Chapter 5
Stochastic Integration

This chapter is at the core of the present book. We start by defining the stochastic integral with respect to a square-integrable continuous martingale, considering first the integral of elementary processes (which play a role analogous to step functions in the theory of the Riemann integral) and then using an isometry between Hilbert spaces to deal with the general case. It is easy to extend the definition of stochastic integrals to continuous local martingales and semimartingales. We then derive the celebrated Itô's formula, which shows that the image of one or several continuous semimartingales under a smooth function is still a continuous semimartingale, whose canonical decomposition is given in terms of stochastic integrals. Itô's formula is the main technical tool of stochastic calculus, and we discuss several important applications of this formula, including Lévy's theorem characterizing Brownian motion as a continuous local martingale with quadratic variation process equal to t, the Burkholder–Davis–Gundy inequalities and the representation of martingales as stochastic integrals in a Brownian filtration. The end of the chapter is devoted to Girsanov's theorem, which deals with the stability of the notions of a martingale and a semimartingale under an absolutely continuous change of probability measure. As an application of Girsanov's theorem, we establish the famous Cameron–Martin formula giving the image of the Wiener measure under a translation by a deterministic function.

5.1 The Construction of Stochastic Integrals

Throughout this chapter, we argue on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$, and we assume that the filtration $(\mathscr{F}_t)$ is complete. Unless otherwise specified, all processes in this chapter are indexed by $\mathbb{R}_+$ and take real values. We often say "continuous martingale" instead of "martingale with continuous sample paths".

J.-F. Le Gall, *Brownian Motion, Martingales, and Stochastic Calculus*,
Graduate Texts in Mathematics 274, DOI 10.1007/978-3-319-31089-3_5

5.1.1 Stochastic Integrals for Martingales Bounded in L^2

We write $\mathbb{H}^2$ for the space of all continuous martingales M which are bounded in L^2 and such that $M_0 = 0$, with the usual convention that two indistinguishable processes are identified. Equivalently, $M \in \mathbb{H}^2$ if and only if M is a continuous local martingale such that $M_0 = 0$ and $E[\langle M, M\rangle_\infty] < \infty$ (Proposition 4.13). By Proposition 3.21, if $M \in \mathbb{H}^2$, we have $M_t = E[M_\infty \mid \mathscr{F}_t]$ where $M_\infty \in L^2$ is the almost sure limit of M_t as $t \to \infty$.

Proposition 4.15 (v) shows that, if $M, N \in \mathbb{H}^2$, the random variable $\langle M, N\rangle_\infty$ is well defined, and we have $E[|\langle M, N\rangle_\infty|] < \infty$. This allows us to define a symmetric bilinear form on $\mathbb{H}^2$ via the formula

$$(M, N)_{\mathbb{H}^2} = E[\langle M, N\rangle_\infty] = E[M_\infty N_\infty],$$

where the second equality comes from Proposition 4.15 (v). Clearly $(M, M)_{\mathbb{H}^2} = 0$ if and only if $M = 0$. The scalar product $(M, N)_{\mathbb{H}^2}$ thus yields a norm on $\mathbb{H}^2$ given by

$$\|M\|_{\mathbb{H}^2} = (M, M)_{\mathbb{H}^2}^{1/2} = E[\langle M, M\rangle_\infty]^{1/2} = E[(M_\infty)^2]^{1/2}.$$

Proposition 5.1 *The space $\mathbb{H}^2$ equipped with the scalar product $(M, N)_{\mathbb{H}^2}$ is a Hilbert space.*

Proof We need to verify that the vector space $\mathbb{H}^2$ is complete for the norm $\|\ \|_{\mathbb{H}^2}$. Let $(M^n)_{n\geq 1}$ be a sequence in $\mathbb{H}^2$ which is Cauchy for that norm. We have then

$$\lim_{m,n\to\infty} E[(M^n_\infty - M^m_\infty)^2] = \lim_{m,n\to\infty} (M^n - M^m, M^n - M^m)_{\mathbb{H}^2} = 0.$$

Consequently, the sequence (M^n_∞) converges in L^2 to a limit, which we denote by Z. On the other hand, Doob's inequality in L^2 (Proposition 3.15 (ii)) and a straightforward passage to the limit show that, for every m, n,

$$E\Big[\sup_{t\geq 0}(M^n_t - M^m_t)^2\Big] \leq 4\, E[(M^n_\infty - M^m_\infty)^2].$$

We thus obtain that

$$\lim_{m,n\to\infty} E\Big[\sup_{t\geq 0}(M^n_t - M^m_t)^2\Big] = 0. \tag{5.1}$$

Hence, for every $t \geq 0$, M^n_t converges in L^2, and we want to argue that the limit yields a process with continuous sample paths. To this end, we use (5.1) to find an increasing sequence $n_k \uparrow \infty$ such that

$$E\Big[\sum_{k=1}^{\infty}\sup_{t\geq 0}|M_t^{n_k}-M_t^{n_{k+1}}|\Big]\leq \sum_{k=1}^{\infty}E\Big[\sup_{t\geq 0}(M_t^{n_k}-M_t^{n_{k+1}})^2\Big]^{1/2}<\infty.$$

