Chapter 1
Numbers and Limits

Abstract This chapter introduces basic concepts and properties of numbers that are necessary prerequisites for defining the calculus concepts of limit, derivative, and integral.

1.1 Inequalities

One cannot exaggerate the importance in calculus of inequalities between numbers. Inequalities are at the heart of the basic notion of convergence, an idea central to calculus. Inequalities can be used to prove the equality of two numbers by showing that one is neither less than nor greater than the other. For example, Archimedes showed that the area of a circle was neither less than nor greater than the area of a triangle with base the circumference and height the radius of the circle.

A different use of inequalities is descriptive. Sets of numbers described by inequalities can be visualized on the number line.

Fig. 1.1 The number line

To say that a is less than b, denoted by $a < b$, means that $b-a$ is positive. On the number line in Fig. 1.1, a would lie to the left of b. Inequalities are often used to describe intervals of numbers. The numbers that satisfy $a < x < b$ are the numbers between a and b, not including the endpoints a and b. This is an example of an *open* interval, which is indicated by round brackets, (a,b).

To say that a is less than or equal to b, denoted by $a \le b$, means that $b-a$ is not negative. The numbers that satisfy $a \le x \le b$ are the numbers between a and b, including the endpoints a and b. This is an example of a *closed* interval, which is indicated by square brackets, $[a,b]$. Intervals that include one endpoint but not

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Fig. 1.2 *Left*: the open interval (a,b). *Center*: the half open interval $(a,b]$. *Right*: the closed interval $[a,b]$

the other are called *half-open* or *half-closed*. For example, the interval $a < x \leq b$ is denoted by $(a,b]$ (Fig. 1.2).

The absolute value $|a|$ of a number a is the distance of a from 0; for a positive, then, $|a| = a$, while for a negative, $|a| = -a$. The absolute value of a difference, $|a-b|$, can be interpreted as the distance between a and b on the number line, or as the length of the interval between a and b (Fig. 1.3).

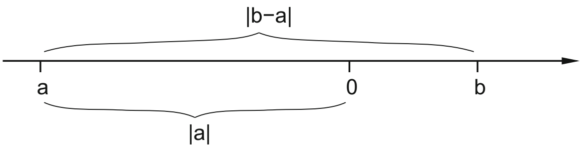


Fig. 1.3 Distances are measured using absolute value

The inequality

$$|a-b| < \varepsilon$$

can be interpreted as stating that the distance between a and b on the number line is less than ε. It also means that the difference between a and b is no more than ε and no less than $-\varepsilon$:

$$-\varepsilon < a-b < \varepsilon. \tag{1.1}$$

In Problem 1.9, we ask you to use some of the properties of inequalities stated in Sect. 1.1a to obtain inequality (1.1).

Example 1.1. The inequality $|x-5| < \frac{1}{2}$ describes the numbers x whose distance from 5 is less than $\frac{1}{2}$. This is the open interval $(4.5, 5.5)$. It also tells us that the difference $x-5$ is between $-\frac{1}{2}$ and $\frac{1}{2}$. See Fig. 1.4. The inequality $|x-5| \leq \frac{1}{2}$ describes the closed interval $[4.5, 5.5]$.

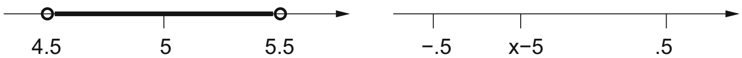


Fig. 1.4 *Left*: the numbers specified by the inequality $|x-5| < \frac{1}{2}$ in Example 1.1. *Right*: the difference $x-5$ is between $-\frac{1}{2}$ and $\frac{1}{2}$

The inequality $|\pi - 3.141| \leq \dfrac{1}{10^3}$ can be interpreted as a statement about the precision of 3.141 as an approximation of π. It tells us that 3.141 is within $\dfrac{1}{10^3}$ of π, and that π is in an interval centered at 3.141 of length $\dfrac{2}{10^3}$.

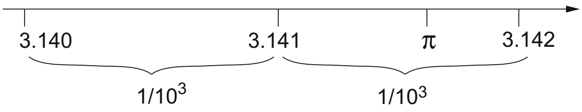


Fig. 1.5 Approximations to π

We can imagine smaller intervals contained inside the larger one in Fig. 1.5, which surround π more closely. Later in this chapter we will see that one way to determine a number is by trapping it within progressively tighter intervals. This process is described by the nested interval theorem in Sect. 1.3c.

We use (a,∞) to denote the set of numbers that are greater than a, and $[a,\infty)$ to denote the set of numbers that are greater than or equal to a. Similarly, $(-\infty,a)$ denotes the set of numbers less than a, and $(-\infty,a]$ denotes those less than or equal to a. See Fig. 1.6.

Fig. 1.6 The intervals $(-\infty,a)$, $(-\infty,a]$, $[a,\infty)$, and (a,∞) are shown from left to right

Example 1.2. The inequality $|x-5| \geq \frac{1}{2}$ describes the numbers whose distance from 5 is greater than or equal to $\frac{1}{2}$. These are the numbers that are in $(-\infty,4.5]$ or in $[5.5,\infty)$. See Fig. 1.7.

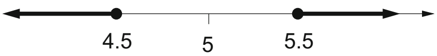


Fig. 1.7 The numbers specified by the inequality in Example 1.2

1.1a Rules for Inequalities

Next we review some rules for handling inequalities.

(a) *Trichotomy:* For any numbers a and b, either $a<b$ or $a=b$ or $b<a$.
(b) *Transitivity:* If $a<b$ and $b<c$, then $a<c$.
(c) *Addition:* If $a<b$ and $c<d$, then $a+c<b+c$ and $a+c<b+d$.
(d) *Multiplication:* If $a<b$ and p is positive, then $pa<pb$, but if $a<b$ and n is negative, then $nb<na$.
(e) *Reciprocal:* If a and b are positive numbers and $a<b$, then $\dfrac{1}{b}<\dfrac{1}{a}$.

The rules for inequalities can be used algebraically to simplify inequalities or to derive new inequalities from old ones. With the exception of trichotomy, these rules are still true if $<$ is replaced by $\leq$. In Problem 1.8 we ask you to use trichotomy to show that if $a\leq b$ and $b\leq a$, then $a=b$.

Example 1.3. If $|x-3|<2$ and $|y-4|<6$, then according to the inequality rule on addition,

$$|x-3|+|y-4|<2+6.$$

Example 1.4. If $0<a<b$, then according to inequality rule on multiplication,

$$a^2<ab \text{ and } ab<b^2.$$

Then by the transitivity rule, $a^2<b^2$.

1.1b The Triangle Inequality

There are two notable inequalities that we use often, the triangle inequality, and the arithmetic–geometric mean inequality. The triangle inequality is as important as it is simple:

$$|a+b|\le|a|+|b|.$$

Try substituting in a few numbers. What does it say, for example when $a=-3$ and $b=1$? It is easy to convince yourself that when a and b are of the same sign, or one of them is zero, equality holds. If a and b have opposite signs, inequality holds.

The triangle inequality can be used to quickly estimate the accuracy of a sum of approximations.

Example 1.5. Using

$$|\pi-3.141|<10^{-3} \text{ and } |\sqrt{2}-1.414|<10^{-3},$$

the inequality addition rule gives $|\pi-3.141|+|\sqrt{2}-1.414|<10^{-3}+10^{-3}$. The triangle inequality then tells us that

$$\begin{aligned}|(\pi+\sqrt{2})-4.555| &= |(\pi-3.141)+(\sqrt{2}-1.414)|\\ &\le|\pi-3.141|+|\sqrt{2}-1.414|\le 2\times10^{-3}.\end{aligned}$$

That is, knowing $\sqrt{2}$ and π within 10^{-3}, we know their sum within 2×10^{-3}.

Another use of the triangle inequality is to relate distances between numbers on the number line. The inequality says that the distance between x and z is less than or equal to the sum of the distance between x and y and the distance between y and z. That is,

$$|z-x|=|(z-y)+(y-x)|\le|z-y|+|y-x|.$$

In Fig. 1.8 we illustrate two cases: in which y is between x and z, and in which it is not.

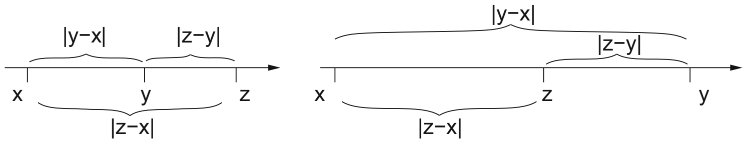


Fig. 1.8 Distances related by the triangle inequality

1.1c The Arithmetic–Geometric Mean Inequality

Next we explore an important but less familiar inequality.

Theorem 1.1. The arithmetic–geometric mean inequality. *The geometric mean of two positive numbers is less than their arithmetic mean:*

$$\sqrt{ab} \le \frac{a+b}{2},$$

with equality only in the case $a = b$.

We refer to this as the "A-G" inequality. The word "mean" is used in the following sense:

(a) The mean lies between the smaller and the larger of the two numbers a and b.
(b) When a and b are equal, their mean is equal to a and b.

You can check that each side of the inequality is a mean in this sense.

A visual proof: Figure 1.9 provides a visual proof that $4ab \le (a+b)^2$. The A-G inequality follows once you divide by 4 and take the square root.

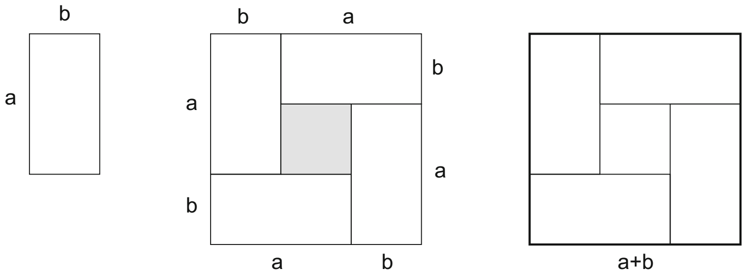


Fig. 1.9 A visual proof that $4ab \le (a+b)^2$, by comparing areas

An algebraic proof: Since the square of any number is positive or zero, it follows that

$$0 \le (a-b)^2 = a^2 - 2ab + b^2,$$

with equality holding only when $a = b$. By adding $4ab$ to both sides, we get

$$4ab \le a^2 + 2ab + b^2 = (a+b)^2,$$

the same inequality we derived visually. Dividing by 4 and taking square roots, we get

$$\sqrt{ab} \le \frac{a+b}{2},$$

with equality holding only when $a = b$.

Example 1.6. The A-G inequality can be used to prove that among all rectangles with the same perimeter, the square has the largest area. See Fig. 1.10. *Proof:* Denote the lengths of the sides of the rectangle by W and L. Its area is WL. The lengths of the sides of the square with the same perimeter are $\frac{W+L}{2}$, and its area is $\left(\frac{W+L}{2}\right)^2$. The inequality

$$WL \le \left(\frac{W+L}{2}\right)^2$$

follows from squaring both sides of the A-G inequality.

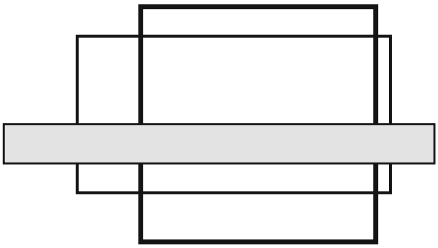


Fig. 1.10 Three rectangles measuring 6 by 6, 8 by 4, and 11 by 1. All have perimeter 24. The areas are 36, 32, and 11, and the square has the largest area. See Example 1.6

The A-G Inequality for n Numbers. The arithmetic and geometric means can be defined for more than two numbers. The arithmetic mean of $a_1, a_2, \ldots, a_n$ is

$$\text{arithmetic mean } = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

The geometric mean of n positive numbers is defined as the nth root of their product:

$$\text{geometric mean } = (a_1 a_2 \cdots a_n)^{1/n}.$$

For n numbers, the A-G inequality is

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n},$$

with equality holding only when $a_1 = a_2 = \cdots = a_n$. As in the case of a rectangle, the A-G inequality for three numbers can be interpreted geometrically: Consider the volume of a box that measures a_1 by a_2 by a_3. Then the inequality states that among all boxes with a given edge sum, the cube has the largest volume.

The proof of the case for n numbers is outlined in Problem 1.17. The key to the proof is understanding how to use the result for two numbers to derive it for four numbers. Curiously, the result for $n = 4$ can then be used to prove the result for $n = 3$. The general proof proceeds in a similar manner. Use the result for $n = 4$ to get the result for $n = 8$, and then use the result for $n = 8$ to get the result for $n = 5$, 6, and 7, and so forth.

