Chapter 7
Brownian Motion and Partial Differential Equations

In this chapter, we use the results of the preceding two chapters to discuss connections between Brownian motion and partial differential equations. After a brief discussion of the heat equation, we focus on the Laplace equation $\Delta u = 0$ and on the relations between Brownian motion and harmonic functions on a domain of $\mathbb{R}^d$. In particular, we give the probabilistic solution of the classical Dirichlet problem in a bounded domain whose boundary satisfies the exterior cone condition. In the case where the domain is a ball, the solution is made explicit by the Poisson kernel, which corresponds to the density of the exit distribution of the ball for Brownian motion. We then discuss recurrence and transience of d-dimensional Brownian motion, and we establish the conformal invariance of planar Brownian motion as a simple corollary of the results of Chap. 5. An important application is the so-called skew-product decomposition of planar Brownian motion, which we use to derive several asymptotic laws, including the celebrated Spitzer theorem on Brownian windings.

7.1 Brownian Motion and the Heat Equation

Throughout this chapter, we let B stand for a d-dimensional Brownian motion that starts from x under the probability measure P_x, for every $x \in \mathbb{R}^d$ (one may use the canonical construction of Sect. 2.2, defining P_x as the image of Wiener measure $W(\mathrm{dw})$ under the translation $\mathrm{w} \mapsto x + \mathrm{w}$). Then $(B_t)_{t\geq 0}$ is a Feller process with semigroup

$$Q_t\varphi(x) = \int_{\mathbb{R}^d} (2\pi t)^{-d/2} \exp(-\frac{|y-x|^2}{2t})\, \varphi(y)\, \mathrm{d}y,$$

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for $\varphi \in B(\mathbb{R}^d)$. We write L for the generator of this Feller process. If ψ is a twice continuously differentiable function on $\mathbb{R}^d$ such that both ψ and $\Delta\psi$ belong to $C_0(\mathbb{R}^d)$, then $\psi \in D(L)$ and $L\psi = \frac{1}{2}\Delta\psi$ (see the end of Sect. 6.2).

If $\varphi \in B(\mathbb{R}^d)$, then, for every fixed $t > 0$, $Q_t\varphi$ can be viewed as the convolution of φ with the C^∞ function

$$p_t(x) = (2\pi t)^{-d/2} \exp(-\frac{|x|^2}{2t}).$$

It follows that $Q_t\varphi$ is also a C^∞ function. Furthermore, if $\varphi \in C_0(\mathbb{R}^d)$, differentiation under the integral sign shows that all derivatives of $Q_t\varphi$ also belong to $C_0(\mathbb{R}^d)$. It follows that we have $Q_t\varphi \in D(L)$ and $L(Q_t\varphi) = \frac{1}{2}\Delta(Q_t\varphi)$, for every $t > 0$.

Theorem 7.1 *Let $\varphi \in C_0(\mathbb{R}^d)$. For every $t > 0$ and $x \in \mathbb{R}^d$, set*

$$u_t(x) = Q_t\varphi(x) = E_x[\varphi(B_t)].$$

Then, the function $(u_t(x))_{t>0,x\in\mathbb{R}^d}$ solves the partial differential equation

$$\frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t,$$

on $(0,\infty) \times \mathbb{R}^d$. Furthermore, for every $x \in \mathbb{R}^d$,

$$\lim_{\substack{s\downarrow 0 \\ y\to x}} u_s(y) = \varphi(x).$$

Proof By the remarks preceding the theorem, we already know that, for every $t > 0$, u_t is a C^∞ function, $u_t \in D(L)$, and $Lu_t = \frac{1}{2}\Delta u_t$. Let $\varepsilon > 0$. By applying Proposition 6.11 to $f = u_\varepsilon$, we get for every $t \geq \varepsilon$,

$$u_t = u_\varepsilon + \int_0^{t-\varepsilon} L(Q_s u_\varepsilon)\,\mathrm{d}s = u_\varepsilon + \int_\varepsilon^t Lu_s\,\mathrm{d}s.$$

Since $Lu_s = Q_{s-\varepsilon}(Lu_\varepsilon)$ depends continuously on $s \in [\varepsilon,\infty)$, it follows that, for $t \geq \varepsilon$,

$$\frac{\partial u_t}{\partial t} = Lu_t = \frac{1}{2}\Delta u_t.$$

The last assertion is just the fact that $Q_s\varphi \longrightarrow \varphi$ as $s \to 0$. □

Remark We could have proved Theorem 7.1 by direct calculations from the explicit form of $Q_t\varphi$, and these calculations imply that the same statement holds if we only assume that φ is bounded and continuous. The above proof however has the advantage of showing the relation between this result and our general study of

Markov processes. It also indicates that similar results will hold for more general equations of the form $\frac{\partial u}{\partial t} = Au$ provided A can be interpreted as the generator of an appropriate Markov process.

Brownian motion can be used to provide probabilistic representations for solutions of many other parabolic partial differential equations. In particular, solutions of equations of the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - v\,u,$$

where v is a nonnegative function on $\mathbb{R}^d$, are expressed via the so-called Feynman–Kac formula: See Exercise 7.28 below for a precise statement.

7.2 Brownian Motion and Harmonic Functions

Let us now turn to connections between Brownian motion and the Laplace equation $\Delta u = 0$. We start with a classical definition.

Definition 7.2 Let D be a domain of $\mathbb{R}^d$. A function $u : D \longrightarrow \mathbb{R}$ is said to be *harmonic* on D if it is twice continuously differentiable and $\Delta u = 0$ on D.

Let D' be a subdomain of D whose closure is contained in D. Consider the stopping time $T := \inf\{t \geq 0 : B_t \notin D'\}$. An application of Itô's formula (justified by the remark preceding Proposition 5.11) shows that, if u is harmonic on D, then for every $x \in D'$, the process

$$u(B_{t\wedge T}) = u(B_0) + \int_0^{t\wedge T} \nabla u(B_s)\cdot \mathrm{d}B_s \tag{7.1}$$

is a local martingale under P_x [here and later, to apply the stochastic calculus results of Chap. 5, we let $(\mathscr{F}_t)$ be the canonical filtration of B completed under P_x].

So, roughly speaking, harmonic functions are functions which when composed with Brownian motion give (local) martingales.

Proposition 7.3 *Let u be harmonic on the domain D. Let D' be a bounded subdomain of D whose closure is contained in D, and consider the stopping time $T := \inf\{t \geq 0 : B_t \notin D'\}$. Then, for every $x \in D'$,*

$$u(x) = E_x[u(B_T)].$$

Proof Since D' is bounded, both u and ∇u are bounded on D', and we also know that $P_x(T < \infty) = 1$ for every $x \in D'$. It follows from (7.1) that $u(B_{t\wedge T})$ is a (true) martingale, and in particular, we have

$$u(x) = E_x[u(B_{t\wedge T})]$$

for every $x \in D'$. By letting $t \to \infty$ and using dominated convergence we get that

$$u(x) = E_x[u(B_T)]$$

for every $x \in D'$. □

The preceding proposition easily leads to the mean value property for harmonic functions. In order to state this property, first recall that the uniform probability measure on the unit sphere, denoted by $\sigma_1(\mathrm{d}y)$, is the unique probability measure on $\{y \in \mathbb{R}^d : |y| = 1\}$ that is invariant under all vector isometries. For every $x \in \mathbb{R}^d$ and $r > 0$, we then let $\sigma_{x,r}(\mathrm{d}y)$ be the image of $\sigma_1(\mathrm{d}y)$ under the mapping $y \mapsto x + ry$.

Proposition 7.4 (Mean value property) *Suppose that u is harmonic on the domain D. Then, for every $x \in D$ and for every $r > 0$ such that the closed ball of radius r centered at x is contained in D, we have*

$$u(x) = \int \sigma_{x,r}(\mathrm{d}y)\, u(y).$$

Proof First observe that, if $T_1 = \inf\{t \geq 0 : |B_t| = 1\}$, the distribution of B_{T_1} under P_0 is invariant under all vector isometries of $\mathbb{R}^d$ (by the invariance properties of Brownian motion stated at the end of Chap. 2) and therefore this distribution is $\sigma_1(dy)$. By scaling and translation invariance, it follows that for every $x \in \mathbb{R}^d$ and $r > 0$, if $T_{x,r} = \inf\{t \geq 0 : |B_t - x| = r\}$, the distribution of $B_{T_{x,r}}$ under P_x is $\sigma_{x,r}$. However, Proposition 7.3 implies that, under the conditions in the proposition, we must have $u(x) = E_x[u(B_{T_{x,r}})]$. The desired result follows. □

We say that a (locally bounded and measurable) function u on D satisfies the mean value property if the conclusion of Proposition 7.4 holds. It turns out that this property characterizes harmonic functions.