The last display implies that, a.s.,

$$\sum_{k=1}^{\infty}\sup_{t\geq 0}|M_t^{n_k}-M_t^{n_{k+1}}|<\infty,$$

and thus the sequence $(M_t^{n_k})_{t\geq 0}$ converges uniformly on $\mathbb{R}_+$, a.s., to a limit denoted by $(M_t)_{t\geq 0}$. On the zero probability set where the uniform convergence does not hold, we take $M_t = 0$ for every $t \geq 0$. Clearly the limiting process M has continuous sample paths and is adapted (here we use the fact that the filtration is complete). Furthermore, from the L^2-convergence of (M_∞^n) to Z, we immediately get by passing to the limit in the identity $M_t^{n_k} = E[M_\infty^{n_k} \mid \mathscr{F}_t]$ that $M_t = E[Z \mid \mathscr{F}_t]$. Hence $(M_t)_{t\geq 0}$ is a continuous martingale and is bounded in L^2, so that $M \in \mathbb{H}^2$. The a.s. uniform convergence of $(M_t^{n_k})_{t\geq 0}$ to $(M_t)_{t\geq 0}$ then ensures that $M_\infty = \lim M_\infty^{n_k} = Z$ a.s. Finally, the L^2-convergence of (M_∞^n) to $Z = M_\infty$ shows that the sequence (M^n) converges to M in $\mathbb{H}^2$. □

We denote the progressive σ-field on $\Omega \times \mathbb{R}_+$ by $\mathscr{P}$ (see the end of Sect. 3.1), and, if $M \in \mathbb{H}^2$, we let $L^2(M)$ be the set of all progressive processes H such that

$$E\Big[\int_0^\infty H_s^2\,\mathrm{d}\langle M,M\rangle_s\Big]<\infty,$$

with the convention that two progressive processes H and H' satisfying this integrability condition are identified if $H_s = H'_s = 0$, $\mathrm{d}\langle M,M\rangle_s$ a.e., a.s. We can view $L^2(M)$ as an ordinary L^2 space, namely

$$L^2(M) = L^2(\Omega\times\mathbb{R}_+,\mathscr{P},\mathrm{d}P\,\mathrm{d}\langle M,M\rangle_s)$$

where $\mathrm{d}P\,\mathrm{d}\langle M,M\rangle_s$ refers to the finite measure on $(\Omega\times\mathbb{R}_+,\mathscr{P})$ that assigns the mass

$$E\Big[\int_0^\infty \mathbf{1}_A(\omega,s)\,\mathrm{d}\langle M,M\rangle_s\Big]$$

to a set $A \in \mathscr{P}$ (the total mass of this measure is $E[\langle M,M\rangle_\infty] = \|M\|^2_{\mathbb{H}^2}$) .

Just like any L^2 space, the space $L^2(M)$ is a Hilbert space for the scalar product

$$(H,K)_{L^2(M)} = E\Big[\int_0^\infty H_sK_s\,\mathrm{d}\langle M,M\rangle_s\Big],$$

and the associated norm is

$$\|H\|_{L^2(M)} = \Big(E\Big[\int_0^\infty H_s^2 \,\mathrm{d}\langle M,M\rangle_s\Big]\Big)^{1/2}.$$

Definition 5.2 An *elementary process* is a progressive process of the form

$$H_s(\omega) = \sum_{i=0}^{p-1} H_{(i)}(\omega)\,\mathbf{1}_{(t_i,t_{i+1}]}(s),$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_p$ and for every $i \in \{0,1,\ldots,p-1\}$, $H_{(i)}$ is a bounded $\mathscr{F}_{t_i}$-measurable random variable.

The set $\mathscr{E}$ of all elementary processes forms a linear subspace of $L^2(M)$. To be precise, we should here say "equivalence classes of elementary processes" (recall that H and H' are identified in $L^2(M)$ if $\|H - H'\|_{L^2(M)} = 0$).

Proposition 5.3 *For every $M \in \mathbb{H}^2$, $\mathscr{E}$ is dense in $L^2(M)$.*

Proof By elementary Hilbert space theory, it is enough to verify that, if $K \in L^2(M)$ is orthogonal to $\mathscr{E}$, then $K = 0$. Assume that $K \in L^2(M)$ is orthogonal to $\mathscr{E}$, and set, for every $t \geq 0$,

$$X_t = \int_0^t K_u \,\mathrm{d}\langle M,M\rangle_u.$$

To see that the integral in the right-hand side makes sense, and defines a finite variation process $(X_t)_{t\geq 0}$, we use the Cauchy–Schwarz inequality to observe that

$$E\Big[\int_0^t |K_u|\,\mathrm{d}\langle M,M\rangle_u\Big] \leq \Big(E\Big[\int_0^t (K_u)^2\,\mathrm{d}\langle M,M\rangle_u\Big]\Big)^{1/2} \times (E[\langle M,M\rangle_\infty])^{1/2}.$$

The right-hand side is finite since $M \in \mathbb{H}^2$ and $K \in L^2(M)$, and thus we have in particular

$$\text{a.s.} \quad \forall t \geq 0, \quad \int_0^t |K_u|\,\mathrm{d}\langle M,M\rangle_u < \infty.$$

By Proposition 4.5 (and Remark (i) following this proposition), $(X_t)_{t\geq 0}$ is well defined as a finite variation process. The preceding bound also shows that $X_t \in L^1$ for every $t \geq 0$.

