Chapter 3
Filtrations and Martingales

In this chapter, we provide a short introduction to the theory of continuous time random processes on a filtered probability space. On the way, we generalize several notions introduced in the previous chapter in the framework of Brownian motion, and we provide a thorough discussion of stopping times. In a second step, we develop the theory of continuous time martingales, and, in particular, we derive regularity results for sample paths of martingales. We finally discuss the optional stopping theorem for martingales and supermartingales, and we give applications to explicit calculations of distributions related to Brownian motion.

3.1 Filtrations and Processes

Throughout this chapter, we consider a probability space $(\Omega, \mathscr{F}, P)$. In this section, we introduce some general notions that will be of constant use later.

Definition 3.1 A *filtration* on $(\Omega, \mathscr{F}, P)$ is a collection $(\mathscr{F}_t)_{0\leq t\leq\infty}$ indexed by $[0, \infty]$ of sub-σ-fields of $\mathscr{F}$, such that $\mathscr{F}_s \subset \mathscr{F}_t$ for every $s \leq t \leq \infty$.

We have thus, for every $0 \leq s < t$,

$$\mathscr{F}_0 \subset \mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}_\infty \subset \mathscr{F} .$$

We also say that $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ is a filtered probability space

Example If B is a Brownian motion, we considered in Chap. 2 the filtration

$$\mathscr{F}_t = \sigma(B_s, 0 \leq s \leq t), \quad \mathscr{F}_\infty = \sigma(B_s, s \geq 0).$$

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J.-F. Le Gall, *Brownian Motion, Martingales, and Stochastic Calculus*, Graduate Texts in Mathematics 274, DOI 10.1007/978-3-319-31089-3_3

More generally, if $X = (X_t, t \geq 0)$ is any random process indexed by $\mathbb{R}_+$, the *canonical filtration* of X is defined by $\mathscr{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$ and $\mathscr{F}_\infty^X = \sigma(X_s, s \geq 0)$.

Let $(\mathscr{F}_t)_{0\leq t\leq\infty}$ be a filtration on $(\Omega, \mathscr{F}, P)$. We set, for every $t \geq 0$

$$\mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s,$$

and $\mathscr{F}_{\infty+} = \mathscr{F}_\infty$. Note that $\mathscr{F}_t \subset \mathscr{F}_{t+}$ for every $t \in [0, \infty]$. The collection $(\mathscr{F}_{t+})_{0\leq t\leq\infty}$ is also a filtration. We say that the filtration $(\mathscr{F}_t)$ is right-continuous if

$$\mathscr{F}_{t+} = \mathscr{F}_t, \qquad \forall t \geq 0.$$

By construction, the filtration $(\mathscr{F}_{t+})$ is right-continuous.

Let $(\mathscr{F}_t)$ be a filtration and let $\mathscr{N}$ be the class of all $(\mathscr{F}_\infty, P)$-negligible sets (i.e. $A \in \mathscr{N}$ if there exists an $A' \in \mathscr{F}_\infty$ such $A \subset A'$ and $P(A') = 0$). The filtration is said to be *complete* if $\mathscr{N} \subset \mathscr{F}_0$ (and thus $\mathscr{N} \subset \mathscr{F}_t$ for every t).

If $(\mathscr{F}_t)$ is not complete, it can be completed by setting $\mathscr{F}_t' = \mathscr{F}_t \vee \sigma(\mathscr{N})$, for every $t \in [0, \infty]$, using the notation $\mathscr{F}_t \vee \sigma(\mathscr{N})$ for the smallest σ-field that contains both $\mathscr{F}_t$ and $\sigma(\mathscr{N})$ (recall that $\sigma(\mathscr{N})$ is the σ-field generated by $\mathscr{N}$). We will often apply this completion procedure to the canonical filtration of a random process $(X_t)_{t\geq 0}$, and call the resulting filtration the *completed canonical filtration* of X. The reader will easily check that all results stated in Chap. 2, where we were considering the canonical filtration of a Brownian motion B, remain valid if instead we deal with the completed canonical filtration. The point is that augmenting a σ-field with negligible sets does not alter independence properties.

Let us turn to random processes, which in this chapter will be indexed by $\mathbb{R}_+$.

Definition 3.2 A process $X = (X_t)_{t\geq 0}$ with values in a measurable space $(E, \mathscr{E})$ is said to be *measurable* if the mapping

$$(\omega, t) \mapsto X_t(\omega)$$

defined on $\Omega \times \mathbb{R}_+$ equipped with the product σ-field $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+)$ is measurable. (We recall that $\mathscr{B}(\mathbb{R}_+)$ stands for the Borel σ-field of $\mathbb{R}$.)

This is stronger than saying that, for every $t \geq 0$, X_t is $\mathscr{F}$-measurable. On the other hand, considering for instance the case where $E = \mathbb{R}$, it is easy to see that if the sample paths of X are continuous, or only right-continuous, the fact that X_t is $\mathscr{F}$-measurable for every t implies that the process is measurable in the previous sense – see the argument in the proof of Proposition 3.4 below.

In the remaining part of this chapter, we fix a filtration $(\mathscr{F}_t)$ on $(\Omega, \mathscr{F}, P)$, and the notions that will be introduced depend on the choice of this filtration.

Definition 3.3 A random process $(X_t)_{t\geq 0}$ with values in a measurable space $(E, \mathscr{E})$ is called *adapted* if, for every $t \geq 0$, X_t is $\mathscr{F}_t$-measurable. This process is said to be *progressive* if, for every $t \geq 0$, the mapping

$$(\omega, s) \mapsto X_s(\omega)$$

defined on $\Omega \times [0,t]$ is measurable for the σ-field $\mathscr{F}_t \otimes \mathscr{B}([0,t])$.

Note that a progressive process is both adapted and measurable (saying that a process is measurable is equivalent to saying that, for every $t \geq 0$, the mapping $(\omega, s) \mapsto X_s(\omega)$ defined on $\Omega \times [0,t]$ is measurable for $\mathscr{F} \otimes \mathscr{B}([0,t])$).

Proposition 3.4 *Let $(X_t)_{t\geq 0}$ be a random process with values in a metric space (E,d) (equipped with its Borel σ-field). Suppose that X is adapted and that the sample paths of X are right-continuous (i.e. for every $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right-continuous). Then X is progressive. The same conclusion holds if one replaces right-continuous by left-continuous.*

Proof We treat only the case of right-continuous sample paths, as the other case is similar. Fix $t > 0$. For every $n \geq 1$ and $s \in [0,t]$, define a random variable X^n_s by setting

$$X^n_s = X_{kt/n} \quad \text{if } s \in [(k-1)t/n, kt/n),\ k \in \{1,\ldots,n\},$$

and $X^n_t = X_t$. The right-continuity of sample paths ensures that, for every $s \in [0,t]$ and $\omega \in \Omega$,

$$X_s(\omega) = \lim_{n\to\infty} X^n_s(\omega).$$

On the other hand, for every Borel subset A of E,

$$\{(\omega,s) \in \Omega \times [0,t] : X^n_s(\omega) \in A\} = (\{X_t \in A\} \times \{t\}) \cup \Big(\bigcup_{k=1}^{n} \Big(\{X_{kt/n} \in A\} \times [\frac{(k-1)t}{n}, \frac{kt}{n})\Big)\Big)$$

which belongs to the σ-field $\mathscr{F}_t \otimes \mathscr{B}([0,t])$. Hence, for every $n \geq 1$, the mapping $(\omega, s) \mapsto X^n_s(\omega)$, defined on $\Omega \times [0,t]$, is measurable for $\mathscr{F}_t \otimes \mathscr{B}([0,t])$. Since a pointwise limit of measurable functions is also measurable, the same measurability property holds for the mapping $(\omega,s) \mapsto X_s(\omega)$ defined on $\Omega \times [0,t]$. It follows that the process X is progressive. □

The progressive σ-field The collection $\mathscr{P}$ of all sets $A \in \mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+)$ such that the process $X_t(\omega) = \mathbf{1}_A(\omega,t)$ is progressive forms a σ-field on $\Omega \times \mathbb{R}_+$, which is called the progressive σ-field. A subset A of $\Omega \times \mathbb{R}_+$ belongs to $\mathscr{P}$ if and only if, for every $t \geq 0$, $A \cap (\Omega \times [0,t])$ belongs to $\mathscr{F}_t \otimes \mathscr{B}([0,t])$.

One then verifies that a process X is progressive if and only if the mapping $(\omega,t) \mapsto X_t(\omega)$ is measurable on $\Omega \times \mathbb{R}_+$ equipped with the σ-field $\mathscr{P}$.

3.2 Stopping Times and Associated σ-Fields

In this section, we extend certain notions that were already introduced in the previous chapter in the framework of Brownian motion. The following definition is just a repetition of the corresponding one in the previous chapter.

Definition 3.5 A random variable $T : \Omega \longrightarrow [0,\infty]$ is a *stopping time* of the filtration $(\mathscr{F}_t)$ if $\{T \le t\} \in \mathscr{F}_t$, for every $t \ge 0$. The *σ-field of the past before T* is then defined by

$$\mathscr{F}_T = \{A \in \mathscr{F}_\infty : \forall t \ge 0,\ A \cap \{T \le t\} \in \mathscr{F}_t\}.$$

The reader will verify that $\mathscr{F}_T$ is indeed a σ-field.

In what follows, "stopping time" will mean stopping time of the filtration $(\mathscr{F}_t)$ unless otherwise specified. If T is a stopping time, we also have $\{T < t\} \in \mathscr{F}_t$ for every $t > 0$, by the same argument as in the Brownian case, and moreover

$$\{T = \infty\} = \Big(\bigcup_{n=1}^{\infty} \{T \le n\}\Big)^c \in \mathscr{F}_\infty.$$

Recall the definition of the filtration $(\mathscr{F}_{t+})$. A stopping time (of the filtration $(\mathscr{F}_t)$) is obviously also a stopping time of the filtration $(\mathscr{F}_{t+})$, but the converse need not be true in general.