Lemma 7.5 *Let u be a locally bounded and measurable function on D that satisfies the mean value property. Then u is harmonic on D.*

Proof Fix $r_0 > 0$ and let D' be the open subset of D consisting of all points whose distance to D^c is greater than r_0. It is enough to prove that u is twice continuously differentiable and $\Delta u = 0$ on D'. Let $h : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a C^∞ function with compact support contained in $(0, r_0)$ and not identically zero. Then, for every $x \in D'$ and every $r \in (0, r_0)$, we have

$$u(x) = \int \sigma_{x,r}(\mathrm{d}y)\, u(y).$$

We multiply both sides of this equality by $r^{d-1}h(r)$ and integrate with respect to Lebesgue measure $\mathrm{d}r$ on $(0, r_0)$. Using the formula for integration in polar coordinates, and agreeing for definiteness that $u = 0$ on D^c, we get, for every $x \in D'$,

$$c\, u(x) = \int_{\{|y|<r_0\}} \mathrm{d}y\, h(|y|)\, u(x+y) = \int_{\mathbb{R}^d} \mathrm{d}z\, h(|z-x|)\, u(z),$$

where $c > 0$ is a constant and we use the fact that $h(|z-x|) = 0$ if $|z-x| \geq r_0$. Since $z \mapsto h(|z|)$ is a C^∞ function, the convolution in the right-hand side of the last display also defines a C^∞ function on D'.

It remains to check that $\Delta u = 0$ on D'. To this end, we use a probabilistic argument (an analytic argument is also easy). By applying Itô's formula to $u(B_t)$ under P_x, we get, for $x \in D'$ and $r \in (0, r_0)$,

$$E_x[u(B_{t\wedge T_{x,r}})] = u(x) + E_x\Big[\int_0^{t\wedge T_{x,r}} \mathrm{d}s\, \Delta u(B_s)\Big].$$

If we let $t \to \infty$, noting that $E_x[T_{x,r}] < \infty$ (cf. example (b) after Corollary 3.24), we get

$$E_x[u(B_{T_{x,r}})] = u(x) + E_x\Big[\int_0^{T_{x,r}} \mathrm{d}s\, \Delta u(B_s)\Big].$$

The mean value property just says that $E_x[u(B_{T_{x,r}})] = u(x)$ and so we have

$$E_x\Big[\int_0^{T_{x,r}} \mathrm{d}s\, \Delta u(B_s)\Big] = 0.$$

Since this holds for any $r \in (0, r_0)$ it follows that $\Delta u(x) = 0$. □

From now on, we assume that the domain D is **bounded**.

Definition 7.6 (Classical Dirichlet problem) Let g be a continuous function on ∂D. A function $u : D \longrightarrow \mathbb{R}$ solves the *Dirichlet problem* in D with boundary condition g, if u is harmonic on D and has boundary condition g, in the sense that, for every $y \in \partial D$, $u(x) \longrightarrow g(y)$ as $x \to y$, $x \in D$.

Recall that $\bar{D}$ stands for the closure of D. If u solves the Dirichlet problem with boundary condition g, the function $\tilde{u}$ defined on $\bar{D}$ by $\tilde{u}(x) = u(x)$ if $x \in D$, and $\tilde{u}(x) = g(x)$ if $x \in \partial D$, is then continuous, hence bounded on $\bar{D}$.

Proposition 7.7 *Let D be a bounded domain, and write $T = \inf\{t \geq 0 : B_t \notin D\}$ for the exit time of Brownian motion from D.*

(i) *Let g be a continuous function on ∂D, and let u be a solution of the Dirichlet problem in D with boundary condition g. Then, for every $x \in D$,*

$$u(x) = E_x[g(B_T)].$$

(ii) *Let g be a bounded measurable function on ∂D. Then the function*

$$u(x) = E_x[g(B_T)], \quad x \in D,$$

is harmonic on D.

Remark Assertion (i) implies that, if a solution to the Dirichlet problem with boundary condition g exists, then it is unique, which is also easy to prove using the mean value property.

Proof

(i) Fix $x \in D$ and $\varepsilon_0 > 0$ such that the ball of radius ε_0 centered at x is contained in D. For every $\varepsilon \in (0, \varepsilon_0)$, let D_ε be the connected component containing x of the open set consisting of all points of D whose distance to D^c is greater than ε. If $T_\varepsilon = \inf\{t \geq 0 : B_t \notin D_\varepsilon\}$, Proposition 7.3 shows that

$$u(x) = E_x[u(B_{T_\varepsilon})].$$

Now observe that $T_\varepsilon \uparrow T$ as $\varepsilon \downarrow 0$ (if T' is the increasing limit of T_ε as $\varepsilon \downarrow 0$, we have $T' \leq T$ and on the other hand $B_{T'} \in \partial D$ by the continuity of sample paths). Using dominated convergence, it follows that $E_x[u(B_{T_\varepsilon})]$ converges to $E_x[g(B_T)]$ as $\varepsilon \to 0$.

(ii) By Lemma 7.5, it is enough to verify that the function $u(x) = E_x[g(B_T)]$ satisfies the mean value property. Recall the notation $T_{x,r} = \inf\{t \geq 0 : |B_t - x| = r\}$ for $x \in \mathbb{R}^d$ and $r > 0$. Fix $x \in D$ and $r > 0$ such that the closed ball of radius r centered at x is contained in D. We apply the strong Markov property in the form given in Theorem 6.17: With the notation of this theorem, we let $\Phi(\mathrm{w})$, for $\mathrm{w} \in C(\mathbb{R}_+, \mathbb{R}^d)$ such that $\mathrm{w}(0) \in D$, be the value of g at the first exit point of w from D (we take $\Phi(\mathrm{w}) = 0$ if w never exits D) and we observe that we have

$$g(B_T) = \Phi(B_t, t \geq 0) = \Phi(B_{T_{x,r}+t}, t \geq 0), \qquad P_x \text{ a.s.}$$

because the paths $(B_t, t \geq 0)$ and $(B_{T_{x,r}+t}, t \geq 0)$ have the same exit point from D. It follows that

$$u(x) = E_x[g(B_T)] = E_x[\Phi(B_{T_{x,r}+t}, t \geq 0)] = E_x[E_{B_{T_{\varepsilon,r}}}[\Phi(B_t, t \geq 0)]] = E_x[u(B_{T_{\varepsilon,r}})].$$

Since we know that the law of $B_{T_{\varepsilon,r}}$ under P_x is $\sigma_{x,r}$, this gives the mean value property. □

Part (i) of Proposition 7.7 tells us that the solution of the Dirichlet problem with boundary condition g, if it exists, is given by the probabilistic formula $u(x) = E_x[g(B_T)]$. On the other hand, for any choice of the (bounded measurable) function g on ∂D, part (ii) tells us that the probabilistic formula yields a function u that is harmonic on D. Even if g is assumed to be continuous, it is however not clear that the function u has boundary condition g, and this need not be true in general (see Exercises 7.24 and 7.25 for examples where the Dirichlet problem has no solution). We state a theorem that gives a partial answer to this question.

If $y \in \partial D$, we say that D satisfies the *exterior cone condition* at y if there exist a (nonempty) open cone $\mathscr{C}$ with apex y and a real $r > 0$ such that the intersection of $\mathscr{C}$ with the open ball of radius r centered at y is contained in D^c. For instance, a convex domain satisfies the exterior cone condition at every point of its boundary.

Theorem 7.8 (Solution of the Dirichlet problem) *Let D be a bounded domain in $\mathbb{R}^d$. Assume that D satisfies the exterior cone condition at every $y \in \partial D$. Then, for every continuous function g on ∂D, the formula*

$$u(x) = E_x[g(B_T)] , \quad \text{where } T = \inf\{t \geq 0 : B_t \notin D\},$$

gives the unique solution of the Dirichlet problem with boundary condition g.

