Chapter 4
Continuous Semimartingales

Continuous semimartingales provide the general class of processes with continuous sample paths for which we will develop the theory of stochastic integration in the next chapter. By definition, a continuous semimartingale is the sum of a continuous local martingale and a (continuous) finite variation process. In the present chapter, we study separately these two classes of processes. We start with some preliminaries about deterministic functions with finite variation, before considering the corresponding random processes. We then define (continuous) local martingales and we construct the quadratic variation of a local martingale, which will play a fundamental role in the construction of stochastic integrals. We explain how properties of a local martingale are related to those of its quadratic variation. Finally, we introduce continuous semimartingales and their quadratic variation processes.

4.1 Finite Variation Processes

In this chapter, all processes are indexed by $\mathbb{R}_+$ and take real values. The first section provides a brief presentation of finite variation processes. We start by discussing functions with finite variation in a deterministic setting.

4.1.1 Functions with Finite Variation

In our discussion of functions with finite variation, we restrict our attention to *continuous* functions, as this is the case of interest in the subsequent developments. Recall that a signed measure on a compact interval $[0, T]$ is the difference of two finite positive measures on $[0, T]$.

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Definition 4.1 Let $T \geq 0$. A continuous function $a : [0,T] \longrightarrow \mathbb{R}$ such that $a(0) = 0$ is said to have *finite variation* if there exists a signed measure μ on $[0,T]$ such that $a(t) = \mu([0,t])$ for every $t \in [0,T]$.

The measure μ is then determined uniquely by a. Since a is continuous and $a(0) = 0$, it follows that μ has no atoms.

Remark The general definition of a function with finite variation does not require continuity nor the condition $a(0) = 0$. We impose these two conditions for convenience.

The decomposition of μ as a difference of two finite positive measures on $[0,T]$ is not unique, but there exists a unique decomposition $\mu = \mu_+ - \mu_-$ such that μ_+ and μ_- are supported on disjoint Borel sets. To get the existence of such a decomposition, start from an arbitrary decomposition $\mu = \mu_1 - \mu_2$, set $\nu = \mu_1 + \mu_2$ and then use the Radon–Nikodym theorem to find two nonnegative Borel functions h_1 and h_2 on $[0,T]$ such that

$$\mu_1(\mathrm{d}t) = h_1(t)\nu(\mathrm{d}t), \quad \mu_2(\mathrm{d}t) = h_2(t)\nu(\mathrm{d}t).$$

Then, if $h(t) = h_1(t) - h_2(t)$, we have

$$\mu(\mathrm{d}t) = h(t)\nu(\mathrm{d}t) = h(t)^+\nu(\mathrm{d}t) - h(t)^-\nu(\mathrm{d}t),$$

which gives the decomposition $\mu = \mu_+ - \mu_-$ with $\mu_+(\mathrm{d}t) = h(t)^+\nu(\mathrm{d}t)$, $\mu_-(\mathrm{d}t) = h(t)^-\nu(\mathrm{d}t)$, and the measures μ_+ and μ_- are supported respectively on the disjoint Borel sets $D_+ = \{t : h(t) > 0\}$ and $D_- = \{t : h(t) < 0\}$. The uniqueness of this decomposition $\mu = \mu_+ - \mu_-$ follows from the fact that we have necessarily, for every $A \in \mathscr{B}([0,T])$,

$$\mu_+(A) = \sup\{\mu(C) : C \in \mathscr{B}([0,T]),\ C \subset A\}.$$

We write $|\mu|$ for the (finite) positive measure $|\mu| = \mu_+ + \mu_-$. The measure $|\mu|$ is called the *total variation* of a. We have $|\mu(A)| \leq |\mu|(A)$ for every $A \in \mathscr{B}([0,T])$. Moreover, the Radon–Nikodym derivative of μ with respect to $|\mu|$ is

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|} = \mathbf{1}_{D_+} - \mathbf{1}_{D_-}.$$

The fact that $a(t) = \mu_+([0,t]) - \mu_-([0,t])$ shows that a is the difference of two monotone nondecreasing continuous functions that vanish at 0 (since μ has no atoms, the same holds for μ_+ of μ_-). Conversely, the difference of two monotone nondecreasing continuous functions that vanish at 0 has finite variation in the sense of the previous definition. Indeed, this follows from the well-known fact that the formula $g(t) = \theta([0,t])$, $t \in [0,T]$ induces a bijection between monotone nondecreasing right-continuous functions $g : [0,T] \longrightarrow \mathbb{R}_+$ and finite positive measures θ on $[0,T]$.

Let $f : [0, T] \longrightarrow \mathbb{R}$ be a measurable function such that $\int_{[0,T]} |f(s)|\, |\mu|(\mathrm{d}s) < \infty$. We set

$$\int_0^T f(s)\, \mathrm{d}a(s) = \int_{[0,T]} f(s)\, \mu(\mathrm{d}s),$$
$$\int_0^T f(s)\, |\mathrm{d}a(s)| = \int_{[0,T]} f(s)\, |\mu|(\mathrm{d}s).$$

Then the bound

$$\left|\int_0^T f(s)\, \mathrm{d}a(s)\right| \le \int_0^T |f(s)|\, |\mathrm{d}a(s)|$$

holds. By restricting a to $[0, t]$ (which amounts to restricting μ, μ_+, μ_-), we can define $\int_0^t f(s)\, \mathrm{d}a(s)$ for every $t \in [0, T]$, and we observe that the function $t \mapsto \int_0^t f(s)\, \mathrm{d}a(s)$ also has finite variation on $[0, T]$ (the associated measure is just $\mu'(\mathrm{d}s) = f(s)\mu(\mathrm{d}s)$).

Proposition 4.2 *For every* $t \in (0, T]$,

$$\int_0^t |\mathrm{d}a(s)| = \sup\left\{\sum_{i=1}^{p} |a(t_i) - a(t_{i-1})|\right\},$$

where the supremum is over all subdivisions $0 = t_0 < t_1 < \cdots < t_p = t$ *of* $[0, t]$. *More precisely, for any increasing sequence* $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ *of subdivisions of* $[0, t]$ *whose mesh tends to* 0, *we have*

$$\lim_{n\to\infty} \sum_{i=1}^{p_n} |a(t_i^n) - a(t_{i-1}^n)| = \int_0^t |\mathrm{d}a(s)|.$$

Remark In the usual presentation of functions with finite variation, one starts from the property that the supremum in the first display of the proposition is finite.

Proof Clearly, it is enough to treat the case $t = T$. The inequality $\ge$ in the first assertion is very easy since, for any subdivision $0 = t_0 < t_1 < \cdots < t_p = T$ of $[0, T]$,

$$|a(t_i) - a(t_{i-1})| = |\mu((t_{i-1}, t_i])| \le |\mu|((t_{i-1}, t_i]), \qquad \forall i \in \{1, \ldots, p\},$$

and

$$\sum_{i=1}^{p} |\mu|((t_{i-1}, t_i]) = |\mu|([0, t]) = \int_0^T |\mathrm{d}a(s)|.$$

In order to get the reverse inequality, it suffices to prove the second assertion. So we consider an increasing sequence $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = T$ of subdivisions of $[0, T]$, whose mesh $\max\{t_i^n - t_{i-1}^n : 1 \leq i \leq p_n\}$ tends to 0. Although we are proving a "deterministic" result, we will use a martingale argument. Leaving aside the trivial case where $|\mu| = 0$, we introduce the probability space $\Omega = [0, T]$, which is equipped with the Borel σ-field $\mathscr{B} = \mathscr{B}([0, T])$ and the probability measure $P(\mathrm{d}s) = (|\mu|([0, T]))^{-1}|\mu|(\mathrm{d}s)$. On this probability space, we consider the discrete filtration $(\mathscr{B}_n)_{n\geq 0}$ such that, for every integer $n \geq 0$, $\mathscr{B}_n$ is the σ-field generated by the intervals $(t_{i-1}^n, t_i^n]$, $1 \leq i \leq p_n$. We then set

$$X(s) = \mathbf{1}_{D_+}(s) - \mathbf{1}_{D_-}(s) = \frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(s),$$

and, for every $n \geq 0$,

$$X_n = E[X \mid \mathscr{B}_n].$$

Properties of conditional expectation show that X_n is constant on every interval $(t_{i-1}^n, t_i^n]$ and takes the value

$$\frac{\mu((t_{i-1}^n, t_i^n])}{|\mu|((t_{i-1}^n, t_i^n])} = \frac{a(t_i^n) - a(t_{i-1}^n)}{|\mu|((t_{i-1}^n, t_i^n])}$$

on this interval. On the other hand, the sequence (X_n) is a closed martingale, with respect to the discrete filtration $(\mathscr{B}_n)$. Since X is measurable with respect to $\mathscr{B} = \bigvee_n \mathscr{B}_n$, this martingale converges to X in L^1, by the convergence theorem for closed discrete martingales (see Appendix A2). In particular,

$$\lim_{n\to\infty} E[|X_n|] = E[|X|] = 1,$$

where the last equality is clear since $|X(s)| = 1$, $|\mu|(\mathrm{d}s)$ a.e. The desired result follows by noting that

$$E[|X_n|] = (|\mu|([0, T]))^{-1} \sum_{i=1}^{p_n} |a(t_i^n) - a(t_{i-1}^n)|,$$

and recalling that $|\mu|([0, T]) = \int_0^T |\mathrm{d}a(s)|$. □

We now give a useful approximation lemma for the integral of a continuous function with respect to a function with finite variation.