Proposition 3.6 *Write $\mathscr{G}_t = \mathscr{F}_{t+}$ for every $t \in [0,\infty]$.*

(i) *A random variable $T : \Omega \longrightarrow [0,\infty]$ is a stopping time of the filtration $(\mathscr{G}_t)$ if and only if $\{T < t\} \in \mathscr{F}_t$ for every $t > 0$. This is also equivalent to saying that $T \wedge t$ is $\mathscr{F}_t$-measurable for every $t > 0$.*

(ii) *Let T be a stopping time of the filtration $(\mathscr{G}_t)$. Then*

$$\mathscr{G}_T = \{A \in \mathscr{F}_\infty : \forall t > 0,\ A \cap \{T < t\} \in \mathscr{F}_t\}.$$

We will write

$$\mathscr{F}_{T+} := \mathscr{G}_T.$$

Proof

(i) Suppose that T is a stopping time of the filtration $(\mathscr{G}_t)$. Then, for every $t > 0$,

$$\{T < t\} = \bigcup_{q \in \mathbb{Q}_+,\, q < t} \{T \le q\} \in \mathscr{F}_t$$

because $\{T \le q\} \in \mathscr{G}_q \subset \mathscr{F}_t$ if $q < t$. Conversely, assume that $\{T < t\} \in \mathscr{F}_t$ for every $t > 0$. Then, for every $t \ge 0$ and $s > t$,

$$\{T \le t\} = \bigcap_{q \in \mathbb{Q}_+,\, t<q<s} \{T < q\} \in \mathscr{F}_s$$

and it follows that $\{T \le t\} \in \mathscr{F}_{t+} = \mathscr{G}_t$.

Then, saying that $T \wedge t$ is $\mathscr{F}_t$-measurable for every $t > 0$ is equivalent to saying that, for every $s < t$, $\{T \le s\} \in \mathscr{F}_t$. Taking a sequence of values of s that increases to t, we see that the latter property implies that $\{T < t\} \in \mathscr{F}_t$, and so T is a stopping time of the filtration $(\mathscr{G}_t)$. Conversely, if T is a stopping time of the filtration $(\mathscr{G}_t)$, we have $\{T \le s\} \in \mathscr{G}_s \subset \mathscr{F}_t$ whenever $s < t$, and thus $T \wedge t$ is $\mathscr{F}_t$-measurable.

(ii) First, if $A \in \mathscr{G}_T$, we have $A \cap \{T \le t\} \in \mathscr{G}_t$ for every $t \ge 0$. Hence, for $t > 0$,

$$A \cap \{T < t\} = \bigcup_{q \in \mathbb{Q}_+,\, q<t} \Big(A \cap \{T \le q\}\Big) \in \mathscr{F}_t$$

since $A \cap \{T \le q\} \in \mathscr{G}_q \subset \mathscr{F}_t$, for every $q < t$.

Conversely, assume that $A \cap \{T < t\} \in \mathscr{F}_t$ for every $t > 0$. Then, for every $t \ge 0$, and $s > t$,

$$A \cap \{T \le t\} = \bigcap_{q \in \mathbb{Q}_+,\, t<q<s} \Big(A \cap \{T < q\}\Big) \in \mathscr{F}_s.$$

In this way, we get that $A \cap \{T \le t\} \in \mathscr{F}_{t+} = \mathscr{G}_t$ and thus $A \in \mathscr{G}_T$.

□

Properties of stopping times and of the associated σ-fields

(a) For every stopping time T, we have $\mathscr{F}_T \subset \mathscr{F}_{T+}$. If the filtration $(\mathscr{F}_t)$ is right-continuous, we have $\mathscr{F}_{T+} = \mathscr{F}_T$.
(b) If $T = t$ is a constant stopping time, $\mathscr{F}_T = \mathscr{F}_t$ and $\mathscr{F}_{T+} = \mathscr{F}_{t+}$.
(c) Let T be a stopping time. Then T is $\mathscr{F}_T$-measurable.
(d) Let T be a stopping time and $A \in \mathscr{F}_\infty$. Set

$$T^A(\omega) = \begin{cases} T(\omega) & \text{if } \omega \in A\,, \\ +\infty & \text{if } \omega \notin A\,. \end{cases}$$

Then $A \in \mathscr{F}_T$ if and only if T^A is a stopping time.
(e) Let S, T be two stopping times such that $S \le T$. Then $\mathscr{F}_S \subset \mathscr{F}_T$ and $\mathscr{F}_{S+} \subset \mathscr{F}_{T+}$.
(f) Let S, T be two stopping times. Then, $S \vee T$ and $S \wedge T$ are also stopping times and $\mathscr{F}_{S \wedge T} = \mathscr{F}_S \cap \mathscr{F}_T$. Furthermore, $\{S \le T\} \in \mathscr{F}_{S \wedge T}$ and $\{S = T\} \in \mathscr{F}_{S \wedge T}$.

(g) If (S_n) is a monotone increasing sequence of stopping times, then $S = \lim \uparrow S_n$ is also a stopping time.

(h) If (S_n) is a monotone decreasing sequence of stopping times, then $S = \lim \downarrow S_n$ is a stopping time of the filtration $(\mathscr{F}_{t+})$, and

$$\mathscr{F}_{S+} = \bigcap_n \mathscr{F}_{S_n+}.$$

(i) If (S_n) is a monotone decreasing sequence of stopping times, which is also stationary (in the sense that, for every ω, there exists an integer $N(\omega)$ such that $S_n(\omega) = S(\omega)$ for every $n \geq N(\omega)$) then $S = \lim \downarrow S_n$ is also a stopping time, and

$$\mathscr{F}_S = \bigcap_n \mathscr{F}_{S_n}.$$

(j) Let T be a stopping time. A function $\omega \mapsto Y(\omega)$ defined on the set $\{T < \infty\}$ and taking values in the measurable set $(E, \mathscr{E})$ is $\mathscr{F}_T$-measurable if and only if, for every $t \geq 0$, the restriction of Y to the set $\{T \leq t\}$ is $\mathscr{F}_t$-measurable.

Remark In property (j) we use the (obvious) notion of $\mathscr{G}$-measurability for a random variable $\omega \mapsto Y(\omega)$ that is defined only on a $\mathscr{G}$-measurable subset of Ω (here $\mathscr{G}$ is a σ-field on Ω). This notion will be used again in Theorem 3.7 below.

Proof (a), (b) and (c) are almost immediate from our definitions. Let us prove the other statements.

(d) For every $t \geq 0$,

$$\{T^A \leq t\} = A \cap \{T \leq t\}$$

and the result follows from the definition of $\mathscr{F}_T$.

(e) It is enough to prove that $\mathscr{F}_S \subset \mathscr{F}_T$. If $A \in \mathscr{F}_S$, we have

$$A \cap \{T \leq t\} = (A \cap \{S \leq t\}) \cap \{T \leq t\} \in \mathscr{F}_t,$$

hence $A \in \mathscr{F}_T$.

(f) We have

$$\{S \wedge T \leq t\} = \{S \leq t\} \cup \{T \leq t\} \in \mathscr{F}_t,$$
$$\{S \vee T \leq t\} = \{S \leq t\} \cap \{T \leq t\} \in \mathscr{F}_t,$$

so that $S \wedge T$ and $S \vee T$ are stopping times.

It follows from (e) that $\mathscr{F}_{S \wedge T} \subset (\mathscr{F}_S \cap \mathscr{F}_T)$. Moreover, if $A \in \mathscr{F}_S \cap \mathscr{F}_T$,

$$A \cap \{S \wedge T \leq t\} = (A \cap \{S \leq t\}) \cup (A \cap \{T \leq t\}) \in \mathscr{F}_t,$$

hence $A \in \mathscr{F}_{S \wedge T}$.

Then, for every $t \geq 0$,

$$\{S \leq T\} \cap \{T \leq t\} = \{S \leq t\} \cap \{T \leq t\} \cap \{S \wedge t \leq T \wedge t\} \in \mathscr{F}_t,$$
$$\{S \leq T\} \cap \{S \leq t\} = \{S \wedge t \leq T \wedge t\} \cap \{S \leq t\} \in \mathscr{F}_t,$$

because $S \wedge t$ and $T \wedge t$ are both $\mathscr{F}_t$-measurable by Proposition 3.6 (i). It follows that $\{S \leq T\} \in \mathscr{F}_S \cap \mathscr{F}_T = \mathscr{F}_{S\wedge T}$. Then $\{S = T\} = \{S \leq T\} \cap \{T \leq S\}$.

(g) For every $t \geq 0$,

$$\{S \leq t\} = \bigcap_n \{S_n \leq t\} \in \mathscr{F}_t.$$

(h) Similarly

$$\{S < t\} = \bigcup_n \{S_n < t\} \in \mathscr{F}_t,$$

and we use Proposition 3.6 (i). Then, by (e), we have $\mathscr{F}_{S+} \subset \mathscr{F}_{S_n+}$ for every n, and conversely, if $A \in \bigcap_n \mathscr{F}_{S_n+}$,

$$A \cap \{S < t\} = \bigcup_n (A \cap \{S_n < t\}) \in \mathscr{F}_t,$$

hence $A \in \mathscr{F}_{S+}$.

(i) In that case, we also have

$$\{S \leq t\} = \bigcup_n \{S_n \leq t\} \in \mathscr{F}_t,$$

and if $A \in \bigcap_n \mathscr{F}_{S_n}$,

$$A \cap \{S \leq t\} = \bigcup_n (A \cap \{S_n \leq t\}) \in \mathscr{F}_t,$$

so that $A \in \mathscr{F}_S$.

