

Chapter 9

Local Times

In this chapter, we apply stochastic calculus to the theory of local times of continuous semimartingales. Roughly speaking, the local time at level a of a semimartingale X is an increasing process that measures the “number of visits” of X at level a . We use the classical Tanaka formulas to construct local times and then to study their regularity properties with respect to the space variable. We show how local times can be used to obtain a generalized version of Itô’s formula, and we establish the so-called density of occupation time formula. We also give several approximations of local times. We then focus on the case of Brownian motion, where we state the classical Trotter theorem as a corollary of our results for general semimartingales, and we derive the famous Lévy theorem identifying the law of the Brownian local time process at level 0. In the last section, we use Brownian local times to prove the Kallianpur–Robbins law that was stated at the end of Chap. 7. This chapter can be read independently of Chaps. 6, 7 and 8, except for the last section that relies on Chap. 7.

9.1 Tanaka’s Formula and the Definition of Local Times

Throughout this chapter, we argue on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, and the filtration $(\mathcal{F}_t)_{t \geq 0}$ is assumed to be complete. Let X be a continuous semimartingale. If f is a twice continuously differentiable function defined on \mathbb{R} , Itô’s formula asserts that $f(X_t)$ is still a continuous semimartingale, and

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X, X \rangle_s.$$

The next proposition shows that this formula can be extended to the case when f is a convex function.

Proposition 9.1 *Let f be a convex function on \mathbb{R} . Then $f(X_t)$ is a semimartingale, and, more precisely, there exists an increasing process A^f such that, for every $t \geq 0$,*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s) dX_s + A_t^f,$$

where $f'_-(x)$ denotes the left-derivative of f at x .

More generally, $f(X_t)$ is a semimartingale if f is a difference of convex functions.

Proof Let h be a nonnegative continuous function on \mathbb{R} such that $h(x) = 0$ if $x \notin [0, 1]$ and $\int_0^1 h(x) dx = 1$. For every integer $n \geq 1$, set $h_n(x) = n h(nx)$. Define a function $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi_n(x) = h_n * f(x) = \int_{\mathbb{R}} h_n(y) f(x - y) dy.$$

Then it is elementary to verify that φ_n is twice continuously differentiable on \mathbb{R} , $\varphi'_n = h_n * f'_-$, and $\varphi_n(x) \rightarrow f(x)$, $\varphi'_n(x) \rightarrow f'_-(x)$ as $n \rightarrow \infty$, for every $x \in \mathbb{R}$. Furthermore, the functions φ_n are also convex, so that $\varphi''_n \geq 0$.

Let $X = M + V$ be the canonical decomposition of the semimartingale X , and consider an integer $K \geq 1$. Introduce the stopping time

$$T_K := \inf\{t \geq 0 : |X_t| + \langle M, M \rangle_t + \int_0^t |dV_s| \geq K\}.$$

From Itô's formula, we have

$$\varphi_n(X_{t \wedge T_K}) = \varphi_n(X_0) + \int_0^{t \wedge T_K} \varphi'_n(X_s) dX_s + \frac{1}{2} \int_0^{t \wedge T_K} \varphi''_n(X_s) d\langle M, M \rangle_s. \quad (9.1)$$

From the definition of T_K , we have $\langle M, M \rangle_{T_K} \leq K$. Noting that the functions φ'_n are uniformly bounded over any compact interval, we get, by a simple application of Proposition 5.8,

$$\int_0^{t \wedge T_K} \varphi'_n(X_s) dX_s \xrightarrow[n \rightarrow \infty]{} \int_0^{t \wedge T_K} f'_-(X_s) dX_s, \quad (9.2)$$

in probability. For every $t \geq 0$, set

$$A_t^{f,K} := f(X_{t \wedge T_K}) - f(X_0) - \int_0^{t \wedge T_K} f'_-(X_s) dX_s. \quad (9.3)$$

Since $\varphi_n(X_0) \rightarrow f(X_0)$ and $\varphi_n(X_{t \wedge T_K}) \rightarrow f(X_{t \wedge T_K})$ as $n \rightarrow \infty$, we deduce from (9.2) and (9.1) that

$$\frac{1}{2} \int_0^{t \wedge T_K} \varphi_n''(X_s) d\langle M, M \rangle_s \xrightarrow[n \rightarrow \infty]{} A_t^{f,K}, \tag{9.4}$$

in probability. By (9.3), the process $(A_t^{f,K})_{t \geq 0}$ has continuous sample paths, and $A_0^{f,K} = 0$. Since $\varphi_n'' \geq 0$, it follows from the convergence (9.4) that the sample paths of $(A_t^{f,K})_{t \geq 0}$ are also nondecreasing. Hence $A^{f,K}$ is an increasing process. Finally, one gets from (9.4) that $A_t^{f,K} = A_t^{f,K'}$ if $K \leq K'$. It follows that there exists an increasing process A^f such that $A_t^{f,K} = A_{t \wedge T_K}^f$ for every $t \geq 0$ and $K \geq 1$. We then get the formula of the proposition by letting $K \rightarrow \infty$ in (9.3). \square

Remark Write f'_+ for the right-derivative of f . An argument similar to the preceding proof shows that there exists an increasing process \tilde{A}^f such that

$$f(X_t) = f(X_0) + \int_0^t f'_+(X_s) dX_s + \tilde{A}_t^f.$$

If f is twice continuously differentiable, $A_t^f = \tilde{A}_t^f = \frac{1}{2} \int_0^t f''(X_s) d\langle X, X \rangle_s$. In general, however, we may have $\tilde{A}_t^f \neq A_t^f$.

The previous proposition leads to an easy definition of the local times of a semimartingale. For every $x \in \mathbb{R}$, we set $\text{sgn}(x) := \mathbf{1}_{\{x > 0\}} - \mathbf{1}_{\{x \leq 0\}}$ (the fact that we define $\text{sgn}(0) = -1$ here plays a significant role).

Proposition 9.2 *Let X be a continuous semimartingale and $a \in \mathbb{R}$. There exists an increasing process $(L_t^a(X))_{t \geq 0}$ such that the following three identities hold:*

$$|X_t - a| = |X_0 - a| + \int_0^t \text{sgn}(X_s - a) dX_s + L_t^a(X), \tag{9.5}$$

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t \mathbf{1}_{\{X_s > a\}} dX_s + \frac{1}{2} L_t^a(X), \tag{9.6}$$

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t \mathbf{1}_{\{X_s \leq a\}} dX_s + \frac{1}{2} L_t^a(X). \tag{9.7}$$

The increasing process $(L_t^a(X))_{t \geq 0}$ is called the local time of X at level a . Furthermore, for every stopping time T , we have $L_t^a(X^T) = L_{t \wedge T}^a(X)$.

We will refer to any of the identities (9.5), (9.6), (9.7) as Tanaka's formula.

Proof We apply Proposition 9.1 to the convex function $f(x) = |x - a|$, noting that $f'_-(x) = \text{sgn}(x - a)$. It follows from Proposition 9.1 that the process $(L_t^a(X))_{t \geq 0}$ defined by

$$L_t^a(X) := |X_t - a| - |X_0 - a| - \int_0^t \text{sgn}(X_s - a) dX_s$$

is an increasing process. We then need to verify that (9.6) and (9.7) also hold. To this end, we apply Proposition 9.1 to the convex functions $f(x) = (x - a)^+$ and $f(x) = (x - a)^-$. It follows that there exist two increasing processes $A^{a,(+)}$ and $A^{a,-}$ such that

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t \mathbf{1}_{\{X_s > a\}} dX_s + A_t^{a,(+)},$$

and

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t \mathbf{1}_{\{X_s \leq a\}} dX_s + A_t^{a,-}.$$

By considering the difference between the last two displays, we immediately get that $A^{a,(+)} = A^{a,-}$. On the other hand, if we add these two displays and compare with (9.5), we get $A_t^{a,(+)} + A_t^{a,-} = L_t^a(X)$. Hence $A_t^{a,(+)} = A_t^{a,-} = \frac{1}{2}L_t^a(X)$.

