm{i}}\right)' = (f \circ g)'(t) = f'(g(t))g'(t) = -(g(t)+\mathrm{i})^{-2}g'(t) = \frac{1}{(t^2+\mathrm{i})^2}2t.$$

9.2c *Integral of Complex-Valued Functions*

Splitting a complex-valued function into its real and imaginary parts is suitable for defining the integral of a complex-valued function.

Definition 9.4. For $f = p + \mathrm{i}q$, where p and q are continuous real-valued functions on $[a,b]$, we set

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b p(t)\,\mathrm{d}t + \mathrm{i}\int_a^b q(t)\,\mathrm{d}t.$$

The properties of integrals of complex-valued functions follow from the definition and the corresponding properties of integrals of real-valued functions that are continuous on the indicated intervals:

- *Additivity:* $\displaystyle\int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t.$

- *Linearity:* For a complex constant k, $\int_a^b kf(t)\,dt = k\int_a^b f(t)\,dt$ and

$$\int_a^b \big(f(t)+g(t)\big)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.$$

- *Fundamental theorem:* $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ and $\displaystyle\int_a^b F'(t)\,dt = F(b)-F(a)$.

Recall that the integrals of p and q in Definition 9.4 are defined as the limits of approximating sums $I_{\text{approx}}(p,[a,b])$ and $I_{\text{approx}}(q,[a,b])$. Let us use the same subdivision of $[a,b]$ and choices of the t_j and define

$$I_{\text{approx}}(f,[a,b]) = I_{\text{approx}}(p,[a,b]) + \mathrm{i}I_{\text{approx}}(q,[a,b]).$$

We can conclude that $I_{\text{approx}}(f,[a,b])$ tends to $\int f(t)\,dt$. That is, the integral of a complex-valued function can be defined in terms of approximating sums. This bears out our original contention that most of the theory we have developed for real-valued functions applies verbatim to complex-valued functions. Another property of the integral of real-valued functions is the following.

- *Upper bound property:* If $|f(t)| \le M$ for every t in $[a,b]$, then $\left|\int_a^b f(t)\,dt\right| \le M(b-a)$.

This inequality, too, remains true for complex-valued functions, and for the same reason: the analogous estimate holds for the approximating sums, with absolute value as defined for complex numbers.

9.2d Functions of a Complex Variable

Next we consider complex-valued functions $f(z) = w$ of a complex variable z. Do such functions have derivatives? In Chap. 3, we introduced the derivative in two ways:

- the rate at which the value of the function changes;
- the slope of the line tangent to the graph of the function.

For functions of a complex variable, the geometric definition as slope is no longer available, but the rate of change definition is still meaningful.

Definition 9.5. A complex-valued function $f(z)$ of a complex variable z is differentiable at z if the difference quotients

$$\frac{f(z+h)-f(z)}{h}$$

tend to a limit as the complex number h tends to zero. The limit is called the derivative of f at z and is denoted by $f'(z)$.

Example 9.19. Let $f(z)=z^2$. Then

$$\frac{f(z+h)-f(z)}{h}=\frac{(z+h)^2-z^2}{h}=\frac{2zh+h^2}{h}=2z+h.$$

As the complex number h tends to 0, $2z+h$ tends to $2z$. So $f'(z)=2z$.

Theorem 9.3. *Every positive integer power of z,*

$$z^m \qquad (m=1,2,3,\ldots),$$

is a differentiable function of the complex variable z, and its derivative is mz^{m-1}.

Proof. According to the binomial theorem, valid for complex numbers,

$$(z+h)^m=z^m+mz^{m-1}h+\cdots+h^m. \tag{9.14}$$

All the terms after the first two have h raised to the power 2 or higher. So

$$\frac{(z+h)^m-z^m}{h}=mz^{m-1}+\cdots,$$

where each of the terms after the plus sign contains h as a factor. It follows that as h tends to zero, the difference quotient above tends to mz^{m-1}. This concludes the proof. □

Since sums and constant multiples of differentiable functions are differentiable, it follows that every polynomial $p(z)$ is a differentiable function of z. Next we show that we can extend Newton's method to find complex roots of polynomials.