Let $0 \leq s < t$, let F be a bounded $\mathscr{F}_s$-measurable random-variable, and let $H \in \mathscr{E}$ be the elementary process defined by $H_r(\omega) = F(\omega)\,\mathbf{1}_{(s,t]}(r)$. Writing $(H,K)_{L^2(M)} = 0$, we get

$$E\Big[F\int_s^t K_u\,\mathrm{d}\langle M,M\rangle_u\Big] = 0.$$

It follows that $E[F(X_t - X_s)] = 0$ for every $s < t$ and every bounded $\mathscr{F}_s$-measurable variable F. Since the process X is adapted and we know that $X_r \in L^1$ for every $r \geq 0$, this implies that X is a (continuous) martingale. On the other hand, X is also a finite variation process and, by Theorem 4.8, this is only possible if $X = 0$. We have thus proved that

$$\int_0^t K_u \, \mathrm{d}\langle M, M\rangle_u = 0 \qquad \forall t \geq 0, \quad \text{a.s.}$$

which implies that, a.s., the signed measure having density K_u with respect to $\mathrm{d}\langle M, M\rangle_u$ is the zero measure, which is only possible if

$$K_u = 0, \qquad \mathrm{d}\langle M, M\rangle_u \text{ a.e.}, \quad \text{a.s.}$$

or equivalently $K = 0$ in $L^2(M)$. □

Recall our notation X^T for the process X stopped at the stopping time T: $X_t^T = X_{t\wedge T}$. If $M \in \mathbb{H}^2$, the fact that $\langle M^T, M^T\rangle_\infty = \langle M, M\rangle_T$ immediately implies that M^T also belongs to $\mathbb{H}^2$. Furthermore, if $H \in L^2(M)$, the process $\mathbf{1}_{[0,T]} H$ defined by $(\mathbf{1}_{[0,T]} H)_s(\omega) = \mathbf{1}_{\{0\leq s\leq T(\omega)\}} H_s(\omega)$ also belongs to $L^2(M)$ (note that $\mathbf{1}_{[0,T]}$ is progressive since it is adapted with left-continuous sample paths).

Theorem 5.4 *Let $M \in \mathbb{H}^2$. For every $H \in \mathscr{E}$ of the form*

$$H_s(\omega) = \sum_{i=0}^{p-1} H_{(i)}(\omega) \, \mathbf{1}_{(t_i,t_{i+1}]}(s),$$

the formula

$$(H \cdot M)_t = \sum_{i=0}^{p-1} H_{(i)} \, (M_{t_{i+1}\wedge t} - M_{t_i\wedge t})$$

defines a process $H \cdot M \in \mathbb{H}^2$. The mapping $H \mapsto H \cdot M$ extends to an isometry from $L^2(M)$ into $\mathbb{H}^2$. Furthermore, $H \cdot M$ is the unique martingale of $\mathbb{H}^2$ that satisfies the property

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle, \qquad \forall N \in \mathbb{H}^2. \tag{5.2}$$

If T is a stopping time, we have

$$(\mathbf{1}_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T. \tag{5.3}$$

We often use the notation

$$(H \cdot M)_t = \int_0^t H_s \, \mathrm{d}M_s$$

and call $H \cdot M$ the stochastic integral of H with respect to M.

Remark The quantity $H \cdot \langle M, N \rangle$ in the right-hand side of (5.2) is an integral with respect to a finite variation process, as defined in Sect. 4.1. The fact that we use a similar notation $H \cdot A$ and $H \cdot M$ for the integrals with respect to a finite variation process A and with respect to a martingale M creates no ambiguity since these two classes of processes are essentially disjoint.

Proof As a preliminary observation, we note that the definition of $H \cdot M$ when $H \in \mathscr{E}$ does not depend on the decomposition chosen for H in the first display of the theorem. Using this remark, one then checks that the mapping $H \mapsto H \cdot M$ is linear. We next verify that this mapping is an isometry from $\mathscr{E}$ (viewed as a subspace of $L^2(M)$) into $\mathbb{H}^2$.

Fix $H \in \mathscr{E}$ of the form given in the theorem, and for every $i \in \{0, 1, \ldots, p-1\}$, set

$$M_t^i = H_{(i)} \, (M_{t_{i+1} \wedge t} - M_{t_i \wedge t}),$$

for every $t \geq 0$. Then a simple verification shows that M^i is a continuous martingale (this was already used in the beginning of the proof of Theorem 4.9), and that this martingale belongs to $\mathbb{H}^2$. It follows that $H \cdot M = \sum_{i=0}^{p-1} M^i$ is also a martingale in $\mathbb{H}^2$. Then, we note that the continuous martingales M^i are orthogonal, and their respective quadratic variations are given by

$$\langle M^i, M^i \rangle_t = H_{(i)}^2 \big(\langle M, M \rangle_{t_{i+1} \wedge t} - \langle M, M \rangle_{t_i \wedge t} \big)$$

(the orthogonality of the martingales M^i as well as the formula of the last display are easily checked, for instance by using the approximations of $\langle M, N \rangle$). We conclude that

$$\langle H \cdot M, H \cdot M \rangle_t = \sum_{i=0}^{p-1} H_{(i)}^2 \big(\langle M, M \rangle_{t_{i+1} \wedge t} - \langle M, M \rangle_{t_i \wedge t} \big) = \int_0^t H_s^2 \, \mathrm{d}\langle M, M \rangle_s.$$

Consequently,

$$\|H \cdot M\|_{\mathbb{H}^2}^2 = E[\langle H \cdot M, H \cdot M \rangle_\infty] = E\Big[\int_0^\infty H_s^2 \, \mathrm{d}\langle M, M \rangle_s \Big] = \|H\|_{L^2(M)}^2.$$

By linearity, this implies that $H \cdot M = H' \cdot M$ if H' is another elementary process that is identified with H in $L^2(M)$. Therefore the mapping $H \mapsto H \cdot M$ makes sense from

$\mathscr{E}$ viewed as a subspace of $L^2(M)$ into $\mathbb{H}^2$. The latter mapping is linear, and, since it preserves the norm, it is an isometry from $\mathscr{E}$ (equipped with the norm of $L^2(M)$) into $\mathbb{H}^2$. Since $\mathscr{E}$ is dense in $L^2(M)$ (Proposition 5.3) and $\mathbb{H}^2$ is a Hilbert space (Proposition 5.1), this mapping can be extended in a unique way to an isometry from $L^2(M)$ into $\mathbb{H}^2$.