The last assertion immediately follows from (9.5) since $\int_0^{t \wedge T} \text{sgn}(X_s - a) dX_s = \int_0^t \text{sgn}(X_s^T - a) dX_s^T$ by properties of the stochastic integral. \square

Let us state the key property of local times. We use the notation $d_s L_s^a(X)$ for the random measure associated with the increasing function $s \mapsto L_s^a(X)$ (i.e. $\int_{[0,t]} d_s L_s^a(X) = L_t^a(X)$).

Proposition 9.3 *Let X be a continuous semimartingale and let $a \in \mathbb{R}$. Then a.s. the random measure $d_s L_s^a(X)$ is supported on $\{s \geq 0 : X_s = a\}$.*

Proof Set $W_t = |X_t - a|$ and note that (9.5) gives $\langle W, W \rangle_t = \langle X, X \rangle_t$ since $|\text{sgn}(x)| = 1$ for every $x \in \mathbb{R}$. By applying Itô's formula to $(W_t)^2$, we get

$$(X_t - a)^2 = W_t^2 = (X_0 - a)^2 + 2 \int_0^t (X_s - a) dX_s + 2 \int_0^t |X_s - a| d_s L_s^a(X) + \langle X, X \rangle_t.$$

Comparing with the result of a direct application of Itô's formula to $(X_t - a)^2$, we get

$$\int_0^t |X_s - a| d_s L_s^a(X) = 0,$$

which gives the desired result. \square

Proposition 9.3 shows that the function $t \mapsto L_t^a(X)$ may only increase when $X_t = a$. So in some sense, $L_t^a(X)$ measures the "number of visits" of the process X at level a before time t (the results of Sect. 9.3 give rigorous versions of this intuitive statement). This also justifies the name "local time".

9.2 Continuity of Local Times and the Generalized Itô Formula

We consider a continuous semimartingale X and write $X = M + V$ for its canonical decomposition. Our first goal is to study the continuity of the local times of X with respect to the space variable a .

It is convenient to write $L^a(X)$ for the random continuous function $(L_t^a(X))_{t \geq 0}$, which we view as a random variable with values in the space $C(\mathbb{R}_+, \mathbb{R}_+)$. As usual, the latter space is equipped with the topology of uniform convergence on every compact set.

Theorem 9.4 *The process $(L^a(X), a \in \mathbb{R})$ with values in $C(\mathbb{R}_+, \mathbb{R}_+)$ has a càdlàg modification, which we consider from now on and for which we keep the same notation $(L^a(X), a \in \mathbb{R})$. Furthermore, if $L^{a-}(X) = (L_t^{a-}(X))_{t \geq 0}$ denotes the left limit of $b \rightarrow L^b(X)$ at a , we have for every $t \geq 0$,*

$$L_t^a(X) - L_t^{a-}(X) = 2 \int_0^t \mathbf{1}_{\{X_s = a\}} dV_s. \quad (9.8)$$

In particular, if X is a continuous local martingale, the process $(L_t^a(X))_{a \in \mathbb{R}, t \geq 0}$ has jointly continuous sample paths.

The proof of the theorem relies on Tanaka's formula and the following technical lemma.

Lemma 9.5 *Let $p \geq 1$. There exists a constant C_p , which only depends on p , such that for every $a, b \in \mathbb{R}$ with $a < b$, we have*

$$E \left[\left(\int_0^t \mathbf{1}_{\{a < X_s \leq b\}} d\langle M, M \rangle_s \right)^p \right] \leq C_p (b - a)^p \left(E[\langle M, M \rangle_t^{p/2}] + E \left[\left(\int_0^t |dV_s| \right)^p \right] \right).$$

For every $a \in \mathbb{R}$, write $Y^a = (Y_t^a)_{t \geq 0}$ for the random variable with values in $C(\mathbb{R}_+, \mathbb{R})$ defined by

$$Y_t^a = \int_0^t \mathbf{1}_{\{X_s > a\}} dM_s.$$

The process $(Y^a, a \in \mathbb{R})$ has a continuous modification.

Proof Let us start with the first assertion. It is enough to prove that the stated bound holds when $a = -u$ and $b = u$ for some $u > 0$ (then take $u = (b - a)/2$ and replace X by $X - (b + a)/2$). Let f be the unique twice continuously differentiable function such that

$$f''(x) = (2 - \frac{|x|}{u})^+,$$

and $f(0) = f'(0) = 0$. Note that we then have $|f'(x)| \leq 2u$ for every $x \in \mathbb{R}$. Since $f'' \geq 0$ and $f''(x) \geq 1$ if $-u \leq x \leq u$, we have

$$\int_0^t \mathbf{1}_{\{-u < X_s \leq u\}} d\langle M, M \rangle_s \leq \int_0^t f''(X_s) d\langle M, M \rangle_s. \quad (9.9)$$

However, by Itô's formula,

$$\frac{1}{2} \int_0^t f''(X_s) d\langle M, M \rangle_s = f(X_t) - f(X_0) - \int_0^t f'(X_s) dX_s. \quad (9.10)$$

Recalling that $|f'| \leq 2u$, we have

$$\begin{aligned} E[|f(X_t) - f(X_0)|^p] &\leq (2u)^p E[|X_t - X_0|^p] \\ &\leq (2u)^p E\left[\left(|M_t - M_0| + \int_0^t |dV_s|\right)^p\right] \\ &\leq C_p (2u)^p \left(E[\langle M, M \rangle_t^{p/2}] + E\left[\left(\int_0^t |dV_s|\right)^p\right]\right), \end{aligned}$$

using the Burkholder–Davis–Gundy inequalities (Theorem 5.16). Here and below, C_p stands for a constant that depends only on p , which may vary from line to line.

Then,

$$\int_0^t f'(X_s) dX_s = \int_0^t f'(X_s) dM_s + \int_0^t f'(X_s) dV_s.$$

We have

$$E\left[\left|\int_0^t f'(X_s) dV_s\right|^p\right] \leq (2u)^p E\left[\left(\int_0^t |dV_s|\right)^p\right]$$

and, using the Burkholder–Davis–Gundy inequalities once again,

$$\begin{aligned} E\left[\left|\int_0^t f'(X_s) dM_s\right|^p\right] &\leq C_p E\left[\left(\int_0^t f'(X_s)^2 d\langle M, M \rangle_s\right)^{p/2}\right] \\ &\leq C_p (2u)^p E[\langle M, M \rangle_t^{p/2}]. \end{aligned}$$

The first assertion of the lemma follows by combining the previous bounds, using (9.9) and (9.10).

Let us turn to the second assertion. We fix $p > 2$. By the Burkholder–Davis–Gundy inequalities, we have for every $a < b$ and every $t \geq 0$,

$$E\left[\sup_{s \leq t} |Y_s^b - Y_s^a|^p\right] \leq C_p E\left[\left(\int_0^t \mathbf{1}_{\{a < X_s \leq b\}} d\langle M, M \rangle_s\right)^{p/2}\right], \tag{9.11}$$

and the right-hand side can be estimated from the first assertion of the lemma. More precisely, for every integer $n \geq 1$, introduce the stopping time

$$T_n := \inf\{t \geq 0 : \langle M, M \rangle_t + \int_0^t |dV_s| \geq n\}.$$

From the first assertion of the lemma with X replaced by the stopped process X^{T_n} , we have, for every $t \geq 0$,

$$E\left[\left(\int_0^{t \wedge T_n} \mathbf{1}_{\{a < X_s \leq b\}} d\langle M, M \rangle_s\right)^{p/2}\right] \leq C_p (n^{p/4} + n^{p/2}) (b - a)^{p/2}.$$

Using (9.11), again with X replaced by X^{T_n} , and letting $t \rightarrow \infty$, we obtain

$$E\left[\sup_{s \geq 0} |Y_{s \wedge T_n}^b - Y_{s \wedge T_n}^a|^p\right] \leq C_p (n^{p/4} + n^{p/2}) (b - a)^{p/2}.$$

Since $p > 2$, we see that we can apply Kolmogorov’s lemma (Theorem 2.9) to get the existence of a continuous modification of the process $a \rightarrow (Y_{s \wedge T_n}^a)_{s \geq 0}$, with values in $C(\mathbb{R}_+, \mathbb{R})$. Write $(Y_s^{(n),a})_{s \geq 0}$ for this continuous modification.