Here is the proof for $n = 4$. Let c, d, e, f be four positive numbers. Denote by a the arithmetic mean of c and d, and denote by b the arithmetic mean of e and f:

$$a = \frac{c+d}{2}, \quad b = \frac{e+f}{2}.$$

By the A-G inequality for two numbers, applied three times, we get

$$\sqrt{cd} \le a, \quad \sqrt{ef} \le b, \tag{1.2}$$

and

$$\sqrt{ab} \le \frac{a+b}{2}. \tag{1.3}$$

Combining inequalities (1.2) and (1.3) gives

$$(cdef)^{1/4} \le \frac{a+b}{2}. \tag{1.4}$$

Since

$$\frac{a+b}{2} = \frac{\frac{c+d}{2} + \frac{e+f}{2}}{2} = \frac{c+d+e+f}{4},$$

we can rewrite inequality (1.4) as

$$(cdef)^{1/4} \le \frac{c+d+e+f}{4},$$

with equality holding only when $a = b$ and when $c = d$ and $e = f$. This completes the argument for four numbers. Next we see how to use the result for four to prove the result for three numbers.

We start with the observation that if a, b, and c are any three numbers, and m is their arithmetic mean,

$$m = \frac{a+b+c}{3}, \tag{1.5}$$

then m is also the arithmetic mean of the four numbers a, b, c, and m:

$$m = \frac{a+b+c+m}{4}.$$

To see this, multiply Eq. (1.5) by 3 and add m to both sides. We get $4m = a+b+c+m$. Dividing by 4 gives the result we claimed. Now apply the A-G inequality to the four numbers a, b, c, and m. We get

$$(abcm)^{1/4} \le m.$$

Raise both sides to the fourth power. We get $abcm \le m^4$. Divide both sides by m and then take the cube root of both sides; we get the desired inequality

$$(abc)^{1/3} \le m = \frac{a+b+c}{3}.$$

This completes the argument for $n = 2$, 3, and 4. The rest of the proof proceeds similarly.

Problems

1.1. Find the numbers that satisfy each inequality, and sketch the solution on a number line.

(a) $|x-3| \le 4$
(b) $|x+50| \le 2$
(c) $1 < |y-7|$
(d) $|3-x| < 4$

1.2. Find the numbers that satisfy each inequality, and sketch the solution on a number line.

(a) $|x-4| < 2$
(b) $|x+4| \le 3$
(c) $|y-9| \ge 2$
(d) $|4-x| < 2$

1.3. Use inequalities to describe the numbers *not* in the interval $[-3,3]$ in two ways:

(a) using an absolute value inequality
(b) using one or more simple inequalities.

1.4. Find the arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$ of the pairs $(a,b) = (5,5),\ (3,7),\ (1,9)$. Sketch a square corresponding to each case, as in the geometric proof. Interpret the pairs as dimensions of a rectangle. Find the perimeter and area of each.

1.5. Find the geometric mean of 2, 4, and 8. Verify that it is less than the arithmetic mean.

1.6. Which inequalities are true for all numbers a and b satisfying $0 < a < b < 1$?

(a) $ab > 1$
(b) $\frac{1}{a} < \frac{1}{b}$
(c) $\frac{1}{b} > 1$
(d) $a+b < 1$
(e) $a^2 < 1$
(f) $a^2+b^2 < 1$
(g) $a^2+b^2 > 1$
(h) $\frac{1}{a} > b$

1.7. You know from algebra that when x and y are positive numbers,

$$(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y}) = x-y.$$

(a) Suppose $x > y > 5$. Show that $\sqrt{x}-\sqrt{y} \leq \frac{1}{4}(x-y)$.
(b) Suppose y is within 0.02 of x. Use the inequality in part (a) to estimate how close $\sqrt{y}$ is to $\sqrt{x}$.

1.8. Use the trichotomy rule to show that if $a \leq b$ and $b \leq a$, then $a = b$.

1.9. Suppose $|b-a| < \varepsilon$. Explain why each of the following items is true.

(a) $0 \leq (b-a) < \varepsilon$ or $0 \leq -(b-a) < \varepsilon$
(b) $-\varepsilon < b-a < \varepsilon$
(c) $a-\varepsilon < b < a+\varepsilon$
(d) $-\varepsilon < a-b < \varepsilon$
(e) $b-\varepsilon < a < b+\varepsilon$

1.10.(a) A rectangular enclosure is to be constructed with 16 m of fence. What is the largest possible area of the enclosure?
(b) If instead of four fenced sides, one side is provided by a large barn wall, what is the largest possible area of the enclosure?

1.11. A shipping company limits the sum of the three dimensions of a rectangular box to 5 m. What are the dimensions of the box that contains the largest possible volume?

1.12. Two pieces of string are measured to within 0.001 m of their true length. The first measures 4.325 m and the second measures 5.579 m. A good estimate for the total length of string is 9.904 m. How accurate is that estimate?

1.13. In this problem we see how the A-G inequality can be used to derive various inequalities. Let x be positive.

(a) Write the A-G inequality for the numbers 1, 1, x, to show that $x^{1/3} \leq \frac{x+2}{3}$.
(b) Similarly, show that $x^{1/n} \leq \frac{x+n-1}{n}$ for every positive integer n.

(c) By letting $x = n$ in the inequality in (b), we get

$$n^{1/n} \le \frac{2n-1}{n}.$$

Explain how it follows that $n^{1/n}$ is always less than 2.

1.14. The *harmonic mean* is defined for positive numbers a and b by

$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

(a) For the cases $(a,b) = (2,3)$ and $(3,3)$, verify that

$$H(a,b) \le G(a,b) \le A(a,b), \tag{1.6}$$

i.e., $\dfrac{2}{\frac{1}{a}+\frac{1}{b}} \le \sqrt{ab} \le \dfrac{a+b}{2}$.

(b) On a trip, a driver goes the first 100 miles at 40 mph, and the second 100 miles at 60 mph. Show that the average speed is the harmonic mean of 40 and 60.

(c) Deduce $H(a,b) \le G(a,b)$ from $G\left(\frac{1}{a},\frac{1}{b}\right) \le A\left(\frac{1}{a},\frac{1}{b}\right)$.

(d) A battery supplies the same voltage V to each of two resistors in parallel in Fig. 1.11. The current I splits as $I = I_1 + I_2$, so that Ohm's law $V = I_1R_1 = I_2R_2$ holds for each resistor. Show that the value R to be used in $V = IR$ is one-half the harmonic mean of R_1 and R_2.

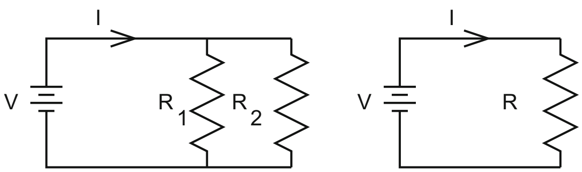


Fig. 1.11 Two resistors in parallel with a battery, and an equivalent circuit with only one resistor. See Problem 1.14

1.15. The product of the numbers 1 through n is the factorial $n! = (1)(2)(3)\cdots(n)$. Their sum is

$$1+2+3+\cdots+n = \frac{1}{2}n(n+1).$$

(a) Show that $(n!)^{1/n} \le n$.

(b) Use the A-G inequality to derive the better result that $(n!)^{1/n} \le \dfrac{n+1}{2}$.

1.16. If we want to know how much the product ab varies when we allow a and b to vary independently, there is a clever algebra trick that helps in this:

$$ab - a_0b_0 = ab - ab_0 + ab_0 - a_0b_0.$$

(a) Show that
$$|ab - a_0b_0| \le |a||b - b_0| + |b_0||a - a_0|.$$
(b) Suppose a and b are in the interval $[0, 10]$, and that a_0 is within 0.001 of a and b_0 is within 0.001 of b. How close is a_0b_0 to ab?

1.17. Here you may finish the proof of the A-G inequality.

(a) Prove the A-G inequality for eight numbers by using twice the A-G mean inequality for four numbers, and combine it with the A-G inequality for two numbers.
(b) Show that if a, b, c, d, and e are any five numbers, and m is their arithmetic mean, then the arithmetic mean of the eight numbers a, b, c, d, e, m, m, and m is again m. Use this and the A-G inequality for eight numbers to prove the A-G inequality for five numbers.
(c) Prove the general case of the A-G inequality by generalizing (a) and (b).

1.18. Another important inequality is due to the French mathematician Cauchy and the German mathematician Schwarz: Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two sets of numbers. Then
$$a_1b_1 + \cdots + a_nb_n \le \sqrt{a_1^2 + \cdots + a_n^2}\sqrt{b_1^2 + \cdots + b_n^2}.$$
Verify each of these steps of the proof:

(a) The roots of the polynomial $p(x) = Px^2 + 2Qx + R$ are $\dfrac{-Q \pm \sqrt{Q^2 - PR}}{P}$.
(b) Show that if $p(x)$ does not take negative values, then $p(x)$ has at most one real root. Show that in this case, $Q^2 \le PR$.
(c) Take $p(x) = (a_1x + b_1)^2 + \cdots + (a_nx + b_n)^2$. Show that
$$P = a_1^2 + \cdots + a_n^2, \qquad Q = a_1b_1 + \cdots + a_nb_n, \qquad \text{and} \quad R = b_1^2 + \cdots + b_n^2.$$
(d) Since $p(x)$ defined above is a sum of squares, it does not take negative values. Therefore, $Q^2 \le PR$. Deduce from this the Cauchy–Schwarz inequality.
(e) Determine the condition for equality to occur.

1.2 Numbers and the Least Upper Bound Theorem

1.2a Numbers as Infinite Decimals

There are two familiar ways of looking at numbers: as infinite decimals and as points on a number line. The integers divide the number line into infinitely many intervals of unit length. If we include the left endpoint of each interval but not the right, we can cover the number line with nonoverlapping intervals such that each number a

belongs to exactly one of them, $n \le a < n+1$. Each interval can be subdivided into ten subintervals of length $\frac{1}{10}$. As before, if we agree to count the left endpoint but not the right as part of each interval, the intervals do not overlap. Our number a belongs to exactly one of these ten subintervals, say to

$$n + \frac{\alpha_1}{10} \le a < n + \frac{\alpha_1 + 1}{10}.$$

This determines the first decimal digit α_1 of a. For example, Fig. 1.12 illustrates how to find the first decimal digit of a number a between 2 and 3.

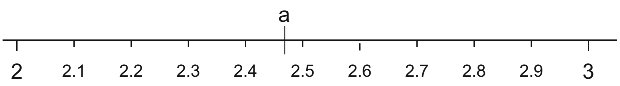


Fig. 1.12 a is in the interval $[2.4, 2.5)$, so $\alpha_1 = 4$

The second decimal digit α_2 is determined similarly, by subdividing the interval $[2.4, 2.5)$ into ten equal subintervals, and so on. Figure 1.13 illustrates the example $\alpha_2 = 7$.

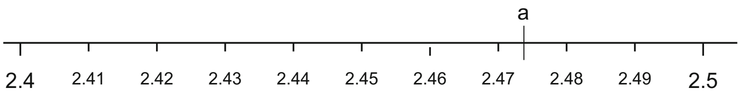


Fig. 1.13 a is in the interval $[2.47, 2.48)$, so $\alpha_2 = 7$

Thus using the representation of a as a point on the number line and the procedure just described, we can find α_k in $a = n.\alpha_1\alpha_2\ldots\alpha_k\ldots$ by determining the appropriate interval in the kth step of this process. Conversely, once we have the decimal representation of a number, we can identify its location on the number line.

Example 1.7. Examining the decimal representation $\frac{31}{39} = 0.7948717\ldots$, we see that

$$0.79487 \le \frac{31}{39} < 0.79488.$$

Repeated Nines in Decimals. The method we described for representing numbers as infinite decimals does not result in decimal fractions that end with infinitely many nines. Nevertheless, such decimals come up when we do arithmetic with infinite decimals. For instance, take the sum

$$\begin{array}{r} 1/3 = 0.333333333\cdots \\ +\ 2/3 = 0.666666666\cdots \\ \hline 1 = 0.999999999\cdots \end{array}$$

Similarly, every infinite decimal ending with all nines is equal to a finite decimal, such as

$$0.39529999999\cdots = 0.3953.$$

Decimals and Ordering. The importance of the infinite decimal representation of numbers lies in the ease with which numbers can be compared. For example, which of the numbers

$$\frac{17}{20}, \quad \frac{31}{39}, \quad \frac{45}{53}, \quad \frac{74}{87}$$

is the largest? To compare them as fractions, we would have to bring them to a common denominator. If we represent the numbers as decimals,

$$\frac{17}{20} = 0.85000\ldots$$
$$\frac{31}{39} = 0.79487\ldots$$
$$\frac{45}{53} = 0.84905\ldots$$
$$\frac{74}{87} = 0.85057\ldots$$

we can tell which number is larger by examining their integer parts and decimal digits, place by place. Then clearly,

$$\frac{31}{39} < \frac{45}{53} < \frac{17}{20} < \frac{74}{87}.$$

1.2b The Least Upper Bound Theorem

The same process we used for comparing four numbers can be used to find the largest number in any finite set of numbers that are represented as decimals. Can we apply a similar procedure to find the largest number in an infinite set S of numbers? Clearly, the set S of positive integers has no largest element. Suppose we rule out sets that contain arbitrarily large numbers and assume that all numbers in S are less than some number k. Such a number k is called an *upper bound* of S.