Lemma 4.3 *If $f : [0, T] \longrightarrow \mathbb{R}$ is a continuous function, and if $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = T$ is a sequence of subdivisions of $[0, T]$ whose mesh tends to 0, we have*

$$\int_0^T f(s)\, \mathrm{d}a(s) = \lim_{n\to\infty} \sum_{i=1}^{p_n} f(t_{i-1}^n)\, (a(t_i^n) - a(t_{i-1}^n)).$$

Proof Let f_n be defined on $[0, T]$ by $f_n(s) = f(t_{i-1}^n)$ if $s \in (t_{i-1}^n, t_i^n]$, $1 \le i \le p_n$, and $f_n(0) = f(0)$. Then,

$$\sum_{i=1}^{p_n} f(t_{i-1}^n)\, (a(t_i^n) - a(t_{i-1}^n)) = \int_{[0,T]} f_n(s)\, \mu(\mathrm{d}s),$$

and the desired result follows by dominated convergence since $f_n(s) \longrightarrow f(s)$ as $n \to \infty$, for every $s \in [0, T]$. □

We say that a function $a : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a finite variation function on $\mathbb{R}_+$ if the restriction of a to $[0, T]$ has finite variation on $[0, T]$, for every $T > 0$. Then there is a unique σ-finite (positive) measure on $\mathbb{R}_+$ whose restriction to every interval $[0, T]$ is the total variation measure of the restriction of a to $[0, T]$, and we write

$$\int_0^\infty f(s)\, |\mathrm{d}a(s)|$$

for the integral of a nonnegative Borel function f on $\mathbb{R}_+$ with respect to this σ-finite measure. Furthermore, we can define

$$\int_0^\infty f(s)\mathrm{d}a(s) = \lim_{T\to\infty} \int_0^T f(s)\mathrm{d}a(s) \in (-\infty, \infty)$$

for any real Borel function f on $\mathbb{R}_+$ such that $\int_0^\infty |f(s)||\mathrm{d}a(s)| < \infty$.

4.1.2 Finite Variation Processes

We now consider random variables and processes defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$.

Definition 4.4 An adapted process $A = (A_t)_{t\ge 0}$ is called a *finite variation process* if all its sample paths are finite variation functions on $\mathbb{R}_+$. If in addition the sample paths are nondecreasing functions, the process A is called an increasing process.

Remark In particular, $A_0 = 0$ and the sample paths of A are continuous – one can define finite variation processes with càdlàg sample paths, but in this book we consider **only the case of continuous sample paths**. Our special convention that the initial value of a finite variation process is 0 will be convenient for certain uniqueness statements.

If A is a finite variation process, the process

$$V_t = \int_0^t |\mathrm{d}A_s|$$

is an increasing process. Indeed, it is clear that the sample paths of V are nondecreasing functions (as well as continuous functions that vanish at $t = 0$). The fact that V_t is an $\mathscr{F}_t$-measurable random variable can be deduced from the second part of Proposition 4.2. Writing $A_t = \frac{1}{2}(V_t + A_t) - \frac{1}{2}(V_t - A_t)$ shows that any finite variation process can be written as the difference of two increasing processes (the converse is obvious).

Proposition 4.5 *Let A be a finite variation process, and let H be a progressive process such that*

$$\forall t \geq 0,\ \forall \omega \in \Omega,\ \int_0^t |H_s(\omega)|\,|\mathrm{d}A_s(\omega)| < \infty.$$

Then the process $H \cdot A = ((H \cdot A)_t)_{t \geq 0}$ defined by

$$(H \cdot A)_t = \int_0^t H_s\,\mathrm{d}A_s$$

is also a finite variation process.

Proof By the observations preceding the statement of Proposition 4.2, we know that the sample paths of $H \cdot A$ are finite variation functions. It remains to verify that the process $H \cdot A$ is adapted. To this end, it is enough to check that, if $t > 0$ is fixed, if $h : \Omega \times [0, t] \longrightarrow \mathbb{R}$ is measurable for the σ-field $\mathscr{F}_t \otimes \mathscr{B}([0, t])$, and if $\int_0^t |h(\omega, s)||\mathrm{d}A_s(\omega)| < \infty$ for every ω, then the variable $\int_0^t h(\omega, s)\mathrm{d}A_s(\omega)$ is $\mathscr{F}_t$-measurable.

If $h(\omega, s) = \mathbf{1}_{(u,v]}(s)\mathbf{1}_\Gamma(\omega)$ with $(u, v] \subset [0, t]$ and $\Gamma \in \mathscr{F}_t$, the result is immediate since $\int_0^t h(\omega, s)\mathrm{d}A_s(\omega) = 1_\Gamma(\omega)\,(A_v(\omega) - A_u(\omega))$ in that case. A monotone class argument (see Appendix A1) then gives the case $h = \mathbf{1}_G$, $G \in \mathscr{F}_t \otimes \mathscr{B}([0, t])$. Finally, in the general case, we observe that we can write h as a pointwise limit of a sequence of simple functions (i.e. finite linear combinations of indicator functions of measurable sets) h_n such that $|h_n| \leq |h|$ for every n, and that we then have $\int_0^t h_n(\omega, s)\mathrm{d}A_s(\omega) \longrightarrow \int_0^t h(\omega, s)\mathrm{d}A_s(\omega)$ by dominated convergence, for every $\omega \in \Omega$. □

Remarks

(i) It happens frequently that instead of the assumption of the proposition we have the weaker assumption

$$\text{a.s.} \quad \forall t \geq 0, \quad \int_0^t |H_s(\omega)|\, |\mathrm{d}A_s(\omega)| < \infty.$$

If the filtration is complete, we can still define $H \cdot A$ as a finite variation process under this weaker assumption. We replace H by the process H' defined by

$$H'_t(\omega) = \begin{cases} H_t(\omega) & \text{if } \int_0^n |H_s(\omega)|\, |\mathrm{d}A_s(\omega)| < \infty, \ \forall n \ , \\ 0 & \text{otherwise.} \end{cases}$$

Thanks to the fact that the filtration is complete, the process H' is still progressive, which allows us to define $H \cdot A = H' \cdot A$. We will use this extension of Proposition 4.5 implicitly in what follows.

(ii) Under appropriate assumptions (if H and K are progressive and $\int_0^t |H_s|\, |\mathrm{d}A_s| < \infty$, $\int_0^t |H_s K_s|\, |\mathrm{d}A_s| < \infty$ for every $t \geq 0$), we have the "associativity" property

$$K \cdot (H \cdot A) = (KH) \cdot A. \tag{4.1}$$

This indeed follows from the analogous deterministic result saying informally that $k(s)\, (h(s)\, \mu(ds)) = (k(s)h(s))\, \mu(ds)$ if $h(s)$ and $k(s)h(s)$ are integrable with respect to the signed measure μ on $[0, t]$.

An important special case of the proposition is the case where $A_t = t$. If H is a progressive process such that

$$\forall t \geq 0, \ \forall \omega \in \Omega, \quad \int_0^t |H_s(\omega)|\, \mathrm{d}s < \infty,$$

the process $\int_0^t H_s\, \mathrm{d}s$ is a finite variation process.

4.2 Continuous Local Martingales

We consider again a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. If T is a stopping time, and if $X = (X_t)_{t \geq 0}$ is an adapted process with continuous sample paths, we will write X^T for process X stopped at T, defined by $X_t^T = X_{t \wedge T}$ for every $t \geq 0$. It is useful to observe that, if S is another stopping time,

$$(X^T)^S = (X^S)^T = X^{S \wedge T}.$$

Definition 4.6 An adapted process $M = (M_t)_{t \geq 0}$ with continuous sample paths and such that $M_0 = 0$ a.s. is called a *continuous local martingale* if there exists a nondecreasing sequence $(T_n)_{n \geq 0}$ of stopping times such that $T_n \uparrow \infty$ (i.e. $T_n(\omega) \uparrow \infty$ for every ω) and, for every n, the stopped process M^{T_n} is a uniformly integrable martingale.