(j) First assume that, for every $t \geq 0$, the restriction of Y to $\{T \leq t\}$ is $\mathscr{F}_t$-measurable. Then, for every measurable subset A of E,

$$\{Y \in A\} \cap \{T \leq t\} \in \mathscr{F}_t.$$

Letting $t \to \infty$, we first obtain that $\{Y \in A\} \in \mathscr{F}_\infty$, and then we deduce from the previous display that $\{Y \in A\} \in \mathscr{F}_T$.

Conversely, if Y is $\mathscr{F}_T$-measurable, $\{Y \in A\} \in \mathscr{F}_T$ and thus $\{Y \in A\} \cap \{T \leq t\} \in \mathscr{F}_t$, giving the desired result. □

Theorem 3.7 *Let $(X_t)_{t\geq 0}$ be a progressive process with values in a measurable space $(E,\mathscr{E})$, and let T be a stopping time. Then the function $\omega \mapsto X_T(\omega) := X_{T(\omega)}(\omega)$, which is defined on the event $\{T < \infty\}$, is $\mathscr{F}_T$-measurable.*

Proof We use property (j) above. Let $t \geq 0$. The restriction to $\{T \leq t\}$ of the function $\omega \mapsto X_T(\omega)$ is the composition of the two mappings

$$\begin{array}{ccc} \{T \leq t\} \ni \omega & \mapsto & (\omega, T(\omega) \wedge t) \\ \mathscr{F}_t & & \mathscr{F}_t \otimes \mathscr{B}([0,t]) \end{array}$$

and

$$\begin{array}{ccc} \Omega \times [0,t] \ni (\omega, s) & \mapsto & X_s(\omega) \\ \mathscr{F}_t \otimes \mathscr{B}([0,t]) & & \mathscr{E} \end{array}$$

which are both measurable (the first one since $T \wedge t$ is $\mathscr{F}_t$-measurable, by Proposition 3.6 (i), and the second one by the definition of a progressive process). It follows that the restriction to $\{T \leq t\}$ of the function $\omega \mapsto X_T(\omega)$ is $\mathscr{F}_t$-measurable, which gives the desired result by property (j). □

Proposition 3.8 *Let T be a stopping time and let S be an $\mathscr{F}_T$-measurable random variable with values in $[0,\infty]$, such that $S \geq T$. Then S is also a stopping time.*

In particular, if T is a stopping time,

$$T_n = \sum_{k=0}^{\infty} \frac{k+1}{2^n} \mathbf{1}_{\{k2^{-n} < T \leq (k+1)2^{-n}\}} + \infty \cdot \mathbf{1}_{\{T=\infty\}}, \qquad n = 0, 1, 2, \ldots$$

defines a sequence of stopping times that decreases to T.

Proof For the first assertion, we write, for every $t \geq 0$,

$$\{S \leq t\} = \{S \leq t\} \cap \{T \leq t\} \in \mathscr{F}_t$$

since $\{S \leq t\}$ is $\mathscr{F}_T$-measurable. The second assertion follows since $T_n \geq T$, and T_n is a function of T, hence $\mathscr{F}_T$-measurable, and $T_n \downarrow T$ as $n \uparrow \infty$ by construction. □

The following proposition will be our main tool to construct stopping times associated with random processes.

Proposition 3.9 *Let $(X_t)_{t\geq 0}$ be an adapted process with values in a metric space (E,d).*

(i) *Assume that the sample paths of X are right-continuous, and let O be an open subset of E. Then*

$$T_O = \inf\{t \geq 0 : X_t \in O\}$$

is a stopping time of the filtration $(\mathscr{F}_{t+})$.

(ii) *Assume that the sample paths of X are continuous, and let F be a closed subset of E. Then*

$$T_F = \inf\{t \geq 0 : X_t \in F\}$$

is a stopping time (of the filtration $(\mathscr{F}_t)$*).*

Proof

(i) For every $t > 0$,

$$\{T_O < t\} = \bigcup_{s \in [0,t) \cap \mathbb{Q}} \{X_s \in O\} \in \mathscr{F}_t,$$

and we use Proposition 3.6 (i).

(ii) For every $t \geq 0$,

$$\{T_F \leq t\} = \left\{ \inf_{0 \leq s \leq t} d(X_s, F) = 0 \right\} = \left\{ \inf_{s \in [0,t] \cap \mathbb{Q}} d(X_s, F) = 0 \right\} \in \mathscr{F}_t.$$

□

3.3 Continuous Time Martingales and Supermartingales

Recall that we have fixed a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. In the remaining part of this chapter, all processes take values in $\mathbb{R}$. The following is an obvious analog of the corresponding definition in discrete time (see Appendix A2 below).

Definition 3.10 An adapted real-valued process $(X_t)_{t \geq 0}$ such that $X_t \in L^1$ for every $t \geq 0$ is called

- a *martingale* if, for every $0 \leq s < t$, $E[X_t \mid \mathscr{F}_s] = X_s$;
- a *supermartingale* if, for every $0 \leq s < t$, $E[X_t \mid \mathscr{F}_s] \leq X_s$;
- a *submartingale* if, for every $0 \leq s < t$, $E[X_t \mid \mathscr{F}_s] \geq X_s$.

If $(X_t)_{t \geq 0}$ is a submartingale, $(-X_t)_{t \geq 0}$ is a supermartingale. For this reason, some of the results below are stated for supermartingales only, but the analogous results for submartingales immediately follow.

If $(X_t)_{t \geq 0}$ is a martingale (resp. a supermartingale, resp. a submartingale), we have $E[X_s] = E[X_t]$ (resp. $E[X_s] \geq E[X_t]$, resp. $E[X_s] \leq E[X_t]$) whenever $0 \leq s \leq t$.

A simple way to construct a martingale is to take a random variable $Z \in L^1$ and to set $X_t = E[Z \mid \mathscr{F}_t]$ for every $t \geq 0$. Not all martingales are of this type, however. Let us turn to an important class of examples.

Important example We say that a process $(Z_t)_{t\geq 0}$ with values in $\mathbb{R}$ or in $\mathbb{R}^d$ has independent increments with respect to the filtration $(\mathscr{F}_t)$ if Z is adapted and if, for every $0 \leq s < t$, $Z_t - Z_s$ is independent of $\mathscr{F}_s$ (for instance, a Brownian motion has independent increments with respect to its canonical filtration). If Z is a real-valued process having independent increments with respect to $(\mathscr{F}_t)$, then

(i) if $Z_t \in L^1$ for every $t \geq 0$, then $\widetilde{Z}_t = Z_t - E[Z_t]$ is a martingale;
(ii) if $Z_t \in L^2$ for every $t \geq 0$, then $Y_t = \widetilde{Z}_t^2 - E[\widetilde{Z}_t^2]$ is a martingale;
(iii) if, for some $\theta \in \mathbb{R}$, we have $E[e^{\theta Z_t}] < \infty$ for every $t \geq 0$, then

$$X_t = \frac{e^{\theta Z_t}}{E[e^{\theta Z_t}]}$$

is a martingale.

Proofs of these facts are very easy. In the second case, we have for every $0 \leq s < t$,

$$\begin{aligned} E[(\widetilde{Z}_t)^2 \mid \mathscr{F}_s] &= E[(\widetilde{Z}_s + \widetilde{Z}_t - \widetilde{Z}_s)^2 \mid \mathscr{F}_s] \\ &= \widetilde{Z}_s^2 + 2\widetilde{Z}_s E[\widetilde{Z}_t - \widetilde{Z}_s \mid \mathscr{F}_s] + E[(\widetilde{Z}_t - \widetilde{Z}_s)^2 \mid \mathscr{F}_s] \\ &= \widetilde{Z}_s^2 + E[(\widetilde{Z}_t - \widetilde{Z}_s)^2] \\ &= \widetilde{Z}_s^2 + E[\widetilde{Z}_t^2] - 2E[\widetilde{Z}_s\widetilde{Z}_t] + E[\widetilde{Z}_s^2] \\ &= \widetilde{Z}_s^2 + E[\widetilde{Z}_t^2] - E[\widetilde{Z}_s^2], \end{aligned}$$

because $E[\widetilde{Z}_s\widetilde{Z}_t] = E[\widetilde{Z}_s E[\widetilde{Z}_t \mid \mathscr{F}_s]] = E[\widetilde{Z}_s^2]$. The desired result follows. In the third case,

$$E[X_t \mid \mathscr{F}_s] = \frac{e^{\theta Z_s}\, E[e^{\theta(Z_t - Z_s)} \mid \mathscr{F}_s]}{E[e^{\theta Z_s}]\, E[e^{\theta(Z_t - Z_s)}]} = \frac{e^{\theta Z_s}}{E[e^{\theta Z_s}]} = X_s,$$

using the fact that $E[e^{\theta(Z_t - Z_s)} \mid \mathscr{F}_s] = E[e^{\theta(Z_t - Z_s)}]$ by independence.

Consider the special case of Brownian motion.

Definition 3.11 A real-valued process $B = (B_t)_{t\geq 0}$ is an $(\mathscr{F}_t)$-Brownian motion if B is a Brownian motion and if B is adapted and has independent increments with respect to $(\mathscr{F}_t)$. Similarly, a process $B = (B_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ is a d-dimensional $(\mathscr{F}_t)$-Brownian motion if B is a d-dimensional Brownian motion and if B is adapted and has independent increments with respect to $(\mathscr{F}_t)$.

Note that if B is a (d-dimensional) Brownian motion and $(\mathscr{F}_t^B)$ is the (possibly completed) canonical filtration of B, then B is a (d-dimensional) $(\mathscr{F}_t^B)$-Brownian motion.