Proof Thanks to Proposition 7.7 (ii), we only need to verify that, for every fixed $y \in \partial D$,

$$\lim_{x \to y, x \in D} u(x) = g(y). \tag{7.2}$$

Let $\varepsilon > 0$. Since g is continuous, we can find $\delta > 0$ such that we have $|g(z)-g(y)| \leq \varepsilon/3$ whenever $z \in \partial D$ and $|z - y| < \delta$. Let $M > 0$ be such that $|g(z)| \leq M$ for every $z \in \partial D$. Then, for every $\eta > 0$,

$$\begin{aligned} |u(x) - g(y)| &\leq E_x[|g(B_T) - g(y)|\mathbf{1}_{\{T \leq \eta\}}] + E_x[|g(B_T) - g(y)|\mathbf{1}_{\{T > \eta\}}] \\ &\leq E_x[|g(B_T) - g(y)|\mathbf{1}_{\{T \leq \eta\}}\mathbf{1}_{\{\sup\{|B_t - x| : t \leq \eta\} \leq \delta/2\}}] \\ &\quad +2M\, P_x\Big(\sup_{t \leq \eta} |B_t - x| > \frac{\delta}{2}\Big) + 2M\, P_x(T > \eta). \end{aligned}$$

Write A_1, A_2, A_3 for the three terms in the sum in the right-hand side of the last display. We assume that $|y-x| < \delta/2$, and we bound successively these three terms.

First note that we have $|B_T - y| \leq |B_T - x| + |y - x| < \delta$ on the event

$$\{T \leq \eta\} \cap \sup\{|B_t - x| : t \leq \eta\} \leq \delta/2\},$$

and our choice of δ ensures that $A_1 \leq \varepsilon/3$.

Then, translation invariance gives

$$A_2 = 2M\, P_0\Big(\sup_{t \leq \eta} |B_t| > \frac{\delta}{2}\Big),$$

which tends to 0 when $\eta > 0$ by the continuity of sample paths. So we can fix $\eta > 0$ so that $A_2 < \varepsilon/3$.

Finally, we claim that we can choose $\alpha \in (0, \delta/2]$ small enough so that we also have $A_3 = 2M\, P_x(T > \eta) < \varepsilon/3$ whenever $|x - y| < \alpha$. It follows that $|u(x) - g(y)| < \varepsilon$, whenever $|x - y| < \alpha$, thus completing the proof of (7.2). Thus it only remains to prove our claim, which is the goal of the next lemma. □

Lemma 7.9 *Under the exterior cone condition, we have for every $y \in \partial D$ and every $\eta > 0$,*

$$\lim_{x \to y, x \in D} P_x(T > \eta) = 0.$$

Proof For every $u \in \mathbb{R}^d$ with $|u| = 1$ and every $\gamma \in (0, 1)$, consider the circular cone

$$\mathscr{C}(u, \gamma) := \{z \in \mathbb{R}^d : z \cdot u > (1-\gamma)|z|\},$$

where $z \cdot u$ stands for the usual scalar product. If $y \in \partial D$ is given, the exterior cone condition means that we can fix $r > 0$, u and γ such that

$$y + (\mathscr{C}(u, \gamma) \cap \mathscr{B}_r) \subset D^c,$$

where $\mathscr{B}_r$ denotes the open ball of radius r centered at 0. To simplify notation, we set $\mathscr{C} = \mathscr{C}(u, \gamma) \cap \mathscr{B}_r$, and also

$$\mathscr{C}' = \mathscr{C}(u, \frac{\gamma}{2}) \cap \mathscr{B}_{r/2}$$

which is the intersection of a smaller cone with $\mathscr{B}_{r/2}$.

For every open subset F of $\mathbb{R}^d$, write $T_F = \inf\{t \geq 0 : B_t \in F\}$. An application of Blumenthal's zero-one law (Theorem 2.13, or rather its easy extension to d-dimensional Brownian motion) along the lines of the proof of Proposition 2.14 (i) shows that $P_0(T_{\mathscr{C}(u,\gamma/2)} = 0) = 1$ and hence $P_0(T_{\mathscr{C}'} = 0) = 1$ by the continuity of sample paths. On the other hand, set $\mathscr{C}'_a = \{z \in \mathscr{C}' : |z| > a\}$, for every $a \in (0, r/2)$. The sets $\mathscr{C}'_a$ increase to $\mathscr{C}'$ as $a \downarrow 0$, and thus we have $T_{\mathscr{C}'_a} \downarrow T_{\mathscr{C}'} = 0$ as $a \downarrow 0$, P_0 a.s. Hence, given any $\beta > 0$ we can fix a small enough so that

$$P_0(T_{\mathscr{C}'_a} \leq \eta) \geq 1 - \beta.$$

Recalling that $y + \mathscr{C} \subset D^c$, we have

$$P_x(T \leq \eta) \geq P_x(T_{y+\mathscr{C}} \leq \eta) = P_0(T_{y-x+\mathscr{C}} \leq \eta).$$

However, a simple geometric argument shows that, as soon as $|y-x|$ is small enough, the shifted cone $y - x + \mathscr{C}$ contains $\mathscr{C}'_a$, and therefore

$$P_x(T \leq \eta) \geq P_0(T_{\mathscr{C}'_a} \leq \eta) \geq 1 - \beta.$$

Since β was arbitrary, this completes the proof. □

Remark The exterior cone condition is only a sufficient condition for the existence (and uniqueness) of a solution to the Dirichlet problem. See e.g. [69] for necessary and sufficient conditions that ensure the existence of a solution for any continuous boundary value.

7.3 Harmonic Functions in a Ball and the Poisson Kernel

Consider again a bounded domain D and a continuous function g on ∂D. Let $T = \inf\{t \geq 0 : B_t \notin D\}$ be the exit time of D by Brownian motion. Proposition 7.7 (i) shows that the solution of the Dirichlet problem in D with boundary condition g, if it exists (which is the case under the assumption of Theorem 7.8), is given by

$$u(x) = E_x[g(B_T)] = \int_{\partial D} \omega(x, \mathrm{d}y)\, g(y),$$

where, for every $x \in D$, $\omega(x, \mathrm{d}y)$ denotes the distribution of B_T under P_x. The measure $\omega(x, \mathrm{d}y)$ is a probability measure on ∂D called the *harmonic measure* of D relative to x. In general, it is hopeless to try to find an explicit expression for the measures $\omega(x, \mathrm{d}y)$. It turns out that, in the case of balls, such an explicit expression is available and makes the representation of solutions of the Dirichet problem more concrete.

From now on, we suppose that $D = \mathscr{B}_1$ is the open unit ball in $\mathbb{R}^d$. We also assume that $d \geq 2$ to avoid trivialities. The boundary $\partial \mathscr{B}_1$ is the unit sphere in $\mathbb{R}^d$.

Definition 7.10 The *Poisson kernel* (of the unit ball) is the function K defined on $\mathscr{B}_1 \times \partial \mathscr{B}_1$ by

$$K(x, y) = \frac{1 - |x|^2}{|y - x|^d},$$

for every $x \in \mathscr{B}_1$ and $y \in \partial \mathscr{B}_1$.

Lemma 7.11 *For every fixed* $y \in \partial \mathscr{B}_1$*, the function* $x \mapsto K(x, y)$ *is harmonic on* $\mathscr{B}_1$.

Proof Set $K_y(x) = K(x, y)$ for $x \in \mathscr{B}_1$. Then K_y is a C^∞ function on $\mathscr{B}_1$. Moreover a (somewhat tedious) direct calculation left to the reader shows that $\Delta K_y = 0$ on $\mathscr{B}_1$. □

In view of deriving further properties of the Poisson kernel, the following lemma about radial harmonic functions will be useful.