More generally, when we do not assume that $M_0 = 0$ a.s., we say that M is a continuous local martingale if the process $N_t = M_t - M_0$ is a continuous local martingale.

In all cases, we say that the sequence of stopping times (T_n) *reduces* M if $T_n \uparrow \infty$ and, for every n, the stopped process M^{T_n} is a uniformly integrable martingale.

Remarks

(i) We do not require in the definition of a continuous local martingale that the variables M_t are in L^1 (compare with the definition of martingales). In particular, the variable M_0 may be any $\mathscr{F}_0$-measurable random variable.

(ii) Any martingale with continuous sample paths is a continuous local martingale (see property (a) below) but the converse is false, and for this reason we will sometimes speak of "true martingales" to emphasize the difference with local martingales. Let us give a few examples of continuous local martingales which are not (true) martingales. If B is an $(\mathscr{F}_t)$-Brownian motion started from 0, and Z is an $\mathscr{F}_0$-measurable random variable, the process $M_t = Z + B_t$ is always a continuous local martingale, but is not a martingale if $E[|Z|] = \infty$. If we require the property $M_0 = 0$, we can also consider $M_t = ZB_t$, which is always a continuous local martingale (see Exercise 4.22) but is not a martingale if $E[|Z|] = \infty$. For a less artificial example, we refer to question (**8**) of Exercise 5.33.

(iii) One can define a notion of local martingale with càdlàg sample paths. In this course, however, we consider only continuous local martingales.

The following properties are easily established.

Properties of continuous local martingales.

(a) A martingale with continuous sample paths is a continuous local martingale, and the sequence $T_n = n$ reduces M.

(b) In the definition of a continuous local martingale starting from 0, one can replace "uniformly integrable martingale" by "martingale" (indeed, one can then observe that $M^{T_n \wedge n}$ is uniformly integrable, and we still have $T_n \wedge n \uparrow \infty$).

(c) If M is a continuous local martingale, then, for every stopping time T, M^T is a continuous local martingale (this follows from Corollary 3.24).

(d) If (T_n) reduces M and if (S_n) is a sequence of stopping times such that $S_n \uparrow \infty$, then the sequence $(T_n \wedge S_n)$ also reduces M (use Corollary 3.24 again).

(e) The space of all continuous local martingales is a vector space (to check stability under addition, note that if M and M' are two continuous local martingales such that $M_0 = 0$ and $M'_0 = 0$, if the sequence (T_n) reduces M and if the sequence (T'_n) reduces M', property (d) shows that the sequence $T_n \wedge T'_n$ reduces $M + M'$).

The next proposition gives three other useful properties of local martingales.

Proposition 4.7

(i) *A nonnegative continuous local martingale M such that $M_0 \in L^1$ is a supermartingale.*
(ii) *A continuous local martingale M such that there exists a random variable $Z \in L^1$ with $|M_t| \leq Z$ for every $t \geq 0$ (in particular a bounded continuous local martingale) is a uniformly integrable martingale.*
(iii) *If M is a continuous local martingale and $M_0 = 0$ (or more generally $M_0 \in L^1$), the sequence of stopping times*

$$T_n = \inf\{t \geq 0 : |M_t| \geq n\}$$

reduces M.

Proof

(i) Write $M_t = M_0 + N_t$. By definition, there exists a sequence (T_n) of stopping times that reduces N. Then, if $s \leq t$, we have for every n,

$$N_{s\wedge T_n} = E[N_{t\wedge T_n} \mid \mathscr{F}_s].$$

We can add on both sides the random variable M_0 (which is $\mathscr{F}_0$-measurable and in L^1 by assumption), and we get

$$M_{s\wedge T_n} = E[M_{t\wedge T_n} \mid \mathscr{F}_s].$$

Since M takes nonnegative values, we can now let n tend to ∞ and apply the version of Fatou's lemma for conditional expectations, which gives

$$M_s \geq E[M_t \mid \mathscr{F}_s].$$

Taking $s = 0$, we get $E[M_t] \leq E[M_0] < \infty$, hence $M_t \in L^1$ for every $t \geq 0$. The previous inequality now shows that M is a supermartingale.

(ii) By the same argument as in (i), we get for $0 \leq s \leq t$,

$$M_{s\wedge T_n} = E[M_{t\wedge T_n} \mid \mathscr{F}_s]. \tag{4.2}$$

Since $|M_{t\wedge T_n}| \leq Z$, we can use dominated convergence to obtain that the sequence $M_{t\wedge T_n}$ converges to M_t in L^1. We can thus pass to the limit $n \to \infty$ in (4.2), and get that $M_s = E[M_t \mid \mathscr{F}_s]$.

(iii) Suppose that $M_0 = 0$. The random times T_n are stopping times by Proposition 3.9. The desired result is an immediate consequence of (ii) since M^{T_n} is a continuous local martingale and $|M^{T_n}| \leq n$. If we only assume that $M_0 \in L^1$, we observe that M^{T_n} is dominated by $n + |M_0|$. □

Remark Considering property (ii) of the proposition, one might expect that a continuous local martingale M such that the collection $(M_t)_{t\geq 0}$ is uniformly integrable (or even a continuous local martingale satisfying the stronger property of being bounded in L^p for some $p > 1$) is automatically a martingale. This is incorrect!! For instance, if B is a three-dimensional Brownian motion started from $x \neq 0$, the process $M_t = 1/|B_t|$ is a continuous local martingale bounded in L^2, but is not a martingale: see Exercise 5.33.

Theorem 4.8 *Let M be a continuous local martingale. Assume that M is also a finite variation process (in particular $M_0 = 0$). Then $M_t = 0$ for every $t \geq 0$, a.s.*

Proof Set

$$\tau_n = \inf\{t \geq 0 : \int_0^t |\mathrm{d}M_s| \geq n\}$$

for every integer $n \geq 0$. By Proposition 3.9, τ_n is a stopping time (recall that $\int_0^t |\mathrm{d}M_s|$ is an increasing process if M is a finite variation process).

Fix $n \geq 0$ and set $N = M^{\tau_n}$. Note that, for every $t \geq 0$,

$$|N_t| = |M_{t\wedge\tau_n}| \leq \int_0^{t\wedge\tau_n} |\mathrm{d}M_s| \leq n.$$

By Proposition 4.7, N is a (bounded) martingale. Let $t > 0$ and let $0 = t_0 < t_1 < \cdots < t_p = t$ be any subdivision of $[0, t]$. Then, from Proposition 3.14, we have

$$\begin{aligned} E[N_t^2] &= \sum_{i=1}^{p} E[(N_{t_i} - N_{t_{i-1}})^2] \\ &\leq E\Big[\Big(\sup_{1\leq i\leq p} |N_{t_i} - N_{t_{i-1}}|\Big) \sum_{i=1}^{p} |N_{t_i} - N_{t_{i-1}}|\Big] \\ &\leq n\, E\Big[\sup_{1\leq i\leq p} |N_{t_i} - N_{t_{i-1}}|\Big] \end{aligned}$$

noting that $\int_0^t |\mathrm{d}N_s| \leq n$ by the definition of τ_n, and using Proposition 4.2.

We now apply the preceding bound to a sequence $0 = t_0^k < t_1^k < \cdots < t_{p_k}^k = t$ of subdivisions of $[0, t]$ whose mesh tends to 0. Using the continuity of sample paths, and the fact that N is bounded (to justify dominated convergence), we get

$$\lim_{k\to\infty} E\Big[\sup_{1\le i\le p_k} |N_{t_i^k} - N_{t_{i-1}^k}|\Big] = 0.$$

We then conclude that $E[N_t^2] = 0$, hence $M_{t\wedge\tau_n} = 0$ a.s. Letting n tend to ∞, we get that $M_t = 0$ a.s. □

4.3 The Quadratic Variation of a Continuous Local Martingale

From now on until the end of this chapter (and in the next chapter), we assume that the filtration $(\mathscr{F}_t)$ is complete. The next theorem will play a very important role in forthcoming developments.

Theorem 4.9 *Let $M = (M_t)_{t\ge0}$ be a continuous local martingale. There exists an increasing process denoted by $(\langle M,M\rangle_t)_{t\ge0}$, which is unique up to indistinguishability, such that $M_t^2 - \langle M,M\rangle_t$ is a continuous local martingale. Furthermore, for every fixed $t > 0$, if $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ is an increasing sequence of subdivisions of $[0,t]$ with mesh tending to 0, we have*

$$\langle M,M\rangle_t = \lim_{n\to\infty} \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 \tag{4.3}$$

in probability. The process $\langle M,M\rangle$ is called the quadratic variation of M.