Let B be an $(\mathscr{F}_t)$-Brownian motion started from 0 (or from any $a \in \mathbb{R}$). Then it follows from the above observations that the processes

$$B_t\ ,\ B_t^2 - t\ ,\ e^{\theta B_t - \frac{\theta^2}{2}t}$$

are martingales with continuous sample paths. The processes $e^{\theta B_t - \frac{\theta^2}{2}t}$ are called exponential martingales of Brownian motion

We can also take, for $f \in L^2(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+), \mathrm{d}t)$,

$$Z_t = \int_0^t f(s)\, \mathrm{d}B_s\,.$$

Properties of Gaussian white noise imply that Z has independent increments with respect to the canonical filtration of B, and thus

$$\int_0^t f(s)\mathrm{d}B_s\,,\ \Big(\int_0^t f(s)\mathrm{d}B_s\Big)^2 - \int_0^t f(s)^2\mathrm{d}s\,,\ \exp\Big(\theta \int_0^t f(s)\mathrm{d}B_s - \frac{\theta^2}{2}\int_0^t f(s)^2\mathrm{d}s\Big)$$

are martingales (with respect to this filtration). One can prove that these martingales have a modification with continuous sample paths – it is enough to do it for the first one, and this will follow from the more general results in Chap. 5 below.

Finally, if $Z = N$ is a Poisson process with parameter λ (and $(\mathscr{F}_t)$ is the canonical filtration of N), it is well known that Z has independent increments, and we get that

$$N_t - \lambda t\,,\ (N_t - \lambda t)^2 - \lambda t,\ \exp(\theta N_t - \lambda t(e^\theta - 1))$$

are martingales. In contrast with the previous examples, these martingales do not have a modification with continuous sample paths.

Proposition 3.12 *Let $(X_t)_{t\geq 0}$ be an adapted process and let $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ be a convex function such that $E[f(X_t)] < \infty$ for every $t \geq 0$.*

(i) *If $(X_t)_{t\geq 0}$ is a martingale, then $(f(X_t))_{t\geq 0}$ is a submartingale.*
(ii) *If $(X_t)_{t\geq 0}$ is a submartingale, and if in addition f is nondecreasing, then $(f(X_t))_{t\geq 0}$ is a submartingale.*

Proof By Jensen's inequality, we have, for $s < t$,

$$E[f(X_t) \mid \mathscr{F}_s] \geq f(E[X_t \mid \mathscr{F}_s]) \geq f(X_s).$$

In the last inequality, we need the fact that f is nondecreasing when (X_t) is only a submartingale. □

Consequences If $(X_t)_{t\geq 0}$ is a martingale, $|X_t|$ is a submartingale and more generally, for every $p \geq 1$, $|X_t|^p$ is a submartingale, provided that we have $E[|X_t|^p] < \infty$ for every $t \geq 0$. If $(X_t)_{t\geq 0}$ is a submartingale, $(X_t)^+ = X_t \vee 0$ is also a submartingale.

Remark If $(X_t)_{t\geq 0}$ is any martingale, Jensen's inequality shows that $E[|X_t|^p]$ is a nondecreasing function of t with values in $[0, \infty]$, for every $p \geq 1$.

Proposition 3.13 *Let $(X_t)_{t\geq 0}$ be a submartingale or a supermartingale. Then, for every $t > 0$,*

$$\sup_{0\leq s\leq t} E[|X_s|] < \infty.$$

Proof It is enough to treat the case where $(X_t)_{t\geq 0}$ is a submartingale. Since $(X_t)^+$ is also a submartingale, we have for every $s \in [0, t]$,

$$E[(X_s)^+] \leq E[(X_t)^+].$$

On the other hand, since X is a submartingale, we also have for $s \in [0, t]$,

$$E[X_s] \geq E[X_0].$$

By combining these two bounds, and noting that $|x| = 2x^+ - x$, we get

$$\sup_{s\in[0,t]} E[|X_s|] \leq 2\,E[(X_t)^+] - E[X_0] < \infty,$$

giving the desired result. □

The next proposition will be very useful in the study of square integrable martingales.

Proposition 3.14 *Let $(M_t)_{t\geq 0}$ be a square integrable martingale (that is, $M_t \in L^2$ for every $t \geq 0$). Let $0 \leq s < t$ and let $s = t_0 < t_1 < \cdots < t_p = t$ be a subdivision of the interval $[s, t]$. Then,*

$$E\Big[\sum_{i=1}^{p}(M_{t_i} - M_{t_{i-1}})^2 \,\Big|\, \mathscr{F}_s\Big] = E[M_t^2 - M_s^2 \mid \mathscr{F}_s] = E[(M_t - M_s)^2 \mid \mathscr{F}_s].$$

In particular,

$$E\Big[\sum_{i=1}^{p}(M_{t_i} - M_{t_{i-1}})^2\Big] = E[M_t^2 - M_s^2] = E[(M_t - M_s)^2].$$

Proof For every $i = 1, \ldots, p$,

$$\begin{aligned}
E[(M_{t_i} - M_{t_{i-1}})^2 \mid \mathscr{F}_s] &= E[E[(M_{t_i} - M_{t_{i-1}})^2 \mid \mathscr{F}_{t_{i-1}}] \mid \mathscr{F}_s] \\
&= E\Big[E[M_{t_i}^2 \mid \mathscr{F}_{t_{i-1}}] - 2M_{t_{i-1}}\,E[M_{t_i} \mid \mathscr{F}_{t_{i-1}}] + M_{t_{i-1}}^2 \,\Big|\, \mathscr{F}_s\Big] \\
&= E\Big[E[M_{t_i}^2 \mid \mathscr{F}_{t_{i-1}}] - M_{t_{i-1}}^2 \,\Big|\, \mathscr{F}_s\Big] \\
&= E[M_{t_i}^2 - M_{t_{i-1}}^2 \mid \mathscr{F}_s]
\end{aligned}$$

and the desired result follows by summing over i. □

Our next goal is to study the regularity properties of sample paths of martingales and supermartingales. We first establish continuous time analogs of classical inequalities in the discrete time setting.

Proposition 3.15

(i) (Maximal inequality) *Let* $(X_t)_{t\geq 0}$ *be a supermartingale with right-continuous sample paths. Then, for every* $t > 0$ *and every* $\lambda > 0$,

$$\lambda\, P\Big(\sup_{0\leq s\leq t} |X_s| > \lambda\Big) \leq E[|X_0|] + 2E[|X_t|].$$

(ii) (Doob's inequality in L^p) *Let* $(X_t)_{t\geq 0}$ *be a martingale with right-continuous sample paths. Then, for every* $t > 0$ *and every* $p > 1$,

$$E\Big[\sup_{0\leq s\leq t} |X_s|^p\Big] \leq \Big(\frac{p}{p-1}\Big)^p E[|X_t|^p].$$

Note that part (ii) of the proposition is useful only if $E[|X_t|^p] < \infty$.

Proof

(i) Fix $t > 0$ and consider a countable dense subset D of $\mathbb{R}_+$ such that $0 \in D$ and $t \in D$. Then $D \cap [0,t]$ is the increasing union of a sequence $(D_m)_{m\geq 1}$ of finite subsets $[0,t]$ of the form $D_m = \{t_0^m, t_1^m, \ldots, t_m^m\}$ where $0 = t_0^m < t_1^m < \cdots < t_m^m = t$. For every fixed m, we can apply the discrete time maximal inequality (see Appendix A2) to the sequence $Y_n = X_{t_{n\wedge m}}$, which is a discrete supermartingale with respect to the filtration $\mathscr{G}_n = \mathscr{F}_{t_{n\wedge m}}$. We get

$$\lambda\, P\Big(\sup_{s\in D_m} |X_s| > \lambda\Big) \leq E[|X_0|] + 2E[|X_t|].$$

Then, we observe that

$$P\Big(\sup_{s\in D_m} |X_s| > \lambda\Big) \uparrow P\Big(\sup_{s\in D\cap[0,t]} |X_s| > \lambda\Big)$$

when $m \uparrow \infty$. We have thus

$$\lambda\, P\Big(\sup_{s\in D\cap[0,t]} |X_s| > \lambda\Big) \leq E[|X_0|] + 2E[|X_t|].$$

Finally, the right-continuity of sample paths (and the fact that $t \in D$) ensures that

$$\sup_{s\in D\cap[0,t]} |X_s| = \sup_{s\in[0,t]} |X_s|. \tag{3.1}$$

Assertion (i) now follows.

(ii) Following the same strategy as in the proof of (i), and using now Doob's inequality in L^p for discrete martingales (see Appendix A2), we get, for every $m \geq 1$,

$$E\Big[\sup_{s\in D_m} |X_s|^p\Big] \leq \Big(\frac{p}{p-1}\Big)^p E[|X_t|^p].$$

Now we just have to let m tend to infinity, using the monotone convergence theorem and then the identity (3.1).

□

Remark If we no longer assume that the sample paths of the supermartingale X are right-continuous, the preceding proof shows that, for every countable dense subset D of $\mathbb{R}_+$, and every $t > 0$,

$$P\Big(\sup_{s\in D\cap[0,t]} |X_s| > \lambda\Big) \leq \frac{1}{\lambda}(E[|X_0|] + 2E[|X_t|]).$$

Letting $\lambda \to \infty$, we have in particular

$$\sup_{s\in D\cap[0,t]} |X_s| < \infty \ , \quad \text{a.s.}$$

Upcrossing numbers Let $f : I \longrightarrow \mathbb{R}$ be a function defined on a subset I of $\mathbb{R}_+$. If $a < b$, the upcrossing number of f along $[a, b]$, denoted by $M^f_{ab}(I)$, is the maximal integer $k \geq 1$ such that there exists a finite increasing sequence $s_1 < t_1 < \cdots < s_k < t_k$ of elements of I such that $f(s_i) \leq a$ and $f(t_i) \geq b$ for every $i \in \{1, \ldots, k\}$ (if, even for $k = 1$, there is no such subsequence, we take $M^f_{ab}(I) = 0$, and if such a subsequence exists for every $k \geq 1$, we take $M^f_{ab}(I) = \infty$). Upcrossing numbers are a convenient tool to study the regularity of functions.