Lemma 7.12 *Let* $0 \leq r_1 < r_2$ *be two real numbers and let* $h : (r_1, r_2) \to \mathbb{R}$ *be a measurable function. The function* $u(x) = h(|x|)$ *is harmonic on the domain* $\{x \in \mathbb{R}^d : r_1 < |x| < r_2\}$ *if and only if there exist two constants a and b such that*

$$h(r) = \begin{cases} a + b \log r & \text{if } d = 2, \\ a + b\, r^{2-d} & \text{if } d \geq 3. \end{cases}$$

Proof Suppose that $u(x) = h(|x|)$ is harmonic on $\{x \in \mathbb{R}^d : r_1 < |x| < r_2\}$. Then u is twice continuously differentiable and so is h. From the expression of the

Laplacian of a radial function, we get that $\Delta u = 0$ if and only if

$$h''(r) + \frac{d-1}{r} h'(r) = 0\ , \quad r \in (r_1, r_2).$$

The solutions of this second order linear differential equations are the functions of the form given in the statement. The lemma follows. □

Recall our notation $\sigma_1(\mathrm{d}y)$ for the uniform probability measure on the unit sphere $\partial\mathscr{B}_1$.

Lemma 7.13 *For every $x \in \mathscr{B}_1$,*

$$\int_{\partial\mathscr{B}_1} K(x, y)\, \sigma_1(\mathrm{d}y) = 1.$$

Proof For every $x \in \mathscr{B}_1$, set

$$F(x) = \int_{\partial\mathscr{B}_1} K(x, y)\, \sigma_1(\mathrm{d}y).$$

Then the preceding lemma implies that F is harmonic on $\mathscr{B}_1$. Indeed, if $x \in \mathscr{B}_1$ and $r < 1-|x|$, Lemma 7.11 and the mean value property imply that, for every $y \in \partial\mathscr{B}_1$,

$$K(x, y) = \int K(z, y)\, \sigma_{x,r}(\mathrm{d}z).$$

Hence, using Fubini's theorem,

$$\begin{aligned}\int F(z)\, \sigma_{x,r}(\mathrm{d}z) &= \int \Big(\int K(z, y)\, \sigma_1(\mathrm{d}y)\Big)\sigma_{x,r}(\mathrm{d}z)\\ &= \int \Big(\int K(z, y)\, \sigma_{x,r}(\mathrm{d}z)\Big)\sigma_1(\mathrm{d}y) = \int K(x, y)\, \sigma_1(\mathrm{d}y) = F(x),\end{aligned}$$

showing that the mean value property holds for F.

If ψ is a vector isometry of $\mathbb{R}^d$, we have $K(\psi(x), \psi(y)) = K(x, y)$ for every $x \in \mathscr{B}_1$ and $y \in \partial B_1$, and the fact that $\sigma_1(\mathrm{d}y)$ is invariant under ψ implies that $F(\psi(x)) = F(x)$ for every $x \in \mathscr{B}_1$. Hence F is a radial harmonic function and Lemma 7.12 (together with the fact that F is bounded in the neighborhood of 0) implies that F is constant. Since $F(0) = 1$, the proof is complete. □

Theorem 7.14 *Let g be a continuous function on $\partial\mathscr{B}_1$. The unique solution of the Dirichlet problem in $\mathscr{B}_1$ with boundary condition g is given by*

$$u(x) = \int_{\partial\mathscr{B}_1} g(y)\, K(x, y)\, \sigma_1(\mathrm{d}y)\ ,\ x \in \mathscr{B}_1.$$

Proof The very same arguments as in the beginning of the proof of Lemma 7.13 show that u is harmonic on $\mathscr{B}_1$. To verify the boundary condition, fix $y_0 \in \partial\mathscr{B}_1$. For every $\delta > 0$, the explicit form of the Poisson kernel shows that, if $x \in \mathscr{B}_1$ and $y \in \partial\mathscr{B}_1$ are such that $|x - y_0| < \delta/2$ and $|y - y_0| > \delta$, then

$$K(x, y) \leq (\frac{2}{\delta})^d (1 - |x|^2).$$

It follows from this bound that, for every $\delta > 0$,

$$\lim_{x \to y_0, x \in \mathscr{B}_1} \int_{\{|y - y_0| > \delta\}} K(x, y)\, \sigma_1(\mathrm{d}y) = 0. \tag{7.3}$$

Then, given $\varepsilon > 0$, we can choose $\delta > 0$ sufficiently small so that the conditions $y \in \partial\mathscr{B}_1$ and $|y - y_0| \leq \delta$ imply $|g(y) - g(y_0)| \leq \varepsilon$. If $M = \sup\{|g(y)| : y \in \partial\mathscr{B}_1\}$, it follows that

$$\begin{aligned} |u(x) - g(y_0)| &= \Big| \int K(x, y)\, (g(y) - g(y_0))\, \sigma_1(\mathrm{d}y) \Big| \\ &\leq 2M \int_{\{|y - y_0| > \delta\}} K(x, y)\, \sigma_1(\mathrm{d}y) + \varepsilon, \end{aligned}$$

using Lemma 7.13 in the first equality, and then our choice of δ. Thanks to (7.3), we now get

$$\limsup_{x \to y_0, x \in \mathscr{B}_1} |u(x) - g(y_0)| \leq \varepsilon.$$

Since ε was arbitrary, this yields the desired boundary condition. □

The preceding theorem allows us to identify the harmonic measures of the unit ball.

Corollary 7.15 *Let $T = \inf\{t \geq 0 : B_t \notin \mathscr{B}_1\}$. For every $x \in \mathscr{B}_1$, the distribution of B_T under P_x has density $K(x, y)$ with respect to $\sigma_1(\mathrm{d}y)$.*

This is immediate since, by combining Proposition 7.7 (i) with Theorem 7.14, we get that, for any continuous function g on $\partial\mathscr{B}_1$,

$$E_x[g(B_T)] = \int_{\partial\mathscr{B}_1} g(y)\, K(x, y)\, \sigma_1(\mathrm{d}y)\ , \quad \forall x \in \mathscr{B}_1.$$

7.4 Transience and Recurrence of Brownian Motion

We consider again a d-dimensional Brownian motion $(B_t)_{t\geq 0}$ that starts from x under the probability measure P_x. We again suppose that $d \geq 2$, since the corresponding results for $d = 1$ have already been derived in the previous chapters.

For every $a \geq 0$, we introduce the stopping time

$$U_a = \inf\{t \geq 0 : |B_t| = a\},$$

with the usual convention $\inf \varnothing = \infty$.

Proposition 7.16 *Suppose that $x \neq 0$, and let ε and R be such that $0 < \varepsilon < |x| < R$. Then,*

$$P_x(U_\varepsilon < U_R) = \begin{cases} \dfrac{\log R - \log |x|}{\log R - \log \varepsilon} & \text{if } d = 2, \\[2ex] \dfrac{R^{2-d} - |x|^{2-d}}{R^{2-d} - \varepsilon^{2-d}} & \text{if } d \geq 3. \end{cases} \tag{7.4}$$

Consequently, we have $P_x(U_0 < \infty) = 0$ and for every $\varepsilon \in (0, |x|)$,

$$P_x(U_\varepsilon < \infty) = \begin{cases} 1 & \text{if } d = 2, \\[2ex] \left(\dfrac{\varepsilon}{|x|}\right)^{d-2} & \text{if } d \geq 3. \end{cases}$$

Proof Write $D_{\varepsilon,R}$ for the annulus $\{y \in \mathbb{R}^d : \varepsilon < |y| < R\}$. Let $u(x)$ be the function defined for $x \in D_{\varepsilon,R}$ that appears in the right-hand side of (7.4). By Lemma 7.12, u is harmonic on $D_{\varepsilon,R}$, and it is also clear that u solves the Dirichlet problem in $D_{\varepsilon,R}$ with boundary condition $g(y) = 0$ if $|y| = R$ and $g(y) = 1$ if $|y| = \varepsilon$. If $T_{\varepsilon,R}$ denotes the first exit time from $D_{\varepsilon,R}$, Proposition 7.7 shows that we must have $u(x) = E_x[g(B_{T_{\varepsilon,R}})]$ for every $x \in D_{\varepsilon,R}$. Formula (7.4) follows since $E_x[g(B_{T_{\varepsilon,R}})] = P_x(U_\varepsilon < U_R)$.

If $R > |x|$ is fixed, the event $\{U_0 < U_R\}$ is (P_x a.s.) contained in $\{U_\varepsilon < U_R\}$, for every $0 < \varepsilon < |x|$. By passing to the limit $\varepsilon \to 0$ in the right-hand side of (7.4), we thus get that $P_x(U_0 < U_R) = 0$. Since $U_R \uparrow \infty$ as $R \uparrow \infty$, it follows that $P_x(U_0 < \infty) = 0$.