Let us immediately mention an important special case. If $M = B$ is an $(\mathscr{F}_t)$-Brownian motion (see Definition 3.11) then B is a martingale with continuous sample paths, hence a continuous local martingale. Then by comparing (4.3) with Proposition 2.16, we get that, for every $t \ge 0$,

$$\langle B,B\rangle_t = t.$$

So the quadratic variation of a Brownian motion is the simplest increasing process one can imagine.

Remarks

(i) We observe that the process $\langle M,M\rangle$ does not depend on the initial value M_0, but only on the increments of M: if $M_t = M_0 + N_t$, we have $\langle M,M\rangle = \langle N,N\rangle$. This is obvious from the second assertion of the theorem, and this will also be clear from the proof that follows.

(ii) In the second assertion of the theorem, it is in fact not necessary to assume that the sequence of subdivisions is increasing.

Proof We start by proving the first assertion. Uniqueness is an easy consequence of Theorem 4.8. Indeed, let A and A' be two increasing processes satisfying the condition given in the statement. Then the process $A_t - A'_t = (M_t^2 - A'_t) - (M_t^2 - A_t)$ is both a continuous local martingale and a finite variation process. It follows that $A - A' = 0$.

In order to prove existence, consider first the case where $M_0 = 0$ and M is bounded (hence M is a true martingale, by Proposition 4.7 (ii)). Fix $K > 0$ and an increasing sequence $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = K$ of subdivisions of $[0, K]$ with mesh tending to 0.

We observe that, for every $0 \le r < s$ and for every bounded $\mathscr{F}_r$-measurable variable Z, the process

$$N_t = Z(M_{s\wedge t} - M_{r\wedge t})$$

is a martingale (the reader is invited to write down the easy proof!). It follows that, for every n, the process

$$X_t^n = \sum_{i=1}^{p_n} M_{t_{i-1}^n}(M_{t_i^n \wedge t} - M_{t_{i-1}^n \wedge t})$$

is a (bounded) martingale. The reason for considering these martingales comes from the following identity, which results from a simple calculation: for every n, for every $j \in \{0, 1, \ldots, p_n\}$,

$$M_{t_j^n}^2 - 2X_{t_j^n}^n = \sum_{i=1}^{j} (M_{t_i^n} - M_{t_{i-1}^n})^2. \tag{4.4}$$

Lemma 4.10 *We have*

$$\lim_{n,m\to\infty} E[(X_K^n - X_K^m)^2] = 0.$$

Proof of the lemma Let us fix $n \le m$ and evaluate the product $E[X_K^n X_K^m]$. This product is equal to

$$\sum_{i=1}^{p_n} \sum_{j=1}^{p_m} E[M_{t_{i-1}^n}(M_{t_i^n} - M_{t_{i-1}^n}) M_{t_{j-1}^m}(M_{t_j^m} - M_{t_{j-1}^m})].$$

In this double sum, the only terms that may be nonzero are those corresponding to indices i and j such that the interval $(t_{j-1}^m, t_j^m]$ is contained in $(t_{i-1}^n, t_i^n]$. Indeed, suppose that $t_i^n \le t_{j-1}^m$ (the symmetric case $t_j^m \le t_{i-1}^n$ is treated in an analogous way).

Then, conditioning on the σ-field $\mathscr{F}_{t^m_{j-1}}$, we have

$$\begin{aligned}&E[M_{t^n_{i-1}}(M_{t^n_i}-M_{t^n_{i-1}})\,M_{t^m_{j-1}}(M_{t^m_j}-M_{t^m_{j-1}})]\\&\quad=E[M_{t^n_{i-1}}(M_{t^n_i}-M_{t^n_{i-1}})\,M_{t^m_{j-1}}\,E[M_{t^m_j}-M_{t^m_{j-1}}\mid\mathscr{F}_{t^m_{j-1}}]]=0.\end{aligned}$$

For every $j=1,\dots,p_m$, write $i_{n,m}(j)$ for the unique index i such that $(t^m_{j-1},t^m_j]\subset(t^n_{i-1},t^n_i]$. It follows from the previous considerations that

$$E[X^n_K X^m_K]=\sum_{1\le j\le p_m,\ i=i_{n,m}(j)}E[M_{t^n_{i-1}}(M_{t^n_i}-M_{t^n_{i-1}})\,M_{t^m_{j-1}}(M_{t^m_j}-M_{t^m_{j-1}})].$$

In each term $E[M_{t^n_{i-1}}(M_{t^n_i}-M_{t^n_{i-1}})\,M_{t^m_{j-1}}(M_{t^m_j}-M_{t^m_{j-1}})]$, we can now decompose

$$M_{t^n_i}-M_{t^n_{i-1}}=\sum_{k:i_{n,m}(k)=i}(M_{t^m_k}-M_{t^m_{k-1}})$$

and we observe that, if k is such that $i_{n,m}(k)=i$ but $k\neq j$,

$$E[M_{t^n_{i-1}}(M_{t^m_k}-M_{t^m_{k-1}})\,M_{t^m_{j-1}}(M_{t^m_j}-M_{t^m_{j-1}})]=0$$

(condition on $\mathscr{F}_{t^m_{k-1}}$ if $k>j$ and on $\mathscr{F}_{t^m_{j-1}}$ if $k<j$). The only case that remains is $k=j$, and we have thus obtained

$$E[X^n_K X^m_K]=\sum_{1\le j\le p_m,\ i=i_{n,m}(j)}E[M_{t^n_{i-1}}M_{t^m_{j-1}}(M_{t^m_j}-M_{t^m_{j-1}})^2].$$

As a special case of this relation, we have

$$E[(X^m_K)^2]=\sum_{1\le j\le p_m}E[M^2_{t^m_{j-1}}(M_{t^m_j}-M_{t^m_{j-1}})^2].$$

Furthermore,

$$\begin{aligned}E[(X^n_K)^2]&=\sum_{1\le i\le p_n}E[M^2_{t^n_{i-1}}(M_{t^n_i}-M_{t^n_{i-1}})^2]\\&=\sum_{1\le i\le p_n}E[M^2_{t^n_{i-1}}E[(M_{t^n_i}-M_{t^n_{i-1}})^2\mid\mathscr{F}_{t^n_{i-1}}]]\\&=\sum_{1\le i\le p_n}E\Big[M^2_{t^n_{i-1}}\sum_{j:i_{n,m}(j)=i}E[(M_{t^m_j}-M_{t^m_{j-1}})^2\mid\mathscr{F}_{t^n_{i-1}}]\Big]\\&=\sum_{1\le j\le p_m,\ i=i_{n,m}(j)}E[M^2_{t^n_{i-1}}(M_{t^m_j}-M_{t^m_{j-1}})^2],\end{aligned}$$

where we have used Proposition 3.14 in the third equality.

If we combine the last three displays, we get

$$E[(X^n_K - X^m_K)^2] = E\Big[\sum_{1\le j\le p_m,\ i=i_{n,m}(j)} (M_{t^n_{i-1}} - M_{t^m_{j-1}})^2\,(M_{t^m_j} - M_{t^m_{j-1}})^2\Big].$$

Using the Cauchy–Schwarz inequality, we then have

$$\begin{aligned} E[(X^n_K - X^m_K)^2] \le E\Big[\sup_{1\le j\le p_m,\ i=i_{n,m}(j)} (M_{t^n_{i-1}} - M_{t^m_{j-1}})^4\Big]^{1/2} \\ \times E\Big[\Big(\sum_{1\le j\le p_m} (M_{t^m_j} - M_{t^m_{j-1}})^2\Big)^2\Big]^{1/2}. \end{aligned}$$

By the continuity of sample paths (together with the fact that the mesh of our subdivisions tends to 0) and dominated convergence, we have

$$\lim_{n,m\to\infty,\ n\le m} E\Big[\sup_{1\le j\le p_m,\ i=i_{n,m}(j)} (M_{t^n_{i-1}} - M_{t^m_{j-1}})^4\Big] = 0.$$

To complete the proof of the lemma, it is then enough to prove the existence of a finite constant C such that, for every m,

$$E\Big[\Big(\sum_{1\le j\le p_m} (M_{t^m_j} - M_{t^m_{j-1}})^2\Big)^2\Big] \le C. \tag{4.5}$$