Definition 1.1. A number k is called an *upper bound* for a set S of numbers if

$$x \le k$$

for every x in S, and we say that S is *bounded above* by k. Analogously, k is called a *lower bound* for S if $k \le x$ for every x in S, and we say that S is *bounded below* by k.

Imagine pegs in the number line at all points of the set S. Let k be an upper bound for S that is to the right of every point of S. Put the point of your pencil at k and move it as far to the left as the pegs will let it go (Fig. 1.14). The point where the pencil gets stuck is also an upper bound of S. There can be no smaller upper bound, for if there were, we could have slid the pencil further to the left. It is the *least upper bound* of S.[1]

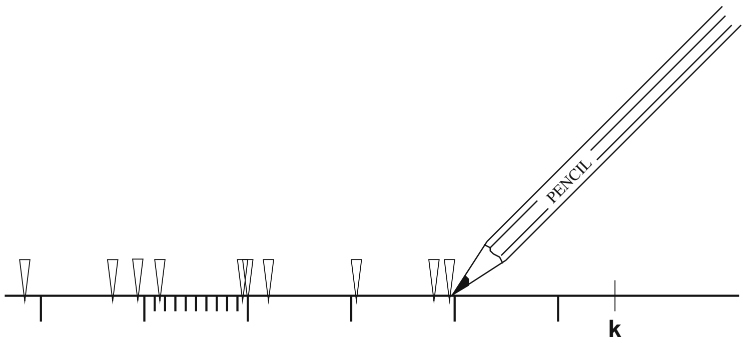


Fig. 1.14 The least upper bound of a bounded set of numbers

This result is so important it deserves restatement and a special name:

Theorem 1.2. The least upper bound theorem. *Every set S of numbers that is bounded above has a least upper bound.*

Proof. We prove the theorem when S is an infinite set of numbers between 0 and 1. The proof of the general case is similar. Examine the first decimal digits of the numbers in S and keep only those with the largest first digit. We call the remaining numbers *eligible* after the first step. Examine the second digits of the numbers that were eligible after the first step and keep only those with the largest second digit. Those are the numbers that are eligible after the second step. Define the number s by setting its jth digit equal to the jth digit of any number that remains eligible after j steps. By construction, s is greater than or equal to every number in S, i.e., s is an upper bound of S.

Next we show that every number that is smaller than s is not an upper bound of S, i.e., s is the smallest, or least, upper bound of S. Let $m = 0.m_1m_2m_3\dots m_n\dots$ be any number smaller than $s = 0.s_1s_2s_3\dots s_n\dots$. Denote by j the first position in which the digits of s and m differ. That means that for $n < j$, $s_n = m_n$. Since m is smaller than s, $m_j < s_j$. At the jth step in our construction of s there was at least one number x in

[1] The story is told that R.L. Moore, a famous mathematician in Texas, asked a student to give a proof or find a counterexample to the statement "Every bounded set of numbers has a largest element." The student came up with a counterexample: the set consisting of the numbers 1 and 2; it has a *larger* element, but no *largest*.

S that agreed with s up through the jth decimal digit. By comparing decimal digits, we see that m is less than x. So m is not an upper bound for S. Since no number less than s is an upper bound for S, s is the least upper bound of S. □

An analogous theorem is true for lower bounds:

Theorem 1.3. The greatest lower bound theorem. *Every set of numbers that is bounded below has a greatest lower bound.*

The least upper bound of set S is also known as the *supremum* of S, and the greatest lower bound as the *infimum* of S, abbreviated as $\sup S$ and $\inf S$ respectively.

The least upper bound theorem is one of the workhorses for proving things in calculus. Here is an example.

Existence of Square Roots. If we think of positive numbers geometrically as representing lengths of intervals and areas of geometric figures such as squares, then it is clear that every positive number p has a square root. It is the length of the edge of a square with area p. We now think of numbers as infinite decimals. We can use the least upper bound theorem to prove that a positive number has a square root. Let us do this for a particular positive number, say

$$p = 5.1.$$

A calculator produces the approximation $\sqrt{5.1} \approx 2.2583$. By squaring, we see that $(2.2583)^2 = 5.09991889$. Let S be the set of numbers a with $a^2 < 5.1$. Then S is not empty, because as we just saw, 2.2583 is in S, and so are 1 and 2 and many other numbers. Also, S is bounded above, for example by 3, because numbers larger than 3 cannot be in S; their squares are too large. The least upper bound theorem says that the set S has a least upper bound; call it r.

We show that $r^2 = 5.1$ by eliminating the possibility that $r^2 > 5.1$ or $r^2 < 5.1$. By squaring,

$$\left(r + \frac{1}{n}\right)^2 = r^2 + \frac{1}{n}\left(2r + \frac{1}{n}\right),$$

we see that the square of a number slightly bigger than r is more than r^2, but not much more when n is sufficiently large. Also,

$$\left(r - \frac{1}{n}\right)^2 = r^2 - \frac{1}{n}\left(2r - \frac{1}{n}\right)$$

shows that the square of a number slightly less than r is less than r^2, but not much less when n is sufficiently large. So, if r^2 is more than 5.1, there is a smaller number of the form $r - \frac{1}{n}$ whose square is also more than 5.1, so r is not the least upper bound of S, a contradiction. If r^2 is less than 5.1, then there is a larger number of the form $r + \frac{1}{n}$ whose square is also less than 5.1, so r is not an upper bound at all, a contradiction. The only other possibility is that $r^2 = 5.1$, and $r = \sqrt{5.1}$.

1.2c Rounding

As a practical matter, comparing two infinite decimal numbers involves rounding. If two decimal numbers with the same integer part have n digits in common, then they differ by less than 10^{-n}. The converse is not true: two numbers can differ by less than 10^{-n} but have no digits in common. For example, the numbers 0.300000 and 0.299999 differ by 10^{-6} but have no digits in common. The operation of *rounding* makes it clear by how much two numbers in decimal form differ.

Rounding a number a to m decimal digits starts with finding the decimal interval of length 10^{-m} that contains a. Then *a rounded down* to m digits is the left endpoint of this interval. Similarly, *a rounded up* to m digits is the right endpoint of the interval. Another way to round a up to m digits is to round a down to m digits and then add 10^{-m}. For example, $\frac{31}{39} = 0.7948717949\ldots$ rounded down to three digits is 0.794, and $\frac{31}{39}$ rounded up to three digits is 0.795.

When calculating, we frequently round numbers up or down. If after rounding, two numbers appear equal, how far apart might they be? Here are two observations about the distance between two numbers a and b and their roundings:

Theorem 1.4. *If a and b are two numbers given in decimal form and if one of the two roundings of a to m digits agrees with one of the two roundings of b to m digits, then $|a - b| < 2 \cdot 10^{-m}$.*

Proof. If a and b rounded down to m digits agree, then a and b are in the same interval of width 10^{-m}, and the difference between them is less than 10^{-m}. In the case that one of these numbers rounded up to m digits agrees with the other number rounded down to m digits, a and b lie in adjacent intervals of length 10^{-m}, and hence a and b differ by less than 2×10^{-m}. □

Similarly, if we know how close a and b are, we can conclude something about their roundings:

Theorem 1.5. *If the distance between a and b is less than 10^{-m}, then one of the roundings of a to m digits agrees with one of the roundings of b to m digits.*

Proof. The interval between a rounded down and a rounded up to m digits contains a and is 10^{-m} wide. Similarly, the interval between b rounded down and b rounded up to m digits contains b and is 10^{-m} wide. Since a and b differ by less than 10^{-m}, these two intervals are either identical or adjacent. In either case, they have at least one endpoint in common, so one of the roundings of a must agree with one of the roundings of b. □

Rounding and Calculation Errors. There are infinitely many real numbers, but calculators and computers have finite capacities to represent them. So numbers are stored by rounding. Calculations of basic arithmetic operations are a source of error due to rounding. Here is an example.

In Archimedes' work *Measurement of a Circle*, he approximated π by computing the perimeters of inscribed and circumscribed regular polygons with n sides. There are recurrence formulas for these estimates. Let p_1 be the perimeter of a regular hexagon inscribed in a unit circle. The length of each side of the hexagon is $s_1 = 1$. Then $p_1 = 6s_1 = 6$. Let p_2 be the perimeter of the regular 12-gon. The length of each side s_2 can be expressed in terms of s_1 using the Pythagorean theorem. We have in Fig. 1.15,

$$D = \frac{1}{2}s_1, \quad C = s_2, \quad \text{and} \quad B = 1 - A.$$

By the Pythagorean theorem, $A = \sqrt{1 - D^2}$ and $C = \sqrt{B^2 + D^2}$. Combining these, we find that

$$s_2 = \sqrt{\left(1 - \sqrt{1 - \left(\frac{1}{2}s_1\right)^2}\right)^2 + \left(\frac{1}{2}s_1\right)^2} = \sqrt{2 - 2\sqrt{1 - \left(\frac{1}{2}s_1\right)^2}}.$$

The same formula can be used to express the side s_n of the polygon of $3(2^n)$ sides in terms of s_{n-1}. The perimeter $p_n = 3(2^n)s_n$ approximates the circumference of the unit circle, 2π. The table in Fig. 1.15 shows that the formula appears to work well through $n = 16$, but after that something goes wrong, as you certainly see by line 29. This is an example of the catastrophic effect of round-off error.

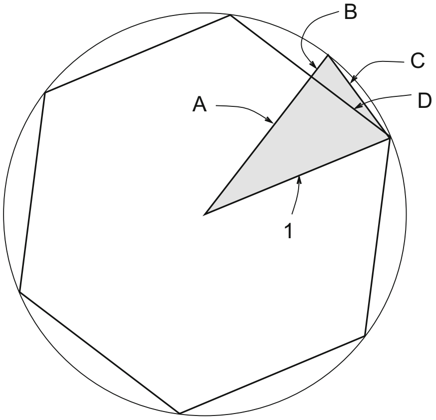
As we will see, many of the key concepts of calculus rely on differences of numbers that are nearly equal, sums of many numbers near zero, or quotients of very small numbers. This example shows that it is unwise to naively implement an algorithm in a computer program without considering the effects of rounding.

Problems

1.19. What would you choose for m in $|\sqrt{3} - 1.7| < 10^{-m}$, and why?

1.20. Find the least upper bound and the greatest lower bound of each of the following sets. Or if it is not possible, explain why.

(a) the interval $(8, 10)$.
(b) the interval $(8, 10]$.



n	s_n	p_n
1	1.000000000000000	6.000000000000000
2	0.517638090205042	6.211657082460500
3	0.261052384440103	6.265257226562474
4	0.130806258460286	6.278700406093744
5	0.065438165643553	6.282063901781060
6	0.032723463252972	6.282904944570689
7	0.016362279207873	6.283115215823244
8	0.008181208052471	6.283167784297872
9	0.004090612582340	6.283180926473523
10	0.002045307360705	6.283184212086097
11	0.001022653813994	6.283185033176309
12	0.000511326923607	6.283185237281579
13	0.000255663463975	6.283185290642431
14	0.000127831731987	6.283185290642431
15	0.000063915865994	6.283185290642431
16	0.000031957932997	6.283185290642431
17	0.000015978971709	6.283187339698854
18	0.000007989482381	6.283184607623475
19	0.000003994762034	6.283217392449608
20	0.000001997367121	6.283173679310083
21	0.000000998711352	6.283348530043515
22	0.000000499355676	6.283348530043515
23	0.000000249788979	6.286145480340079
24	0.000000125559416	6.319612329882269
25	0.000000063220273	6.363961030678928
26	0.000000033320009	6.708203932499369
27	0.000000021073424	8.485281374238571
28	0.000000014901161	12.000000000000000
29	0.000000000000000	0.000000000000000

Fig. 1.15 *Left*: the regular hexagon and part of the 12-gon inscribed in the *circle*. *Right*: calculated values for the edge lengths s_n and perimeters p_n of the inscribed $3(2^n)$-gon. Note that as n increases, the exact value of p_n approaches $2\pi = 6.2831853071795\ldots$

(c) the nonpositive integers.
(d) the set of four numbers $\frac{30}{279}, \frac{29}{263}, \frac{59}{525}, \frac{1}{9}$.
(e) the set $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$.

1.21. Take the unit square, and by connecting the midpoints of opposite sides, divide it into $2^2 = 4$ subsquares, each of side 2^{-1}. Repeat this division for each subsquare, obtaining $2^4 = 16$ squares whose sides have length 2^{-2}. Continue this process so that after n steps, there are 2^{2n} squares, each having sides of length 2^{-n}. See Fig. 1.16. With the lower left corner as center and radius 1, inscribe a unit quarter circle into the square. Denote by a_n the total area of those squares that at the nth step of the process, lie entirely inside the quarter circle. For example, $a_1 = 0$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{2}$.