Then, if $1 \leq n < m$, for every fixed a , we have $Y_s^{(n),a} = Y_{s \wedge T_n}^{(m),a}$ for every $s \geq 0$, a.s. By a continuity argument, the latter equality holds simultaneously for every $a \in \mathbb{R}$ and every $s \geq 0$, outside a single set of probability zero. It follows that we

can define a process $(\tilde{Y}^a, a \in \mathbb{R})$ with values in $C(\mathbb{R}_+, \mathbb{R})$, with continuous sample paths, such that, for every $n \geq 1$, $Y_s^{(n),a} = \tilde{Y}_{s \wedge T_n}^a$ for every $a \in \mathbb{R}$ and every $s \geq 0$, a.s. The process $(\tilde{Y}^a, a \in \mathbb{R})$ is the desired continuous modification. \square

Remark By applying the bound of Lemma 9.5 to X^{T_n} (with T_n as in the previous proof) and letting a tend to b , we get that, for every $b \in \mathbb{R}$,

$$\int_0^t \mathbf{1}_{\{X_s=b\}} d\langle M, M \rangle_s = 0$$

for every $t \geq 0$, a.s. Consequently, using Proposition 4.12, we also have

$$\int_0^t \mathbf{1}_{\{X_s=b\}} dM_s = 0, \quad (9.12)$$

for every $t \geq 0$, a.s.

Proof of Theorem 9.4 With a slight abuse of notation, we still write $(Y^a, a \in \mathbb{R})$ for the continuous modification obtained in the second assertion of Lemma 9.5. We also let $(Z^a, a \in \mathbb{R})$ be the process with values in $C(\mathbb{R}_+, \mathbb{R})$ defined by

$$Z_t^a = \int_0^t \mathbf{1}_{\{X_s>a\}} dV_s.$$

By Tanaka's formula, we have for every fixed $a \in \mathbb{R}$,

$$L_t^a = 2\left((X_t - a)^+ - (X_0 - a)^+ - Y_t^a - Z_t^a\right), \text{ for every } t \geq 0, \text{ a.s.}$$

The right-hand side of the last display provides the desired càdlàg modification. Indeed, the process

$$a \mapsto \left((X_t - a)^+ - (X_0 - a)^+ - Y_t^a\right)_{t \geq 0}$$

has continuous sample paths, and on the other hand the process $a \mapsto Z^a$ has càdlàg sample paths: For every $a_0 \in \mathbb{R}$, the dominated convergence theorem shows that

$$\begin{aligned} \int_0^t \mathbf{1}_{\{X_s>a\}} dV_s &\xrightarrow{a \downarrow a_0} \int_0^t \mathbf{1}_{\{X_s>a_0\}} dV_s, \\ \int_0^t \mathbf{1}_{\{X_s>a\}} dV_s &\xrightarrow{a \uparrow a_0, a < a_0} \int_0^t \mathbf{1}_{\{X_s \geq a_0\}} dV_s, \end{aligned}$$

uniformly on every compact time interval. The previous display also shows that the jump $Z^{a_0} - Z^{a_0^-}$ is given by

$$Z_t^{a_0} - Z_t^{a_0^-} = - \int_0^t \mathbf{1}_{\{X_s = a_0\}} dV_s,$$

and this completes the proof of the theorem. □

From now on, we only deal with the càdlàg modification of local times obtained in Theorem 9.4.

Remark To illustrate Theorem 9.4, set $W_t = |X_t|$, which is also a semimartingale by Tanaka’s formula (9.5). By (9.6) applied to W_t , we have

$$\begin{aligned} W_t &= (W_t)^+ = |X_0| + \int_0^t \mathbf{1}_{\{|X_s| > 0\}} (\text{sgn}(X_s) dX_s + dL_s^0(X)) + \frac{1}{2} L_t^0(W) \\ &= |X_0| + \int_0^t \text{sgn}(X_s) dX_s + \int_0^t \mathbf{1}_{\{X_s = 0\}} dX_s + \frac{1}{2} L_t^0(W), \end{aligned}$$

noting that $\int_0^t \mathbf{1}_{\{|X_s| > 0\}} dL_s^0(X) = 0$ by the support property of local time (Proposition 9.3). Comparing the resulting formula with (9.5) written with $a = 0$, we get

$$L_t^0(W) = 2L_t^0(X) - 2 \int_0^t \mathbf{1}_{\{X_s = 0\}} dX_s = L_t^0(X) + L_t^{0-}(X),$$

using (9.8). The formula $L_t^0(W) = L_t^0(X) + L_t^{0-}(X)$ is a special case of the more general formula $L_t^a(W) = L_t^a(X) + L_t^{(-a)-}(X)$, for every $a \geq 0$, which is easily deduced from Corollary 9.7 below. We note that the support property of local time implies $L_t^a(W) = 0$ for every $a < 0$, and in particular $L_t^{0-}(W) = 0$. We leave it as an exercise for the reader to verify that formula (9.8) applied to $L_t^0(W) - L_t^{0-}(W)$ gives a result which is consistent with the preceding expression for $L_t^0(W)$.

We will now give an extension of Itô’s formula (in the case where it is applied to a function of a single semimartingale). If f is a convex function on \mathbb{R} , the left derivative f'_- is a left-continuous monotone nondecreasing function, and there exists a unique Radon measure $f''(dy)$ on \mathbb{R}_+ such that $f''([a, b]) = f'_-(b) - f'_-(a)$, for every $a < b$. One can also interpret f'' as the second derivative of f in the sense of distributions. Note that $f''(da) = f''(a)da$ if f is twice continuously differentiable. If f is now a difference of convex functions, that is, $f = f_1 - f_2$ where both f_1 and f_2 are convex, we can still make sense of $\int f''(dy) \varphi(y) = \int f_1''(dy) \varphi(y) - \int f_2''(dy) \varphi(y)$ for any bounded measurable function φ supported on a compact interval of \mathbb{R} .

The next theorem identifies the increasing process A_t^f that appeared in Proposition 9.1.

Theorem 9.6 (Generalized Itô formula) *Let f be a difference of convex functions on \mathbb{R} . Then, for every $t \geq 0$,*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s) dX_s + \frac{1}{2} \int_{\mathbb{R}} L_t^a(X) f''(da).$$

Remark By Proposition 9.2 and a continuity argument, we have

$$L_t^a(X) = 0 \text{ for every } a \notin \left[\min_{0 \leq s \leq t} X_s, \max_{0 \leq s \leq t} X_s \right], \quad \text{a.s.}$$

and furthermore the function $a \mapsto L_t^a(X)$ is bounded. Together with the observations preceding the statement of the theorem, this shows that the integral $\int_{\mathbb{R}} L_t^a(X) f''(da)$ makes sense.