Let A be a constant such that $|M_t| \le A$ for every $t \ge 0$. Expanding the square and using Proposition 3.14 twice, we have

$$\begin{aligned} &E\Big[\Big(\sum_{1\le j\le p_m} (M_{t^m_j} - M_{t^m_{j-1}})^2\Big)^2\Big] \\ &= E\Big[\sum_{1\le j\le p_m} (M_{t^m_j} - M_{t^m_{j-1}})^4\Big] + 2E\Big[\sum_{1\le j<k\le p_m} (M_{t^m_j} - M_{t^m_{j-1}})^2 (M_{t^m_k} - M_{t^m_{k-1}})^2\Big] \\ &\le 4A^2 E\Big[\sum_{1\le j\le p_m} (M_{t^m_j} - M_{t^m_{j-1}})^2\Big] \\ &\qquad + 2\sum_{j=1}^{p_m-1} E\Big[(M_{t^m_j} - M_{t^m_{j-1}})^2 E\Big[\sum_{k=j+1}^{p_m} (M_{t^m_k} - M_{t^m_{k-1}})^2 \,\Big|\, \mathscr{F}_{t^m_j}\Big]\Big] \\ &= 4A^2 E\Big[\sum_{1\le j\le p_m} (M_{t^m_j} - M_{t^m_{j-1}})^2\Big] \\ &\qquad + 2\sum_{j=1}^{p_m-1} E\Big[(M_{t^m_j} - M_{t^m_{j-1}})^2\, E[(M_K - M_{t^m_j})^2 \mid \mathscr{F}_{t^m_j}]\Big] \end{aligned}$$

$$\leq 12A^2 E\Big[\sum_{1\leq j\leq p_m} (M_{t_j^m} - M_{t_{j-1}^m})^2\Big]$$
$$= 12A^2 E[(M_K - M_0)^2]$$
$$\leq 48A^4$$

which gives the bound (4.5) with $C = 48A^4$. This completes the proof. □

We now return to the proof of the theorem. Thanks to Doob's inequality in L^2 (Proposition 3.15 (ii)), and to Lemma 4.10, we have

$$\lim_{n,m\to\infty} E\Big[\sup_{t\leq K}(X_t^n - X_t^m)^2\Big] = 0. \tag{4.6}$$

In particular, for every $t \in [0, K]$, $(X_t^n)_{n\geq 1}$ is a Cauchy sequence in L^2 and thus converges in L^2. We want to argue that the limit yields a process Y indexed by $[0, K]$ with continuous sample paths. To see this, we note that (4.6) allows us find a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that, for every $k \geq 1$,

$$E\Big[\sup_{t\leq K}(X_t^{n_{k+1}} - X_t^{n_k})^2\Big] \leq 2^{-k}.$$

This implies that

$$E\Big[\sum_{k=1}^{\infty} \sup_{t\leq K} |X_t^{n_{k+1}} - X_t^{n_k}|\Big] < \infty$$

and thus

$$\sum_{k=1}^{\infty} \sup_{t\leq K} |X_t^{n_{k+1}} - X_t^{n_k}| < \infty\,, \quad \text{a.s.}$$

Consequently, except on the negligible set $\mathscr{N}$ where the series in the last display diverges, the sequence of random functions $(X_t^{n_k}, 0 \leq t \leq K)$ converges uniformly on $[0, K]$ as $k \to \infty$, and the limiting random function is continuous by uniform convergence. We can thus set $Y_t(\omega) = \lim X_t^{n_k}(\omega)$, for every $t \in [0, K]$, if $\omega \in \Omega\backslash\mathscr{N}$, and $Y_t(\omega) = 0$, for every $t \in [0, K]$, if $\omega \in \mathscr{N}$. The process $(Y_t)_{0\leq t\leq K}$ has continuous sample paths and Y_t is $\mathscr{F}_t$-measurable for every $t \in [0, K]$ (here we use the fact that the filtration is complete, which ensures that $\mathscr{N} \in \mathscr{F}_t$ for every $t \geq 0$). Furthermore, since the L^2-limit of $(X_t^n)_{n\geq 1}$ must coincide with the a.s. limit of a subsequence, Y_t is also the limit of X_t^n in L^2, for every $t \in [0, K]$, and we can pass to the limit in the martingale property for X^n, to obtain that $E[Y_t \mid \mathscr{F}_s] = Y_s$ for every $0 \leq s \leq t \leq K$. It follows that $(Y_{t\wedge K})_{t\geq 0}$ is a martingale with continuous sample paths.

On the other hand, the identity (4.4) shows that the sample paths of the process $M_t^2-2X_t^n$ are nondecreasing along the finite sequence $(t_i^n, 0 \le i \le p_n)$. By passing to the limit $n \to \infty$ along the sequence $(n_k)_{k\ge 1}$, we get that the sample paths of $M_t^2 - 2Y_t$ are nondecreasing on $[0,K]$, except maybe on the negligible set $\mathscr{N}$. For every $t \in [0,K]$, we set $A_t^{(K)} = M_t^2 - 2Y_t$ on $\Omega\backslash\mathscr{N}$, and $A_t^{(K)} = 0$ on $\mathscr{N}$. Then $A_0^{(K)} = 0$, $A_t^{(K)}$ is $\mathscr{F}_t$-measurable for every $t \in [0,K]$, $A^{(K)}$ has nondecreasing continuous sample paths, and $(M_{t\wedge K}^2 - A_{t\wedge K}^{(K)})_{t\ge 0}$ is a martingale.

We apply the preceding considerations with $K = \ell$, for every integer $\ell \ge 1$, and we get a process $(A_t^{(\ell)})_{0\le t\le \ell}$. We then observe that, for every $\ell \ge 1$, $A_{t\wedge\ell}^{(\ell+1)} = A_{t\wedge\ell}^{(\ell)}$ for every $t \ge 0$, a.s., by the uniqueness argument explained at the beginning of the proof. It follows that we can define an increasing process $\langle M,M\rangle$ such that $\langle M,M\rangle_t = A_t^{(\ell)}$ for every $t \in [0,\ell]$ and every $\ell \ge 1$, a.s., and clearly $M_t^2 - \langle M,M\rangle_t$ is a martingale.

In order to get (4.3), we observe that, if $K > 0$ and the sequence of subdivisions $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = K$ are fixed as in the beginning of the proof, the process $A_{t\wedge K}^{(K)}$ must be indistinguishable from $\langle M,M\rangle_{t\wedge K}$, again by the uniqueness argument (we know that both $M_{t\wedge K}^2 - A_{t\wedge K}^{(K)}$ and $M_{t\wedge K}^2 - \langle M,M\rangle_{t\wedge K}$ are martingales). In particular, we have $\langle M,M\rangle_K = A_K^{(K)}$ a.s. Then, from (4.4) with $j = p_n$, and the fact that X_K^n converges in L^2 to $Y_K = \frac{1}{2}(M_K^2 - A_K^{(K)})$, we get that

$$\lim_{n\to\infty} \sum_{j=1}^{p_n} (M_{t_j^n} - M_{t_{j-1}^n})^2 = \langle M,M\rangle_K$$

in L^2. This completes the proof of the theorem in the case when $M_0 = 0$ and M is bounded.

Let us consider the general case. Writing $M_t = M_0 + N_t$, so that $M_t^2 = M_0^2 + 2\,M_0N_t + N_t^2$, and noting that M_0N_t is a continuous local martingale (see Exercise 4.22), we see that we may assume that $M_0 = 0$. We then set

$$T_n = \inf\{t \ge 0 : |M_t| \ge n\}$$

and we can apply the bounded case to the stopped martingales M^{T_n}. Set $A^{[n]} = \langle M^{T_n}, M^{T_n}\rangle$. The uniqueness part of the theorem shows that the processes $A_{t\wedge T_n}^{[n+1]}$ and $A_t^{[n]}$ are indistinguishable. It follows that there exists an increasing process A such that, for every n, the processes $A_{t\wedge T_n}$ and $A_t^{[n]}$ are indistinguishable. By construction, $M_{t\wedge T_n}^2 - A_{t\wedge T_n}$ is a martingale for every n, which precisely implies that $M_t^2 - A_t$ is a continuous local martingale. We take $\langle M,M\rangle_t = A_t$, which completes the proof of the existence part of the theorem.

Finally, to get (4.3), it suffices to consider the case $M_0 = 0$. The bounded case then shows that (4.3) holds if M and $\langle M,M\rangle_t$ are replaced respectively by M^{T_n} and $\langle M,M\rangle_{t\wedge T_n}$ (even with convergence in L^2). Then it is enough to observe that, for every $t > 0$, $P(t \le T_n)$ converges to 1 when $n \to \infty$. $\square$

Proposition 4.11 *Let M be a continuous local martingale and let T be a stopping time. Then we have a.s. for every $t \geq 0$,*

$$\langle M^T, M^T \rangle_t = \langle M, M \rangle_{t \wedge T}.$$

This follows from the fact that $M^2_{t\wedge T} - \langle M, M \rangle_{t\wedge T}$ is a continuous local martingale (cf. property (c) of continuous local martingales).