Let us verify property (5.2). We fix $N \in \mathbb{H}^2$. We first note that, if $H \in L^2(M)$, the Kunita–Watanabe inequality (Proposition 4.18) shows that

$$E\Big[\int_0^\infty |H_s|\,|\mathrm{d}\langle M,N\rangle_s|\Big] \le \|H\|_{L^2(M)}\,\|N\|_{\mathbb{H}^2} < \infty$$

and thus the variable $\int_0^\infty H_s \mathrm{d}\langle M,N\rangle_s = (H\cdot\langle M,N\rangle)_\infty$ is well defined and in L^1. Consider first the case where H is an elementary process of the form given in the theorem, and define the continuous martingales M^i, $0 \le i \le p-1$, as previously. Then, for every $i \in \{0,1,\ldots,p-1\}$,

$$\langle H\cdot M, N\rangle = \sum_{i=0}^{p-1} \langle M^i, N\rangle$$

and we have

$$\langle M^i, N\rangle_t = H_{(i)}\big(\langle M,N\rangle_{t_{i+1}\wedge t} - \langle M,N\rangle_{t_i\wedge t}\big).$$

It follows that

$$\langle H\cdot M, N\rangle_t = \sum_{i=0}^{p-1} H_{(i)}\big(\langle M,N\rangle_{t_{i+1}\wedge t} - \langle M,N\rangle_{t_i\wedge t}\big) = \int_0^t H_s\,\mathrm{d}\langle M,N\rangle_s$$

which gives (5.2) when $H \in \mathscr{E}$. We then observe that the linear mapping $X \mapsto \langle X,N\rangle_\infty$ is continuous from $\mathbb{H}^2$ into L^1. Indeed, by the Kunita–Watanabe inequality,

$$E[|\langle X,N\rangle_\infty|] \le E[\langle X,X\rangle_\infty]^{1/2} E[\langle N,N\rangle_\infty]^{1/2} = \|N\|_{\mathbb{H}^2}\,\|X\|_{\mathbb{H}^2}.$$

If $(H^n)_{n\ge1}$ is a sequence in $\mathscr{E}$, such that $H^n \to H$ in $L^2(M)$, we have therefore

$$\langle H\cdot M, N\rangle_\infty = \lim_{n\to\infty} \langle H^n\cdot M, N\rangle_\infty = \lim_{n\to\infty} (H^n\cdot\langle M,N\rangle)_\infty = (H\cdot\langle M,N\rangle)_\infty,$$

where the convergences hold in L^1, and the last equality again follows from the Kunita–Watanabe inequality by writing

$$E\Big[\Big|\int_0^\infty (H^n_s - H_s)\,\mathrm{d}\langle M,N\rangle_s\Big|\Big] \le E[\langle N,N\rangle_\infty]^{1/2}\,\|H^n - H\|_{L^2(M)}.$$

We have thus obtained the identity $\langle H \cdot M, N\rangle_\infty = (H \cdot \langle M, N\rangle)_\infty$, but replacing N by the stopped martingale N^t in this identity also gives $\langle H \cdot M, N\rangle_t = (H \cdot \langle M, N\rangle)_t$, which completes the proof of (5.2).

It is easy to see that (5.2) characterizes $H \cdot M$ among the martingales of $\mathbb{H}^2$. Indeed, if X is another martingale of $\mathbb{H}^2$ that satisfies the same identity, we get, for every $N \in \mathbb{H}^2$,

$$\langle H \cdot M - X, N\rangle = 0.$$

Taking $N = H \cdot M - X$ and using Proposition 4.12 we obtain that $X = H \cdot M$.

It remains to verify (5.3). Using the properties of the bracket of two continuous local martingales, we observe that, if $N \in \mathbb{H}^2$,

$$\langle (H \cdot M)^T, N\rangle_t = \langle H \cdot M, N\rangle_{t\wedge T} = (H \cdot \langle M, N\rangle)_{t\wedge T} = (\mathbf{1}_{[0,T]}\, H \cdot \langle M, N\rangle)_t$$

which shows that the stopped martingale $(H \cdot M)^T$ satisfies the characteristic property of the stochastic integral $(\mathbf{1}_{[0,T]} H) \cdot M$. The first equality in (5.3) follows. The second one is proved analogously, writing

$$\langle H \cdot M^T, N\rangle = H \cdot \langle M^T, N\rangle = H \cdot \langle M, N\rangle^T = \mathbf{1}_{[0,T]}\, H \cdot \langle M, N\rangle.$$

This completes the proof of the theorem. □

Remark We could have used the relation (5.2) to *define* the stochastic integral $H \cdot M$, observing that the mapping $N \mapsto E[(H \cdot \langle M, N\rangle)_\infty]$ yields a continuous linear form on $\mathbb{H}^2$, and thus there exists a unique martingale $H \cdot M$ in $\mathbb{H}^2$ such that

$$E[(H \cdot \langle M, N\rangle)_\infty] = (H \cdot M, N)_{\mathbb{H}^2} = E[\langle H \cdot M, N\rangle_\infty].$$