In the next lemma, the notation

$$\lim_{s\downarrow\downarrow t} f(s) \qquad (\text{resp. } \lim_{s\uparrow\uparrow t} f(s)\ )$$

means

$$\lim_{s\downarrow t, s>t} f(s) \qquad (\text{resp. } \lim_{s\uparrow t, s<t} f(s)\ ).$$

We say that $g : \mathbb{R}_+ \to \mathbb{R}$ is càdlàg (for the French "continue à droite avec des limites à gauche") if g is right-continuous and has left-limits at every $t > 0$.

Lemma 3.16 *Let D be a countable dense subset of $\mathbb{R}_+$ and let f be a real function defined on D. We assume that, for every $T \in D$,*

(i) *the function f is bounded on $D \cap [0, T]$;*
(ii) *for all rationals a and b such that $a < b$,*

$$M^f_{ab}(D \cap [0, T]) < \infty.$$

Then, the right-limit

$$f(t+) := \lim_{s \downarrow\downarrow t, s \in D} f(s)$$

exists for every real $t \geq 0$, *and similarly the left-limit*

$$f(t-) := \lim_{s \uparrow\uparrow t, s \in D} f(s)$$

exists for every real $t > 0$. *Furthermore, the function* $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ *defined by* $g(t) = f(t+)$ *is càdlàg.*

We omit the proof of this analytic lemma. It is important to note that the right and left-limits $f(t+)$ and $f(t-)$ are defined for every $t \geq 0$ ($t > 0$ in the case of $f(t-)$) and not only for $t \in D$.

Theorem 3.17 *Let* $(X_t)_{t \geq 0}$ *be a supermartingale, and let* D *be a countable dense subset of* $\mathbb{R}_+$.

(i) *For almost every* $\omega \in \Omega$, *the restriction of the function* $s \mapsto X_s(\omega)$ *to the set* D *has a right-limit*

$$X_{t+}(\omega) := \lim_{s \downarrow\downarrow t, s \in D} X_s(\omega) \tag{3.2}$$

at every $t \in [0, \infty)$, *and a left-limit*

$$X_{t-}(\omega) := \lim_{s \uparrow\uparrow t, s \in D} X_s(\omega)$$

at every $t \in (0, \infty)$.

(ii) *For every* $t \in \mathbb{R}_+$, $X_{t+} \in L^1$ *and*

$$X_t \geq E[X_{t+} \mid \mathscr{F}_t],$$

with equality if the function $t \longrightarrow E[X_t]$ *is right-continuous (in particular if* X *is a martingale). The process* $(X_{t+})_{t \geq 0}$ *is a supermartingale with respect to the filtration* $(\mathscr{F}_{t+})$. *It is a martingale if* X *is a martingale.*

Remark For the last assertions of (ii), we need $X_{t+}(\omega)$ to be defined for *every* $\omega \in \Omega$ and not only outside a negligible set. As we will see in the proof, we can just take $X_{t+}(\omega) = 0$ when the limit in (3.2) does not exist.

Proof

(i) Fix $T \in D$. By the remark following Proposition 3.15, we have

$$\sup_{s \in D \cap [0,T]} |X_s| < \infty \ , \quad \text{a.s.}$$

As in the proof of Proposition 3.15, we can choose a sequence $(D_m)_{m\geq 1}$ of finite subsets of D that increase to $D \cap [0,T]$ and are such that $0, T \in D_m$. Doob's upcrossing inequality for discrete supermartingales (see Appendix A2) gives, for every $a < b$ and every $m \geq 1$,

$$E[M^X_{ab}(D_m)] \leq \frac{1}{b-a} E[(X_T - a)^-].$$

We let $m \to \infty$ and get by monotone convergence

$$E[M^X_{ab}(D \cap [0,T])] \leq \frac{1}{b-a} E[(X_T - a)^-] < \infty.$$

We thus have

$$M^X_{ab}([0,T] \cap D) < \infty \ , \quad \text{a.s.}$$

Set

$$N = \bigcup_{T\in D} \Big(\Big\{ \sup_{t\in D\cap[0,T]} |X_t| = \infty \Big\} \bigcup \Big(\bigcup_{a,b\in\mathbb{Q}, a<b} \{M^X_{ab}(D \cap [0,T]) = \infty\} \Big) \Big). \tag{3.3}$$

Then $P(N) = 0$ by the preceding considerations. On the other hand, if $\omega \notin N$, the function $D \ni t \mapsto X_t(\omega)$ satisfies all assumptions of Lemma 3.16. Assertion (i) now follows from this lemma.

(ii) To define $X_{t+}(\omega)$ for every $\omega \in \Omega$ and not only on $\Omega \backslash N$, we set

$$X_{t+}(\omega) = \begin{cases} \lim\limits_{s\downarrow\downarrow t, s\in D} X_s(\omega) & \text{if the limit exists} \\ 0 & \text{otherwise.} \end{cases}$$

With this definition, X_{t+} is $\mathscr{F}_{t+}$-measurable.

Fix $t \geq 0$ and choose a sequence $(t_n)_{n\geq 0}$ in D such that t_n decreases strictly to t as $n \to \infty$. Then, by construction, we have a.s.

$$X_{t+} = \lim_{n\to\infty} X_{t_n}.$$

Set $Y_k = X_{t_{-k}}$ for every integer $k \leq 0$. Then Y is a backward supermartingale with respect to the (backward) discrete filtration $\mathscr{H}_k = \mathscr{F}_{t_{-k}}$ (see Appendix A2). From Proposition 3.13, we have $\sup_{k\leq 0} E[|Y_k|] < \infty$. The convergence theorem for backward supermartingales (see Appendix A2) then implies that the sequence X_{t_n} converges to X_{t+} in L^1. In particular, $X_{t+} \in L^1$.

Thanks to the L^1-convergence, we can pass to the limit $n \to \infty$ in the inequality $X_t \geq E[X_{t_n} \mid \mathscr{F}_t]$, and we get

$$X_t \geq E[X_{t_+} \mid \mathscr{F}_t]$$

(we use the fact that the conditional expectation is continuous for the L^1-norm, and it is important to realize that an a.s. convergence would not be sufficient to warrant this passage to the limit). Furthermore, thanks again to the L^1-convergence, we have $E[X_{t+}] = \lim E[X_{t_n}]$. Thus, if the function $s \longrightarrow E[X_s]$ is right-continuous, we must have $E[X_t] = E[X_{t+}] = E[E[X_{t_+} \mid \mathscr{F}_t]]$, and the inequality $X_t \geq E[X_{t_+} \mid \mathscr{F}_t]$ then forces $X_t = E[X_{t_+} \mid \mathscr{F}_t]$.

We already noticed that X_{t+} is $\mathscr{F}_{t+}$-measurable. Let $s < t$ and let $(s_n)_{n\geq 0}$ be a sequence in D that decreases strictly to s. We may assume that $s_n \leq t_n$ for every n. Then as previously X_{s_n} converges to X_{s+} in L^1, and thus, if $A \in \mathscr{F}_{s+}$, which implies $A \in \mathscr{F}_{s_n}$ for every n, we have

$$E[X_{s+}\mathbf{1}_A] = \lim_{n\to\infty} E[X_{s_n}\mathbf{1}_A] \geq \lim_{n\to\infty} E[X_{t_n}\mathbf{1}_A] = E[X_{t+}\mathbf{1}_A] = E[E[X_{t+} \mid \mathscr{F}_{s+}]\mathbf{1}_A].$$

Since this inequality holds for every $A \in \mathscr{F}_{s+}$, and since X_{s+} and $E[X_{t+} \mid \mathscr{F}_{s+}]$ are both $\mathscr{F}_{s+}$-measurable, it follows that $X_{s+} \geq E[X_{t+} \mid \mathscr{F}_{s+}]$. Finally, if X is a martingale, inequalities can be replaced by equalities in the previous considerations.

□

Theorem 3.18 *Assume that the filtration $(\mathscr{F}_t)$ is right-continuous and complete. Let $X = (X_t)_{t\geq 0}$ be a supermartingale, such that the function $t \longrightarrow E[X_t]$ is right-continuous. Then X has a modification with càdlàg sample paths, which is also an $(\mathscr{F}_t)$-supermartingale.*

Proof Let D be a countable dense subset of $\mathbb{R}_+$ as in Theorem 3.17. Let N be the negligible set defined in (3.3). We set, for every $t \geq 0$,

$$Y_t(\omega) = \begin{cases} X_{t+}(\omega) & \text{if } \omega \notin N \\ 0 & \text{if } \omega \in N. \end{cases}$$

Lemma 3.16 then shows that the sample paths of Y are càdlàg.