Proposition 4.12 *Let M be a continuous local martingale such that $M_0 = 0$. Then we have $\langle M, M \rangle = 0$ if and only if $M = 0$.*

Proof Suppose that $\langle M, M \rangle = 0$. Then M_t^2 is a nonnegative continuous local martingale and, by Proposition 4.7 (i), M_t^2 is a supermartingale, hence $E[M_t^2] \leq E[M_0^2] = 0$, so that $M_t = 0$ for every t. The converse is obvious. □

The next theorem shows that properties of a continuous local martingale are closely related to those of its quadratic variation. If A is an increasing process, A_∞ denotes the increasing limit of A_t as $t \to \infty$ (this limit always exists in $[0, \infty]$).

Theorem 4.13 *Let M be a continuous local martingale with $M_0 \in L^2$.*

(i) *The following are equivalent:*

  (a) *M is a (true) martingale bounded in L^2.*
  (b) *$E[\langle M, M \rangle_\infty] < \infty$.*

  Furthermore, if these properties hold, the process $M_t^2 - \langle M, M \rangle_t$ is a uniformly integrable martingale, and in particular $E[M_\infty^2] = E[M_0^2] + E[\langle M, M \rangle_\infty]$.

(ii) *The following are equivalent:*

  (a) *M is a (true) martingale and $M_t \in L^2$ for every $t \geq 0$.*
  (b) *$E[\langle M, M \rangle_t] < \infty$ for every $t \geq 0$.*

Furthermore, if these properties hold, the process $M_t^2 - \langle M, M \rangle_t$ is a martingale.

Remark In property (a) of (i) (or of (ii)), it is essential to suppose that M is a martingale, and not only a continuous local martingale. Doob's inequality used in the following proof is not valid in general for a continuous local martingale!

Proof

(i) Replacing M by $M - M_0$, we may assume that $M_0 = 0$ in the proof. Let us first assume that M is a martingale bounded in L^2. Doob's inequality in L^2 (Proposition 3.15 (ii)) shows that, for every $T > 0$,

$$E\Big[\sup_{0 \leq t \leq T} M_t^2\Big] \leq 4\, E[M_T^2].$$

By letting T go to ∞, we have

$$E\Big[\sup_{t \geq 0} M_t^2\Big] \leq 4 \sup_{t \geq 0} E[M_t^2] =: C < \infty.$$

Set $S_n = \inf\{t \geq 0 : \langle M, M \rangle_t \geq n\}$. Then the continuous local martingale $M^2_{t\wedge S_n} - \langle M, M \rangle_{t\wedge S_n}$ is dominated by the variable

$$\sup_{s\geq 0} M_s^2 + n,$$

which is integrable. From Proposition 4.7 (ii), we get that this continuous local martingale is a uniformly integrable martingale, hence

$$E[\langle M, M \rangle_{t\wedge S_n}] = E[M^2_{t\wedge S_n}] \leq E\Big[\sup_{s\geq 0} M_s^2\Big] \leq C.$$

By letting n, and then t tend to infinity, and using monotone convergence, we get $E[\langle M, M \rangle_\infty] \leq C < \infty$.

Conversely, assume that $E[\langle M, M \rangle_\infty] < \infty$. Set $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$. Then the continuous local martingale $M^2_{t\wedge T_n} - \langle M, M \rangle_{t\wedge T_n}$ is dominated by the variable

$$n^2 + \langle M, M \rangle_\infty,$$

which is integrable. From Proposition 4.7 (ii) again, this continuous local martingale is a uniformly integrable martingale, hence, for every $t \geq 0$,

$$E[M^2_{t\wedge T_n}] = E[\langle M, M \rangle_{t\wedge T_n}] \leq E[\langle M, M \rangle_\infty] =: C' < \infty.$$

By letting $n \to \infty$ and using Fatou's lemma, we get $E[M_t^2] \leq C'$, so that the collection $(M_t)_{t\geq 0}$ is bounded in L^2. We have not yet verified that $(M_t)_{t\geq 0}$ is a martingale. However, the previous bound on $E[M^2_{t\wedge T_n}]$ shows that the sequence $(M_{t\wedge T_n})_{n\geq 1}$ is uniformly integrable, and therefore converges both a.s. and in L^1 to M_t, for every $t \geq 0$. Recalling that M^{T_n} is a martingale (Proposition 4.7 (iii)), the L^1-convergence allows us to pass to the limit $n \to \infty$ in the martingale property $E[M_{t\wedge T_n} \mid \mathscr{F}_s] = M_{s\wedge T_n}$, for $0 \leq s < t$, and to get that M is a martingale.

Finally, if properties (a) and (b) hold, the continuous local martingale $M^2 - \langle M, M \rangle$ is dominated by the integrable variable

$$\sup_{t\geq 0} M_t^2 + \langle M, M \rangle_\infty$$

and is therefore (by Proposition 4.7 (ii)) a uniformly integrable martingale.

(ii) It suffices to apply (i) to $(M_{t\wedge a})_{t\geq 0}$ for every choice of $a \geq 0$. □

4.4 The Bracket of Two Continuous Local Martingales

Definition 4.14 If M and N are two continuous local martingales, the *bracket* $\langle M, N\rangle$ is the finite variation process defined by setting, for every $t \geq 0$,

$$\langle M, N\rangle_t = \frac{1}{2}(\langle M+N, M+N\rangle_t - \langle M, M\rangle_t - \langle N, N\rangle_t).$$

Let us state a few easy properties of the bracket.

Proposition 4.15

(i) *$\langle M, N\rangle$ is the unique (up to indistinguishability) finite variation process such that $M_t N_t - \langle M, N\rangle_t$ is a continuous local martingale.*

(ii) *The mapping $(M, N) \mapsto \langle M, N\rangle$ is bilinear and symmetric.*

(iii) *If $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ is an increasing sequence of subdivisions of $[0, t]$ with mesh tending to 0, we have*

$$\lim_{n\to\infty} \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})(N_{t_i^n} - N_{t_{i-1}^n}) = \langle M, N\rangle_t$$

in probability.

(iv) *For every stopping time T, $\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M, N\rangle_{t\wedge T}$.*

(v) *If M and N are two martingales (with continuous sample paths) bounded in L^2, $M_t N_t - \langle M, N\rangle_t$ is a uniformly integrable martingale. Consequently, $\langle M, N\rangle_\infty$ is well defined as the almost sure limit of $\langle M, N\rangle_t$ as $t \to \infty$, is integrable, and satisfies*

$$E[M_\infty N_\infty] = E[M_0 N_0] + E[\langle M, N\rangle_\infty].$$

Proof (i) follows from the analogous characterization in Theorem 4.9 (uniqueness follows from Theorem 4.8). Similarly (iii) is a consequence of the analogous assertion in Theorem 4.9. (ii) follows from (iii), or can be proved directly via the uniqueness argument. We can then get (iv) as a consequence of property (iii), noting that this property implies, for every $0 \leq s \leq t$, a.s.,

$$\begin{aligned} &\langle M^T, N^T\rangle_t = \langle M^T, N\rangle_t = \langle M, N\rangle_t && \text{on } \{T \geq t\}, \\ &\langle M^T, N^T\rangle_t - \langle M^T, N^T\rangle_s = \langle M^T, N\rangle_t - \langle M^T, N\rangle_s = 0 && \text{on } \{T \leq s < t\}. \end{aligned}$$

Finally, (v) follows as a consequence of Theorem 4.13 (i). □

Remark A consequence of (iv) is the fact that $M^T(N - N^T)$ is a continuous local martingale, which is not so easy to prove directly.

Proposition 4.16 *Let B and B' be two independent $(\mathscr{F}_t)$-Brownian motions. Then $\langle B, B'\rangle_t = 0$ for every $t \geq 0$.*

Proof By subtracting the initial values, we may assume that $B_0 = B'_0 = 0$. We then observe that the process $X_t = \frac{1}{\sqrt{2}}(B_t + B'_t)$ is a martingale, as a linear combination of martingales. By checking the finite-dimensional marginals of X, we verify that X is also a Brownian motion (notice that we do not claim that X is an $(\mathscr{F}_t)$-Brownian motion). Proposition 2.16 implies that $\langle X, X\rangle_t = t$, and, using the bilinearity of the bracket, it follows that $\langle B, B'\rangle_t = 0$. □

Definition 4.17 Two continuous local martingales M and N are said to be *orthogonal* if $\langle M, N\rangle = 0$, which holds if and only if MN is a continuous local martingale.