Using the notation introduced at the end of Theorem 5.4, we can rewrite (5.2) in the form

$$\langle \int_0^\cdot H_s \mathrm{d}M_s, N\rangle_t = \int_0^t H_s\, \mathrm{d}\langle M, N\rangle_s.$$

We interpret this by saying that the stochastic integral "commutes" with the bracket. Let us immediately mention a very important consequence. If $M \in \mathbb{H}^2$, and $H \in L^2(M)$, two successive applications of (5.2) give

$$\langle H \cdot M, H \cdot M\rangle = H \cdot (H \cdot \langle M, M\rangle) = H^2 \cdot \langle M, M\rangle,$$

using the "associativity property" (4.1) of integrals with respect to finite variation processes. Put differently, the quadratic variation of the continuous martingale $H \cdot M$ is

$$\langle\int_0^\cdot H_s \mathrm{d}M_s, \int_0^\cdot H_s \mathrm{d}M_s\rangle_t = \int_0^t H_s^2 \,\mathrm{d}\langle M, M\rangle_s. \tag{5.4}$$

More generally, if N is another martingale of $\mathbb{H}^2$ and $K \in L^2(N)$, the same argument gives

$$\langle\int_0^\cdot H_s \mathrm{d}M_s, \int_0^\cdot K_s \mathrm{d}N_s\rangle_t = \int_0^t H_s K_s \,\mathrm{d}\langle M, N\rangle_s. \tag{5.5}$$

The following "associativity" property of stochastic integrals, which is analogous to property (4.1) for integrals with respect to finite variation processes, is very useful.

Proposition 5.5 *Let $H \in L^2(M)$. If K is a progressive process, we have $KH \in L^2(M)$ if and only if $K \in L^2(H \cdot M)$. If the latter properties hold,*

$$(KH) \cdot M = K \cdot (H \cdot M).$$

Proof Using property (5.4), we have

$$E\Big[\int_0^\infty K_s^2 H_s^2 \,\mathrm{d}\langle M, M\rangle_s\Big] = E\Big[\int_0^\infty K_s^2 \,\mathrm{d}\langle H \cdot M, H \cdot M\rangle_s\Big],$$

which gives the first assertion. For the second one, we write for $N \in \mathbb{H}^2$,

$$\langle (KH) \cdot M, N\rangle = KH \cdot \langle M, N\rangle = K \cdot (H \cdot \langle M, N\rangle) = K \cdot \langle H \cdot M, N\rangle$$

and, by the uniqueness statement in (5.2), this implies that $(KH) \cdot M = K \cdot (H \cdot M)$. □

Moments of stochastic integrals. Let $M \in \mathbb{H}^2$, $N \in \mathbb{H}^2$, $H \in L^2(M)$ and $K \in L^2(N)$. Since $H \cdot M$ and $K \cdot N$ are martingales in $\mathbb{H}^2$, we have, for every $t \in [0, \infty]$,

$$E\Big[\int_0^t H_s \,\mathrm{d}M_s\Big] = 0 \tag{5.6}$$

$$E\Big[\Big(\int_0^t H_s \mathrm{d}M_s\Big)\Big(\int_0^t K_s \mathrm{d}N_s\Big)\Big] = E\Big[\int_0^t H_s K_s \,\mathrm{d}\langle M, N\rangle_s\Big], \tag{5.7}$$

using Proposition 4.15 (v) and (5.5) to derive (5.7). In particular,

$$E\Big[\Big(\int_0^t H_s \,\mathrm{d}M_s\Big)^2\Big] = E\Big[\int_0^t H_s^2 \,\mathrm{d}\langle M, M\rangle_s\Big]. \tag{5.8}$$

Furthermore, since $H \cdot M$ is a (true) martingale, we also have for every $0 \le s < t \le \infty$,

$$E\Big[\int_0^t H_r \,\mathrm{d}M_r \,\Big|\, \mathscr{F}_s\Big] = \int_0^s H_r \,\mathrm{d}M_r, \tag{5.9}$$

or equivalently

$$E\Big[\int_s^t H_r \,\mathrm{d}M_r \,\Big|\, \mathscr{F}_s\Big] = 0$$

with an obvious notation for $\int_s^t H_r \,\mathrm{d}M_r$. It is important to observe that these formulas (and particularly (5.6) and (5.8)) may no longer hold for the extensions of stochastic integrals that we will now describe.

5.1.2 Stochastic Integrals for Local Martingales

We will now use the identities (5.3) to extend the definition of $H \cdot M$ to an arbitrary continuous local martingale. If M is a continuous local martingale, we write $L^2_{\mathrm{loc}}(M)$ (resp. $L^2(M)$) for the set of all progressive processes H such that

$$\int_0^t H_s^2 \,\mathrm{d}\langle M, M\rangle_s < \infty, \ \forall t \ge 0, \text{ a.s.} \quad \text{(resp. such that } E\Big[\int_0^\infty H_s^2 \,\mathrm{d}\langle M, M\rangle_s\Big] < \infty).$$

For future reference, we note that $L^2(M)$ (with the same identifications as in the case where $M \in \mathbb{H}^2$) can again be viewed as an "ordinary" L^2-space and thus has a Hilbert space structure.