Proof By linearity, it suffices to treat the case when f is convex. Furthermore, by simple “localization” arguments, we can assume that f'' is a finite measure supported on the interval $[-K, K]$ for some $K > 0$. By adding an affine function to f , we can also assume that $f = 0$ on $(-\infty, -K]$. Then, it is elementary to verify that, for every $x \in \mathbb{R}$,

$$f(x) = \int_{\mathbb{R}} (x - a)^+ f''(da),$$

and

$$f'_-(x) = \int \mathbf{1}_{\{a < x\}} f''(da). \quad (9.13)$$

Tanaka’s formula gives, for every $a \in \mathbb{R}$,

$$(X_t - a)^+ = (X_0 - a)^+ + Y_t^a + Z_t^a + \frac{1}{2} L_t^a(X),$$

where we use the notation of the proof of Theorem 9.4 (and we recall that $(Y^a, a \in \mathbb{R})$ stands for the continuous modification obtained in Lemma 9.5). We can integrate the latter equality with respect to the finite measure $f''(da)$ and we get

$$f(X_t) = f(X_0) + \int Y_t^a f''(da) + \int Z_t^a f''(da) + \frac{1}{2} \int L_t^a(X) f''(da).$$

By Fubini’s theorem,

$$\begin{aligned} \int Z_t^a f''(da) &= \int \left(\int_0^t \mathbf{1}_{\{X_s > a\}} dV_s \right) f''(da) = \int_0^t \left(\int \mathbf{1}_{\{X_s > a\}} f''(da) \right) dV_s \\ &= \int_0^t f'_-(X_s) dV_s. \end{aligned}$$

So the proof will be complete if we can also verify that

$$\int Y_t^a f''(da) = \int_0^t f'_-(X_s) dM_s. \tag{9.14}$$

This identity should be viewed as a kind of Fubini theorem involving a stochastic integral. To provide a rigorous justification, it is convenient to introduce the stopping times $T_n := \inf\{s \geq 0 : \langle M, M \rangle_s \geq n\}$, for every $n \geq 1$. Recalling (9.13), we see that our claim (9.14) will follow if we can verify that, for every $n \geq 1$, we have a.s.

$$\int \left(\int_0^{t \wedge T_n} \mathbf{1}_{\{X_s > a\}} dM_s \right) f''(da) = \int_0^{t \wedge T_n} \left(\int \mathbf{1}_{\{X_s > a\}} f''(da) \right) dM_s, \tag{9.15}$$

where in the left-hand side we agree that we consider the continuous modification of $a \mapsto \int_0^{t \wedge T_n} \mathbf{1}_{\{X_s > a\}} dM_s$ provided by Lemma 9.5. It is straightforward to verify that the left-hand side of (9.15) defines a martingale M_t^f in \mathbb{H}^2 , and furthermore, for any other martingale N in \mathbb{H}^2 ,

$$\begin{aligned} E[\langle M^f, N \rangle_\infty] &= E[M_\infty^f N_\infty] = E\left[\int \left(\int_0^{T_n} \mathbf{1}_{\{X_s > a\}} d\langle M, N \rangle_s \right) f''(da) \right] \\ &= E\left[\int_0^{T_n} \left(\int \mathbf{1}_{\{X_s > a\}} f''(da) \right) d\langle M, N \rangle_s \right] \\ &= E\left[\left(\int_0^{T_n} \left(\int \mathbf{1}_{\{X_s > a\}} f''(da) \right) dM_s \right) N_\infty \right]. \end{aligned}$$

By a duality argument in \mathbb{H}^2 , this suffices to verify that M_t^f coincides with the martingale of \mathbb{H}^2 in the right-hand side of (9.15). This completes the proof. \square

The following corollary is even more important than the preceding theorem.

Corollary 9.7 (Density of occupation time formula) *We have almost surely, for every $t \geq 0$ and every nonnegative measurable function φ on \mathbb{R} ,*

$$\int_0^t \varphi(X_s) d\langle X, X \rangle_s = \int_{\mathbb{R}} \varphi(a) L_t^a(X) da.$$

More generally, we have a.s. for any nonnegative measurable function F on $\mathbb{R}_+ \times \mathbb{R}$,

$$\int_0^\infty F(s, X_s) d\langle X, X \rangle_s = \int_{\mathbb{R}} da \int_0^\infty F(s, a) d_s L_s^a(X).$$

Proof Fix $t \geq 0$ and consider a nonnegative continuous function φ on \mathbb{R} with compact support. Let f be a twice continuously differentiable function on \mathbb{R} such that $f'' = \varphi$. Note that f is convex since $\varphi \geq 0$. By comparing Itô's formula applied

to $f(X_t)$ and the formula of Theorem 9.6, we immediately get that a.s.

$$\int_0^t \varphi(X_s) d\langle X, X \rangle_s = \int_{\mathbb{R}} \varphi(a) L_t^a(X) da.$$

This formula holds simultaneously (outside a set of probability zero) for every $t \geq 0$ (by a continuity argument) and for every function φ belonging to a countable dense subset of the set of all nonnegative continuous functions on \mathbb{R} with compact support. This suffices to conclude that a.s. for every $t \geq 0$, the random measure

$$A \mapsto \int_0^t \mathbf{1}_A(X_s) d\langle X, X \rangle_s$$

has density $(L_t^a(X))_{a \in \mathbb{R}}$ with respect to Lebesgue measure on \mathbb{R} . This gives the first assertion of the corollary. It follows that the formula in the second assertion holds when F is of the type

$$F(s, a) = \mathbf{1}_{[u, v]}(s) \mathbf{1}_A(a)$$

where $0 \leq u \leq v$ and A is a Borel subset of \mathbb{R} . Hence, a.s. the σ -finite measures

$$B \longrightarrow \int_0^\infty \mathbf{1}_B(s, X_s) d\langle X, X \rangle_s$$

and

$$B \longrightarrow \int_{\mathbb{R}} da \int_0^\infty \mathbf{1}_B(s, a) d_s L_s^a(X)$$

take the same value for B of the form $B = [u, v] \times A$, and this implies that the two measures coincide. \square

If $X = M + V$ is a continuous semimartingale, then an immediate application of the density of occupation time formula gives, for every $b \in \mathbb{R}$,

$$\int_0^t \mathbf{1}_{\{X_s = b\}} d\langle M, M \rangle_s = \int_{\mathbb{R}} \mathbf{1}_{\{b\}}(a) L_t^a(X) da = 0.$$

This property has already been derived after the proof of Lemma 9.5. On the other hand, there may exist values of b such that

$$\int_0^t \mathbf{1}_{\{X_s = b\}} dV_s \neq 0,$$

and these values of b correspond to discontinuities of the local time with respect to the space variable, as shown by Theorem 9.4.

Corollary 9.8 *If X is of the form $X_t = X_0 + V_t$, where V is a finite variation process, then $L_t^a(X) = 0$ for all $a \in \mathbb{R}$ and $t \geq 0$.*

Proof From the density of occupation time formula and the fact that $\langle X, X \rangle = 0$, we get $\int_{\mathbb{R}} \varphi(a) L_t^a(X) da = 0$ for any nonnegative measurable function φ , and the desired result follows. \square

Remark We could have derived the last corollary directly from Tanaka’s formula.

9.3 Approximations of Local Times

Our first approximation result is an easy consequence of the density of occupation time formula.

Proposition 9.9 *Let X be a continuous semimartingale. Then a.s. for every $a \in \mathbb{R}$ and $t \geq 0$,*

$$L_t^a(X) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{a \leq X_s \leq a+\varepsilon\}} d\langle X, X \rangle_s.$$

Proof By the density of occupation time formula,

$$\frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{a \leq X_s \leq a+\varepsilon\}} d\langle X, X \rangle_s = \frac{1}{\varepsilon} \int_a^{a+\varepsilon} L_t^b(X) db,$$

and the result follows from the right-continuity of $b \mapsto L_t^b(X)$ at a (Theorem 9.4). \square

Remark The same argument gives

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{a-\varepsilon \leq X_s \leq a+\varepsilon\}} d\langle X, X \rangle_s = \frac{1}{2} (L_t^a(X) + L_t^{a-}(X)).$$

The quantity $\tilde{L}_t^a(X) := \frac{1}{2} (L_t^a(X) + L_t^{a-}(X))$ is sometimes called the symmetric local time of the semimartingale X . Note that the density of occupation time formula remains true if $L_t^a(X)$ is replaced by $\tilde{L}_t^a(X)$ (indeed, $\tilde{L}_t^a(X)$ and $L_t^a(X)$ may differ in at most countably many values of a). The generalized Itô formula (Theorem 9.6) also remains true if $L_t^a(X)$ is replaced by $\tilde{L}_t^a(X)$, provided the left-derivative f'_- is replaced by $\frac{1}{2}(f'_+ + f'_-)$. Similar observations apply to Tanaka’s formulas.

As a consequence of the preceding proposition and Lemma 9.5, we derive a useful bound on moments of local times.