(a) Is the set S of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ bounded above? If so, find an upper bound.
(b) Does S have a least upper bound? If so, what number do you think the least upper bound is?

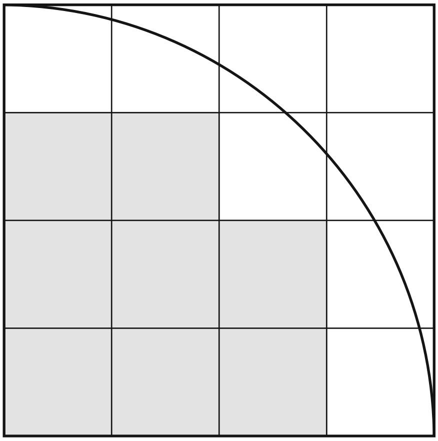


Fig. 1.16 The square in Problem 1.21. Area a_3 is shaded

1.22. Use rounding to add these two numbers so that the error in the sum is not more than 10^{-9}:

$$\begin{array}{r} 0.1234567898765432104567898765432101 \\ +\ 9.1111112221112221118765432104567892 \\ \hline \end{array}$$

1.23. Tell how, in principle, to add the two numbers having the indicated pattern of decimals:

$$\begin{array}{r} 0.101100111000111100001111100000\cdots \\ +\ 0.898989898989898989898989898989\cdots \\ \hline \end{array}$$

Does your explanation involve rounding?

1.24. Show that the least upper bound of a set S is *unique*. That is, if x_1 and x_2 both are least upper bounds of S, then $x_1 = x_2$.

Hint: Recall that given any numbers a and b, exactly one of the following holds: $a < b$, $a > b$, or $a = b$.

1.3 Sequences and Their Limits

In Sect. 1.2a we described numbers as infinite decimals. That is a very good theoretical description, but not a practical one. How long would it take to write down infinitely many decimal digits of a number, and where would we write them?

For an alternative practical description of numbers, we borrow from engineering the idea of *tolerance*. When engineers specify the size of an object to be used in a design, they give its magnitude, say 3 m. But they also realize that nothing built by human beings is exact, so they specify the error they can tolerate, say 1 mm, and still use the object. This means that to be usable, the object has to be no larger than 3.001 m and no smaller than 2.999 m.

This tolerable error is called *tolerance* and is usually denoted by the Greek letter ε. By its very nature, tolerance is a positive quantity, i.e., $\varepsilon > 0$. Here are some examples of tolerable errors, or tolerances, in approximating π:

$$|\pi - 3.14159| < 10^{-5}$$
$$|\pi - 3.141592| < 10^{-6}$$
$$|\pi - 3.14159265| < 10^{-8}$$
$$|\pi - 3.14159265358979| < 10^{-14}$$

Notice that the smaller tolerances pinpoint the number π within a smaller interval.

To determine a number a, we must be able to give an approximation to a for any tolerance ε, no matter how small. Suppose we take a sequence of tolerances ε_n tending to zero, and suppose that for each n, we have an approximation a_n that is within ε_n of a. The approximations a_n form an infinite sequence of numbers a_1, a_2, a_3, ... that tend to a in the sense that the difference between a_n and a tends to zero as n grows larger and larger. This leads to the general concept of the *limit of a sequence*.

Definition 1.2. A list of numbers is called a *sequence*. The numbers are called the terms of the sequence. We say that an infinite sequence a_1, a_2, a_3, ..., a_n, ... *converges* to the number a (is *convergent*) if given any tolerance $\varepsilon > 0$, no matter how small, there is a whole number N, dependent on ε, such that for all $n > N$, a_n differs from a by less than ε:

$$|a_n - a| < \varepsilon.$$

The number a is called the *limit* of the sequence $\{a_n\}$, and we write

$$\lim_{n\to\infty} a_n = a.$$

A sequence that has no limit *diverges* (is *divergent*).

A note on terminology and history: When the distinguished Polish mathematician Antoni Zygmund, author of the text *Trigonometric Series*, came as a refugee to America, he was eager to learn about his adopted country. Among other things, he asked an American friend to explain baseball to him, a game totally unknown in Europe. He received a lengthy lecture. His only comment was that the World Series should be called the World Sequence. As we will see later, the word "series" in mathematics refers to the *sum* of the terms of a sequence.

Because many numbers are known only through a sequence of approximations, a question that arises immediately, and will be with us throughout this calculus course, is this: How can we decide whether a given sequence converges, and if it does converge, what is its limit? Each case has to be analyzed individually, but there are some rules of arithmetic for convergent sequences.

Theorem 1.6. *Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences,*

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b.$$

Then

(a) $\lim_{n\to\infty}(a_n + b_n) = a + b.$
(b) $\lim_{n\to\infty}(a_n b_n) = ab.$
(c) If a is not zero, then for n large enough, $a_n \neq 0$, and $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}.$

These rules certainly agree with our experience of computation with decimals. They assert that if the numbers a_n and b_n are close to the numbers a and b, then their sum, product, and reciprocals are close to the sum, product, and reciprocals of a and b themselves. In Problem 1.33 we show you how to prove these properties of convergent sequences.

Next we give some examples of convergent sequences.

Example 1.8. $a_n = \frac{1}{n}$. For any tolerance ε no matter how small, $\frac{1}{n}$ is within ε of 0 once n is greater than $\frac{1}{\varepsilon}$. So

$$\left|\frac{1}{n} - 0\right| < \varepsilon \qquad \text{for } n > \frac{1}{\varepsilon}, \text{ and } \lim_{n\to\infty} \frac{1}{n} = 0.$$

Example 1.9. $a_n = \frac{1}{2^n}$. Since $2^n > n$ when $n > 2$, we see that $\frac{1}{2^n} < \frac{1}{n} < \varepsilon$ if n is large enough. So

$$\left|\frac{1}{2^n} - 0\right| < \varepsilon \qquad \text{for } n \text{ sufficiently large, and } \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

In these two examples the limit is zero, a rather simple number. Let us look at a very simple sequence whose limit is not zero.

Example 1.10. The limit of the constant sequence $a_n = 5$ for all $n = 1, 2, 3, \ldots$ is 5. The terms of the sequence do not differ from 5, so no matter how small ε is, $|a_n - 5| < \varepsilon$.

Here is a slightly more complicated example.

Example 1.11. Using algebra to rewrite the terms of the sequence, we obtain

$$\lim_{n\to\infty}\frac{5n+7}{n+1}=\lim_{n\to\infty}\left(\frac{5n+5}{n+1}+\frac{2}{n+1}\right)=\lim_{n\to\infty}\left(5+\frac{2}{n+1}\right).$$

Now by Theorem 1.6,

$$\lim_{n\to\infty}\left(5+\frac{2}{n+1}\right)=\lim_{n\to\infty}5+2\lim_{n\to\infty}\frac{1}{n+1}=5+2(0)=5.$$

As we just saw, the arithmetic rules for convergent sequences can help us evaluate limits of sequences by reducing them to known ones. The next theorem gives us a different way to use the behavior of known sequences to show convergence.

Theorem 1.7. The squeeze theorem. *Suppose that for all $n > N$,*

$$a_n \le b_n \le c_n,$$

and that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a$. Then $\lim_{n\to\infty} b_n = a$.

Proof. Subtracting a from the inequalities, we get

$$a_n - a \le b_n - a \le c_n - a.$$

Let $\varepsilon > 0$ be any tolerance. Since $\{a_n\}$ and $\{c_n\}$ have limit a, there is a number N_1 such that when $n > N_1$, a_n is within ε of a, and there is a number N_2 such that when $n > N_2$, c_n is within ε of a. Let M be the largest of N, N_1, and N_2. Then when $n > M$, we get

$$-\varepsilon < a_n - a \le b_n - a \le c_n - a < \varepsilon.$$

So for the middle term, we see that $|b_n - a| < \varepsilon$. This shows that b_n converges to a. ☐

Example 1.12. Suppose $\frac{1}{2^n} \le a_n \le \frac{1}{n}$ for $n > 2$. Since $\lim_{n\to\infty}\frac{1}{2^n} = \lim_{n\to\infty}\frac{1}{n} = 0$, by the squeeze theorem, $\lim_{n\to\infty} a_n = 0$ as well.

Example 1.13. Suppose $|a_n| \le |b_n|$ and $\lim_{n\to\infty} b_n = 0$. By the squeeze theorem applied to

$$0 \le |a_n| \le |b_n|,$$

we see that $\lim_{n\to\infty}|a_n| = 0$. It is also true that $\lim_{n\to\infty} a_n = 0$, since the distance between a_n and 0 is equal to the distance between $|a_n|$ and 0, which can be made arbitrarily small by taking n large enough.

1.3a Approximation of $\sqrt{2}$

Now let us apply what we have learned to construct a sequence of numbers that converges to the square root of 2. Let us start with an approximation s. How can we find a better one? The product of the numbers s and $\frac{2}{s}$ is 2. It follows that $\sqrt{2}$ lies between these two numbers, for if both were greater than $\sqrt{2}$, their product would be greater than 2, and if both of them were less than $\sqrt{2}$, their product would be less than 2. So a good guess for a better approximation is the arithmetic mean of the two numbers,

$$\text{new approximation} = \frac{s + \frac{2}{s}}{2}.$$

By the A-G inequality, this is greater than the geometric mean of the two numbers,

$$\sqrt{s\left(\frac{2}{s}\right)} < \frac{s + \frac{2}{s}}{2}.$$

This shows that our new approximation is greater than the square root of 2.

We generate a sequence of approximations $s_1, s_2, \ldots$ as follows:

$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right). \tag{1.7}$$

Starting with, say, $s_1 = 2$, we get

$$\begin{aligned}
s_1 &= 2\\
s_2 &= 1.5\\
s_3 &= 1.41666666666666\ldots\\
s_4 &= 1.41421568627451\ldots\\
s_5 &= 1.41421356237469\ldots\\
s_6 &= 1.41421356237309\ldots
\end{aligned}$$

The first twelve digits of s_5 and s_6 are the same. We surmise that they are the first twelve digits of $\sqrt{2}$. Squaring s_5, we get

$$s_5^2 \approx 2.00000000000451,$$

gratifyingly close to 2. So the numerical evidence suggests that the sequence $\{s_n\}$ defined above converges to $\sqrt{2}$. We are going to prove that this is so.

How much does s_{n+1} differ from $\sqrt{2}$?

$$s_{n+1} - \sqrt{2} = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right) - \sqrt{2}.$$

Let us bring the fractions on the right side to a common denominator:

$$s_{n+1} - \sqrt{2} = \frac{1}{2s_n}\left(s_n^2 + 2 - 2s_n\sqrt{2}\right).$$

We recognize the expression in parentheses as a perfect square, $(s_n - \sqrt{2})^2$. So we can rewrite the above equation as

$$s_{n+1} - \sqrt{2} = \frac{1}{2s_n}\left(s_n - \sqrt{2}\right)^2.$$

Next we rewrite the right side giving

$$s_{n+1} - \sqrt{2} = \frac{1}{2}\left(s_n - \sqrt{2}\right)\left(\frac{s_n - \sqrt{2}}{s_n}\right).$$

Since s_n is greater than $\sqrt{2}$, the factor $\left(\frac{s_n - \sqrt{2}}{s_n}\right)$ is less than one. Therefore, dropping it gives the inequality

$$0 < s_{n+1} - \sqrt{2} < \frac{1}{2}\left(s_n - \sqrt{2}\right).$$

Applying this repeatedly gives

$$0 < s_{n+1} - \sqrt{2} < \frac{1}{2^n}\left(s_1 - \sqrt{2}\right).$$

We have shown in the previous example that the sequence $\frac{1}{2^n}$ tends to the limit zero. It follows from Theorem 1.7, the squeeze theorem, that $s_{n+1} - \sqrt{2}$ tends to zero. This concludes the proof that $\lim_{n\to\infty} s_n = \sqrt{2}$.

1.3b Sequences and Series

One of the most useful tools for proving that a sequence converges to a limit is the monotone convergence theorem, which we discuss next.

Definition 1.3. A sequence $\{a_n\}$ is called *increasing* if $a_n \le a_{n+1}$. It is *decreasing* if $a_n \ge a_{n+1}$. The sequence is *monotonic* if it is either increasing or decreasing.

Definition 1.4. A sequence $\{a_n\}$ is called *bounded* if all numbers in the sequence are contained in some interval $[-B,B]$, so that $|a_n| \le B$. Every such number B is a *bound*.

If $a_n < K$ for all n, we say that $\{a_n\}$ is *bounded above* by K. If $K < a_n$ for all n, we say that $\{a_n\}$ is *bounded below* by K.