The random variable X_{t+} is $\mathscr{F}_{t+}$-measurable, and thus $\mathscr{F}_t$-measurable since the filtration is right-continuous. As the negligible set N belongs to $\mathscr{F}_\infty$, the completeness of the filtration ensures that Y_t is $\mathscr{F}_t$-measurable. By Theorem 3.17 (ii), we have for every $t \geq 0$,

$$X_t = E[X_{t+} \mid \mathscr{F}_t] = X_{t+} = Y_t, \quad \text{a.s.}$$

because X_{t+} is $\mathscr{F}_t$-measurable. Consequently, Y is a modification of X. The process Y is adapted to the filtration $(\mathscr{F}_t)$. Since Y is a modification of X the inequality $E[X_t \mid \mathscr{F}_s] \leq X_s$, for $0 \leq s < t$, implies that the same inequality holds for Y. □

Remarks

(i) Let us comment on the assumptions of the theorem. A simple example shows that our assumption that the filtration is right-continuous is necessary. Take

$\Omega = \{-1, 1\}$, with the probability measure P defined by $P(\{1\}) = P(\{-1\}) = 1/2$. Let ε be the random variable $\varepsilon(\omega) = \omega$, and let the process $(X_t)_{t \geq 0}$ be defined by $X_t = 0$ if $0 \leq t \leq 1$, and $X_t = \varepsilon$ if $t > 1$. Then it is easy to verify that X is a martingale with respect to its canonical filtration $(\mathscr{F}_t^X)$ (which is complete since there are no nonempty negligible sets!). On the other hand, no modification of X can be right-continuous at $t = 1$. This does not contradict the theorem since the filtration is not right-continuous ($\mathscr{F}_{1+}^X \neq \mathscr{F}_1^X$).

(ii) Similarly, to show that the right-continuity of the mapping $t \longrightarrow E[X_t]$ is needed, we can just take $X_t = f(t)$, where f is any nonincreasing deterministic function. If f is not right-continuous, no modification of X can have right-continuous sample paths.

3.4 Optional Stopping Theorems

We start with a convergence theorem for supermartingales.

Theorem 3.19 *Let X be a supermartingale with right-continuous sample paths. Assume that the collection $(X_t)_{t \geq 0}$ is bounded in L^1. Then there exists a random variable $X_\infty \in L^1$ such that*

$$\lim_{t \to \infty} X_t = X_\infty, \quad a.s.$$

Proof Let D be a countable dense subset of $\mathbb{R}_+$. From the proof of Theorem 3.17, we have, for every $T \in D$ and $a < b$,

$$E[M_{ab}^X(D \cap [0, T])] \leq \frac{1}{b-a} E[(X_T - a)^-].$$

By monotone convergence, we get, for every $a < b$,

$$E[M_{ab}^X(D)] \leq \frac{1}{b-a} \sup_{t \geq 0} E[(X_t - a)^-] < \infty,$$

since the collection $(X_t)_{t \geq 0}$ is bounded in L^1. Hence, a.s. for all rationals $a < b$, we have $M_{ab}^X(D) < \infty$. This implies that the limit

$$X_\infty := \lim_{D \ni t \to \infty} X_t \tag{3.4}$$

exists a.s. in $[-\infty, \infty]$. We can in fact exclude the values $+\infty$ and $-\infty$, since Fatou's lemma gives

$$E[|X_\infty|] \leq \liminf_{D \ni t \to \infty} E[|X_t|] < \infty,$$

and we get that $X_\infty \in L^1$. The right-continuity of sample paths (which we have not yet used) allows us to remove the restriction $t \in D$ in the limit (3.4). □

Under the assumptions of Theorem 3.19, the convergence of X_t towards X_∞ may not hold in L^1. The next result gives, in the case of a martingale, necessary and sufficient conditions for the convergence to also hold in L^1.

Definition 3.20 A martingale $(X_t)_{t\geq 0}$ is said to be *closed* if there exists a random variable $Z \in L^1$ such that, for every $t \geq 0$,

$$X_t = E[Z \mid \mathscr{F}_t].$$

Theorem 3.21 *Let X be a martingale with right-continuous sample paths. Then the following properties are equivalent:*

(i) *X is closed;*
(ii) *the collection $(X_t)_{t\geq 0}$ is uniformly integrable;*
(iii) *X_t converges a.s. and in L^1 as $t \to \infty$.*

Moreover, if these properties hold, we have $X_t = E[X_\infty \mid \mathscr{F}_t]$ for every $t \geq 0$, where $X_\infty \in L^1$ is the a.s. limit of X_t as $t \to \infty$.

Proof The fact that (i)⇒(ii) is easy: If $Z \in L^1$, the collection of all random variables $E[Z \mid \mathscr{G}]$, when $\mathscr{G}$ varies over sub-σ-fields of $\mathscr{F}$, is uniformly integrable. If (ii) holds, in particular the collection $(X_t)_{t\geq 0}$ is bounded in L^1 and Proposition 3.19 implies that X_t converges a.s. to X_∞. By uniform integrability, the latter convergence also holds in L^1. Finally, if (iii) holds, for every $s \geq 0$, we can pass to the limit $t \to \infty$ in the equality $X_s = E[X_t \mid \mathscr{F}_s]$ (using the fact that the conditional expectation is continuous for the L^1-norm), and we get $X_s = E[X_\infty \mid \mathscr{F}_s]$. □

We will now use the optional stopping theorems for discrete martingales and supermartingales in order to establish similar results in the continuous time setting. Let $(X_t)_{t\geq 0}$ be a martingale or a supermartingale with right-continuous sample paths, and such that X_t converges a.s. as $t \to \infty$ to a random variable denoted by X_∞. Then, for every stopping time T, we write X_T for the random variable

$$X_T(\omega) = \mathbf{1}_{\{T(\omega)<\infty\}} X_{T(\omega)}(\omega) + \mathbf{1}_{\{T(\omega)=\infty\}} X_\infty(\omega).$$

Compare with Theorem 3.7, where the random variable X_T was only defined on the subset $\{T < \infty\}$ of Ω. With this definition, the random variable X_T is still $\mathscr{F}_T$-measurable: Use Theorem 3.7 and the easily verified fact that $\mathbf{1}_{\{T=\infty\}} X_\infty$ is $\mathscr{F}_T$-measurable.

Theorem 3.22 (Optional stopping theorem for martingales) *Let $(X_t)_{t\geq 0}$ be a uniformly integrable martingale with right-continuous sample paths. Let S and T be two stopping times with $S \leq T$. Then X_S and X_T are in L^1 and*

$$X_S = E[X_T \mid \mathscr{F}_S].$$

In particular, for every stopping time S, we have

$$X_S = E[X_\infty \mid \mathscr{F}_S],$$

and

$$E[X_S] = E[X_\infty] = E[X_0].$$

Proof Set, for every integer $n \geq 0$,

$$T_n = \sum_{k=0}^{\infty} \frac{k+1}{2^n} \mathbf{1}_{\{k2^{-n} < T \leq (k+1)2^{-n}\}} + \infty \cdot \mathbf{1}_{\{T=\infty\}}$$

and similarly

$$S_n = \sum_{k=0}^{\infty} \frac{k+1}{2^n} \mathbf{1}_{\{k2^{-n} < S \leq (k+1)2^{-n}\}} + \infty \cdot \mathbf{1}_{\{S=\infty\}}.$$

By Proposition 3.8, (T_n) and (S_n) are two sequences of stopping times that decrease respectively to T and to S. Moreover, we have $S_n \leq T_n$ for every $n \geq 0$.

Now observe that, for every fixed n, $2^n S_n$ and $2^n T_n$ are stopping times of the discrete filtration $\mathscr{H}_k^{(n)} := \mathscr{F}_{k/2^n}$, and $Y_k^{(n)} := X_{k/2^n}$ is a discrete martingale with respect to this filtration. From the optional stopping theorem for uniformly integrable discrete martingales (see Appendix A2) we get that $Y_{2^n S_n}^{(n)}$ and $Y_{2^n T_n}^{(n)}$ are in L^1, and

$$X_{S_n} = Y_{2^n S_n}^{(n)} = E[Y_{2^n T_n}^{(n)} \mid \mathscr{H}_{2^n S_n}^{(n)}] = E[X_{T_n} \mid \mathscr{F}_{S_n}]$$

(here we need to verify that $\mathscr{H}_{2^n S_n}^{(n)} = \mathscr{F}_{S_n}$, but this is straightforward).

Let $A \in \mathscr{F}_S$. Since $\mathscr{F}_S \subset \mathscr{F}_{S_n}$, we have $A \in \mathscr{F}_{S_n}$ and thus

$$E[\mathbf{1}_A X_{S_n}] = E[\mathbf{1}_A X_{T_n}].$$

By the right-continuity of sample paths, we get a.s.

$$X_S = \lim_{n\to\infty} X_{S_n} \ , \ X_T = \lim_{n\to\infty} X_{T_n}.$$

These limits also hold in L^1. Indeed, thanks again to the optional stopping theorem for uniformly integrable discrete martingales, we have $X_{S_n} = E[X_\infty \mid \mathscr{F}_{S_n}]$ for every n, and thus the sequence (X_{S_n}) is uniformly integrable (and the same holds for the sequence (X_{T_n})).

The L^1-convergence implies that X_S and X_T belong to L^1, and also allows us to pass to the limit $n \to \infty$ in the equality $E[\mathbf{1}_A X_{S_n}] = E[\mathbf{1}_A X_{T_n}]$ in order to get

$$E[\mathbf{1}_A X_S] = E[\mathbf{1}_A X_T].$$

Since this holds for every $A \in \mathscr{F}_S$, and since the variable X_S is $\mathscr{F}_S$-measurable (by the remarks before the theorem), we conclude that

$$X_S = E[X_T \mid \mathscr{F}_S],$$

which completes the proof. □

We now give two corollaries of Theorem 3.22.

Corollary 3.23 *Let $(X_t)_{t\geq 0}$ be a martingale with right-continuous sample paths, and let $S \leq T$ be two bounded stopping times. Then X_S and X_T are in L^1 and*

$$X_S = E[X_T \mid \mathscr{F}_S] .$$

Proof Let $a \geq 0$ such that $S \leq T \leq a$. We apply Theorem 3.22 to the martingale $(X_{t\wedge a})_{t\geq 0}$ which is closed by X_a. □

The second corollary shows that a martingale (resp. a uniformly integrable martingale) stopped at an arbitrary stopping time remains a martingale (resp. a uniformly integrable martingale). This result will play an important role in the next chapters.