Corollary 9.10 *Let $p \geq 1$. There exists a constant C_p such that, for any continuous semimartingale X with canonical decomposition $X = M + V$, we have for every $a \in \mathbb{R}$ and $t \geq 0$,*

$$E[(L_t^a(X))^p] \leq C_p \left(E[(\langle M, M \rangle_t)^{p/2}] + E\left[\left(\int_0^t |dV_s|\right)^p\right] \right).$$

Proof This readily follows from the bound of Lemma 9.5, using the approximation of $L_t^a(X)$ in Proposition 9.9 and Fatou’s lemma. \square

We next turn to the upcrossing approximation of local time. We first need to introduce some notation. We let X be a continuous semimartingale, and $\varepsilon > 0$. We then introduce two sequences $(\sigma_n^\varepsilon)_{n \geq 1}$ and $(\tau_n^\varepsilon)_{n \geq 1}$ of stopping times, which are defined inductively by

$$\sigma_1^\varepsilon := \inf\{t \geq 0 : X_t = 0\}, \quad \tau_1^\varepsilon := \inf\{t \geq \sigma_1^\varepsilon : X_t = \varepsilon\},$$

and, for every $n \geq 1$,

$$\sigma_{n+1}^\varepsilon := \inf\{t \geq \tau_n^\varepsilon : X_t = 0\}, \quad \tau_{n+1}^\varepsilon := \inf\{t > \sigma_n^\varepsilon : X_t = \varepsilon\}.$$

We then define the upcrossing number of X along $[0, \varepsilon]$ before time t by

$$N_\varepsilon^X(t) = \text{Card}\{n \geq 1 : \tau_n^\varepsilon \leq t\}.$$

This notion has already been introduced in Sect. 3.3 with a slightly different presentation.

Proposition 9.11 *We have, for every $t \geq 0$,*

$$\varepsilon N_\varepsilon^X(t) \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2} L_t^0(X)$$

in probability.

Proof To simplify notation, we write L_s^0 instead of $L_s^0(X)$ in this proof. We first use Tanaka’s formula to get, for every $n \geq 1$,

$$(X_{\tau_n^\varepsilon \wedge t})^+ - (X_{\sigma_n^\varepsilon \wedge t})^+ = \int_{\sigma_n^\varepsilon \wedge t}^{\tau_n^\varepsilon \wedge t} \mathbf{1}_{\{X_s > 0\}} dX_s + \frac{1}{2} (L_{\tau_n^\varepsilon \wedge t}^0 - L_{\sigma_n^\varepsilon \wedge t}^0).$$

We sum the last identity over all $n \geq 1$ to get

$$\sum_{n=1}^{\infty} ((X_{\tau_n^\varepsilon \wedge t})^+ - (X_{\sigma_n^\varepsilon \wedge t})^+) = \int_0^t \left(\sum_{n=1}^{\infty} \mathbf{1}_{(\sigma_n^\varepsilon, \tau_n^\varepsilon]}(s) \right) \mathbf{1}_{\{X_s > 0\}} dX_s + \frac{1}{2} \sum_{n=1}^{\infty} (L_{\tau_n^\varepsilon \wedge t}^0 - L_{\sigma_n^\varepsilon \wedge t}^0). \tag{9.16}$$

Note that there are only finitely many values of n such that $\tau_n^\varepsilon \leq t$, and that the interversion of the series and the stochastic integral is justified by approximating the series with finite sums and using Proposition 5.8 (the required domination is obvious since the integrands are bounded by 1).

Consider the different terms in (9.16). Since the local time L^0 does not increase on intervals of the type $[\tau_n^\varepsilon, \sigma_{n+1}^\varepsilon)$ (nor on $[0, \sigma_1^\varepsilon)$), we have

$$\sum_{n=1}^\infty (L_{\tau_n^\varepsilon \wedge t}^0 - L_{\sigma_n^\varepsilon \wedge t}^0) = \sum_{n=1}^\infty (L_{\sigma_{n+1}^\varepsilon \wedge t}^0 - L_{\sigma_n^\varepsilon \wedge t}^0) = L_t^0.$$

Then, noting that $(X_{\tau_n^\varepsilon \wedge t})^+ - (X_{\sigma_n^\varepsilon \wedge t})^+ = \varepsilon$ if $\tau_n^\varepsilon \leq t$, we have

$$\sum_{n=1}^\infty ((X_{\tau_n^\varepsilon \wedge t})^+ - (X_{\sigma_n^\varepsilon \wedge t})^+) = \varepsilon N_\varepsilon^X(t) + u(\varepsilon),$$

where $0 \leq u(\varepsilon) \leq \varepsilon$.

From (9.16) and the last two displays, the result of the proposition will follow if we can verify that

$$\int_0^t \left(\sum_{n=1}^\infty \mathbf{1}_{(\sigma_n^\varepsilon, \tau_n^\varepsilon]}(s) \right) \mathbf{1}_{\{X_s > 0\}} dX_s \xrightarrow[\varepsilon \rightarrow 0]{} 0$$

in probability. This is again a consequence of Proposition 5.8, since

$$0 \leq \left(\sum_{n=1}^\infty \mathbf{1}_{(\sigma_n^\varepsilon, \tau_n^\varepsilon]}(s) \right) \mathbf{1}_{\{X_s > 0\}} \leq \mathbf{1}_{\{0 < X_s \leq \varepsilon\}}$$

and $\mathbf{1}_{\{0 < X_s \leq \varepsilon\}} \rightarrow 0$ as $\varepsilon \rightarrow 0$. □

9.4 The Local Time of Linear Brownian Motion

Throughout this section, $(B_t)_{t \geq 0}$ is a real Brownian motion started from 0 and (\mathcal{F}_t) is the (completed) canonical filtration of B .

The following theorem, which is known as Trotter’s theorem, is essentially a restatement of the results of the previous sections in the special case of real Brownian motion. Still the importance of the result justifies this repetition. We write $\text{supp}(\mu)$ for the topological support of a finite measure μ on \mathbb{R}_+ .

Theorem 9.12 (Trotter) *There exists a (unique) process $(L_t^a(B))_{a \in \mathbb{R}, t \geq 0}$, whose sample paths are continuous functions of the pair (a, t) , such that, for every fixed $a \in \mathbb{R}$, $(L_t^a(B))_{t \geq 0}$ is an increasing process, and, a.s. for every $t \geq 0$, for every*

nonnegative measurable function φ on \mathbb{R} ,

$$\int_0^t \varphi(B_s) ds = \int_{\mathbb{R}} \varphi(a) L_t^a(B) da.$$

Furthermore, a.s. for every $a \in \mathbb{R}$,

$$\text{supp}(d_s L_s^a(B)) \subset \{s \geq 0 : B_s = a\}, \quad (9.17)$$

and this inclusion is an equality with probability one if a is fixed.

Proof The first assertion follows by applying Theorem 9.4 and Corollary 9.7 to $X = B$, noting that $\langle B, B \rangle_t = t$. We have already seen that the inclusion (9.17) holds with probability one if a is fixed, hence simultaneously for all rationals, a.s. A continuity argument allows us to get that (9.17) holds simultaneously for all $a \in \mathbb{R}$ outside a single set of probability zero. Indeed, suppose that for some $a \in \mathbb{R}$ and $0 \leq s < t$ we have $L_t^a(B) > L_s^a(B)$ and $B_r \neq a$ for every $r \in [s, t]$. Then we can find a rational $b \in \mathbb{R}$ sufficiently close to a such that the same properties hold when a is replaced by b , giving a contradiction.

Finally, let us verify that (9.17) is an a.s. equality if $a \in \mathbb{R}$ is fixed. So let us fix $a \in \mathbb{R}$, and for every rational $q \geq 0$, set

$$H_q := \inf\{t \geq q : B_t = a\}.$$

Our claim will follow if we can verify that a.s. for every $\varepsilon > 0$, $L_{H_q+\varepsilon}^a(B) > L_{H_q}^a(B)$. Using the strong Markov property at time H_q , it suffices to prove that, if B' is a real Brownian motion started from a , we have $L_\varepsilon^a(B') > 0$, for every $\varepsilon > 0$, a.s. Clearly we can take $a = 0$. We then observe that we have

$$L_\varepsilon^0(B) \stackrel{(d)}{=} \sqrt{\varepsilon} L_1^0(B),$$

by an easy scaling argument (use for instance the approximations of the previous section). Also $P(L_1^0(B) > 0) > 0$ since $E[L_1^0(B)] = E[|B_1|]$ by Tanaka's formula. An application of Blumenthal's zero-one law (Theorem 2.13) to the event

$$A := \bigcap_{n=1}^{\infty} \{L_{2^{-n}}^0(B) > 0\} = \lim_{n \uparrow \infty} \uparrow \{L_{2^{-n}}^0(B) > 0\}$$

completes the proof. \square

Remark Theorem 9.12 remains true with a similar proof for an arbitrary (possibly random) initial value B_0 .