Example 1.14. The sequence $a_n = (-1)^n$,

$$a_1 = -1,\ a_2 = 1,\ -1,\ 1,\ -1,\ \ldots,$$

is bounded, since $|a_n| = |(-1)^n| = 1$. The sequence is also bounded above by 2 and bounded below by -3.

When showing that a sequence is bounded it is not necessary to find the smallest bound. A larger bound is often easier to verify.

Example 1.15. The sequence $\{5 + \frac{2}{n+1}\}$ is bounded. Since

$$0 \le \frac{2}{n+1} \le 1 \qquad (n = 1,2,3,\ldots),$$

we can see that $\left|5 + \frac{2}{n+1}\right| \le 6$. It is also true that $\left|5 + \frac{2}{n+1}\right| \le 100$.

The next theorem, which we help you prove in Problem 1.35, shows that being bounded is necessary for sequence convergence.

Theorem 1.8. *Every convergent sequence is bounded.*

The next theorem gives a very powerful and fundamental tool for proving sequence convergence.

Theorem 1.9. *An increasing sequence that is bounded converges to a limit.*

Proof. The proof is very similar to the proof of the existence of the least upper bound of a bounded set. We take the case that the sequence consists of positive numbers. For if not, $|a_n| < b$ for some b and the augmented sequence $\{a_n + b\}$ is an increasing sequence that consists of positive numbers. By Theorem 1.6, if the augmented sequence converges to the limit c, the original sequence converges to $c - b$.

Denote by w_n the integer part of a_n. Since the original sequence is increasing, so is the sequence of their integer parts. Since the original sequence is bounded, so are their integer parts. Therefore, w_{n+1} is greater than w_n for only a finite number of n. It follows that all integer parts w_n are equal for all n greater than some number N.

Denote by w the value of w_n for n greater than N. Next we look at the first decimal digit of a_n for n greater than N:

$$a_n = w.d_n \dots .$$

Since the a_n form an increasing sequence, so do the digits d_n. It follows that the digits d_n are all equal for n greater than some number $N(1)$.

Denote this common value of d_n by c_1. Proceeding in this manner, we see that there is a number $N(k)$ such that for n greater than $N(k)$, the integer part and the first k digits of a_n are equal. Let us denote these common digits by $c_1, c_2, \dots, c_k$, and denote by a the number whose integer part is w and whose kth digit is c_k for all k. Then for n greater than $N(k)$, a_n differs from a by less than 10^{-k}; this proves that the sequence $\{a_n\}$ converges to a. □

We claim that a decreasing, bounded sequence $\{b_n\}$ converges to a limit. To see this, define its negative, the sequence $a_n = -b_n$. This is a bounded *increasing* sequence, and therefore converges to a limit a. The sequence b_n then converges to $-a$. Theorem 1.9 and the analogous theorem for decreasing bounded sequences are often expressed as a single theorem:

Theorem 1.10. The monotone convergence theorem. *A bounded monotone sequence converges to a limit.*

Existence of Square Roots. The monotone convergence theorem is another of the workhorses of calculus. To illustrate its power, we show now how to use it to give a proof, different from the one in Sect. 1.2b, that every positive number has a square root. To keep notation to a minimum, we shall construct the square root of the number 2.

Denote as before by s_n the members of the sequence defined by

$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right). \tag{1.8}$$

We have pointed out earlier that for $n > 1$, s_n is greater than $\sqrt{2}$. Therefore, $\dfrac{2}{s_n}$ is less than $\sqrt{2}$, and hence less than s_n. It follows from Eq. (1.8) that

$$s_{n+1} < \frac{s_n + s_n}{2} = s_n.$$

This shows that $\{s_n\}$ is a decreasing sequence of positive numbers. We appeal to the monotone convergence theorem to conclude that the sequence $\{s_n\}$ converges to a limit. Denote this limit by s. We shall show that s is $\sqrt{2}$.

According to Theorem 1.6, the limit of the sequence on the right side of Eq. (1.8) is $\dfrac{1}{2}\left(s + \dfrac{2}{s}\right)$. This is equal to s, the limit of the left side of Eq. (1.8): $s = \dfrac{1}{2}\left(s + \dfrac{2}{s}\right)$. Multiply this equation by $2s$ to obtain $2s^2 = s^2 + 2$. Therefore $s^2 = 2$.

Geometric Sequences and Series. We define geometric sequences as follows.

Definition 1.5. Sequences of numbers that follow the pattern of multiplying by a fixed number to get the next term are called *geometric* sequences, or geometric progressions.

Example 1.16. The geometric sequences $1, 2, 4, 8, \dots, 2^n, \dots,$

$$\frac{1}{3}, -\frac{1}{6}, \frac{1}{12}, -\frac{1}{24}, \dots, \frac{1}{3}\left(-\frac{1}{2}\right)^n, \dots, \quad \text{and} \quad 0.1, 0.01, 0.001, 0.0001, \dots, (0.1)^n, \dots$$

may be abbreviated $\{2^n\}$, $\{\frac{1}{3}\left(-\frac{1}{2}\right)^n\}$, $\{(0.1)^n\}$, $n = 0, 1, 2, \dots$.

Theorem 1.11. Geometric sequence. *The sequence* $\{r^n\}$

(a) converges if $|r| < 1$, *and in this case,* $\lim_{n\to\infty} r^n = 0$,
(b) converges if $r = 1$, *and in this case,* $\lim_{n\to\infty} 1^n = 1$,
(c) diverges for $r > 1$ *and for* $r \le -1$.

Proof. (a) If $0 \le r < 1$, then $\{r^n\}$ is a decreasing sequence that is bounded, $|r^n| \le 1$. Therefore, by the monotone convergence theorem it converges to a limit a. The sequence $r, r^2, r^3, \dots$ has the same limit as $1, r, r^2, r^3, \dots$, and so by Theorem 1.6,

$$a = \lim_{n\to\infty} r^{n+1} = \lim_{n\to\infty} rr^n = r \lim_{n\to\infty} r^n = ra,$$

and $a(r-1) = 0$. Now since $r \ne 1$, $a = 0$.
If $-1 < r < 0$, then each power r^n has the same distance to 0 as $|r|^n$, so again the limit is 0.

(b) For any tolerance ε, $|1^n - 1| = 0 < \varepsilon$, so the limit is clearly 1.

(c) To show that $\{r^n\}$ diverges for $|r| > 1$ or $r = -1$, suppose the limit exists: $\lim_{n\to\infty} r^n = a$. By the argument in part (a), the limit must be 0. But this is not possible. We know that $|r^n - 0| \ge 1$ for all n, i.e., the distance between r^n and 0 is always at least 1, so r^n does not tend to 0. □

Example 1.17. Recall that $n!$ denotes the product of the first n positive integers, $(1)(2)\cdots(n)$. We shall show that for every number b,

$$\lim_{n\to\infty} \frac{b^n}{n!} = 0.$$

Take an integer $N > |b|$, and decompose every integer n greater than N as $n = N + k$. Then $\frac{b^n}{n!} = \frac{b^N}{N!}\frac{b}{N+1}\cdots\frac{b}{N+k}$. The first factor $\frac{b^N}{N!}$ is a fixed number, and the other k factors each have absolute value less than $r = \dfrac{|b|}{N+1} < 1$. Since r^k tends to 0, the limit is 0 by the squeeze argument used in Example 1.12.

Definition 1.6. The numbers of a sequence $\{a_n\}$ can be added to make a new sequence $\{s_n\}$:

$$
\begin{aligned}
s_1 &= a_1 = \sum_{j=1}^{1} a_j \\
s_2 &= a_1 + a_2 = \sum_{j=1}^{2} a_j \\
&\dots \\
s_n &= a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j \\
&\dots
\end{aligned}
$$

called the sequence of *partial sums* of the *series* $\sum_{n=1}^{\infty} a_n$. If the limit

$$\sum_{j=1}^{\infty} a_j = a_1 + a_2 + a_3 + \cdots = \lim_{n\to\infty} s_n$$

exists, the series *converges*. Otherwise, it *diverges*. The numbers a_n are called the *terms* of the series.

Example 1.18. Take all $a_j = 1$, which gives the series $\sum_{j=1}^{\infty} 1$. The nth partial sum is $s_n = 1 + \cdots + 1 = n$. Since the s_n are not bounded, the series diverges by Theorem 1.8.

Example 1.18 suggests the following necessary condition for convergence.

Theorem 1.12. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let $s_n = a_1 + \cdots + a_n$. Since $\sum_{n=1}^{\infty} a_n$ converges, the limit $\lim_{n\to\infty} s_n = L$ exists, and for the shifted sequence $\lim_{n\to\infty} s_{n-1} = L$ as well. According to Theorem 1.6,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = L - L = 0.$$

□

Example 1.19. The series $\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)$ diverges, because $\lim_{n\to\infty}\frac{n}{2n+1}=\frac{1}{2}$.

The following is one of the best-known and most beloved series.

Theorem 1.13. Geometric series. *If* $|x|<1$*, the sequence of partial sums*

$$s_n = 1+x+x^2+x^3+\cdots+x^n.$$

converges, and

$$\lim_{n\to\infty} s_n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad (|x|<1). \tag{1.9}$$

If $|x|\geq 1$*, the series diverges.*

Proof. By algebra, we see that $s_n(1-x)=(1+x+x^2+x^3+\cdots+x^n)(1-x)=1-x^{n+1}$. Therefore,

$$s_n = \frac{1-x^{n+1}}{1-x}, \qquad (x\neq 1). \tag{1.10}$$

According to Theorem 1.11, for $|x|<1$ we have $\lim_{n\to\infty} x^n = 0$ and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(1+x+x^2+x^3+\cdots+x^n) = \lim_{n\to\infty}\frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}.$$

If $|x|\geq 1$, then x^n does not approach zero, so according to Theorem 1.12, the series diverges. □

Comparing Series. Next we show how to use monotone convergence and the arithmetic properties of sequences to determine convergence of some series. Consider the series

$$(a)\ \sum_{n=0}^{\infty}\frac{1}{2^n+1}, \qquad (b)\ \sum_{n=1}^{\infty}\frac{1}{2^n-1}.$$

For series (a), the numbers $\frac{1}{2^n+1}$ are positive, so the partial sums $\sum_{n=0}^{m}\frac{1}{2^n+1}$ form an increasing sequence. Since

$$\frac{1}{2^n+1} < \left(\frac{1}{2}\right)^n,$$

the partial sums satisfy $\sum_{n=0}^{m} \frac{1}{2^n+1} < \sum_{n=0}^{m} \left(\frac{1}{2}\right)^n$. Since $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ converges, its sequence of partial sums is bounded,

$$0 \leq \sum_{n=0}^{m} \frac{1}{2^n+1} < \sum_{n=0}^{m} \left(\frac{1}{2}\right)^n \leq 2.$$

By the monotone convergence theorem, the sequence of partial sums of $\sum_{n=0}^{\infty} \frac{1}{2^n+1}$ converges, so the series converges.

For series (b), the numbers $\frac{1}{2^n-1}$ are positive, so the partial sums $\sum_{n=1}^{m} \frac{1}{2^n-1}$ form an increasing sequence. Note that $\frac{1}{2^n-1}$ is not less than $\frac{1}{2^n}$ (it is slightly greater), so we cannot set up a comparison as we did for series (a). We look instead at the limit of the ratio of the terms,

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = 1.$$

Since the limit of the ratio is 1, the ratios eventually all get close to 1. So for every $R > 1$, there is a sufficiently large N such that

$$\frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} < R$$

for all $n > N$. Therefore, $\frac{1}{2^n-1} < R\left(\frac{1}{2}\right)^n$ for all $n > N$. Since the partial sums of the series $R \sum_{n=N+1}^{\infty} \left(\frac{1}{2}\right)^n$ are bounded, so are the partial sums of $\sum_{n=N+1}^{\infty} \frac{1}{2^n-1}$. So this series converges. But then so does $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$.

The arguments used for series (a) and (b) can be used to obtain the next two comparison theorems, which we ask you to prove in Problem 1.45.

Theorem 1.14. Comparison theorem. *Suppose that for all n,*

$$0 \leq a_n \leq b_n.$$

If $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Theorem 1.15. Limit comparison theorem. *Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be series of positive terms. If* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and is a positive number, then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $\sum_{n=1}^{\infty} b_n$ *converges.*

The comparison theorems are stated for terms that are positive or not negative. The next theorem is a handy result that sometimes allows us to use these theorems to deduce convergence of series with negative terms.