Corollary 3.24 *Let $(X_t)_{t\geq 0}$ be a martingale with right-continuous sample paths, and let T be a stopping time.*

(i) *The process $(X_{t\wedge T})_{t\geq 0}$ is still a martingale.*
(ii) *Suppose in addition that the martingale $(X_t)_{t\geq 0}$ is uniformly integrable. Then the process $(X_{t\wedge T})_{t\geq 0}$ is also a uniformly integrable martingale, and more precisely we have for every $t \geq 0$,*

$$X_{t\wedge T} = E[X_T \mid \mathscr{F}_t]. \tag{3.5}$$

Proof We start with the proof of (ii). Note that $t \wedge T$ is a stopping time by property (f) of stopping times. By Theorem 3.22, $X_{t\wedge T}$ and X_T are in L^1, and we also know that $X_{t\wedge T}$ is $\mathscr{F}_{t\wedge T}$-measurable, hence $\mathscr{F}_t$-measurable since $\mathscr{F}_{t\wedge T} \subset \mathscr{F}_t$. So in order to get (3.5), it is enough to prove that, for every $A \in \mathscr{F}_t$,

$$E[\mathbf{1}_A X_T] = E[\mathbf{1}_A X_{t\wedge T}].$$

Let us fix $A \in \mathscr{F}_t$. First, we have trivially

$$E[\mathbf{1}_{A\cap\{T\leq t\}} X_T] = E[\mathbf{1}_{A\cap\{T\leq t\}} X_{t\wedge T}]. \tag{3.6}$$

On the other hand, by Theorem 3.22, we have

$$X_{t\wedge T} = E[X_T \mid \mathscr{F}_{t\wedge T}],$$

and we notice that we have both $A \cap \{T > t\} \in \mathscr{F}_t$ and $A \cap \{T > t\} \in \mathscr{F}_T$ (the latter as a straightforward consequence of the definition of $\mathscr{F}_T$), so that $A \cap \{T > t\} \in \mathscr{F}_t \cap \mathscr{F}_T = \mathscr{F}_{t\wedge T}$. Using the preceding display, we obtain

$$E[\mathbf{1}_{A\cap\{T>t\}}\, X_T] = E[\mathbf{1}_{A\cap\{T>t\}}\, X_{t\wedge T}].$$

By adding this equality to (3.6), we get the desired result.

To prove (i), we just need to apply (ii) to the (uniformly integrable) martingale $(X_{t\wedge a})_{a\geq 0}$, for any choice of $a \geq 0$. □

Applications Above all, the optional stopping theorem is a powerful tool for explicit calculations of probability distributions. Let us give a few important and typical examples of such applications (several other examples can be found in the exercises of this and the following chapters). Let B be a real Brownian motion started from 0. We know that B is a martingale with continuous sample paths with respect to its canonical filtration. For every real a, set $T_a = \inf\{t \geq 0 : B_t = a\}$. Recall that $T_a < \infty$ a.s.

(a) *Law of the exit point from an interval*. For every $a < 0 < b$, we have

$$P(T_a < T_b) = \frac{b}{b-a} \quad , \quad P(T_b < T_a) = \frac{-a}{b-a}.$$

To get this result, consider the stopping time $T = T_a \wedge T_b$ and the stopped martingale $M_t = B_{t\wedge T}$ (this is a martingale by Corollary 3.24). Then, $|M|$ is bounded above by $b \vee |a|$, and the martingale M is thus uniformly integrable. We can apply Theorem 3.22 and we get

$$0 = E[M_0] = E[M_T] = b\,P(T_b < T_a) + a\,P(T_a < T_b).$$

Since we also have $P(T_b < T_a) + P(T_a < T_b) = 1$, the desired result follows. In fact the proof shows that the result remains valid if we replace Brownian motion by a martingale with continuous sample paths and initial value 0, provided we know that this process exits (a, b) a.s.

(b) *First moment of exit times*. For every $a > 0$, consider the stopping time $U_a = \inf\{t \geq 0 : |B_t| = a\}$. Then

$$E[U_a] = a^2.$$

To verify this, consider the martingale $M_t = B_t^2 - t$. By Corollary 3.24, $M_{t\wedge U_a}$ is still a martingale, and therefore $E[M_{t\wedge U_a}] = E[M_0] = 0$, giving $E[(B_{t\wedge U_a})^2] = E[t \wedge U_a]$. Then, on one hand, $E[t \wedge U_a]$ converges to $E[U_a]$ as

$t \to \infty$ by monotone convergence, on the other hand, $E[(B_{t\wedge U_a})^2]$ converges to $E[(B_{U_a})^2] = a^2$ as $t \to \infty$, by dominated convergence (note that $(B_{t\wedge U_a})^2 \le a^2$). The stated result follows. We may observe that we have $E[U_a] < \infty$ in contrast with the property $E[T_a] = \infty$, which was noticed in Chap. 2.

(c) *Laplace transform of hitting times*. We now fix $a > 0$ and our goal is to compute the Laplace transform of T_a. For every $\lambda \in \mathbb{R}$, we can consider the exponential martingale

$$N_t^\lambda = \exp(\lambda B_t - \frac{\lambda^2}{2} t).$$

Suppose first that $\lambda > 0$. By Corollary 3.24, the stopped process $N^\lambda_{t\wedge T_a}$ is still a martingale, and we immediately see that this martingale is bounded above by $e^{\lambda a}$, hence uniformly integrable. By applying the last assertion of Theorem 3.22 to this martingale and to the stopping time $S = T_a$ (or to $S = \infty$) we get

$$e^{\lambda a} E[e^{-\frac{\lambda^2}{2} T_a}] = E[N^\lambda_{T_a}] = E[N^\lambda_0] = 1 \; .$$

Replacing λ by $\sqrt{2\lambda}$, we conclude that, for every $\lambda > 0$,

$$E[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}}. \tag{3.7}$$

(This formula could also be deduced from the knowledge of the density of T_a, see Corollary 2.22.) As an instructive example, one may try to reproduce the preceding line of reasoning, using now the martingale N_t^λ for $\lambda < 0$: one gets an absurd result, which can be explained by the fact that the stopped martingale $N^\lambda_{t\wedge T_a}$ is **not uniformly integrable** when $\lambda < 0$. When applying Theorem 3.22, it is crucial to always verify the uniform integrability of the martingale. In most cases, this is done by verifying that the (stopped) martingale is bounded.

(d) *Laplace transform of exit times from an interval.* With the notation of (b), we have for every $a > 0$ and every $\lambda > 0$,

$$E[\exp(-\lambda U_a)] = \frac{1}{\cosh(a\sqrt{2\lambda})}.$$

To see this, first note that U_a and B_{U_a} are independent since, using the symmetry property of Brownian motion,

$$E[\mathbf{1}_{\{B_{U_a}=a\}} \exp(-\lambda U_a)] = E[\mathbf{1}_{\{B_{U_a}=-a\}} \exp(-\lambda U_a)] = \frac{1}{2} E[\exp(-\lambda U_a)].$$

Then the claimed formula is proved by the same method as in (c), writing $E[N^\lambda_{U_a}] = E[N^\lambda_0] = 1$ and noting that the application of the optional stopping theorem is justified by the fact that $N^\lambda_{t\wedge U_a}$ is bounded above by $e^{\lambda a}$. See also Exercise 3.27 for a more general formula.

We end this chapter with the optional stopping theorem for nonnegative supermartingales. This result will be useful in later applications to Markov processes. We first note that, if $(Z_t)_{t\geq 0}$ is a nonnegative supermartingale with right-continuous sample paths, $(Z_t)_{t\geq 0}$ is automatically bounded in L^1 since $E[Z_t] \leq E[Z_0]$, and by Theorem 3.19, Z_t converges a.s. to a random variable $Z_\infty \in L^1$ as $t \to \infty$. As explained before Theorem 3.22, we can thus make sense of Z_T for any (finite or not) stopping time T.

Theorem 3.25 *Let $(Z_t)_{t\geq 0}$ be a nonnegative supermartingale with right-continuous sample paths. Let U and V be two stopping times such that $U \leq V$. Then, Z_U and Z_V are in L^1, and*

$$Z_U \geq E[Z_V \mid \mathscr{F}_U].$$

Remark This implies that $E[Z_U] \geq E[Z_V]$, and since $Z_U = Z_V = Z_\infty$ on the event $\{U = \infty\}$, it also follows that

$$E[\mathbf{1}_{\{U<\infty\}}\, Z_U] \geq E[\mathbf{1}_{\{U<\infty\}}\, Z_V] \geq E[\mathbf{1}_{\{V<\infty\}}\, Z_V].$$