Finally, we have also $P_x(U_\varepsilon < \infty) = \lim P_x(U_\varepsilon < U_R)$ as $R \to \infty$, and by letting $R \to \infty$ in the right-hand side of (7.4) we get the stated formula for $P_x(U_\varepsilon < \infty)$. □

Remark The reader will compare formula (7.4) with the exit distribution from an interval for real Brownian motion that was derived in Chap. 4 (example (a) after Corollary 3.24). We could have proved (7.4) in a way similar to what we did for its one-dimensional analog, by applying the optional stopping theorem to the local martingale $\log|B_t|$ (if $d = 2$) or $|B_t|^{2-d}$ (if $d = 3$). See Exercise 5.33.

For every $y \in \mathbb{R}^d$, set $\tau_y = \inf\{t \geq 0 : B_t = y\}$, so that in particular $\tau_0 = U_0$. The property $P_x(\tau_0 < \infty) = 0$ for $x \neq 0$ implies that $P_x(\tau_y < \infty) = 0$ whenever $y \neq x$, by translation invariance. This means that the probability for Brownian motion to visit a fixed point other than its starting point is zero: one says that points are *polar* for d-dimensional Brownian motion with $d \geq 2$ (see Exercise 7.25 for more about polar sets).

If $\mathbf{m}$ denotes Lebesgue measure on $\mathbb{R}^d$, it follows from Fubini's theorem that

$$E_x[\mathbf{m}(\{B_t, t \geq 0\})] = E_x\Big[\int_{\mathbb{R}^d} \mathrm{d}y\, \mathbf{1}_{\{\tau_y<\infty\}}\Big] = \int_{\mathbb{R}^d} \mathrm{d}y\, P_x(\tau_y < \infty) = 0,$$

and therefore $\mathbf{m}(\{B_t, t \geq 0\}) = 0$, P_x a.s. One can nonetheless prove that the Hausdorff dimension of the curve $\{B_t, t \geq 0\}$ is equal to 2 in any dimension $d \geq 2$ (see e.g. [62]). In some sense, this shows that the planar Brownian curve is "not so far" from having positive Lebesgue measure.

Theorem 7.17

(i) *In dimension $d = 2$, Brownian motion is recurrent, meaning that almost surely, for every nonempty open subset O of $\mathbb{R}^d$, the set $\{t \geq 0 : B_t \in O\}$ is unbounded.*
(ii) *In dimension $d \geq 3$, Brownian motion is transient, meaning that*

$$\lim_{t\to\infty} |B_t| = \infty\,, \quad a.s.$$

Proof

(i) It is enough to prove that the statement holds when O is an open ball of rational radius centered at a point with rational coordinates. So it suffices to consider a fixed open ball $\mathscr{B}$ and we may assume that $\mathscr{B}$ is centered at 0 and that the starting point of B is $x \neq 0$. By Proposition 7.16 we know that Brownian motion will never hit 0 (so that $\inf\{|B_r| : 0 \leq r \leq t\} > 0$ for every $t \geq 0$, a.s.) but still will hit any open ball centered at 0. It follows that B must visit $\mathscr{B}$ at arbitrarily large times, a.s.
(ii) Again we can assume that the starting point of B is $x \neq 0$. Since the function $y \mapsto |y|^{2-d}$ is harmonic on $\mathbb{R}^d\backslash\{0\}$, and since we saw that B does not hit 0, we get that $|B_t|^{2-d}$ is a local martingale and hence a supermartingale by Proposition 4.7. By Theorem 3.19 (and the fact that a positive supermartingale is automatically bounded in L^1), we know that $|B_t|^{2-d}$ converges a.s. as $t \to \infty$. The a.s. limit must be zero (otherwise the curve $\{B_t : t \geq 0\}$ would be bounded!) and this says exactly that $|B_t|$ converges to ∞ as $t \to \infty$. □

Remark In dimension $d = 2$, one can (slightly) reinforce the recurrence property by saying that a.s. for every nonempty open subset O of $\mathbb{R}^2$, the Lebesgue measure of $\{t \geq 0 : B_t \in O\}$ is infinite. This follows by a straightforward application of the strong Markov property, and we omit the details.

7.5 Planar Brownian Motion and Holomorphic Functions

In this section, we concentrate on the planar case $d = 2$, and we write $B_t = (X_t, Y_t)$ for a two-dimensional Brownian motion. It will be convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, so that $B_t = X_t + \mathrm{i}Y_t$, and we sometimes say that B is a *complex Brownian motion*. As previously B starts from z under the probability P_z, for every $z \in \mathbb{C}$.

If $\Phi : \mathbb{C} \longrightarrow \mathbb{C}$ is a holomorphic function, the real and imaginary parts of Φ are harmonic functions, and thus we know that the real and imaginary parts of $\Phi(B_t)$ are continuous local martingales. In fact, much more is true.

Theorem 7.18 *Let $\Phi : \mathbb{C} \longrightarrow \mathbb{C}$ be a nonconstant holomorphic function. For every $t \geq 0$, set*

$$C_t = \int_0^t |\Phi'(B_s)|^2 \, \mathrm{d}s.$$

Let $z \in \mathbb{C}$. There exists a complex Brownian motion Γ that starts from $\Phi(z)$ under P_z, such that

$$\Phi(B_t) = \Gamma_{C_t}\,, \quad \textit{for every } t \geq 0,\ P_z \textit{ a.s.}$$

In other words, the image of complex Brownian motion under a holomorphic function is a time-changed complex Brownian motion. This is the **conformal invariance property** of planar Brownian motion. It is possible (and useful for many applications) to extend Theorem 7.18 to the case where Φ is defined and holomorphic in a domain D of $\mathbb{C}$ (such that $z \in D$). A similar representation then holds for $\Phi(B_t)$ up to the first exit time of D (see e.g. [18]).

Proof Let g and h stand respectively for the real and imaginary parts of Φ. Since g and h are harmonic, an application of Itô's formula gives under P_z,

$$g(B_t) = g(z) + \int_0^t \frac{\partial g}{\partial x}(B_s)\,\mathrm{d}X_s + \int_0^t \frac{\partial g}{\partial y}(B_s)\,\mathrm{d}Y_s$$

and similarly

$$h(B_t) = h(z) + \int_0^t \frac{\partial h}{\partial x}(B_s)\,\mathrm{d}X_s + \int_0^t \frac{\partial h}{\partial y}(B_s)\,\mathrm{d}Y_s.$$

So $M_t = g(B_t)$ and $N_t = h(B_t)$ are local martingales. Moreover, the Cauchy–Riemann equations

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}\,,\ \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}$$

give

$$\langle M, N \rangle_t = 0$$

and

$$\langle M, M \rangle_t = \langle N, N \rangle_t = \int_0^t |\Phi'(B_s)|^2 \, \mathrm{d}s = C_t.$$

The recurrence of planar Brownian motion implies that $C_\infty = \infty$ a.s. (take a ball $\mathscr{B}$ where $|\Phi'|$ is bounded below by a positive constant, and note that the total time spent by B in the ball $\mathscr{B}$ is a.s. infinite). We can then apply Proposition 5.15 to $M_t - g(z)$ and $N_t - h(z)$ under P_z, and we find two independent real Brownian motions β and γ started from 0 such that $M_t = g(z) + \beta_{C_t}$ and $N_t = h(z) + \gamma_{C_t}$, for every $t \geq 0$, a.s. The desired result follows by setting $\Gamma_t = \Phi(z) + \beta_t + \mathrm{i}\gamma_t$. □

We will apply the conformal invariance property of planar Brownian motion to its decomposition in polar coordinates, which is known as the **skew-product representation**.

Theorem 7.19 *Let* $z \in \mathbb{C}\backslash\{0\}$ *and write* $z = \exp(r + \mathrm{i}\theta)$ *where* $r \in \mathbb{R}$ *and* $\theta \in (-\pi, \pi]$. *There exist two independent linear Brownian motions* β *and* γ *that start respectively from* r *and from* θ *under* P_z, *such that we have* P_z *a.s. for every* $t \geq 0$,

$$B_t = \exp(\beta_{H_t} + \mathrm{i}\,\gamma_{H_t}),$$

where

$$H_t = \int_0^t \frac{\mathrm{d}s}{|B_s|^2}.$$

Proof The "natural" method for proving Theorem 7.19 would be to apply a generalized version of Theorem 7.18 to a suitable determination of the complex logarithm. This, however, leads to some technical difficulties, and for this reason we will argue differently.