Newton's Method for Complex Roots. Newton's method for estimating a real root of a real function $f(x)=0$ relied on finding a root of the linear approximation to f at a previous estimate. We carry this idea into the present setting of complex roots.

Suppose z_{old} is an approximate root of $p(z) = 0$. Denote by h the difference between the exact zero z of p and z_{old}:

$$h = z - z_{\text{old}}.$$

Using Eq. (9.14), we see that

$$0 = p(z) = p(z_{\text{old}} + h) = p(z_{\text{old}}) + p'(z_{\text{old}})h + \text{error}, \tag{9.15}$$

where the error is less than a constant times $|h|^2$. Let the new approximation be

$$z_{\text{new}} = z_{\text{old}} - \frac{p(z_{\text{old}})}{p'(z_{\text{old}})}, \tag{9.16}$$

as we did in Sect. 5.3a. We use (9.15) to express

$$p(z_{\text{old}}) = -p'(z_{\text{old}})h - \text{error}.$$

Setting this in (9.16), we get

$$z_{\text{new}} = z_{\text{old}} + h + \frac{\text{error}}{p'(z_{\text{old}})}.$$

Since h was defined as $z - z_{\text{old}}$, we can rewrite this relation as

$$z_{\text{new}} - z = \frac{\text{error}}{p'(z_{\text{old}})}.$$

If $p'(z_{\text{old}})$ is bounded away from zero (which is a basic assumption for Newton's method to work), we can rewrite this relation as the inequality

$$|z - z_{\text{new}}| < (\text{constant})h^2 = (\text{constant})|z - z_{\text{old}}|^2.$$

If z_{old} is so close to the exact root z that the quantity $(\text{constant})|z - z_{\text{old}}|$ is less than 1, then the new approximation is closer to the exact root than the old one. Repeating this process produces a sequence of approximations that converge to the exact root with extraordinary rapidity.

Example 9.20. We have seen that the three cube roots of 1 are 1, $\dfrac{1+\sqrt{3}i}{2}$, and $\dfrac{1-\sqrt{3}i}{2}$. Let us see what Newton's method produces. For $f(z) = z^3 - 1$, Newton's iteration is

$$z_{\text{new}} = z - \frac{f(z)}{f'(z)} = z - \frac{z^3 - 1}{3z^2}.$$

Table 9.1 shows the results starting from three different initial states, and these are sketched in Fig. 9.7.

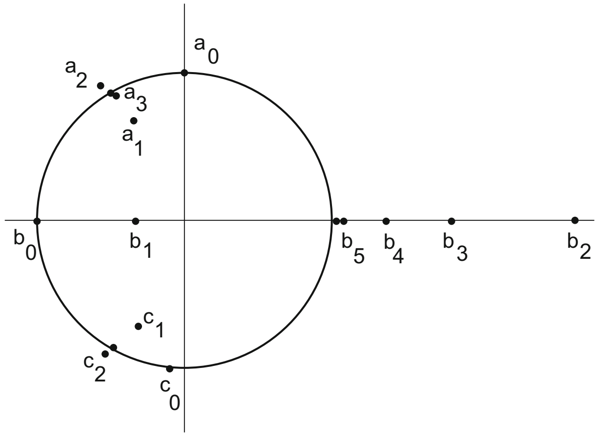

Fig. 9.7 Newton's method used to approximate three roots of $f(z) = z^3 - 1$, starting from $a_0 = \mathrm{i}$, $b_0 = -1$, and $c_0 = -0.1 - \mathrm{i}$. See Example 9.20

Table 9.1 Newton's method applied in Example 9.20 to $z^3 - 1 = 0$, from three starting values

n	a_n	b_n	c_n
0	i	−1	−0.1 − i
1	−0.33333 + 0.66667i	−0.33333	−0.39016 − 0.73202i
2	−0.58222 + 0.92444i	2.7778	−0.53020 − 0.89017i
3	−0.50879 + 0.86817i	1.8951	−0.50135 − 0.86647i
4	−0.50007 + 0.86598i	1.3562	−0.50000 − 0.86602i
5	−0.50000 + 0.86603i	1.0854	−0.50000 − 0.86603i
6	−0.50000 + 0.86603i	1.0065	−0.50000 − 0.86603i
7	−0.50000 + 0.86603i	1.0000	−0.50000 − 0.86603i