Theorem 1.16. *If* $\sum_{j=1}^{\infty} |a_j|$ *converges, then* $\sum_{j=1}^{\infty} a_j$ *also converges.*

Proof. Since $0 \le a_n + |a_n|$, the partial sums $s_m = \sum_{j=1}^{m} (a_n + |a_n|)$ are increasing. Since $a_n + |a_n| \le 2|a_n|$, s_m is less than the mth partial sum of $\sum_{j=1}^{\infty} 2|a_j|$, which converges. The sequence of partial sums s_m is increasing and bounded. Therefore, $\sum_{j=1}^{\infty} (a_n + |a_n|)$ converges. Since $a_j = (a_j + |a_j|) - |a_j|$, $\sum_{j=1}^{\infty} a_j$ converges by Theorem 1.6. □

Example 1.20. The series $\sum_{n=1}^{\infty} \frac{1}{(-2)^n n}$ does not have positive terms, however $\sum_{n=1}^{\infty} \left|\frac{1}{(-2)^n n}\right| = \sum_{n=1}^{\infty} \frac{1}{2^n n}$ does. Since $\frac{1}{2^n n} < \left(\frac{1}{2}\right)^n$, the series $\sum_{n=1}^{\infty} \left|\frac{1}{(-2)^n n}\right|$ converges by comparison with the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. According to Theorem 1.16, $\sum_{n=1}^{\infty} \frac{1}{(-2)^n n}$ converges.

The next two examples show that the converse of Theorem 1.16 is not true. That is, a series $\sum_{n=1}^{\infty} a_n$ may converge while the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example 1.21. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$. This is known as the "harmonic series." Its sequence of partial sums is

$$s_1 = 1, \quad s_2 = s_1 + \frac{1}{2} = 1 + \frac{1}{2}, \quad s_3 = s_2 + \frac{1}{3} = 1 + \frac{1}{2} + \frac{1}{3}, \ldots.$$

By grouping the terms, we can see that

$$\begin{aligned} s_4 &= \quad s_2 + \tfrac{1}{3} + \tfrac{1}{4} &&> 1.5 + \tfrac{2}{4} &&= 2 \\ s_8 &= s_4 + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} &&> 2 + \tfrac{4}{8} &&= 2.5 \\ s_{16} &= \quad s_8 + \tfrac{1}{9} + \cdots + \tfrac{1}{16} &&> 2.5 + \tfrac{8}{16} &&= 3 \end{aligned}$$

and so forth. So s_n is an increasing sequence that is not bounded.

The harmonic series diverges. It is easy to see that the difference between its successive partial sums, $s_{n+1} - s_n = \frac{1}{n+1}$, can be made as small as we like by taking n large. However, this is *not* enough to ensure convergence of the series. We will revisit the harmonic series when we study improper integrals. The harmonic series is a good example of a series where the terms of the series $a_1, a_2, \ldots, a_n, \ldots$ decrease to zero and the series $\sum_{n=1}^{\infty} a_n$ diverges.

Now let us alternate the signs of the terms, to see that the resulting series does converge. Consider the series

$$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{1}{n}. \tag{1.11}$$

The sequence of even partial sums $s_2, s_4, s_6, \ldots, s_{2k}, \ldots$ is an increasing sequence, since $s_{2k+2} - s_{2k} = \dfrac{1}{2k+1} - \dfrac{1}{2k+2} > 0$. It is bounded above by

$$1 > s_{2k} = 1 + \left(-\frac{1}{2} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{2k-2} + \frac{1}{2k-1}\right) - \frac{1}{2k}.$$

The sequence of odd partial sums $s_1, s_3, s_5, \ldots, s_{2k+1}, \ldots$ is a decreasing sequence, since $s_{2(k+1)+1} - s_{2k+1} = -\dfrac{1}{2k+2} + \dfrac{1}{2k+2+1} < 0$. It is bounded below by

$$\frac{1}{2} < s_{2k+1} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \frac{1}{2k+1}.$$

By the monotone convergence theorem, both $\{s_{2k}\}$ and $\{s_{2k+1}\}$ converge. Define $\lim_{k\to\infty} s_{2k} = L_1$ and $\lim_{k\to\infty} s_{2k+1} = L_2$. Then

$$L_2 - L_1 = \lim_{k\to\infty} (s_{2k+1} - s_{2k}) = \lim_{k\to\infty} \frac{1}{2k+1} = 0.$$

Therefore, s_k converges to $L_1 = L_2$, and $\sum_{n=1}^{\infty} (-1)^{(n-1)} \dfrac{1}{n}$ converges.

The same argument can be used to obtain the following more general result. We guide you through the steps in Problem 1.46.

Theorem 1.17. Alternating series theorem. *If*

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots \geq 0$$

and $\lim_{n\to\infty} a_n = 0$*, then* $\sum_{n=1}^{\infty}(-1)^n a_n$ *converges.*

Definition 1.7. If $\sum_{i=1}^{\infty} a_i$ converges and $\sum_{i=1}^{\infty} |a_i|$ diverges, we say that $\sum_{i=1}^{\infty} a_i$ converges *conditionally*. If $\sum_{i=1}^{\infty} |a_i|$ converges, we say that $\sum_{i=1}^{\infty} a_i$ converges *absolutely*.

Example 1.22. In the convergent geometric series

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$$

replace any number of plus signs by minus signs. According to Definition 1.7, the resulting series converges absolutely.

Example 1.23. We showed that $\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n}$ converges, and in Example 1.21, we showed that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Therefore, $\sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n}$ converges conditionally.

Example 1.24. Because $\frac{1}{\sqrt{n}}$ decreases to 0, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$ converges by the alternating series theorem, Theorem 1.17. Since $\sqrt{n} \leq n$, $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$, and the partial sums are given by

$$s_m = \sum_{n=1}^{m}\frac{1}{\sqrt{n}} \geq \sum_{n=1}^{m}\frac{1}{n}.$$

By Example 1.21, the partial sums of the harmonic series are not bounded. Therefore, the s_m are not bounded either; this shows that $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges. Thus $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}}$ converges conditionally.

Further Comparisons. Consider the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}. \tag{1.12}$$

Because the terms $a_n = \frac{n}{2^n}$ are positive, the sequence of partial sums is increasing. If the sequence of partial sums is bounded, then the series converges. Let us look at a few partial sums:

$$\begin{aligned} s_1 &= 0.5 \\ s_2 &= 1 \\ s_3 &= 1.375 \\ s_4 &= 1.625. \end{aligned}$$

We clearly need better information than this. Trying a limit comparison with $\sum_{n=1}^{\infty} \frac{1}{2^n}$ yields

$$\lim_{n\to\infty} \frac{\frac{n}{2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} n,$$

which does not exist, and so such a comparison is not helpful. We notice that when n is large, the terms $\frac{n}{2^n}$ grow by roughly a factor of $\frac{1}{2}$, i.e.,

$$\lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{n}\frac{1}{2} = \frac{1}{2}.$$

This suggests comparing the series with a geometric series. Let r be any number greater than $\frac{1}{2}$ and less than 1. Since $\frac{a_{n+1}}{a_n}$ tends to $\frac{1}{2}$, there is some N such that for $n > N$,

$$\frac{a_{n+1}}{a_n} < r.$$

Multiply by a_n to get $a_{n+1} < ra_n$. Repeating the process gives

$$a_{N+k} < ra_{N+k-1} < \cdots < r^k a_N.$$

Since the geometric series $\sum_{k=0}^{\infty} r^k$ converges, the partial sums of the series $\sum_{k=0}^{\infty} a_{N+k}$ are bounded,

$$\sum_{k=0}^{m} a_{N+k} \le a_N \sum_{k=0}^{m} r^k.$$

Therefore, the partial sums of our series $\sum_{n=1}^{\infty} a_n$ are bounded, and it converges. The idea behind this example leads to the next theorem.

Theorem 1.18. Ratio test. *Suppose that* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$. *Then*

(a) If $L < 1$, *the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

(b) If $L > 1$, *the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

The case $L = 1$ *gives no information.*

In Problem 1.48, we ask you to prove the theorem by extending the argument we used for $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Example 1.25. Let us determine whether these series converge:

$$\sum_{n=1}^{\infty} \frac{n^5}{2^n}, \qquad \sum_{n=1}^{\infty} \frac{(-2)^n}{n!}, \qquad \sum_{n=1}^{\infty} \frac{2^n}{n^2}.$$

(a) $\lim_{n\to\infty} \frac{\frac{(n+1)^5}{2^{n+1}}}{\frac{n^5}{2^n}} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^5 = \frac{1}{2}$, so $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$ converges by the ratio test.

(b) $\lim_{n\to\infty} \left|\frac{\frac{(-2)^{n+1}}{(n+1)!}}{\frac{(-2)^n}{n!}}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0$, so $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ converges absolutely by the ratio test.

(c) $\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^2 = 2$, so $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges by the ratio test.

Example 1.26. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, as we know from Example 1.21. This is a case in which the ratio test would have offered no information, since

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}$$

tends to 1.

We end this section with two very fundamental properties, the nested interval property and Cauchy's criterion for sequence convergence.

1.3c Nested Intervals

We can use the monotone convergence theorem to prove the *nested interval property* of numbers:

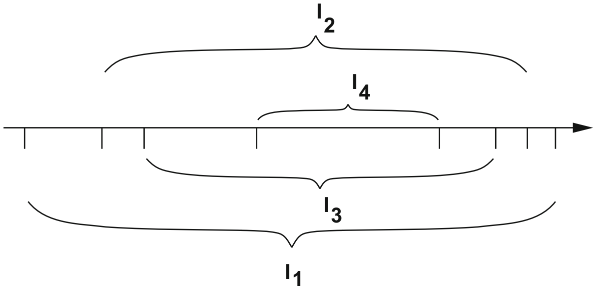


Fig. 1.17 Nested intervals

Theorem 1.19. Nested interval theorem. *If $I_1, I_2, I_3, \dots$ is a sequence of closed intervals that are "nested," that is, each I_n contains I_{n+1} (Fig. 1.17), then the intervals I_n have at least one point in common. If the lengths of the intervals tend to 0, there is exactly one point in common.*

Proof. Denote by a_n and b_n the left and right endpoints of I_n. The assumption of nesting means that the sequence $\{a_n\}$ increases, the sequence $\{b_n\}$ decreases, and that each b_n is greater than every a_n. So by the monotone convergence theorem, $\{a_n\}$ converges to some limit a, and $\{b_n\}$ to some limit b. By the way we constructed the limits, we know that $a_n \le a$ and $b \le b_n$. Now, a cannot be greater than b, for otherwise, some a_n would be larger than some b_m, violating the nesting assumption. So a must be less than or equal to b, and $a_n \le a \le b \le b_n$. If the lengths of the intervals I_n tend to zero as n tends to infinity, then the distance between a and b must be zero, and the intervals I_n have exactly one point in common, namely the point $a = b$. If the lengths of the I_n do not tend to zero, then $[a,b]$ belongs to all the intervals I_n, and the intervals have many points in common. □

The AGM of Two Numbers. In Sect. 1.1c, we saw that given two numbers $0 < g < a$, their arithmetic mean is $\frac{a+g}{2} = a_1$, their geometric mean is $\sqrt{ag} = g_1$, and $g \le g_1 \le a_1 \le a$. Repeat the process of taking means, letting $\frac{a_1+g_1}{2} = a_2$ and $\sqrt{a_1 g_1} = g_2$. Then

$$g \le g_1 \le g_2 \le a_2 \le a_1 \le a.$$

Continuing in this manner, we obtain a sequence of nested closed intervals, $[g_n, a_n]$.

Let us look at what happens to the width of the interval between g and a and how that compares to the width of the interval between $\sqrt{ag} = g_1$ and $\frac{a+g}{2} = a_1$. Using a little algebraic manipulation, we see that

$$\frac{a+g}{2} - \sqrt{ag} = \frac{a-g}{2} + g - \sqrt{ag} \leq \frac{a-g}{2},$$

since $g \leq \sqrt{ag}$. Thus

$$a_1 - g_1 \leq \frac{1}{2}(a-g).$$

This means that the interval between the arithmetic and geometric means of a and g is less than or equal to half the length of the interval between a and g. This reduces the width of the intervals $[g_n, a_n]$ by at least half at each step, and so the lengths of the intervals tend to 0.

The nested interval theorem says that this process squeezes in on exactly one number. This number is called the *arithmetic–geometric mean* of a and g and is denoted by $\mathrm{AGM}(a,g)$:

$$\mathrm{AGM}(a,g) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} g_n.$$

The AGM may seem like a mathematical curiosity. Gauss invented the AGM and used it to give a very fast algorithm for computing π.

1.3d Cauchy Sequences

Sometimes the terms of a sequence appear to cluster tightly about a point on the number line, but we do not know the specific number they seem to be approaching. We present now a very general and very useful criterion for such a sequence to have a limit.

Definition 1.8. Cauchy's criterion. A sequence of numbers $\{a_n\}$ is called a *Cauchy sequence* if given any tolerance ε, no matter how small, there is an integer N such that for all integers n and m greater than N, a_n and a_m differ from each other by less than ε.

Examples of Cauchy sequences abound; you will see in Problem 1.52 that every convergent sequence is a Cauchy sequence. The next example shows how to verify directly that the sequence $\{\frac{1}{n}\}$ is a Cauchy sequence.