In particular, two independent $(\mathscr{F}_t)$-Brownian motions are orthogonal martingales, by Proposition 4.16.

If M and N are two orthogonal martingales bounded in L^2, we have $E[M_tN_t] = E[M_0N_0]$, and even $E[M_SN_S] = E[M_0N_0]$ for any stopping time S. This follows from Theorem 3.22, using property (v) of Proposition 4.15.

Proposition 4.18 (Kunita–Watanabe) *Let M and N be two continuous local martingales and let H and K be two measurable processes. Then, a.s.,*

$$\int_0^\infty |H_s|\,|K_s|\,|\mathrm{d}\langle M, N\rangle_s| \leq \Big(\int_0^\infty H_s^2\,\mathrm{d}\langle M, M\rangle_s\Big)^{1/2}\Big(\int_0^\infty K_s^2\,\mathrm{d}\langle N, N\rangle_s\Big)^{1/2}.$$

Proof Only in this proof, we use the special notation $\langle M, N\rangle_s^t = \langle M, N\rangle_t - \langle M, N\rangle_s$ for $0 \leq s \leq t$. The first step of the proof is to observe that we have a.s. for every choice of the rationals $s < t$ (and also by continuity for every reals $s < t$),

$$|\langle M, N\rangle_s^t| \leq \sqrt{\langle M, M\rangle_s^t}\,\sqrt{\langle N, N\rangle_s^t}.$$

Indeed, this follows from the approximations of $\langle M, M\rangle$ and $\langle M, N\rangle$ given in Theorem 4.9 and in Proposition 4.15 respectively (note that these approximations are easily extended to the increments of $\langle M, M\rangle$ and $\langle M, N\rangle$), together with the Cauchy–Schwarz inequality. From now on, we fix ω such that the inequality of the last display holds for every $s < t$, and we argue with this value of ω (the remaining part of the argument is "deterministic").

We then observe that we also have, for every $0 \leq s \leq t$,

$$\int_s^t |d\langle M, N\rangle_u| \leq \sqrt{\langle M, M\rangle_s^t}\sqrt{\langle N, N\rangle_s^t}. \tag{4.7}$$

Indeed, we use Proposition 4.2, noting that, for any subdivision $s = t_0 < t_1 < \cdots < t_p = t$, we can bound

$$\begin{aligned}\sum_{i=1}^{p} |\langle M, N\rangle_{t_{i-1}}^{t_i}| &\leq \sum_{i=1}^{p} \sqrt{\langle M, M\rangle_{t_{i-1}}^{t_i}} \sqrt{\langle N, N\rangle_{t_{i-1}}^{t_i}} \\ &\leq \Big(\sum_{i=1}^{p} \langle M, M\rangle_{t_{i-1}}^{t_i}\Big)^{1/2} \Big(\sum_{i=1}^{p} \langle N, N\rangle_{t_{i-1}}^{t_i}\Big)^{1/2} \\ &= \sqrt{\langle M, M\rangle_s^t} \sqrt{\langle N, N\rangle_s^t}.\end{aligned}$$

We then get that, for every bounded Borel subset A of $\mathbb{R}_+$,

$$\int_A |\mathrm{d}\langle M, N\rangle_u| \leq \sqrt{\int_A \mathrm{d}\langle M, M\rangle_u} \sqrt{\int_A \mathrm{d}\langle N, N\rangle_u}.$$

When $A = [s, t]$, this is the bound (4.7). If A is a finite union of intervals, this follows from (4.7) and another application of the Cauchy–Schwarz inequality. A monotone class argument shows that the inequality of the last display remains valid for any bounded Borel set A (here we use a version of the monotone class lemma that is different from the one in Appendix A1: precisely, a class of sets which is stable under increasing and decreasing sequential limits and which contains an algebra of sets must contain the σ-field generated by this algebra – see the first chapter of [64]).

Next let $h = \sum_{i=1}^{p} \lambda_i \mathbf{1}_{A_i}$ and $k = \sum_{i=1}^{p} \mu_i \mathbf{1}_{A_i}$ be two nonnegative simple functions on $\mathbb{R}_+$ with bounded support contained in $[0, K]$, for some $K > 0$. Here $A_1, \ldots, A_p$ is a measurable partition of $[0, K]$, and $\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_p$ are reals (we can always assume that h and k are expressed in terms of the same partition). Then,

$$\begin{aligned}\int h(s)k(s)|\mathrm{d}\langle M, N\rangle_s| &= \sum_{i=1}^{p} \lambda_i \mu_i \int_{A_i} |\mathrm{d}\langle M, N\rangle_s| \\ &\leq \Big(\sum_{i=1}^{p} \lambda_i^2 \int_{A_i} \mathrm{d}\langle M, M\rangle_s\Big)^{1/2} \Big(\sum_{i=1}^{p} \mu_i^2 \int_{A_i} \mathrm{d}\langle N, N\rangle_s\Big)^{1/2} \\ &= \Big(\int h(s)^2 \mathrm{d}\langle M, M\rangle_s\Big)^{1/2} \Big(\int k(s)^2 \mathrm{d}\langle N, N\rangle_s\Big)^{1/2},\end{aligned}$$

which gives the desired inequality for simple functions. Since every nonnegative Borel function is a monotone increasing limit of simple functions with bounded support, an application of the monotone convergence theorem completes the proof.

□

4.5 Continuous Semimartingales

We now introduce the class of processes for which we will develop the theory of stochastic integrals.

Definition 4.19 A process $X = (X_t)_{t\geq 0}$ is a *continuous semimartingale* if it can be written in the form

$$X_t = M_t + A_t,$$

where M is a continuous local martingale and A is a finite variation process.

The decomposition $X = M + A$ is then unique up to indistinguishability thanks to Theorem 4.8. We say that this is the *canonical decomposition* of X.

By construction, continuous semimartingales have continuous sample paths. It is possible to define a notion of semimartingale with càdlàg sample paths, but in this book, we will only deal with *continuous* semimartingales, and for this reason we sometimes omit the word continuous.

Definition 4.20 Let $X = M + A$ and $Y = M' + A'$ be the canonical decompositions of two continuous semimartingales X and Y. The *bracket* $\langle X, Y\rangle$ is the finite variation process defined by

$$\langle X, Y\rangle_t = \langle M, M'\rangle_t.$$

In particular, we have $\langle X, X\rangle_t = \langle M, M\rangle_t$.

Proposition 4.21 *Let* $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ *be an increasing sequence of subdivisions of* $[0, t]$ *whose mesh tends to* 0. *Then,*

$$\lim_{n\to\infty} \sum_{i=1}^{p_n} (X_{t_i^n} - X_{t_{i-1}^n})(Y_{t_i^n} - Y_{t_{i-1}^n}) = \langle X, Y\rangle_t$$

in probability.

Proof We treat the case where $X = Y$ and leave the general case to the reader. We have

$$\sum_{i=1}^{p_n} (X_{t_i^n} - X_{t_{i-1}^n})^2 = \sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})^2 + \sum_{i=1}^{p_n} (A_{t_i^n} - A_{t_{i-1}^n})^2$$
$$+2\sum_{i=1}^{p_n} (M_{t_i^n} - M_{t_{i-1}^n})(A_{t_i^n} - A_{t_{i-1}^n}).$$

By Theorem 4.9,

$$\lim_{n\to\infty}\sum_{i=1}^{p_n}(M_{t_i^n}-M_{t_{i-1}^n})^2=\langle M,M\rangle_t=\langle X,X\rangle_t,$$

in probability. On the other hand,

$$\begin{aligned}\sum_{i=1}^{p_n}(A_{t_i^n}-A_{t_{i-1}^n})^2 &\le \Big(\sup_{1\le i\le p_n}|A_{t_i^n}-A_{t_{i-1}^n}|\Big)\sum_{i=1}^{p_n}|A_{t_i^n}-A_{t_{i-1}^n}|\\ &\le \Big(\int_0^t |\mathrm{d}A_s|\Big)\sup_{1\le i\le p_n}|A_{t_i^n}-A_{t_{i-1}^n}|,\end{aligned}$$

which tends to 0 a.s. when $n\to\infty$ by the continuity of sample paths of A. The same argument shows that

$$\Big|\sum_{i=1}^{p_n}(A_{t_i^n}-A_{t_{i-1}^n})(M_{t_i^n}-M_{t_{i-1}^n})\Big|\le\Big(\int_0^t|\mathrm{d}A_s|\Big)\sup_{1\le i\le p_n}|M_{t_i^n}-M_{t_{i-1}^n}|$$

tends to 0 a.s. □

Exercises

In the following exercises, processes are defined on a probability space $(\Omega,\mathscr{F},P)$ *equipped with a complete filtration* $(\mathscr{F}_t)_{t\in[0,\infty]}$.