Proof In the first step of the proof, we make the extra assumption that U and V are bounded and we verify that we then have $E[Z_U] \geq E[Z_V]$. Let $p \geq 1$ be an integer such that $U \leq p$ and $V \leq p$. For every integer $n \geq 0$, set

$$U_n = \sum_{k=0}^{p2^n-1} \frac{k+1}{2^n}\, \mathbf{1}_{\{k2^{-n}<U\leq(k+1)2^{-n}\}}\,, \quad V_n = \sum_{k=0}^{p2^n-1} \frac{k+1}{2^n}\, \mathbf{1}_{\{k2^{-n}<V\leq(k+1)2^{-n}\}}$$

in such a way (by Proposition 3.8) that (U_n) and (V_n) are two sequences of bounded stopping times that decrease respectively to U and V, and additionally we have $U_n \leq V_n$ for every $n \geq 0$. The right-continuity of sample paths ensures that $Z_{U_n} \longrightarrow Z_U$ and $Z_{V_n} \longrightarrow Z_V$ a.s. as $n \to \infty$. Then, by the optional stopping theorem for discrete supermartingales in the case of bounded stopping times (see Appendix A2), with respect to the filtration $(\mathscr{F}_{k/2^{n+1}})_{k\geq 0}$, we have for every $n \geq 0$,

$$Z_{U_{n+1}} \geq E[Z_{U_n} \mid \mathscr{F}_{U_{n+1}}].$$

Setting $Y_n = Z_{U_{-n}}$ and $\mathscr{H}_n = \mathscr{F}_{U_{-n}}$, for every integer $n \leq 0$, we get that the sequence $(Y_n)_{n\leq 0}$ is a backward supermartingale with respect to the filtration $(\mathscr{H}_n)_{n\leq 0}$. Since, for every $n \geq 0$, $E[Z_{U_n}] \leq E[Z_0]$ (by another application of the discrete optional stopping theorem), the sequence $(Y_n)_{n\leq 0}$ is bounded in L^1, and by the convergence theorem for backward supermartingales (see Appendix A2), it converges in L^1. Hence the convergence of Z_{U_n} to Z_U also holds in L^1 and similarly the convergence of Z_{V_n} to Z_V holds in L^1. Since $U_n \leq V_n$, by yet another application of the discrete optional stopping theorem, we have $E[Z_{U_n}] \geq E[Z_{V_n}]$. Using the L^1-convergence of Z_{U_n} and Z_{V_n} we can pass to the limit $n \to \infty$ and obtain that $E[Z_U] \geq E[Z_V]$ as claimed.

Let us prove the statement of the theorem (no longer assuming that U and V are bounded). By the first step of the proof applied to the stopping times 0 and $U \wedge p$, we have $E[Z_{U\wedge p}] \leq E[Z_0]$ for every $p \geq 1$, and Fatou's lemma gives $E[Z_U] \leq E[Z_0] < \infty$ and similarly $E[Z_V] < \infty$. Fix $A \in \mathscr{F}_U \subset \mathscr{F}_V$ and recall our notation U^A for the stopping time defined by $U^A(\omega) = U(\omega)$ if $\omega \in A$ and $U^A(\omega) = \infty$ otherwise (cf. property (d) of stopping times). By the first part of the proof, we have, for every $p \geq 1$,

$$E[Z_{U^A\wedge p}] \geq E[Z_{V^A\wedge p}].$$

By writing each of these two expectations as a sum of expectations over the sets A^c, $A \cap \{U \leq p\}$ and $A \cap \{U > p\}$, and noting that $U > p$ implies $V > p$, we get

$$E[Z_U\, \mathbf{1}_{A\cap\{U\leq p\}}] \geq E[Z_V\, \mathbf{1}_{A\cap\{U\leq p\}}].$$

Letting $p \to \infty$ then gives

$$E[Z_U\, \mathbf{1}_{A\cap\{U<\infty\}}] \geq E[Z_V\, \mathbf{1}_{A\cap\{U<\infty\}}].$$

On the other hand, the equality $E[Z_U\, \mathbf{1}_{A\cap\{U=\infty\}}] = E[Z_V\, \mathbf{1}_{A\cap\{U=\infty\}}]$ is trivial and by adding it to the preceding display, we obtain

$$E[Z_U\, \mathbf{1}_A] \geq E[Z_V\, \mathbf{1}_A] = E[E[Z_V \mid \mathscr{F}_U]\, \mathbf{1}_A].$$

Since this holds for every $A \in \mathscr{F}_U$ and Z_U is $\mathscr{F}_U$-measurable, the desired result follows. □

Exercises

In the following exercises, processes are defined on a probability space $(\Omega, \mathscr{F}, P)$ equipped with a complete filtration $(\mathscr{F}_t)_{t\in[0,\infty]}$.

Exercise 3.26

1. Let M be a martingale with continuous sample paths such that $M_0 = x \in \mathbb{R}_+$. We assume that $M_t \geq 0$ for every $t \geq 0$, and that $M_t \to 0$ when $t \to \infty$, a.s. Show that, for every $y > x$,

$$P\Big(\sup_{t\geq 0} M_t \geq y\Big) = \frac{x}{y}.$$

2. Give the law of

$$\sup_{t \le T_0} B_t$$

when B is a Brownian motion started from $x > 0$ and $T_0 = \inf\{t \ge 0 : B_t = 0\}$.

3. Assume now that B is a Brownian motion started from 0, and let $\mu > 0$. Using an appropriate exponential martingale, show that

$$\sup_{t \ge 0}(B_t - \mu t)$$

is exponentially distributed with parameter 2μ.

Exercise 3.27 Let B be an $(\mathscr{F}_t)$-Brownian motion started from 0. Recall the notation $T_x = \inf\{t \ge 0 : B_t = x\}$, for every $x \in \mathbb{R}$. We fix two real numbers a and b with $a < 0 < b$, and we set

$$T = T_a \wedge T_b\ .$$

1. Show that, for every $\lambda > 0$,

$$E[\exp(-\lambda T)] = \frac{\cosh(\frac{b+a}{2}\sqrt{2\lambda})}{\cosh(\frac{b-a}{2}\sqrt{2\lambda})}.$$

(*Hint*: One may consider a martingale of the form

$$M_t = \exp\Big(\sqrt{2\lambda}(B_t - \alpha) - \lambda t\Big) + \exp\Big(-\sqrt{2\lambda}(B_t - \alpha) - \lambda t\Big)$$

with a suitable choice of α.)

2. Show similarly that, for every $\lambda > 0$,

$$E[\exp(-\lambda T)\,\mathbf{1}_{\{T=T_a\}}] = \frac{\sinh(b\sqrt{2\lambda})}{\sinh((b-a)\sqrt{2\lambda})}.$$

3. Recover the formula for $P(T_a < T_b)$ from question (**2**).

Exercise 3.28 Let $(B_t)_{t \ge 0}$ be an $(\mathscr{F}_t)$-Brownian motion started from 0. Let $a > 0$ and

$$\sigma_a = \inf\{t \ge 0 : B_t \le t - a\}.$$

1. Show that σ_a is a stopping time and that $\sigma_a < \infty$ a.s.

2. Using an appropriate exponential martingale, show that, for every $\lambda \geq 0$,

$$E[\exp(-\lambda\sigma_a)] = \exp(-a(\sqrt{1+2\lambda}-1)).$$

The fact that this formula remains valid for $\lambda \in [-\frac{1}{2}, 0[$ can be obtained via an argument of analytic continuation.

3. Let $\mu \in \mathbb{R}$ and $M_t = \exp(\mu B_t - \frac{\mu^2}{2}t)$. Show that the stopped martingale $M_{\sigma_a \wedge t}$ is closed if and only if $\mu \leq 1$. (*Hint*: This martingale is closed if and only if $E[M_{\sigma_a}] = 1$.)

Exercise 3.29 Let $(Y_t)_{t\geq 0}$ be a uniformly integrable martingale with continuous sample paths, such that $Y_0 = 0$. We set $Y_\infty = \lim_{t\to\infty} Y_t$. Let $p \geq 1$ be a fixed real number. We say that Property (P) holds for the martingale Y if there exists a constant C such that, for every stopping time T, we have

$$E[|Y_\infty - Y_T|^p \mid \mathscr{F}_T] \leq C.$$

1. Show that Property (P) holds for Y if Y_∞ is bounded.

2. Let B be an $(\mathscr{F}_t)$-Brownian motion started from 0. Show that Property (P) holds for the martingale $Y_t = B_{t\wedge 1}$. (*Hint*: One may observe that the random variable $\sup_{t\leq 1} |B_t|$ is in L^p.)

3. Show that Property (P) holds for Y, with the constant C, if and only if, for any stopping time T,

$$E[|Y_T - Y_\infty|^p] \leq C\,P[T < \infty].$$

(*Hint*: It may be useful to consider the stopping times T^A defined for $A \in \mathscr{F}_T$ in property (d) of stopping times.)

4. We assume that Property (P) holds for Y with the constant C. Let S be a stopping time and let Y^S be the stopped martingale defined by $Y_t^S = Y_{t\wedge S}$ (see Corollary 3.24). Show that Property (P) holds for Y^S with the same constant C. One may start by observing that, if S and T are stopping times, one has $Y_T^S = Y_{S\wedge T} = Y_S^T = E[Y_T \mid \mathscr{F}_S]$.

5. We assume in this question and the next one that Property (P) holds for Y with the constant $C = 1$. Let $a > 0$, and let $(R_n)_{n\geq 0}$ be the sequence of stopping times defined by induction by

$$R_0 = 0\,, \qquad R_{n+1} = \inf\{t \geq R_n : |Y_t - Y_{R_n}| \geq a\} \qquad (\inf\varnothing = \infty).$$

Show that, for every integer $n \geq 0$,

$$a^p\, P(R_{n+1} < \infty) \leq P(R_n < \infty).$$

6. Infer that, for every $x > 0$,

$$P\Big(\sup_{t\geq 0} Y_t > x\Big) \leq 2^p\, 2^{-px/2}.$$

Notes and Comments

This chapter gives a brief presentation of the so-called general theory of processes. We limited ourselves to the notions that are needed in the remaining part of this book, but the interested reader can consult the treatise of Dellacherie and Meyer [13, 14] for more about this subject. Most of the martingale theory presented in Sections 3 and 4 goes back to Doob [15]. A comprehensive study of the theory of continuous time martingales can be found in [14]. The applications of the optional stopping theorem to Brownian motion are very classical. For other applications in the same vein, see in particular the book [70] by Revuz and Yor. Exercise 3.29 is taken from the theory of BMO martingales, see e.g. [18, Chapter 7].