Theorem 5.6 *Let M be a continuous local martingale. For every $H \in L^2_{\mathrm{loc}}(M)$, there exists a unique continuous local martingale with initial value 0, which is denoted by $H \cdot M$, such that, for every continuous local martingale N,*

$$\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle. \tag{5.10}$$

If T is a stopping time, we have

$$(\mathbf{1}_{[0,T]} H) \cdot M = (H \cdot M)^T = H \cdot M^T. \tag{5.11}$$

If $H \in L^2_{\mathrm{loc}}(M)$ and K is a progressive process, we have $K \in L^2_{\mathrm{loc}}(H \cdot M)$ if and only if $HK \in L^2_{\mathrm{loc}}(M)$, and then

$$H \cdot (K \cdot M) = HK \cdot M. \tag{5.12}$$

Finally, if $M \in \mathbb{H}^2$, and $H \in L^2(M)$, the definition of $H \cdot M$ is consistent with that of Theorem 5.4.

Proof We may assume that $M_0 = 0$ (in the general case, we write $M = M_0 + M'$ and we just set $H \cdot M = H \cdot M'$, noting that $\langle M, N\rangle = \langle M', N\rangle$ for every continuous local martingale N). Also we may assume that the property $\int_0^t H_s^2 \, \mathrm{d}\langle M, M\rangle_s < \infty$ for every $t \geq 0$ holds for every $\omega \in \Omega$ (on the negligible set where this fails we may replace H by 0).

For every $n \geq 1$, set

$$T_n = \inf\{t \geq 0 : \int_0^t (1 + H_s^2) \, \mathrm{d}\langle M, M\rangle_s \geq n\},$$

so that (T_n) is a sequence of stopping times that increase to $+\infty$. Since

$$\langle M^{T_n}, M^{T_n}\rangle_t = \langle M, M\rangle_{t \wedge T_n} \leq n,$$

the stopped martingale M^{T_n} is in $\mathbb{H}^2$ (Theorem 4.13). Furthermore, we also have

$$\int_0^\infty H_s^2 \, \mathrm{d}\langle M^{T_n}, M^{T_n}\rangle_s = \int_0^{T_n} H_s^2 \, \mathrm{d}\langle M, M\rangle_s \leq n.$$

Hence, $H \in L^2(M^{T_n})$, and the definition of $H \cdot M^{T_n}$ makes sense by Theorem 5.4. Moreover, by property (5.3), we have, if $m > n$,

$$H \cdot M^{T_n} = (H \cdot M^{T_m})^{T_n}.$$

It follows that there exists a unique process denoted by $H \cdot M$ such that, for every n,

$$(H \cdot M)^{T_n} = H \cdot M^{T_n}.$$

Clearly $H \cdot M$ has continuous sample paths and is also adapted since $(H \cdot M)_t = \lim (H \cdot M^{T_n})_t$. Since the processes $(H \cdot M)^{T_n}$ are martingales in $\mathbb{H}^2$, we get that $H \cdot M$ is a continuous local martingale.

Then, to verify (5.10), we may assume that N is a continuous local martingale such that $N_0 = 0$. For every $n \geq 1$, set $T'_n = \inf\{t \geq 0 : |N_t| \geq n\}$, and $S_n = T_n \wedge T'_n$. Then, noting that $N^{T'_n} \in \mathbb{H}^2$, we have

$$\begin{aligned}
\langle H \cdot M, N\rangle^{S_n} &= \langle (H \cdot M)^{T_n}, N^{T'_n}\rangle \\
&= \langle H \cdot M^{T_n}, N^{T'_n}\rangle \\
&= H \cdot \langle M^{T_n}, N^{T'_n}\rangle \\
&= H \cdot \langle M, N\rangle^{S_n} \\
&= (H \cdot \langle M, N\rangle)^{S_n},
\end{aligned}$$

which gives the equality $\langle H\cdot M, N\rangle = H\cdot\langle M, N\rangle$. since $S_n \uparrow \infty$ as $n \to \infty$. The fact that this equality (written for every continuous local martingale N) characterizes $H \cdot M$ among continuous local martingales with initial value 0 is derived from Proposition 4.12 as in the proof of Theorem 5.4.

The property (5.11) is then obtained by the very same arguments as in the proof of property (5.3) in Theorem 5.4 (these arguments only depended on the characteristic property (5.2) which we have just extended in (5.10)). Similarly, the proof of (5.12) is analogous to the proof of Proposition 5.5.

Finally, if $M \in \mathbb{H}^2$ and $H \in L^2(M)$, the equality $\langle H \cdot M, H \cdot M\rangle = H^2 \cdot \langle M, M\rangle$ follows from (5.10), and implies that $H \cdot M \in \mathbb{H}^2$. Then the characteristic property (5.2) shows that the definitions of Theorems 5.4 and 5.6 are consistent. □

In the setting of Theorem 5.6, we will again write

$$(H \cdot M)_t = \int_0^t H_s \, \mathrm{d}M_s.$$

It is worth pointing out that formulas (5.4) and (5.5) **remain valid** when M and N are continuous local martingales and $H \in L^2_{\mathrm{loc}}(M)$, $K \in L^2_{\mathrm{loc}}(N)$. Indeed, these formulas immediately follow from (5.10).

Connection with the Wiener integral Suppose that B is an $(\mathscr{F}_t)$-Brownian motion, and $h \in L^2(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+), \mathrm{d}t)$ is a deterministic square integrable function. We can then define the Wiener integral $\int_0^t h(s)\mathrm{d}B_s = G(f\,\mathbf{1}_{[0,t]})$, where G is the Gaussian white noise associated with B (see the end of Sect. 2.1). It is easy to verify that this integral coincides with the stochastic integral $(h \cdot B)_t$, which makes sense by viewing h as a (deterministic) progressive process. This is immediate when h is a simple function, and the general case follows from a density argument.