We may assume that $z = 1$ (and thus $r = \theta = 0$). The general case can be reduced to that one using scaling and rotational invariance of Brownian motion. Let $\Gamma_t = \Gamma_t^1 + \mathrm{i}\,\Gamma_t^2$ be a complex Brownian motion started from 0. By Theorem 7.18, we have a.s. for every $t \geq 0$,

$$\exp(\Gamma_t) = Z_{C_t}, \tag{7.5}$$

where Z is a complex Brownian motion started from 1, and for every $t \geq 0$,

$$C_t = \int_0^t \exp(2\,\Gamma_s^1) \, \mathrm{d}s.$$

Let $(H_s, s \geq 0)$ be the inverse function of $(C_t, t \geq 0)$, so that, by the formula for the derivative of an inverse function,

$$H_s = \int_0^s \exp(-2\,\Gamma^1_{H_u})\,\mathrm{d}u = \int_0^s \frac{\mathrm{d}u}{|Z_u|^2},$$

using the fact that $\exp(\Gamma^1_{H_u}) = |Z_u|$ in the last equality. By (7.5) with $t = H_s$, we now get

$$Z_s = \exp(\Gamma^1_{H_s} + \mathrm{i}\,\Gamma^2_{H_s}).$$

This is the desired result (since Γ^1 and Γ^2 are independent linear Brownian motions started from 0) except we did not get it for B but for the complex Brownian motion Z introduced in the course of the argument.

To complete the proof, we argue as follows. Write $\arg B_t$ for the continuous determination of the argument of B_t such that $\arg B_0 = 0$ (this makes sense since we know that B does not visit 0, a.s.). The statement of Theorem 7.19 (with $z = 1$) is equivalent to saying that, if we set

$$\beta_t = \log\,|B_{\inf\{s\geq 0:\int_0^s |B_u|^{-2}\mathrm{d}u>t\}}|,$$
$$\gamma_t = \arg B_{\inf\{s\geq 0:\int_0^s |B_u|^{-2}\mathrm{d}u>t\}},$$

then β and γ are two independent real Brownian motions started from 0. Note that β and γ are deterministic functions of B, and so their law must be the same if we replace B by the complex Brownian motion Z. This gives the desired result. □

Let us briefly comment on the skew-product representation. By writing H_t as the inverse of its inverse, we get

$$H_t = \inf\{s \geq 0 : \int_0^s \exp(2\beta_u)\,\mathrm{d}u > t\}, \tag{7.6}$$

and it follows that

$$\log|B_t| = \beta_{\inf\{s\geq 0:\int_0^s \exp(2\beta_u)\,\mathrm{d}u>t\}},$$

showing that $|B|$ is completely determined by the linear Brownian motion β. This is related to the fact that $|B_t|$ is a Markov process, namely a two-dimensional Bessel process (cf. Exercise 6.24, and Sect. 8.4.3 for a brief discussion of Bessel processes).

On the other hand, write $\theta_t = \arg B_t = \gamma_{H_t}$. Then θ_t is not a Markov process: At least intuitively, this can be understood by the fact that the past of θ up to time t gives information on the current value of $|B_t|$ (for instance if θ_t oscillates very rapidly just before t this indicates that $|B_t|$ should be small) and therefore on the future evolution of the process θ.

7.6 Asymptotic Laws of Planar Brownian Motion

In this section, we apply the skew-product decomposition to certain asymptotic results for planar Brownian motion. We fix the starting point $z \in \mathbb{C}\backslash\{0\}$ (we will often take $z = 1$) and for simplicity we write P instead of P_z. We keep the notation $\theta_t = \arg B_t$ for a continuous determination of the argument of B_t. Although the process θ_t is not a Markov process, the fact that it can be written as a linear Brownian motion time-changed by an independent increasing process allows one to derive a lot of information about its path properties. For instance, since $H_t \longrightarrow \infty$ as $t \to \infty$, we immediately get from Proposition 2.14 that, a.s.,

$$\limsup_{t\to\infty} \theta_t = +\infty,$$
$$\liminf_{t\to\infty} \theta_t = -\infty.$$

One may then ask about the typical size of θ_t when t is large. This is the celebrated Spitzer theorem on the winding number of planar Brownian motion.

Theorem 7.20 *Let $(\theta_t, t \geq 0)$ be a continuous determination of the argument of the complex Brownian motion B started from $z \in \mathbb{C}\backslash\{0\}$. Then*

$$\frac{2}{\log t}\,\theta_t$$

converges in distribution as $t \to \infty$ to a standard symmetric Cauchy distribution. In other words, for every $x \in \mathbb{R}$,

$$\lim_{t\to\infty} P\Big(\frac{2}{\log t}\,\theta_t \leq x\Big) = \int_{-\infty}^{x} \frac{\mathrm{d}y}{\pi(1+y^2)}.$$

Before proving the theorem, we will establish a key lemma. Without loss of generality, we may assume that $z = 1$ and $\theta_0 = 0$. We use the notation of Theorem 7.19, so that β and γ are two independent linear Brownian motions started from 0.

Lemma 7.21 *For every $\lambda > 0$, consider the scaled Brownian motion $\beta^{(\lambda)}_t = \frac{1}{\lambda}\,\beta_{\lambda^2 t}$, for every $t \geq 0$, and set $T^{(\lambda)}_1 = \inf\{t \geq 0 : \beta^{(\lambda)}_t = 1\}$. Then*

$$\frac{4}{(\log t)^2}\,H_t - T_1^{((\log t)/2)} \underset{t\to\infty}{\longrightarrow} 0$$

in probability.

Remark This shows in particular that $4(\log t)^{-2}H_t$ converges in distribution to the law of the hitting time of 1 by a linear Brownian motion started from 0.

Proof For every $a > 0$, set $T_a = \inf\{t \geq 0 : \beta_t = a\}$ and, for every $\lambda > 0$, $T_a^{(\lambda)} = \inf\{t \geq 0 : \beta_t^{(\lambda)} = a\}$. For the sake of simplicity, we write $\lambda_t = (\log t)/2$ throughout the proof and always assume that $t > 1$. We first verify that, for every $\varepsilon > 0$,

$$P\Big((\lambda_t)^{-2} H_t > T_{1+\varepsilon}^{(\lambda_t)}\Big) \underset{t\to\infty}{\longrightarrow} 0. \tag{7.7}$$

To this end, recall formula (7.6), which shows that

$$\begin{aligned}\{(\lambda_t)^{-2} H_t > T_{1+\varepsilon}^{(\lambda_t)}\} &= \Big\{ \int_0^{(\lambda_t)^2 T_{1+\varepsilon}^{(\lambda_t)}} \exp(2\beta_u)\,\mathrm{d}u < t \Big\} \\ &= \Big\{ \frac{1}{2\lambda_t} \log \int_0^{(\lambda_t)^2 T_{1+\varepsilon}^{(\lambda_t)}} \exp(2\beta_u)\,\mathrm{d}u < 1 \Big\}, \end{aligned} \tag{7.8}$$

since $2\lambda_t = \log t$. From the change of variables $u = (\lambda_t)^2 v$ in the integral, we get

$$\frac{1}{2\lambda_t} \log \int_0^{(\lambda_t)^2 T_{1+\varepsilon}^{(\lambda_t)}} \exp(2\beta_u)\,\mathrm{d}u = \frac{\log \lambda_t}{\lambda_t} + \frac{1}{2\lambda_t} \log \int_0^{T_{1+\varepsilon}^{(\lambda_t)}} \exp(2\lambda_t \beta_v^{(\lambda_t)})\,\mathrm{d}v.$$