9.2e The Exponential Function of a Complex Variable

We now turn to the function $C(t) = \cos t + \mathrm{i}\sin t$. Its image in the complex plane lies on the unit circle. The function C is differentiable, and $C'(t) = -\sin t + \mathrm{i}\cos t$. A moment's observation discloses that

$$C' = \mathrm{i}C.$$

We now recall that for a real, $\mathrm{e}^{at} = P(t)$ satisfies the differential equation

$$P' = aP.$$

This suggests how to define $\mathrm{e}^{\mathrm{i}t}$.

Definition 9.6.

$$e^{it} = \cos t + i\sin t.$$

The functional equation of the exponential function suggests that

$$e^{x+iy} = e^x e^{iy};$$

combining this with Definition 9.6, we arrive at the definition of the exponential of a complex number.

Definition 9.7. If $x,\ y$ are real numbers, we define

$$e^{x+iy} = e^x\bigl(\cos y + i\sin y\bigr).$$

The exponential function as defined above has complex input, complex output, and it has all the usual properties of the exponential function, as we shall now show.

Theorem 9.4. *For all complex numbers z and w,*

$$e^{z+w} = e^z e^w.$$

Proof. Set $z = x + iy$ and $w = u + iv$. Then by definition, we get

$$e^z e^w = e^x(\cos y + i\sin y)e^u(\cos v + i\sin v).$$

Using the functional equation of the exponential function for real inputs and arithmetic properties of complex numbers yields

$$\begin{aligned} e^z e^w &= e^{x+u}(\cos y + i\sin y)(\cos v + i\sin v) \\ &= e^{x+u}\bigl((\cos y\cos v - \sin y\sin v) + i(\cos y\sin v + \sin y\cos v)\bigr). \end{aligned}$$

Similarly, using the addition formulas for sine and cosine, we conclude that

$$e^z e^w = e^{x+u}(\cos(y+v) + i\sin(y+v)) = e^{x+u+i(y+v)}.$$

We conclude that $e^z e^w = e^{z+w}$, as asserted. □

Theorem 9.5. Differential equation. *For every complex number c,*

$$P(t) = e^{ct}$$

satisfies

$$P'(t) = cP(t).$$

Proof. Set $c = a + \mathrm{i}b$, where a and b are real. By the definition of complex exponentials,

$$\mathrm{e}^{ct} = \mathrm{e}^{at}(\cos(bt) + \mathrm{i}\sin(bt)).$$

The derivative of this is, by the product rule,

$$a\mathrm{e}^{at}(\cos(bt) + \mathrm{i}\sin(bt)) + \mathrm{e}^{at}(-b\sin(bt) + \mathrm{i}b\cos(bt)).$$

This is the same as $a\mathrm{e}^{ct} + \mathrm{i}b\mathrm{e}^{ct} = (a + \mathrm{i}b)\mathrm{e}^{ct} = c\mathrm{e}^{ct}$, as was to be proved. □

Example 9.21. Let us show that $y = \mathrm{e}^{\mathrm{i}t}$ satisfies the differential equation

$$y'' + y = 0.$$

We have $y' = \mathrm{i}\mathrm{e}^{\mathrm{i}t}$, so $y'' = \mathrm{i}^2\mathrm{e}^{\mathrm{i}t} = -\mathrm{e}^{\mathrm{i}t} = -y$ and $y'' + y = 0$.