Let us now discuss the extension of the moment formulas that we stated above in the setting of Theorem 5.4. Let M be a continuous local martingale, $H \in L^2_{\mathrm{loc}}(M)$ and $t \in [0, \infty]$. Then, **under the condition**

$$E\Big[\int_0^t H_s^2 \, \mathrm{d}\langle M, M\rangle_s\Big] < \infty, \tag{5.13}$$

we can apply Theorem 4.13 to $(H \cdot M)^t$, and get that $(H \cdot M)^t$ is a martingale of $\mathbb{H}^2$. It follows that properties (5.6) and (5.8) still hold:

$$E\Big[\int_0^t H_s \, \mathrm{d}M_s\Big] = 0, \qquad E\Big[\Big(\int_0^t H_s \, \mathrm{d}M_s\Big)^2\Big] = E\Big[\int_0^t H_s^2 \, \mathrm{d}\langle M, M\rangle_s\Big],$$

and similarly (5.9) is valid for $0 \le s \le t$. In particular (case $t = \infty$), if $H \in L^2(M)$, the continuous local martingale $H \cdot M$ is in $\mathbb{H}^2$ and its terminal value satisfies

$$E\Big[\Big(\int_0^\infty H_s \, \mathrm{d}M_s\Big)^2\Big] = E\Big[\int_0^\infty H_s^2 \, \mathrm{d}\langle M, M\rangle_s\Big].$$

If the condition (5.13) does not hold, the previous formulas may fail. However, we always have the bound

$$E\Big[\Big(\int_0^t H_s\,\mathrm{d}M_s\Big)^2\Big] \le E\Big[\int_0^t H_s^2\,\mathrm{d}\langle M,M\rangle_s\Big]. \tag{5.14}$$

Indeed, if the right-hand side is finite, this is an equality by the preceding observations. If the right-hand side is infinite, the bound is trivial.

5.1.3 Stochastic Integrals for Semimartingales

We finally extend the definition of stochastic integrals to continuous semimartingales. We say that a progressive process H is *locally bounded* if

$$\forall t \ge 0, \quad \sup_{s\le t} |H_s| < \infty, \quad \text{a.s.}$$

In particular, any adapted process with continuous sample paths is a locally bounded progressive process. If H is (progressive and) locally bounded, then for every finite variation process V, we have

$$\forall t \ge 0, \quad \int_0^t |H_s|\,|\mathrm{d}V_s| < \infty, \quad \text{a.s.}$$

and similarly $H \in L^2_{\mathrm{loc}}(M)$ for every continuous local martingale M.

Definition 5.7 Let X be a continuous semimartingale and let $X = M + V$ be its canonical decomposition. If H is a locally bounded progressive process, the *stochastic integral* $H \cdot X$ is the continuous semimartingale with canonical decomposition

$$H\cdot X = H\cdot M + H\cdot V,$$

and we write

$$(H\cdot X)_t = \int_0^t H_s\,\mathrm{d}X_s.$$

Properties

(i) The mapping $(H,X) \mapsto H\cdot X$ is bilinear.
(ii) $H\cdot(K\cdot X) = (HK)\cdot X$, if H and K are progressive and locally bounded.
(iii) For every stopping time T, $(H\cdot X)^T = H\mathbf{1}_{[0,T]}\cdot X = H\cdot X^T$.
(iv) If X is a continuous local martingale, resp. if X is a finite variation process, then the same holds for $H\cdot X$.

(v) If H is of the form $H_s(\omega) = \sum_{i=0}^{p-1} H_{(i)}(\omega)\,\mathbf{1}_{(t_i,t_{i+1}]}(s)$, where $0 = t_0 < t_1 < \cdots < t_p$, and, for every $i \in \{0, 1, \ldots, p-1\}$, $H_{(i)}$ is $\mathscr{F}_{t_i}$-measurable, then

$$(H \cdot X)_t = \sum_{i=0}^{p-1} H_{(i)}\,(X_{t_{i+1}\wedge t} - X_{t_i\wedge t}).$$

We can restate the "associativity" property (ii) by saying that, if $Y_t = \int_0^t K_s \mathrm{d}X_s$ then

$$\int_0^t H_s\,\mathrm{d}Y_s = \int_0^t H_s K_s\,\mathrm{d}X_s.$$

Properties (i)–(iv) easily follow from the results obtained when X is a continuous local martingale, resp. a finite variation process. As for property (v), we first note that it is enough to consider the case where $X = M$ is a continuous local martingale with $M_0 = 0$, and by stopping M at suitable stopping times (and using (5.11)), we can even assume that M is in $\mathbb{H}^2$. There is a minor difficulty coming from the fact that the variables $H_{(i)}$ are not assumed to be bounded (and therefore we cannot directly use the construction of the integral of elementary processes). To circumvent this difficulty, we set, for every $n \geq 1$,

$$T_n = \inf\{t \geq 0 : |H_t| \geq n\} = \inf\{t_i : |H_{(i)}| \geq n\} \quad (\text{where } \inf\varnothing = \infty).$$