We then note that, for every fixed $t > 1$, the quantity in the right-hand side has the same distribution as

$$\frac{\log \lambda_t}{\lambda_t} + \frac{1}{2\lambda_t} \log \int_0^{T_{1+\varepsilon}} \exp(2\lambda_t \beta_v)\,\mathrm{d}v \tag{7.9}$$

since for any $\lambda > 0$ the scaled Brownian motion $(\beta_t^{(\lambda)})_{t\geq 0}$ has the same distribution as $(\beta_t)_{t\geq 0}$. We then use the simple analytic fact stating that, for any continuous function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$, for any $s > 0$,

$$\frac{1}{2\lambda} \log \int_0^s \exp(2\lambda f(v))\,\mathrm{d}v \underset{\lambda\to\infty}{\longrightarrow} \sup_{0\leq r\leq s} f(r).$$

We leave the proof as an exercise for the reader. It follows that

$$\frac{1}{2\lambda} \log \int_0^{T_{1+\varepsilon}} \exp(2\lambda \beta_v)\,\mathrm{d}v \underset{\lambda\to\infty}{\longrightarrow} \sup_{0\leq r\leq T_{1+\varepsilon}} \beta_r = 1+\varepsilon,$$

a.s., and so the quantity in (7.9) converges to $1+\varepsilon$, a.s. as $t \to \infty$. Thus,

$$\frac{1}{2\lambda_t} \log \int_0^{(\lambda_t)^2 T_{1+\varepsilon}^{(\lambda_t)}} \exp(2\beta_u)\,\mathrm{d}u \underset{t\to\infty}{\longrightarrow} 1+\varepsilon$$

in probability. Hence the probability of the event in the right-hand side of (7.8) tends to 0, proving that (7.7) holds. The very same arguments show that, for every $\varepsilon \in (0, 1)$,

$$P\Big((\lambda_t)^{-2} H_t < T_{1-\varepsilon}^{(\lambda_t)}\Big) \underset{t\to\infty}{\longrightarrow} 0.$$

The desired result now follows, noting that $T_{1-\varepsilon}^{(\lambda_t)} < T_1^{(\lambda_t)} < T_{1+\varepsilon}^{(\lambda_t)}$ and that $T_{1+\varepsilon}^{(\lambda_t)} - T_{1-\varepsilon}^{(\lambda_t)}$ has the same distribution as $T_{1+\varepsilon} - T_{1-\varepsilon}$, which tends to 0 in probability when $\varepsilon \to 0$ (clearly, $T_{1-\varepsilon} \uparrow T_1$ as $\varepsilon \to 0$, and on the other hand $T_{1+\varepsilon} \downarrow T_1$ a.s. as $\varepsilon \to 0$, as a consequence of the strong Markov property at time T_1 and Proposition 2.14 (i)). □

Proof of Theorem 7.20 We keep the notation introduced in the preceding proof and also consider, for every $\lambda > 0$, the scaled Brownian motion $\gamma_t^{(\lambda)} = \frac{1}{\lambda}\gamma_{\lambda^2 t}$. Recalling our notation $\lambda_t = (\log t)/2$ for $t > 1$, we have

$$\frac{2}{\log t}\,\theta_t = \frac{1}{\lambda_t}\,\gamma_{H_t} = \gamma^{(\lambda_t)}_{(\lambda_t)^{-2}H_t}.$$

It then follows from Lemma 7.21 (using also the fact that the linear Brownian motions $\gamma^{(\lambda)}$ all have the same distribution) that

$$\frac{2}{\log t}\,\theta_t - \gamma^{(\lambda_t)}_{T_1^{(\lambda_t)}} \underset{t\to\infty}{\longrightarrow} 0,$$

in probability.

To complete the proof, we just have to notice that, for every fixed $\lambda > 0$, $\gamma^{(\lambda)}_{T_1^{(\lambda)}}$ has the standard symmetric Cauchy distribution. Indeed, since $(\beta^{(\lambda)}, \gamma^{(\lambda)})$ is a pair of independent linear Brownian motions started from 0, this variable has the same distribution as γ_{T_1}, and its characteristic distribution is computed by conditioning first with respect to T_1, and then using the Laplace transform of T_1 found in Example (c) after Corollary 3.24,

$$E[\exp(\mathrm{i}\xi\gamma_{T_1})] = E[\exp(-\frac{1}{2}\,\xi^2 T_1)] = \exp(-|\xi|),$$

which we recognize as the characteristic function of the Cauchy distribution. □

The skew-product decomposition and Lemma 7.21 can be used to derive other asymptotic laws. We know that the planar Brownian motion B started from $z \neq 0$ does not hit 0 a.s., but on the other hand the recurrence property ensures that $\min\{|B_s| : 0 \le s \le t\}$ tends to 0 as $t \to \infty$. One may then ask about the typical size of $\min\{|B_s| : 0 \le s \le t\}$ when t is large: In other words, at which speed does planar Brownian motion approach a point different from its starting point?

Proposition 7.22 *Consider the planar Brownian motion B started from $z \neq 0$. Then, for every $a > 0$,*

$$\lim_{t\to\infty} P\Big(\min_{0\le s\le t} |B_s| \le t^{-a/2}\Big) = \frac{1}{1+a}.$$

For instance, the probability that Brownian motion started from a nonzero initial value comes within distance $1/t$ from the origin before time t converges to $1/3$ as $t \to \infty$, a result which was not so easy to guess!

Proof Without loss of generality, we take $z = 1$. We keep the notation introduced in the proofs of Lemma 7.21 and Theorem 7.20. We observe that

$$\log\Big(\min_{0\le s\le t} |B_s|\Big) = \min_{0\le s\le t} \beta_{H_s} = \min_{0\le s\le H_t} \beta_s.$$

It follows that

$$\frac{2}{\log t}\log\Big(\min_{0\le s\le t} |B_s|\Big) = \frac{1}{\lambda_t}\min_{0\le s\le H_t} \beta_s = \min_{0\le s\le (\lambda_t)^{-2}} \beta_s^{(\lambda_t)}.$$

By Lemma 7.21,

$$\min_{0\le s\le (\lambda_t)^{-2}} \beta_s^{(\lambda_t)} - \min_{0\le s\le T_1^{(\lambda_t)}} \beta_s^{(\lambda_t)} \underset{t\to\infty}{\longrightarrow} 0$$

in probability. We conclude that we have the following convergence in distribution,

$$\frac{2}{\log t}\log\Big(\min_{0\le s\le t} |B_s|\Big) \underset{t\to\infty}{\longrightarrow} \min_{0\le s\le T_1} \beta_s,$$

where β is a linear Brownian motion started from 0 and $T_1 = \inf\{s \ge 0 : \beta_s = 1\}$. To complete the argument, note that

$$P\Big(\min_{0\le s\le T_1} \beta_s \le -a\Big) = P(T_{-a} < T_1),$$

if $T_{-a} = \inf\{s \ge 0 : \beta_s = -a\}$, and that $P(T_{-a} < T_1) = (1+a)^{-1}$ (cf. Sect. 3.4). □

As a last application of the skew-product decomposition, we state the Kallianpur–Robbins asymptotic law for the time spent by Brownian motion in a ball. Here the initial value can be arbitrary.

Theorem 7.23 *Let $z \in \mathbb{C}$ and $R > 0$. Then, under P_z,*

$$\frac{2}{\log t} \int_0^t \mathbf{1}_{\{|B_s|<R\}} \, \mathrm{d}s$$

converges in distribution as $t \to \infty$ to an exponential distribution with mean R^2.

We postpone our proof of Theorem 7.23 to the end of Chap. 9 since it relies in part on the theory of local times developed in that chapter.

Exercises

In all exercises, $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion starting from x under the probability measure P_x. Except in Exercise 7.28, we always assume that $d \geq 2$.

Exercise 7.24 Let $\mathscr{B}_1$ be the open unit ball of $\mathbb{R}^d$ ($d \geq 2$), and $\mathscr{B}_1^* = \mathscr{B}_1 \backslash \{0\}$. Let g be the continuous function defined on $\partial \mathscr{B}_1^*$ by $g(x) = 0$ if $|x| = 1$ and $g(0) = 1$. Prove that the Dirichlet problem in $\mathscr{B}_1^*$ with boundary condition g has no solution.

Exercise 7.25 (*Polar sets*) Throughout this exercise, we consider a nonempty compact subset K of $\mathbb{R}^d$ ($d \geq 2$). We set $T_K = \inf\{t \geq 0 : B_t \in K\}$. We say that K is *polar* if there exists an $x \in K^c$ such that $P_x(T_K < \infty) = 0$.