Theorem 9.6. Series representation. *For every complex number z,*

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \cdots .$$

Proof. We saw in Problem 1.55 of Chap. 1 that the theorem is true in the case of $z = 1$. Certainly it is true when $z = 0$. We proved it for all real z in Sect. 4.3a, and in Problem 9.20, we suggested a method to show that the partial sums are a Cauchy sequence. For the sake of simplicity, we prove this now only for pure imaginary z. Assume $z = \mathrm{i}b$, where b is real. Substituting $z = \mathrm{i}b$ into the series on the right, we get

$$1 + \mathrm{i}b - \frac{b^2}{2} + \cdots + \frac{(\mathrm{i}b)^n}{n!} + \cdots .$$

The powers i^n have period 4, i.e., we have

$$\mathrm{i}^0 = 1, \qquad \mathrm{i}^1 = \mathrm{i}, \qquad \mathrm{i}^2 = -1, \qquad \mathrm{i}^3 = -\mathrm{i},$$

and then the pattern is repeated. This shows that the terms of even order are real, and the odd ones are purely imaginary. Next, we assume that it is valid to rearrange the terms of this series:

$$\begin{aligned} &1 + \mathrm{i}b - \frac{b^2}{2} + \cdots + \frac{(\mathrm{i}b)^n}{n!} + \cdots \\ &= \left(1 - \frac{b^2}{2} + \frac{b^4}{24} - \cdots + \frac{(-1)^m b^{2m}}{(2m)!} + \cdots\right) \\ &+ \mathrm{i}\left(b - \frac{b^3}{6} + \frac{b^5}{120} + \cdots + \frac{(-1)^m b^{2m+1}}{(2m+1)!} + \cdots .\right). \end{aligned}$$

The real and imaginary parts are $\cos b$ and $\sin b$; see Sect. 4.3a and Eq. (4.20). So our series is

$$1 + \mathrm{i}b - \frac{b^2}{2} + \cdots + \frac{(\mathrm{i}b)^n}{n!} + \cdots. = \cos b + \mathrm{i}\sin b,$$

which is $\mathrm{e}^{\mathrm{i}b}$, as was to be proved. □

Set $t = 2\pi$ in the definition of the exponential function, and observe that

$$\mathrm{e}^{\mathrm{i}2\pi} = \cos(2\pi) + \mathrm{i}\sin(2\pi) = 1.$$

More generally, since $\cos(2\pi n) = 1,\ \sin(2\pi n) = 0,$

$$\mathrm{e}^{\mathrm{i}2\pi n} = \cos(2\pi n) + \mathrm{i}\sin(2\pi n) = 1$$

for every integer n.

Now set $t = \pi$ in the definition. Since $\cos\pi = -1$, $\sin\pi = 0$, we get that

$$\mathrm{e}^{\mathrm{i}\pi} = \cos\pi + \mathrm{i}\sin\pi = -1.$$

This can be rewritten in the form[3] $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$.

The Derivative of the Exponential Function of a Complex Variable. We have shown in Theorem 9.3 that polynomials $f(z)$ are differentiable for complex z. We shall show now that so is the function e^z.

Theorem 9.7. *The function* e^z *is differentiable, and its derivative is* e^z.

Proof. We have to show that the difference quotient

$$\frac{\mathrm{e}^{z+h} - \mathrm{e}^z}{h}$$

tends to e^z as h tends to 0. Using the functional equation for the exponential function, $\mathrm{e}^{z+h} = \mathrm{e}^z\mathrm{e}^h$, we can write the difference quotient as

$$\mathrm{e}^z\frac{\mathrm{e}^h - 1}{h}.$$

So what has to be proved is that $\dfrac{\mathrm{e}^h - 1}{h}$ tends to 1 as h tends to zero. Decompose h into its real and imaginary parts: $h = x + \mathrm{i}y$. Then

$$\mathrm{e}^h - 1 = \mathrm{e}^x(\cos y + \mathrm{i}\sin y) - 1.$$

[3] It is worth pointing out to those interested in number mysticism that this relation contains the most important numbers and symbols of mathematics: $0, 1, \mathrm{i}, \pi, \mathrm{e}, +$, and $=$.