Example 1.27. Let ε be any tolerance and let N be a whole number greater than $\frac{1}{\varepsilon}$. Let $m > N$ and $n > N$. By the triangle inequality,

$$\left|\frac{1}{m}-\frac{1}{n}\right| \leq \frac{1}{m}+\frac{1}{n} \leq \frac{1}{N}+\frac{1}{N} < 2\varepsilon.$$

This can be made as small as desired. In fact, we can achieve the tolerance

$$\left|\frac{1}{m}-\frac{1}{n}\right| < \varepsilon$$

if we take n and m greater than $\frac{2}{\varepsilon}$. So $\{\frac{1}{n}\}$ is a Cauchy sequence.

Theorem 1.20. *Every Cauchy sequence converges.*

The proof has four steps. First, we show that *every* sequence has a monotone subsequence. This first step is worth recognizing as a "lemma," a key stepping-stone in the argument. Second, we show that every Cauchy sequence is bounded (and hence every subsequence as well). Third, we recognize that a Cauchy sequence has a monotone subsequence that is bounded and therefore converges. Fourth, we show that a Cauchy sequence converges to the same limit as its monotone subsequence.

Lemma 1.1. *Every infinite sequence of numbers has an infinite monotonic subsequence.*

Proof (of the Lemma). Let $a_1, a_2, \ldots$ be any sequence. We shall show that it contains either an increasing or a decreasing subsequence. We start by trying to construct an increasing subsequence. Start with a_1 and take any subsequent term in the original sequence that is greater than or equal to a_1 as the next term of the subsequence. Continue in this fashion. If we can continue indefinitely, we have the desired increasing subsequence. Suppose, on the other hand, that we get stuck after a finite number of steps at a_j because $a_n < a_j$ for all $n > j$. Then we try again to construct an increasing sequence, starting this time with a_{j+1}. If we can continue ad infinitum, we have an increasing subsequence. If, on the other hand, we get stuck at a_k because $a_n < a_k$ for all $n > k$, then we can again try to construct an increasing subsequence starting at a_{k+1}. Proceeding in this fashion, we either have success at some point, or an infinite sequence of failures. In the second case, the sequence of points $a_j, a_k, \ldots$ where the failures occurred constitutes a decreasing subsequence. This completes the proof of the lemma. □

Proof (of the Theorem). The lemma we just proved guarantees that a Cauchy sequence has a monotone subsequence. Next, we show that a Cauchy sequence is bounded. Being Cauchy ensures that there exists an N such that the terms from a_N onward are all within 1 of each other. This means that the largest of the numbers

$$1+a_1,\ 1+a_2,\ \ldots,\ 1+a_N$$

is an upper bound for $\{a_n\}$, and the smallest of the numbers

$$-1+a_1,\ -1+a_2,\ \ldots,\ -1+a_N$$

is a lower bound of $\{a_n\}$. Now by the monotone convergence theorem, the monotone subsequence of $\{a_n\}$ converges to a limit a.

Next we show that not only a subsequence but the whole sequence converges to a. Let us write a_m for an element of the subsequence and a_n for any element of the sequence, and $a-a_n = a-a_m+a_m-a_n$. The triangle inequality gives

$$|a-a_n| \le |a-a_m| + |a_m-a_n|.$$

The first term on the right is less than any prescribed ε for m large, because the subsequence converges to a. The second is less than any prescribed ε for m and n both large, because this is a Cauchy sequence. This proves that the whole sequence also converges to a. □

Cauchy's criterion for convergence, that given any tolerance there is a place in the sequence beyond which *all* the terms are within that tolerance of each other, is a stronger requirement than just requiring that the difference between one term and the next tend to 0. For example, we saw in Example 1.21 that $s_{n+1}-s_n$ tends to 0 but s_n does not converge.

Problems

1.25. Find all the definitions in this section and copy them onto your paper. Illustrate each one with an example.

1.26. Find all the theorems in this section and copy them onto your paper. Illustrate each one with an example.

1.27. Find the first four approximations s_1,s_2,s_3,s_4 to $\sqrt{3}$ using $s_1 = 1$ as a first approximation and iterating

$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{3}{s_n}\right).$$

What happens if you use $s_1 = 2$ instead to start?

1.28. In approximating $\sqrt{2}$, we used the fact that if $w_{n+1} < \frac{1}{2}w_n$ holds for each n, then

$$w_{n+1} < \frac{1}{2^n}w_1.$$

Explain why this is true.

1.29. We have said that if s is larger than $\sqrt{2}$, then $\frac{2}{s}$ is smaller than $\sqrt{2}$. Show that this is true.

1.30. If a number s is larger than the cube root $\sqrt[3]{2}$, is it true that $\frac{2}{s^2}$ is smaller?

1.31. Show that if $2 < s^2 < 2+p$, then $\sqrt{2} < s < \sqrt{2}+q$, where $q = \frac{p}{2^{1.5}}$.

1.32. Consider the sequences $a_n = -3n+1$ and $b_n = 3n + \frac{2}{n}$. If we carelessly try to write $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$, what does it seem to say? What goes wrong in this example?

1.33. Justify the following steps in the proof of parts of Theorem 1.6. Suppose that $\{a_n\}$ and $\{b_n\}$ are convergent sequences,

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b.$$

(a) We want to prove that the sequence of sums $\{a_n + b_n\}$ converges and that $\lim_{n\to\infty}(a_n+b_n) = a+b$. Let $\varepsilon > 0$ be any tolerance. Show that:

(i) There is a number N_1 such that for all $n > N_1$, a_n is within ε of a, and there is a number N_2 such that for all $n > N_2$, b_n is within ε of b. Set N to be the larger of the two numbers N_1 and N_2. Then for $n > N$, $|a_n - a| < \varepsilon$ and $|b_n - b| < \varepsilon$.

(ii) For every n,

$$|(a_n+b_n)-(a+b)| \le |a_n - a| + |b_n - b|.$$

(iii) For all $n > N$, $a_n + b_n$ is within 2ε of $a+b$.

(iv) We have demonstrated that for $n > N$, $|(a_n+b_n)-(a+b)| \le 2\varepsilon$. Explain why this completes the proof.

(b) We want to prove that if a is not 0, then all but a finite number of the a_n differ from 0 and

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}.$$

Let $\varepsilon > 0$ be any tolerance. Show that:

(i) There is a number N such that when $n > N$, a_n is within ε of a.

(ii) There is a number M such that for $n > M$, $a_n \ne 0$ and $\left|\frac{1}{a_n}\right|$ is bounded by some α.

(iii) For n larger than both M and N,

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a-a_n}{a_n a}\right| = |a-a_n|\left|\frac{1}{a_n}\right|\frac{1}{|a|} < \varepsilon\frac{\alpha}{|a|}.$$

Hence $\frac{1}{a_n}$ converges to $\frac{1}{a}$.

1.34. Solve $x^2 - x - 1 = 0$ as follows. Restate the equation as $x = 1 + \frac{1}{x}$, which suggests the sequence of approximations

$$x_0 = 1, \quad x_1 = 1 + \frac{1}{x_0}, \quad x_2 = 1 + \frac{1}{x_1}, \quad , \ldots .$$

Explain the following items to prove that the sequence converges to a solution.

(a) $x_0 < x_2 < x_1$
(b) $x_0 < x_2 < x_4 < \cdots < x_5 < x_3 < x_1$
(c) The even sequence x_{2k} increases to a limit L, and the odd sequence x_{2k+1} decreases to a limit $R \geq L$.
(d) The distances $(x_{2k+3} - x_{2k+2})$ satisfy $(x_{2k+3} - x_{2k+2}) < \frac{1}{x_2^4}(x_{2k+1} - x_{2k})$.
(e) $R = L = \lim_{k\to\infty} x_k$ is a solution to $x^2 - x - 1 = 0$.

1.35. Suppose a sequence $\{a_n\}$ converges to a. Explain each of the following items, which prove Theorem 1.8.

(a) There is a number N such that for all $n > N$, $|a_n - a| < 1$.
(b) For all $n > N$, $|a_n| \leq |a| + 1$. *Hint*: $a_n = a + (a_n - a)$.
(c) Let α be the largest of the numbers

$$|a_1|, \ |a_2|, \ \ldots |a_N|, \ |a| + 1.$$

Then $|a_k| \leq \alpha$ for $k = 1, 2, 3, \ldots$.
(d) $\{a_n\}$ is bounded.

1.36. This problem explores the sum notation. Write out each finite sum.

(a) $\sum_{n=1}^{5} a_n$
(b) $\sum_{k=2}^{4} \frac{3}{k}$
(c) $\sum_{j=2}^{6} b_{j-1}$
(d) Rewrite the expression $t^2 + 2t^3 + 3t^4$ in the sum notation.
(e) Explain why $\sum_{n=1}^{10} n^2 = 105 + \sum_{n=3}^{9} n^2$, and why $\sum_{n=2}^{20} a_n = \sum_{k=0}^{18} a_{k+2}$.

1.37. Partial sums $s_1 = a_1, \ s_2 = a_1 + a_2$, and so forth are known to be given by $s_n = \frac{n}{n+2}$. Find $a_1, \ a_2$, and $\sum_{n=1}^{\infty} a_n$.

1.38. Use relation (1.10) to evaluate the sum $\sum_{k=0}^{n} \frac{1}{7^k}$.

1.39. Find the limit as n tends to infinity of $\frac{5}{7}+\frac{25}{49}+\frac{125}{343}+\cdots+\frac{5^n}{7^n}$.

1.40. Find the limit as n tends to infinity of $\frac{5}{7}+\frac{5}{49}+\frac{5}{343}+\cdots+\frac{5}{7^n}$.

1.41. Suppose the ratio test indicates that $\sum_{n=0}^{\infty} a_n$ converges. Use the ratio test to show that $\sum_{n=0}^{\infty} na_n$ also converges. What can you say about $\sum_{n=0}^{\infty}(-1)^n n^5 a_n$?

1.42. Why does the series $\sum_{n=1}^{\infty}\frac{n^2}{n^2+1}$ diverge?

1.43. Show that the infinite series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by verifying the following steps:

(a) $\frac{1}{n^2}<\frac{1}{n(n-1)}$ (b) $\frac{1}{n(n-1)}=\frac{1}{n-1}-\frac{1}{n}$ (c) $\sum_{n=2}^{k}\frac{1}{n(n-1)}=1-\frac{1}{k}$

1.44. For what numbers t does the sequence

$$s_n = 1-2t+2^2t^2-2^3t^3+\cdots+(-2)^nt^n$$

converge? What is the limit for those t?

1.45. Carry out the following steps to prove the comparison theorems, Theorems 1.14 and 1.15.

(a) Let $\{a_n\}$ and $\{b_n\}$ be sequences for which $0 \le b_n \le a_n$. Use the monotone convergence theorem to show that if $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} b_n$ also converges.

(b) Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers for which $\lim_{n\to\infty}\frac{a_n}{b_n}$ exists and is a positive number, say L. First show that for n sufficiently large, $a_n \le (L+1)b_n$. Then explain why the convergence of $\sum_{n=N}^{\infty}(L+1)b_n$ implies that of $\sum_{n=0}^{\infty} a_n$.

1.46. Let $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots \ge 0$ be a sequence with $\lim_{n\to\infty} a_n = 0$. Let

$$s_n = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1}a_n.$$

(a) Explain why $a_{2k+1}-a_{2k+2} \ge 0$, and why $-a_{2k}+a_{2k+1} \le 0$.
(b) Explain why $s_2, s_4, s_6, \ldots, s_{2k}, \ldots$ converges, and why $s_1, s_3, s_5, \ldots, s_{2k+1}, \ldots$ converges.

(c) Show that $\lim_{k\to\infty}(s_{2k+1} - s_{2k}) = 0$.

(d) Explain why $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

1.47. Determine which of the following series converge absolutely, which converge, and which diverge.

(a) $\sum_{n=0}^{\infty}\frac{(-2)^n+1}{3^n}$

(b) $\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n}}$

(c) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[4]{n}}$

(d) $\sum_{n=0}^{\infty}\frac{n}{\sqrt{n^2+1}}$

(e) $\sum_{n=0}^{\infty}\frac{n}{\sqrt{n^4+1}}$

(f) $\sum_{n=1}^{\infty}\frac{n^2}{(1.5)^n}$

1.48. We used series (1.12) to motivate the ratio test, Theorem 1.18. Extend the argument used in the example to create a proof of the theorem.

1.49. Determine which of the following series converge absolutely, which converge, and which diverge.

(a) $1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}+\frac{1}{128}-\frac{1}{256}+\cdots$

(b) $\sum_{n=1}^{\infty}10^{-n^2}$

(c) $\sum_{n=1}^{\infty}\frac{b^n}{n!}$ Are there any restrictions on b?