1. Using the strong Markov property as in the proof of Proposition 7.7 (ii), prove that the function $x \mapsto P_x(T_K < \infty)$ is harmonic on every connected component of K^c.
2. From now on until question **4.**, we assume that K is polar. Prove that K^c is connected, and that the property $P_x(T_K < \infty) = 0$ holds for every $x \in K^c$. (*Hint*: Observe that $\{x \in K^c : P_x(T_K < \infty) = 0\}$ is both open and closed).
3. Let D be a bounded domain containing K, and $D' = D \backslash K$. Prove that any bounded harmonic function h on D' can be extended to a harmonic function on D. Does this remain true if the word "bounded" is replaced by "positive"?
4. Set $g(x) = 0$ if $x \in \partial D$ and $g(x) = 1$ if $x \in \partial D' \backslash \partial D$. Prove that the Dirichlet problem in D' with boundary condition g has no solution. (Note that this generalizes the result of Exercise 7.24.)
5. If $\alpha \in (0, d]$, we say that the compact set K has zero α-dimensional Hausdorff measure if, for every $\varepsilon > 0$, we can find an integer $N_\varepsilon \geq 1$ and N_ε open balls $\mathscr{B}_{(1)}, \ldots, \mathscr{B}_{(N_\varepsilon)}$ with respective radii $r_{(1)}, \ldots r_{(N_\varepsilon)}$, such that K is contained in the union $\mathscr{B}_{(1)} \cup \cdots \cup \mathscr{B}_{(N_\varepsilon)}$, and

$$\sum_{j=1}^{N_\varepsilon} (r_j)^\alpha \leq \varepsilon.$$

Prove that if $d \geq 3$ and K has zero $d-2$-dimensional Hausdorff measure then K is polar.

Exercise 7.26 In this exercise, $d \geq 3$. Let K be a compact subset of the open unit ball of $\mathbb{R}^d$, and $T_K := \inf\{t \geq 0 : B_t \in K\}$. We assume that $D := \mathbb{R}^d \backslash K$ is connected. We also consider a function g defined and continuous on K. The goal of the exercise is to determine all functions $u : \bar{D} \to \mathbb{R}$ that satisfy:

(P) u is bounded and continuous on $\bar{D}$, harmonic on D, and $u(y) = g(y)$ if $y \in \partial D$.

(This is the Dirichlet problem in D, but in contrast with Sect. 7.3 above, D is unbounded here.) We fix an increasing sequence $(R_n)_{n\geq 1}$ of reals, with $R_1 \geq 1$ and $R_n \uparrow \infty$ as $n \to \infty$. For every $n \geq 1$, we set $T_{(n)} := \inf\{t \geq 0 : |B_t| \geq R_n\}$.

1. Suppose that u satisfies (P). Prove that, for every $n \geq 1$ and every $x \in D$ such that $|x| < R_n$,
$$u(x) = E_x[g(B_{T_K})\,\mathbf{1}_{\{T_K \leq T_{(n)}\}}] + E_x[u(B_{T_{(n)}})\,\mathbf{1}_{\{T_{(n)} \leq T_K\}}].$$
2. Show that, by replacing the sequence $(R_n)_{n\geq 1}$ with a subsequence if necessary, we may assume that there exists a constant $\alpha \in \mathbb{R}$ such that, for every $x \in D$,
$$\lim_{n\to\infty} E_x[u(B_{T_{(n)}})] = \alpha,$$
and that we then have
$$\lim_{|x|\to\infty} u(x) = \alpha.$$
3. Show that, for every $x \in D$,
$$u(x) = E_x[g(B_{T_K})\,\mathbf{1}_{\{T_K < \infty\}}] + \alpha\, P_x(T_K = \infty).$$
4. Assume that D satisfies the exterior cone condition at every $y \in \partial D$ (this is defined in the same way as when D is bounded). Show that, for any choice of $\alpha \in \mathbb{R}$, the formula of question **3.** gives a solution of the problem (P).

Exercise 7.27 Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic function. Use planar Brownian motion to prove that the set $\{f(z) : z \in \mathbb{C}\}$ is dense in $\mathbb{C}$. (Much more is true, since Picard's little theorem asserts that the complement of $\{f(z) : z \in \mathbb{C}\}$ in $\mathbb{C}$ contains at most one point: This can also be proved using Brownian motion, but the argument is more involved, see [12].)

Exercise 7.28 (*Feynman–Kac formula for Brownian motion*) This is a continuation of Exercise 6.26 in Chap. 6. With the notation of this exercise, we assume that $E = \mathbb{R}^d$ and $X_t = B_t$. Let v be a nonnegative function in $C_0(\mathbb{R}^d)$, and assume that v is continuously differentiable with bounded first derivatives. As in Exercise 6.26, set, for every $\varphi \in B(\mathbb{R}^d)$,
$$Q_t^*\varphi(x) = E_x\Big[\varphi(X_t)\, \exp\Big(-\int_0^t v(X_s)\,\mathrm{d}s\Big)\Big].$$

1. Using the formula derived in question **2.** of Exercise 6.26, prove that, for every $t > 0$, and every $\varphi \in C_0(\mathbb{R}^d)$, the function $Q_t^* \varphi$ is twice continuously differentiable on $\mathbb{R}^d$, and that $Q_t^* \varphi$ and its partial derivatives up to order 2 belong to $C_0(\mathbb{R}^d)$. Conclude that $Q_t^* \varphi \in D(L)$.
2. Let $\varphi \in C_0(\mathbb{R}^d)$ and set $u_t(x) = Q_t^* \varphi(x)$ for every $t > 0$ and $x \in \mathbb{R}^d$. Using question **3.** of Exercise 6.26, prove that, for every $x \in \mathbb{R}^d$, the function $t \mapsto u_t(x)$ is continuously differentiable on $(0, \infty)$, and

$$\frac{\partial u_t}{\partial t} = \frac{1}{2} \Delta u_t - v\, u_t.$$

Exercise 7.29 In this exercise $d = 2$ and $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$. Let $\alpha \in (0, \pi)$, and consider the open cone

$$\mathscr{C}_\alpha = \{r\, \mathrm{e}^{\mathrm{i}\theta} : r > 0, \theta \in (-\alpha, \alpha)\}.$$

Set $T := \inf\{t \geq 0 : B_t \notin \mathscr{C}_\alpha\}$.

1. Show that the law of $\log |B_T|$ under P_1 is the law of $\beta_{\inf\{t \geq 0: |\gamma_t| = \alpha\}}$, where β and γ are two independent linear Brownian motions started from 0.
2. Verify that, for every $\lambda \in \mathbb{R}$,

$$E_1[\mathrm{e}^{\mathrm{i}\lambda \log |B_T|}] = \frac{1}{\cosh(\alpha\lambda)}.$$

Notes and Comments

Connections between Brownian motion and partial differential equations have been known for a long time and motivated the study of this random process. A survey of the partial differential equations that can be solved in terms of Brownian motion can be found in the book of Durrett [18, Chapter 8]. The representation of Theorem 7.1 (written in terms of the Gaussian density) goes back to the ninetieth century and the work of Fourier and Laplace – see the references in [49]. The beautiful relations between Brownian motion and harmonic functions were discovered and studied by Kakutani [45, 46], and Hunt [33, 34] later studied the connections between potential theory and transient Markov processes (see the Blumenthal–Getoor book [5] for more on this topic). Nice accounts of the links between Brownian motion and classical potential theory can be found in the books by Port and Stone [69] and Doob [16] (see also Itô and McKean [42], Chung [9], and Chapters 3 and 8 of [62]). The conformal invariance of planar Brownian motion was stated by Lévy [54] with a very sketchy proof. Davis' paper [12] is a nice survey of relations between planar Brownian motion and analytic functions, see also Durrett's book [18], and the paper [28] by Getoor and Sharpe for a notion of conformal martingale that plays in martingale theory a role similar to that of analytic functions. Spitzer's theorem was obtained in the classical paper [75], and the Kallianpur–Robbins law was derived in [48]. A number of remarkable properties of planar Brownian motion had already been observed by Lévy [53] in 1940. We refer to Pitman and Yor [68] for a systematic study of asymptotic laws of planar Brownian motion. Our presentation closely follows [52], where other applications of the skew-product decomposition can be found.