We now turn to distributional properties of local times of Brownian motion.

- Proposition 9.13** (i) *Let $a \in \mathbb{R} \setminus \{0\}$ and $T_a := \inf\{t \geq 0 : B_t = a\}$. Then $L_{T_a}^0(B)$ has an exponential distribution with mean $2|a|$.*
 (ii) *Let $a > 0$ and $U_a := \inf\{t \geq 0 : |B_t| = a\}$. Then $L_{U_a}^0(B)$ has an exponential distribution with mean a .*

Proof

- (i) By simple scaling and symmetry arguments, it is enough to take $a = 1$. We then observe that $L_\infty^0(B) = \infty$ a.s. Indeed, the scaling argument of the preceding proof shows that $L_\infty^0(B)$ has the same distribution as $\lambda L_\infty^0(B)$, for any $\lambda > 0$, and we have also seen that $L_\infty^0(B) > 0$ a.s. Fix $s > 0$ and set

$$\tau := \inf\{t \geq 0 : L_t^0(B) \geq s\},$$

so that τ is a stopping time of the filtration (\mathcal{F}_t) . Furthermore, $B_\tau = 0$ by the support property of local time. By the strong Markov property,

$$B'_t := B_{\tau+t}$$

is a Brownian motion started from 0, which is also independent of \mathcal{F}_τ . Proposition 9.9 gives, for every $t \geq 0$,

$$L_t^0(B') = L_{\tau+t}^0(B) - s.$$

On the event $\{L_{T_1}^0(B) \geq s\} = \{\tau \leq T_1\}$, we thus have

$$L_{T_1}^0(B) - s = L_{T_1-\tau}^0(B') = L_{T_1'}^0(B'),$$

where $T_1' := \inf\{t \geq 0 : B'_t = 1\}$. Since the event $\{\tau \leq T_1\}$ is \mathcal{F}_τ -measurable and B' is independent of \mathcal{F}_τ , we get that the conditional distribution of $L_{T_1}^0(B) - s$ knowing that $L_{T_1}^0(B) \geq s$ is the same as the unconditional distribution of $L_{T_1'}^0(B')$. This implies that the distribution of $L_{T_1}^0(B)$ is exponential.

Finally, Tanaka's formula (9.6) shows that $\frac{1}{2}E[L_{t \wedge T_1}^0] = E[(B_{t \wedge T_1})^+]$. As $t \rightarrow \infty$, $E[L_{t \wedge T_1}^0]$ converges to $E[L_{T_1}^0]$ by monotone convergence and $E[(B_{t \wedge T_1})^+]$ converges to $E[(B_{T_1})^+]$ by dominated convergence, since $0 \leq (B_{t \wedge T_1})^+ \leq 1$. This shows that $E[L_{T_1}^0] = 2$, as desired.

- (ii) The argument is exactly similar. We now use Tanaka's formula (9.5) to verify that $E[L_{U_a}^0(B)] = a$. □

Remark One can give an alternative proof of the proposition using stochastic calculus. To get (ii), for instance, use Itô's formula to verify that, for every $\lambda > 0$,

$$(1 + \lambda|B_t|) \exp(-\lambda L_t^0(B))$$

is a continuous local martingale, which is bounded on $[0, U_a]$. An application of the optional stopping theorem then shows that $E[\exp(-\lambda L_{U_a}^0(B))] = (1 + \lambda a)^{-1}$.

The previous proof has the advantage of explaining the appearance of the exponential distribution.

For every $t \geq 0$, we set

$$S_t := \sup_{0 \leq s \leq t} B_s, \quad I_t := \inf_{0 \leq s \leq t} B_s.$$

Theorem 9.14 (Lévy) *The two processes $(S_t, S_t - B_t)_{t \geq 0}$ and $(L_t^0(B), |B_t|)_{t \geq 0}$ have the same distribution.*

Remark By an obvious symmetry argument, the pair $(-I_t, B_t - I_t)_{t \geq 0}$ also has the same distribution as $(S_t, S_t - B_t)_{t \geq 0}$.

Proof By Tanaka's formula, for every $t \geq 0$,

$$|B_t| = -\beta_t + L_t^0(B), \tag{9.18}$$

where

$$\beta_t = - \int_0^t \operatorname{sgn}(B_s) dB_s.$$

Since $\langle \beta, \beta \rangle_t = t$, Theorem 5.12 ensures that β is a real Brownian motion started from 0. We then claim that, for every $t \geq 0$,

$$L_t^0(B) = \sup\{\beta_s : s \leq t\}.$$

The fact that $L_t^0(B) \geq \sup\{\beta_s : s \leq t\}$ is immediate since (9.18) shows that $L_t^0(B) \geq L_s^0(B) \geq \beta_s$ for every $s \in [0, t]$. To get the reverse inequality, write γ_t for the last zero of B before time t . By the support property of local time, $L_t^0(B) = L_{\gamma_t}^0(B)$, and using (9.18), $L_{\gamma_t}^0(B) = \beta_{\gamma_t} \leq \sup\{\beta_s : s \leq t\}$.

We have thus proved a.s.

$$(L_t^0(B), |B_t|)_{t \geq 0} = (\sup\{\beta_s : s \leq t\}, \sup\{\beta_s : s \leq t\} - \beta_t)_{t \geq 0},$$

and since $(\beta_s)_{s \geq 0}$ and $(B_s)_{s \geq 0}$ have the same distribution, the pair in the right-hand side has the same distribution as $(S_t, S_t - B_t)_{t \geq 0}$. \square

Theorem 9.14 has several interesting consequences. For every $t \geq 0$, S_t has the same law as $|B_t|$ (Theorem 2.21), and thus the same holds for $L_t^0(B)$. From the explicit formula (2.2) for the density of (S_t, B_t) , we also get the density of the pair $(L_t^0(B), B_t)$.

For every $s \geq 0$, set

$$\tau_s := \inf\{t \geq 0 : L_t^0(B) > s\}.$$

The process $(\tau_s)_{s \geq 0}$ is called the inverse local time (at 0) of the Brownian motion B . By construction, $(\tau_s)_{s \geq 0}$ has càdlàg increasing sample paths. From Lévy’s Theorem 9.14, one gets that

$$(\tau_s)_{s \geq 0} \stackrel{(d)}{=} (\tilde{T}_s)_{s \geq 0},$$

where, for every $s \geq 0$, $\tilde{T}_s := \inf\{t \geq 0 : B_t > s\}$ (it is easy to verify that, for every $s \geq 0$, $\tilde{T}_s = T_s$ a.s., but $(\tilde{T}_s)_{s \geq 0}$ has càdlàg sample paths, which is not the case for $(T_s)_{s \geq 0}$).

The same application of the strong Markov property as in the proof of Proposition 9.13 shows that $(\tau_s)_{s \geq 0}$ has stationary independent increments – compare with Exercise 2.26. Furthermore, using the invariance of Brownian motion under scaling, we have for every $\lambda > 0$,

$$(\tau_{\lambda s})_{s \geq 0} \stackrel{(d)}{=} (\lambda^2 \tau_s)_{s \geq 0}.$$

The preceding properties can be summarized by saying that $(\tau_s)_{s \geq 0}$ is a stable subordinator with index 1/2 (a subordinator is a Lévy process with nondecreasing sample paths).

The interest of considering the process $(\tau_s)_{s \geq 0}$ comes in part from the following proposition.