(d) $\sum_{n=0}^{\infty}\frac{n^{1/n}}{3^n}$ *Hint*: See Problem 1.13.

(e) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{\sqrt{n}}+\frac{1}{2^n}\right)$

(f) $\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{\sqrt{n}}+\frac{1}{2}\right)$

1.50. Suppose $\sum_{n=0}^{\infty}a_n^2$ and $\sum_{n=0}^{\infty}b_n^2$ both converge. Use the Cauchy–Schwarz inequality (see Problem 1.18) to explain the following.

(a) The partial sums $\sum_{n=0}^{k}a_nb_n$ satisfy $\sum_{n=0}^{k}a_nb_n \le \sqrt{\sum_{n=0}^{\infty}a_n^2}\sqrt{\sum_{n=0}^{\infty}b_n^2}$.

(b) If the numbers a_n and b_n are nonnegative, then $\sum_{n=0}^{\infty} a_n b_n$ converges.

(c) $\sum_{n=0}^{\infty} a_n b_n$ converges absolutely.

1.51. Let us set a_n equal to the n-place decimal expansion of some number x. For example, if $x = \sqrt{2}$, then we have $a_1 = 1.4$, $a_2 = 1.41$, $a_3 = 1.414$, etc. Is a_n a Cauchy sequence?

1.52. Prove that every convergent sequence is a Cauchy sequence.

1.4 The Number e

In your study of geometry you have come across the curious number denoted by the Greek letter π. The first six digits of π are

$$\pi = 3.14159\ldots.$$

In this section, we shall define another number of great importance, denoted by the letter e. This number is called Euler's constant in honor of the great mathematician who introduced it; its first six digits are

$$\mathrm{e} = 2.71828\ldots.$$

The number e is defined as the limit of the sequence

$$e_n = \left(1 + \frac{1}{n}\right)^n$$

as n tends to infinity. To make this a legitimate definition, we have to prove that this sequence has a limit.

Financial Motivation. Before we turn to the proof, let us first give a financial motivation for considering this limit. Suppose you invest one dollar at the interest rate of 100% per year. If the interest is compounded annually, a year later you will receive two dollars, that is, the original dollar invested plus another dollar for interest. If interest is compounded semiannually, you will receive at the end of the year $(1.5)^2 = 2.25$ dollars. That is 50% interest after six months, giving you a value of \$1.50, followed by another six months during which you earn 50% interest on your \$1.50. If interest is compounded n times a year, you will receive at the end of the year $\left(1 + \frac{1}{n}\right)^n$ dollars. The more frequently interest is compounded, the higher your return. This suggests the importance of the number e and indicates that the sequence $\left(1 + \frac{1}{n}\right)^n$ is increasing. Later, we shall show how e can be used to study arbitrary interest rates.

Monotonicity of e_n. Let us do a few numerical experiments before trying to prove anything about the sequence $\{e_n\}$. Using a calculator, we calculate the first ten terms of the sequence rounded down to three digits:

$$\begin{aligned} e_1 &= 2.000 \\ e_2 &= 2.250 \\ e_3 &= 2.370 \\ e_4 &= 2.441 \\ e_5 &= 2.488 \\ e_6 &= 2.521 \\ e_7 &= 2.546 \\ e_8 &= 2.565 \\ e_9 &= 2.581 \\ e_{10} &= 2.593 \end{aligned}$$

We notice immediately that this sequence of ten numbers is in increasing order. Just to check, let us do a few further calculations:

$$\begin{aligned} e_{100} &= 2.704 \\ e_{1000} &= 2.716 \\ e_{10000} &= 2.718 \end{aligned}$$

These numbers confirm our financial intuition that more frequent compounding results in a larger annual return, and that $\left(1+\frac{1}{n}\right)^n$ is an increasing sequence.

We shall now give a nonfinancial argument that the sequence e_n increases. We use the A-G inequality for $n+1$ numbers:

$$\left(a_1 a_2 \cdots a_{n+1}\right)^{1/(n+1)} \le \frac{1}{n+1}\left(a_1 + a_2 + \cdots + a_{n+1}\right).$$

Take the $n+1$ numbers

$$\underbrace{\left(1+\frac{1}{n}\right), \ldots, \left(1+\frac{1}{n}\right)}_{n \text{ times}}, \ 1.$$

Their product is $\left(1+\dfrac{1}{n}\right)^n$, and their sum is $n\left(1+\dfrac{1}{n}\right)+1 = n+2$. So their geometric mean is $\left(1+\dfrac{1}{n}\right)^{n/(n+1)}$, and their arithmetic mean is $\dfrac{n+2}{n+1} = 1+\dfrac{1}{n+1}$. According to the A-G inequality,

$$\left(1+\frac{1}{n}\right)^{n/(n+1)} < 1+\frac{1}{n+1};$$

raising both sides to the power $n+1$ gives

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1},$$

proving that e_n is less than e_{n+1}. Therefore, the sequence $\{e_n\}$ is increasing.

Boundedness of e_n. To conclude that the sequence converges, we have to show that it is bounded. To accomplish this, we look at another sequence, $\{f_n\}$, defined as

$$f_n = \left(1+\frac{1}{n}\right)^{n+1}.$$

Using a calculator, we calculate the first ten terms of this sequence, rounded up to three digits:

$$\begin{aligned} f_1 &= 4.000\\ f_2 &= 3.375\\ f_3 &= 3.161\\ f_4 &= 3.052\\ f_5 &= 2.986\\ f_6 &= 2.942\\ f_7 &= 2.911\\ f_8 &= 2.887\\ f_9 &= 2.868\\ f_{10} &= 2.854 \end{aligned}$$

These ten numbers are decreasing, suggesting that the whole infinite sequence f_n is decreasing. Further test calculations offer more evidence:

$$\begin{aligned} f_{100} &= 2.732\\ f_{1000} &= 2.720\\ f_{10000} &= 2.719 \end{aligned}$$

Here is an intuitive demonstration that f_n is a decreasing sequence: Suppose you borrow \$1 from your family at no interest. If you return all that you owe a year later, you have nothing left. But if you return half of what you owe twice a year, you have left $(0.5)^2 = 0.25$. If you return a third of what you owe three times a year, at the end of the year you have left $(2/3)^3$. If you return $(1/n)-$th of what you owe n times a year, you end up with $\left(1-\frac{1}{n}\right)^n$ at the end of the year. This is an intuitive demonstration that the sequence $\{\left(1-\frac{1}{n}\right)^n\}$ is increasing. It follows that the sequence of reciprocals is decreasing:

$$\frac{1}{\left(1-\frac{1}{n}\right)^n} = \left(\frac{n}{n-1}\right)^n = \left(1+\frac{1}{m}\right)^{m+1}, \quad \text{where } m = n-1.$$

In Problem 1.54, we guide you through a nonfinancial argument for the inequality $f_n > f_{n+1}$.

The number f_n is $1+\frac{1}{n}$ raised to the power $n+1$; it is larger than e_n, which is $1+\frac{1}{n}$ raised to the lower power n:

$$e_n < f_n.$$

Since $\{f_n\}$ is a decreasing sequence, it follows that

$$e_n < f_n < f_1 = 4.$$

This proves that the sequence $\{e_n\}$ is monotonically increasing and bounded. It follows from the monotone convergence theorem, Theorem 1.10, that $\{e_n\}$ converges to a limit; this limit is called e.

The sequence $\{f_n\}$ is monotone decreasing and is bounded below by zero. Therefore it, too, tends to a limit; call it f. Next we show that f equals e. Each f_n is greater than e_n, so it follows that f is not less than e. To see that they are equal, we estimate the difference of f_n and e_n:

$$f_n - e_n = \left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}-1\right) = \frac{e_n}{n}.$$

As we have seen, e_n is less than 4. Therefore, it follows that

$$f_n - e_n < \frac{4}{n}.$$

Since e is greater than e_n and f is less than f_n, it follows that also $f - \mathrm{e}$ is less than $\frac{4}{n}$. Since this is true for all n, $f - \mathrm{e}$ must be zero.

Even though the sequences e_n and f_n both converge to e, our calculations show that e_{1000} and f_{1000} are accurate only to two decimal places. The sequences $\{e_n\}$ and $\{f_n\}$ converge very slowly to e. Calculus can be used to develop sequences that converge more rapidly to e. In Sect. 4.3a, we show how to use knowledge of calculus to develop a sequence

$$g_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

that converges to e much more rapidly. In fact, g_9 gives e correct to six decimal places. In Problem 1.55, we lead you through an argument, which does not use calculus, to show that g_n converges to e. In Sect. 10.4, we shed some light on the sequences e_n and f_n and offer another way to use calculus to improve on them.

Problems

1.53. Explain the following items, which prove that $\lim_{n\to\infty} n^{1/n} = 1$.

(a) Use the fact that the sequence $e_n = \left(1+\frac{1}{n}\right)^n$ increases to e to show that $\left(1+\frac{1}{n-1}\right)^n < 6$.
(b) Deduce that the sequence $n^{1/n}$ is decreasing when $n \geq 6$.
(c) $1 \leq n^{1/n}$. Therefore, $r = \lim_{n\to\infty} n^{1/n}$ exists.
(d) Consider $(2n)^{1/(2n)}$ to show that $r > 1$ is not possible.

1.54. Apply the A-G inequality to the $n+1$ numbers $\left(1-\frac{1}{n}\right), \ldots, \left(1-\frac{1}{n}\right), 1$ to conclude that

$$\left(1-\frac{1}{n}\right)^{n/(n+1)} < \frac{n}{n+1}.$$

Take this inequality to the power $n+1$ and take its reciprocal to conclude that

$$\left(1+\frac{1}{n-1}\right)^n = f_{n-1} > f_n = \left(1+\frac{1}{n}\right)^{n+1}.$$

1.55. Set $g_n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$. Here are the first ten values:

$$\begin{aligned}
g_0 &= 1\\
g_1 &= 2\\
g_2 &= 2.5\\
g_3 &= 2.6666666666666\\
g_4 &= 2.7083333333333\\
g_5 &= 2.7166666666666\\
g_6 &= 2.7180555555555\\
g_7 &= 2.7182539682539\\
g_8 &= 2.7182787698412\\
g_9 &= 2.7182815255731
\end{aligned}$$

We know that $e_n = \left(1+\frac{1}{n}\right)^n$ converges to the limit $\mathrm{e} = 2.718\ldots$ as n tends to infinity. Explain the following steps, which show that g_n also converges to e, that is,

$$\mathrm{e} = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

(a) $n!$ is greater than 2^{n-1}. Explain why $g_n < 1+1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}$. Explain why $g_n < 3$, and why g_n tends to a limit.

(b) Recall the binomial theorem $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. Let $a = \frac{1}{n}$ and $b = 1$ and show that

$$e_n = 1 + \sum_{k=1}^{n} \frac{n(n-1)\cdots(n-(k-1))}{k!} \frac{1}{n^k}.$$

Show that the kth term in e_n is less than the kth term in g_n. Use this to conclude that $e_n < g_n$.

In parts (c) and (d), we show that for large n, e_n is not much less than g_n.

(c) Write the difference $g_n - e_n$ as

$$g_n - e_n = \sum_{k=2}^{\infty} \frac{n^k - n(n-1)\cdots(n-(k-1))}{n^k k!},$$

and explain why this is less than $\sum_{k=2}^{\infty} \frac{n^k - (n-k)^k}{n^k k!} = \sum_{k=2}^{\infty} \frac{1 - \left(1 - \frac{k}{n}\right)^k}{k!}$.

(d) In this last part you need to explain through a sequence of steps, outlined below, why $g_n - e_n$ is less than $\frac{4}{n}$ and hence tends to 0. First recall that for $0 < x < 1$, we have the inequality

$$1 - x^k = (1-x)(1 + x + \cdots + x^{k-1}) < (1-x)k.$$

Let $x = 1 - \frac{k}{n}$. Explain why $g_n - e_n < \frac{1}{n} \sum_{k=2}^{\infty} \frac{k^2}{k!}$. Using the convention that $0! = 1$, we have $\frac{k^2}{k!} = \frac{k}{k-1} \frac{1}{(k-2)!}$. Recall that $\frac{1}{(k-2)!}$ is less than $\frac{1}{2^{k-2}}$. Explain why

$$g_n - e_n < \frac{1}{n} \sum_{k=2}^{\infty} \frac{2}{2^{k-2}}.$$

Explain why $g_n - e_n < \frac{4}{n}$, and why this completes the proof that the sequences g_n and e_n have the same limit.

1.56. Let us find a way to calculate e to a tolerance of 10^{-20}. Let g_n denote the numbers in Problem 1.55. Explain why

$$\mathrm{e} - g_n < \frac{1}{n!} \left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots \right).$$

Then explain how that gives $\mathrm{e} - g_n < \frac{1}{n!} \frac{1}{n}$. Finally, what would you take n to be so that g_n approximates e within 10^{-20}?