As h approaches 0, both x and y approach 0 as well. Use the linear approximations of the exponential and trigonometric functions for x and y near zero:

$$e^x = 1 + x + r_1, \quad \cos y = 1 + r_2, \quad \sin y = y + r_3,$$

where the remainders r_1, r_2, and r_3 are less than a constant times $x^2 + y^2 = |h|^2$. Using these approximations, we get

$$\begin{aligned} e^h - 1 &= (1 + x + r_1)\big(1 + r_2 + i(y + r_3)\big) - 1 \\ &= x + iy + xiy + \text{remainder} = h + ixy + \text{remainder}, \end{aligned}$$

where the absolute value of the remainder is less than a constant times $|h|^2$. According to the A-G inequality, it is also true that the absolute value $|ixy|$ is equal to

$$|ixy| = |x||y| \le \frac{1}{2}\left(|x|^2 + |y|^2\right) = \frac{1}{2}|h|^2.$$

Therefore, $e^h - 1 = h + \text{remainder}$, where the absolute value of the remainder is less than a constant times $|h|^2$. Dividing by h, we see that

$$\frac{e^h - 1}{h} = 1 + \frac{\text{remainder}}{h}$$

tends to 1, as claimed. □

So far, all the functions $f(z)$ that we have considered, i.e., polynomials $p(z)$ and the exponential e^z, have been differentiable with respect to z. However, there are simple functions that are not differentiable. Here is an example.

Example 9.22. Let $f(z) = \overline{z}$. Then

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}$$

if the limit exists. For real $h = x + 0i \neq 0$, we have $\overline{h} = h$, so $\frac{\overline{h}}{h} = 1$. For imaginary $h = 0 + yi \neq 0$, we have $\overline{h} = -yi$ and $\frac{\overline{h}}{h} = -1$. Therefore, the limit does not exist, and f is not differentiable.

Problems

9.21. Differentiate the following complex functions of a real variable t:

(a) $e^t + i \sin t$

(b) $\dfrac{1}{t - i} + \dfrac{1}{t + i}$

(c) $\mathrm{i}e^{t^2}$

(d) $\mathrm{i}\sin t + \dfrac{1}{t+3+\mathrm{i}}$

9.22. Differentiate the following complex functions of a complex variable z:

(a) $2\mathrm{i} - z^2$

(b) $z^3 - z + 5e^z$

9.23. Express $\cos t$ and $\sin t$ in terms of $e^{\mathrm{i}t}$.

9.24. Compute the complex integrals:

(a) $\displaystyle\int_0^1 e^{\mathrm{i}s}\,\mathrm{d}s$

(b) $\displaystyle\int_0^{\pi/2} (\cos s + \mathrm{i}\sin s)\,\mathrm{d}s$

9.25. Since we have defined e^z for every complex z, we can extend the definition of the hyperbolic cosine to every complex number. Write that definition, and then verify the identity $\cosh(\mathrm{i}t) = \cos t$ for all real numbers t.

9.26. For a real and positive and z complex, write a definition of a^z by expressing a as $e^{\log a}$. Show that $a^{z+w} = a^z a^w$.

9.27. Find the value of the integral $\displaystyle\int_0^\infty e^{\mathrm{i}kx-x}\,\mathrm{d}x$, k real.

9.28. In this exercise we ask you to test experimentally Newton's method (9.16) for finding complex roots of an algebraic equation $p(z) = 0$. Take for p the cubic polynomial

$$p(z) = z^3 + z^2 + z - \mathrm{i}.$$

(a) Write a computer program that constructs a sequence $z_1, z_2, \ldots$ according to Newton's method, stopping when both the real and imaginary parts of z_{n+1} and z_n differ by less than 10^{-6}, or when n exceeds 30.

(b) Show that $p(z) \neq 0$ if $|z| > 2$ or if $|z| < \frac{1}{2}$.
Hint: Use the triangle inequality in the form

$$|a-b| \geq |a| - |b|$$

to show that $|z^3 + z^2 + z - \mathrm{i}| \geq 1 - |z^3| - |z^2| - |z|$, which is positive if $|z| < \frac{1}{2}$. Similarly, show that if $|z| > 2$, then the z^3 term is the most important to consider, because its absolute value is greater than that of the sum of the other three terms.

(c) Starting with the first approximation

$$z_0 = 0.35 + 0.35\mathrm{i},$$

construct the sequence z_n by Newton's method, and determine whether it converges to a solution of $p(z) = 0$.

(d) Find all solutions of $p(z) = 0$ by starting with different choices for z.