Proposition 9.15 *We have a.s.*

$$\{t \geq 0 : B_t = 0\} = \{\tau_s : s \geq 0\} \cup \{\tau_{s-} : s \in D\}$$

where D is the countable set of jump times of $(\tau_s)_{s \geq 0}$.

Proof We know from (9.17) that a.s.

$$\text{supp}(d_t L_t^0(B)) \subset \{t \geq 0 : B_t = 0\}.$$

It follows that any time t of the form $t = \tau_s$ or $t = \tau_{s-}$ must belong to the zero set of B . Conversely, recalling that (9.17) is an a.s. equality for $a = 0$, we also get that, a.s. for every t such that $B_t = 0$, we have either $L_{t+\varepsilon}^0(B) > L_t^0(B)$ for every $\varepsilon > 0$, or, if $t > 0$, $L_t^0(B) > L_{t-\varepsilon}^0(B)$ for every $\varepsilon > 0$ with $\varepsilon \leq t$ (or both simultaneously), which implies that we have $t = \tau_{L_t^0(B)}$ or $t = \tau_{L_t^0(B)-}$. \square

As a consequence of Proposition 9.15, the connected components of the complement of the zero set $\{t \geq 0 : B_t = 0\}$ are exactly the intervals (τ_{s-}, τ_s) for $s \in D$. These connected components are called the *excursion intervals* (away from 0). For every $s \in D$, the associated *excursion* is defined by

$$e_s(t) := B_{(\tau_{s-+t}) \wedge \tau_s}, \quad t \geq 0.$$

The goal of excursion theory is to describe the distribution of the excursion process, that is, of the collection $(e_s)_{s \in D}$. This study, however, goes beyond the scope of the present book, see in particular [4, 70, 72].

9.5 The Kallianpur–Robbins Law

In this section, we use local times to give a short proof of the Kallianpur–Robbins law for planar Brownian motion, which was stated at the end of Chap. 7 as Theorem 7.23. Let us recall the notation we need. We let B stand for a complex Brownian motion, and for simplicity we assume that $B_0 = 1$ (the general case will then follow, for instance by applying the strong Markov property at the first hitting time of the unit circle). According to Theorem 7.19, we can write $|B_t| = \exp(\beta_{H_t})$, where β is a real Brownian motion started from 0, and

$$H_t = \int_0^t \frac{ds}{|B_s|^2} = \inf\{s \geq 0 : \int_0^s \exp(2\beta_u) du > t\}.$$

For every $\lambda > 0$, we also consider the scaled Brownian motion $\beta_t^{(\lambda)} = \frac{1}{\lambda} \beta_{\lambda^2 t}$, and for $t > 1$ we use the notation $\lambda_t = (\log t)/2$.

We aim at proving that, for every $R > 0$,

$$\frac{2}{\log t} \int_0^t \mathbf{1}_{\{|B_s| < R\}} ds$$

converges in distribution as $t \rightarrow \infty$ to an exponential distribution with mean R^2 . To this end, we write, for every fixed $t > 1$,

$$\begin{aligned} \frac{2}{\log t} \int_0^t \mathbf{1}_{\{|B_s| < R\}} ds &= \frac{1}{\lambda_t} \int_0^t \mathbf{1}_{\{\beta_{H_s} < \log R\}} ds \\ &= \frac{1}{\lambda_t} \int_0^{H_t} \mathbf{1}_{\{\beta_u < \log R\}} \exp(2\beta_u) du \\ &= \lambda_t \int_0^{(\lambda_t)^{-2} H_t} \mathbf{1}_{\{\beta_u^{(\lambda_t)} < (\lambda_t)^{-1} \log R\}} \exp(2\lambda_t \beta_u^{(\lambda_t)}) du \\ &= \lambda_t \int_{-\infty}^{(\lambda_t)^{-1} \log R} \exp(2\lambda_t a) L_{(\lambda_t)^{-2} H_t}^a(\beta^{(\lambda_t)}) da \\ &= \int_0^R L_{(\lambda_t)^{-2} H_t}^{(\lambda_t)^{-1} \log r}(\beta^{(\lambda_t)}) r dr. \end{aligned}$$

In the second last equality, we applied the density of occupation time formula (Corollary 9.7) to the Brownian motion $\beta^{(\lambda_t)}$, and in the last one we used the change of variables $r = e^{\lambda_t a}$. As $t \rightarrow \infty$, $(\lambda_t)^{-1} \log r \rightarrow 0$, for every $r > 0$, and Lemma 7.21 also tells us that $(\lambda_t)^{-2} H_t - T_1^{(\lambda_t)}$ converges in probability to 0, with the notation $T_1^{(\lambda)} = \inf\{s \geq 0 : \beta_s^{(\lambda)} = 1\}$. From the joint continuity of Brownian local times (Theorem 9.12), we then get that, for every $\varepsilon \in (0, R)$,

$$\sup_{\varepsilon \leq r \leq R} \left| L_{(\lambda_t)^{-2} H_t}^{(\lambda_t)^{-1} \log r}(\beta^{(\lambda_t)}) - L_{T_1^{(\lambda_t)}}^0(\beta^{(\lambda_t)}) \right| \xrightarrow[t \rightarrow \infty]{} 0,$$

in probability. By combining this with the previous display, we obtain that

$$\left| \frac{2}{\log t} \int_0^t \mathbf{1}_{\{|B_s| < R\}} ds - \frac{R^2}{2} L_{T_1^{(\lambda_t)}}^0(\beta^{(\lambda_t)}) \right| \xrightarrow[t \rightarrow \infty]{} 0,$$

in probability. To complete the proof, we just note that the law of $L_{T_1^{(\lambda)}}^0(\beta^{(\lambda)})$ does not depend on $\lambda > 0$, and is exponential with mean 2, by Proposition 9.13.

Remark The preceding proof shows that the limiting exponential variable in Theorem 7.23 does not depend on the choice of R , in the sense that we can obtain a joint convergence by taking several values of R , with the same exponential variable in the limit, up to multiplicative constants. This can also be deduced from the Chacon–Ornstein ergodic theorem, which implies that the same limit in distribution holds more generally for the occupation time of an arbitrary compact subset K of \mathbb{C} , the constant R^2 then being replaced by π^{-1} times the Lebesgue measure of K . Our method of proof also shows that the convergence in the Kallianpur–Robbins theorem holds jointly with that of windings in Spitzer’s theorem (Theorem 7.20) and the limiting joint distribution is the law of $(\frac{R^2}{2} L_{T_1}^0(\beta), \gamma_{T_1})$, where β and γ are independent real Brownian motions started from 0, and $T_1 = \inf\{t \geq 0 : \beta_t = 1\}$.

Exercises

Exercise 9.16 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone increasing function, and assume that f is a difference of convex functions. Let X be a semimartingale and consider the semimartingale $Y_t = f(X_t)$. Prove that, for every $a \in \mathbb{R}$,

$$L_t^a(Y) = f'_+(a) L_t^a(X), \quad L_t^{a-}(Y) = f'_-(a) L_t^{a-}(X).$$

In particular, if X is a Brownian motion, the local times of $f(X)$ are continuous in the space variable if and only if f is continuously differentiable.

Exercise 9.17 Let M be a continuous local martingale such that $\langle M, M \rangle_\infty = \infty$ a.s., and let B be the Brownian motion associated with M via the Dambis–Dubins–Schwarz theorem (Theorem 5.13). Prove that, a.s. for every $a \geq 0$ and $t \geq 0$,

$$L_t^a(M) = L_{\langle M, M \rangle_t}^a(B).$$

Exercise 9.18 Let X be a continuous semimartingale, and assume that X can be written in the form

$$X_t = X_0 + \int_0^t \sigma(\omega, s) dB_s + \int_0^t b(\omega, s) ds,$$

where B is a Brownian motion and σ and b are progressive and locally bounded. Assume that $\sigma(\omega, s) \neq 0$ for Lebesgue a.e. $s \geq 0$, a.s. Show that the local times $L_t^a(X)$ are jointly continuous in the pair (a, t) .