Exercise 4.22 Let U be an $\mathscr{F}_0$-measurable real random variable, and let M be a continuous local martingale. Show that the process $N_t=UM_t$ is a continuous local martingale. (*This result was used in the construction of the quadratic variation of a continuous local martingale.*)

Exercise 4.23

1. Let M be a (true) martingale with continuous sample paths, such that $M_0=0$. We assume that $(M_t)_{t\ge 0}$ is also a Gaussian process. Show that, for every $t\ge 0$ and every $s>0$, the random variable $M_{t+s}-M_t$ is independent of $\sigma(M_r,0\le r\le t)$.
2. Under the assumptions of question 1., show that there exists a continuous monotone nondecreasing function $f:\mathbb{R}_+\to\mathbb{R}_+$ such that $\langle M,M\rangle_t=f(t)$ for every $t\ge 0$.

Exercise 4.24 Let M be a continuous local martingale with $M_0 = 0$.

1. For every integer $n \geq 1$, we set $T_n = \inf\{t \geq 0 : |M_t| = n\}$. Show that, a.s.

$$\Big\{\lim_{t\to\infty} M_t \text{ exists and is finite}\Big\} = \bigcup_{n=1}^{\infty} \{T_n = \infty\} \subset \{\langle M, M\rangle_\infty < \infty\}.$$

2. We set $S_n = \inf\{t \geq 0 : \langle M, M\rangle_t = n\}$ for every $n \geq 1$. Show that, a.s.,

$$\{\langle M, M\rangle_\infty < \infty\} = \bigcup_{n=1}^{\infty} \{S_n = \infty\} \subset \Big\{\lim_{t\to\infty} M_t \text{ exists and is finite}\Big\},$$

and conclude that

$$\Big\{\lim_{t\to\infty} M_t \text{ exists and is finite}\Big\} = \{\langle M, M\rangle_\infty < \infty\} \quad , \text{ a.s.}$$

Exercise 4.25 For every integer $n \geq 1$, let $M^n = (M^n_t)_{t\geq 0}$ be a continuous local martingale with $M_0 = 0$. We assume that

$$\lim_{n\to\infty} \langle M^n, M^n\rangle_\infty = 0$$

in probability.

1. Let $\varepsilon > 0$, and, for every $n \geq 1$, let

$$T^n_\varepsilon = \inf\{t \geq 0 : \langle M^n, M^n\rangle_t \geq \varepsilon\}.$$

Justify the fact that T^n_ε is a stopping time, then prove that the stopped continuous local martingale

$$M^{n,\varepsilon}_t = M^n_{t\wedge T^n_\varepsilon} , \qquad \forall t \geq 0 ,$$

is a true martingale bounded in L^2.

2. Show that

$$E\Big[\sup_{t\geq 0} |M^{n,\varepsilon}_t|^2\Big] \leq 4\,\varepsilon^2.$$

3. Writing, for every $a > 0$,

$$P\Big(\sup_{t\geq 0} |M^n_t| \geq a\Big) \leq P\Big(\sup_{t\geq 0} |M^{n,\varepsilon}_t| \geq a\Big) + P(T^n_\varepsilon < \infty),$$

show that

$$\lim_{n\to\infty}\Big(\sup_{t\geq 0}|M_t^n|\Big) = 0$$

in probability.

Exercise 4.26

1. Let A be an increasing process (adapted, with continuous sample paths and such that $A_0 = 0$) such that $A_\infty < \infty$ a.s., and let Z be an integrable random variable. We assume that, for every stopping time T,

$$E[A_\infty - A_T] \leq E[Z\,\mathbf{1}_{\{T<\infty\}}].$$

Show, by introducing an appropriate stopping time, that, for every $\lambda > 0$,

$$E[(A_\infty - \lambda)\,\mathbf{1}_{\{A_\infty>\lambda\}}] \leq E[Z\,\mathbf{1}_{\{A_\infty>\lambda\}}].$$

2. Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a continuously differentiable monotone increasing function such that $f(0) = 0$ and set $F(x) = \int_0^x f(t)\mathrm{d}t$ for every $x \geq 0$. Show that, under the assumptions of question 1., one has

$$E[F(A_\infty)] \leq E[Zf(A_\infty)].$$

(*Hint*: It may be useful to observe that $F(x) = xf(x) - \int_0^x \lambda f'(\lambda)\,d\lambda$ for every $x \geq 0$.)

3. Let M be a (true) martingale with continuous sample paths and bounded in L^2 such that $M_0 = 0$, and let M_∞ be the almost sure limit of M_t as $t \to \infty$. Show that the assumptions of question 1. hold when $A_t = \langle M, M\rangle_t$ and $Z = M_\infty^2$. Infer that, for every real $q \geq 1$,

$$E[(\langle M, M\rangle_\infty)^{q+1}] \leq (q+1)\,E[(\langle M, M\rangle_\infty)^q\,M_\infty^2].$$

4. Let $p \geq 2$ be a real number such that $E[(\langle M, M\rangle_\infty)^p] < \infty$. Show that

$$E[(\langle M, M\rangle_\infty)^p] \leq p^p\,E[|M_\infty|^{2p}].$$

5. Let N be a continuous local martingale such that $N_0 = 0$, and let T be a stopping time such that the stopped martingale N^T is uniformly integrable. Show that, for every real $p \geq 2$,

$$E[(\langle N, N\rangle_T)^p] \leq p^p\,E[|N_T|^{2p}].$$

Give an example showing that this result may fail if N^T is not uniformly integrable.

Exercise 4.27 Let $(X_t)_{t\geq 0}$ be an adapted process with continuous sample paths and taking nonnegative values. Let $(A_t)_{t\geq 0}$ be an increasing process (adapted, with continuous sample paths and such that $A_0 = 0$). We consider the following condition:

(D) For every bounded stopping time T, we have $E[X_T] \leq E[A_T]$.

1. Show that, if M is a square integrable martingale with continuous sample paths and $M_0 = 0$, the condition (D) holds for $X_t = M_t^2$ and $A_t = \langle M, M\rangle_t$.
2. Show that the conclusion of the previous question still holds if one only assumes that M is a continuous local martingale with $M_0 = 0$.
3. We set $X_t^* = \sup_{s\leq t} X_s$. Show that, under the condition (D), we have, for every bounded stopping time S and every $c > 0$,

$$P(X_S^* \geq c) \leq \frac{1}{c} E[A_S].$$

(*Hint*: One may apply (D) to $T = S \wedge R$, where $R = \inf\{t \geq 0 : X_t \geq c\}$.)

4. Infer that, still under the condition (D), one has, for every (finite or not) stopping time S,

$$P(X_S^* > c) \leq \frac{1}{c} E[A_S]$$

(when S takes the value ∞, we of course define $X_\infty^* = \sup_{s\geq 0} X_s$).

5. Let $c > 0$ and $d > 0$, and $S = \inf\{t \geq 0 : A_t \geq d\}$. Let T be a stopping time. Noting that

$$\{X_T^* > c\} \subset \Big(\{X_{T\wedge S}^* > c\} \cup \{A_T \geq d\}\Big)$$

show that, under the condition (D), one has

$$P(X_T^* > c) \leq \frac{1}{c} E[A_T \wedge d] + P(A_T \geq d).$$

6. Use questions (**2**) and (**5**) to verify that, if $M^{(n)}$ is a sequence of continuous local martingales and T is a stopping time such that $\langle M^{(n)}, M^{(n)}\rangle_T$ converges in probability to 0 as $n \to \infty$, then,

$$\lim_{n\to\infty} \Big(\sup_{s\leq T} |M_s^{(n)}|\Big) = 0 \,, \quad \text{in probability.}$$

Notes and Comments

The book [14] of Dellacherie and Meyer is again an excellent reference for the topics of this chapter, in the more general setting of local martingales and semimartingales with càdlàg sample paths. See also [72] and [49] (in particular, a discussion of the elementary theory of finite variation processes can be found in [72]). The notion of a local martingale appeared in Itô and Watanabe [43] in 1965. The notion of a semimartingale seems to be due to Fisk [25] in 1965, who used the name "quasimartingales". See also Meyer [60]. The classical approach to the quadratic variation of a continuous (local) martingale is based on the Doob–Meyer decomposition theorem [58], see e.g. [49]. Our more elementary presentation is inspired by [70].