Exercise 9.19 Let X be a continuous semimartingale. Show that the property

$$\text{supp}(d_s L_s^a(X)) \subset \{s \geq 0 : X_s = a\}$$

holds simultaneously for all $a \in \mathbb{R}$, outside a single set of probability zero.

Exercise 9.20 Let B be a Brownian motion started from 0. Show that a.s. there exists an $a \in \mathbb{R}$ such that the inclusion $\text{supp}(d_s L_s^a(B)) \subset \{s \geq 0 : B_s = a\}$ is not an equality. (*Hint*: Consider the maximal value of B over $[0, 1]$.)

Exercise 9.21 Let B be a Brownian motion started from 0. Note that

$$\int_0^\infty \mathbf{1}_{\{B_s > 0\}} ds = \infty$$

a.s. and set, for every $t \geq 0$,

$$A_t = \int_0^t \mathbf{1}_{\{B_s > 0\}} ds, \quad \sigma_t = \inf\{s \geq 0 : A_s > t\}.$$

1. Verify that the process

$$\gamma_t = \int_0^{\sigma_t} \mathbf{1}_{\{B_s > 0\}} dB_s$$

is a Brownian motion in an appropriate filtration.

2. Show that the process $\Lambda_t = L_{\sigma_t}^0(B)$ has nondecreasing and continuous sample paths, and that the support of the measure $d_s \Lambda_s$ is contained in $\{s : B_{\sigma_s} = 0\}$.
3. Show that the process $(B_{\sigma_t})_{t \geq 0}$ has the same distribution as $(|B_t|)_{t \geq 0}$.

Exercise 9.22 (*Skew Brownian motion*) Let $\alpha, \beta > 0$ and consider the function $g(x) = \alpha \mathbf{1}_{\{x \geq 0\}} - \beta \mathbf{1}_{\{x < 0\}}$. Let X be a continuous semimartingale such that

$$X_t = \int_0^t g(X_s) dB_s, \tag{9.19}$$

where B is a Brownian motion.

1. Set $\varphi(x) = \frac{1}{\alpha} x \mathbf{1}_{\{x \geq 0\}} - \frac{1}{\beta} x \mathbf{1}_{\{x < 0\}}$, and $Y_t = \varphi(X_t)$. Prove that $Y_t = \varphi(X_t)$ solves the equation

$$Y_t = B_t + \frac{1}{2} \left(1 - \frac{\alpha}{\beta}\right) L_t^0(Y)$$

(use the result of Exercise 9.16).

2. Compute $L_t^0(Y) - L_t^{0-}(Y)$ in terms of $L_t^0(Y)$, in two different ways.
3. Starting from a Brownian motion β with $\beta_0 = 0$, set

$$A_t = \int_0^t \frac{ds}{g(\beta_s)^2}, \quad \sigma_t = \inf\{s \geq 0 : A_s > t\}.$$

Verify that the process $X_t = \beta_{\sigma_t}$ satisfies the equation (9.19) in an appropriate filtration and with an appropriate Brownian motion B .

Exercise 9.23 Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a real integrable function ($\int_{\mathbb{R}} |g(x)| dx < \infty$). Let B be a Brownian motion started from 0, and set

$$A_t = \int_0^t g(B_s) ds.$$

1. Justify the fact that the integral defining A_t makes sense, and verify that, for every $c > 0$ and every $u \geq 0$, $A_{c^2 u}$ has the same distribution as

$$c^2 \int_0^u g(c B_s) ds.$$

2. Prove that

$$\frac{1}{\sqrt{t}} A_t \xrightarrow[t \rightarrow \infty]{(d)} \left(\int g(x) dx \right) |N|,$$

where N is $\mathcal{N}(0, 1)$.

Exercise 9.24 Let σ and b be two locally bounded measurable functions on $\mathbb{R}_+ \times \mathbb{R}$, and consider the stochastic differential equation

$$E(\sigma, b) \quad dX_t = \sigma(t, X_t) dB_t + b(t, X_t) dt.$$

Let X and X' be two solutions of $E(\sigma, b)$, on the same filtered probability space and with the same Brownian motion B .

1. Suppose that $L_t^0(X - X') = 0$ for every $t \geq 0$. Show that both $X_t \vee X'_t$ and $X_t \wedge X'_t$ are also solutions of $E(\sigma, b)$. (*Hint*: Write $X_t \vee X'_t = X_t + (X'_t - X_t)^+$, and use Tanaka's formula.)
2. Suppose that $\sigma(t, x) = 1$ for every t, x . Show that the assumption in question 1. holds automatically. Suppose in addition that weak uniqueness holds for $E(\sigma, b)$. Show that, if $X_0 = X'_0 = x \in \mathbb{R}$, the two processes X and X' are indistinguishable.

Exercise 9.25 (*Another look at the Yamada–Watanabe criterion*) Let ρ be a nondecreasing function from $[0, \infty)$ into $[0, \infty)$ such that, for every $\varepsilon > 0$,

$$\int_0^\varepsilon \frac{du}{\rho(u)} = \infty.$$

Consider then the one-dimensional stochastic differential equation

$$E(\sigma, b) \quad dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

where one assumes that the functions σ and b satisfy the conditions

$$(\sigma(x) - \sigma(y))^2 \leq \rho(|x - y|), \quad |b(x) - b(y)| \leq K|x - y|,$$

for every $x, y \in \mathbb{R}$, with a constant $K < \infty$. Our goal is use local times to give a short proof of pathwise uniqueness for $E(\sigma, b)$ (this is slightly stronger than the result of Exercise 8.14).

1. Let Y be a continuous semimartingale such that, for every $t > 0$,

$$\int_0^t \frac{d\langle Y, Y \rangle_s}{\rho(|Y_s|)} < \infty, \quad \text{a.s.}$$

Prove that $L_t^0(Y) = 0$ for every $t \geq 0$, a.s.

2. Let X and X' be two solutions of $E(\sigma, b)$ on the same filtered probability space and with the same Brownian motion B . By applying question 1. to $Y = X - X'$, prove that $L_t^0(X - X') = 0$ for every $t \geq 0$, a.s., and therefore,

$$\begin{aligned} |X_t - X'_t| &= |X_0 - X'_0| + \int_0^t (\sigma(X_s) - \sigma(X'_s)) \operatorname{sgn}(X_s - X'_s) dB_s \\ &\quad + \int_0^t (b(X_s) - b(X'_s)) \operatorname{sgn}(X_s - X'_s) ds. \end{aligned}$$

3. Using Gromwall's lemma, prove that if $X_0 = X'_0$ then $X_t = X'_t$ for every $t \geq 0$, a.s.

Notes and Comments

The local time of Brownian motion was first discussed by Lévy [54] under the name “mesure du voisinage”. In 1958, Trotter [80] established the joint continuity of Brownian local times viewed as densities of the occupation measure. Tanaka [79] obtained the formulas of Proposition 9.2 in the Brownian case. The local time of semimartingales was discussed by Meyer [61], who derived Theorem 9.6 in this general setting (after the earlier work of Tanaka [79] in the Brownian setting). Yor [84] then developed the powerful approach that leads to Theorem 9.4. The upcrossing approximation of local time (Proposition 9.11) is due to Itô and McKean [42, Chapter 2] for Brownian motion, and was extended to semimartingales by El Karoui [22]. Other approximation results for the Brownian local time were obtained by Lévy [54] (see [42] and [70]). Theorem 9.14 is essentially due to Lévy [54], but our proof is based on an argument from Skorokhod [74] (see [23] for a related study of the so-called “reflection problem” in the semimartingale setting). For further properties of local times, the reader may consult Chapter V of [70] or Chapter 6 of [49], as well as the classical book [42] of Itô and McKean. Local times are also a key ingredient of excursion theory, which is treated in the general setting of Markov processes in Blumenthal’s book [4]. Exercise 9.22 deals with the singular stochastic equation first studied by Harrison and Shepp [31], whose solution is the so-called skew Brownian motion. Exercise 9.23 gives the one-dimensional version of the Kallianpur–Robbins law, which can also be found in [48]. Exercise 9.25 is from [51].