It is easy to verify that T_n is a stopping time, and we have $T_n \uparrow \infty$ as $n \to \infty$. Furthermore, we have for every n,

$$H_s\,\mathbf{1}_{[0,T_n]}(s) = \sum_{i=0}^{p-1} H^n_{(i)}\,\mathbf{1}_{(t_i,t_{i+1}]}(s)$$

where the random variables $H^n_{(i)} = H_{(i)}\,\mathbf{1}_{\{T_n > t_i\}}$ satisfy the same properties as the $H_{(i)}$'s and additionally are bounded by n. Hence $H\,\mathbf{1}_{[0,T_n]}$ is an elementary process, and by the very definition of the stochastic integral with respect to a martingale of $\mathbb{H}^2$, we have

$$(H \cdot M)_{t\wedge T_n} = (H\,\mathbf{1}_{[0,T_n]} \cdot M)_t = \sum_{i=0}^{p-1} H^n_{(i)}\,(M_{t_{i+1}\wedge t} - M_{t_i\wedge t}).$$

The desired result now follows by letting n tend to infinity.

5.1.4 Convergence of Stochastic Integrals

We start by giving a "dominated convergence theorem" for stochastic integrals.

Proposition 5.8 *Let $X = M + V$ be the canonical decomposition of a continuous semimartingale X, and let $t > 0$. Let $(H^n)_{n\geq 1}$ and H be locally bounded progressive processes, and let K be a nonnegative progressive process. Assume that the following properties hold a.s.:*

(i) $H^n_s \longrightarrow H_s$ *as* $n \to \infty$, *for every* $s \in [0, t]$;
(ii) $|H^n_s| \leq K_s$, *for every* $n \geq 1$ *and* $s \in [0, t]$;
(iii) $\int_0^t (K_s)^2 \, \mathrm{d}\langle M, M\rangle_s < \infty$ *and* $\int_0^t K_s \, |\mathrm{d}V_s| < \infty$.

Then,

$$\int_0^t H^n_s \, \mathrm{d}X_s \underset{n\to\infty}{\longrightarrow} \int_0^t H_s \, \mathrm{d}X_s$$

in probability.

Remarks

(a) Assertion (iii) holds automatically if K is locally bounded.
(b) Instead of assuming that (i) and (ii) hold for *every* $s \in [0, t]$ (a.s.), it is enough to assume that these conditions hold for $\mathrm{d}\langle M, M\rangle_s$-a.e. $s \in [0, t]$ and for $|\mathrm{d}V_s|$-a.e. $s \in [0, t]$, a.s. This will be clear from the proof.

Proof The a.s. convergence

$$\int_0^t H^n_s \, \mathrm{d}V_s \underset{n\to\infty}{\longrightarrow} \int_0^t H_s \, \mathrm{d}V_s$$

follows from the usual dominated convergence theorem. So we just have to verify that $\int_0^t H^n_s \, \mathrm{d}M_s$ converges in probability to $\int_0^t H_s \, \mathrm{d}M_s$. For every integer $p \geq 1$, consider the stopping time

$$T_p := \inf\{r \in [0, t] : \int_0^r (K_s)^2 \mathrm{d}\langle M, M\rangle_s \geq p\} \wedge t,$$

and observe that $T_p = t$ for all large enough p, a.s., by assumption (iii). Then, the bound (5.14) gives

$$E\Big[\Big(\int_0^{T_p} H^n_s \, \mathrm{d}M_s - \int_0^{T_p} H_s \, \mathrm{d}M_s\Big)^2\Big] \leq E\Big[\int_0^{T_p} (H^n_s - H_s)^2 \mathrm{d}\langle M, M\rangle_s\Big],$$

which tends to 0 as $n \to \infty$, by dominated convergence, using assumptions (i) and (ii) and the fact that $\int_0^{T_p} (K_s)^2 \, \mathrm{d}\langle M, M\rangle_s \leq p$. Since $P(T_p = t)$ tends to 1 as $p \to \infty$, the desired result follows. □

We apply the preceding proposition to an approximation result in the case of continuous integrands, which will be useful in the next section.

Proposition 5.9 *Let X be a continuous semimartingale, and let H be an adapted process with continuous sample paths. Then, for every $t > 0$, for every sequence $0 = t_0^n < \cdots < t_{p_n}^n = t$ of subdivisions of $[0, t]$ whose mesh tends to 0, we have*

$$\lim_{n\to\infty} \sum_{i=0}^{p_n-1} H_{t_i^n}(X_{t_{i+1}^n} - X_{t_i^n}) = \int_0^t H_s \, \mathrm{d}X_s,$$

in probability.

Proof For every $n \geq 1$, define a process H^n by

$$H_s^n = \begin{cases} H_{t_i^n} & \text{if } t_i^n < s \leq t_{i+1}^n, \ \text{ for every } i \in \{0, 1, \dots, p_n - 1\} \\ H_0 & \text{if } s = 0 \\ 0 & \text{if } s > t. \end{cases}$$

Note that H^n is progressive. We then observe that all assumptions of Proposition 5.8 hold if we take

$$K_s = \max_{0 \leq r \leq s} |H_s|,$$

which is a locally bounded progressive process. Hence, we conclude that

$$\int_0^t H_s^n \, \mathrm{d}X_s \underset{n\to\infty}{\longrightarrow} \int_0^t H_s \, \mathrm{d}X_s$$

in probability. This gives the desired result since, by property (v) in Sect. 5.1.3, we have

$$\int_0^t H_s^n \, \mathrm{d}X_s = \sum_{i=0}^{p_n-1} H_{t_i^n}(M_{t_{i+1}^n} - M_{t_i^n}).$$

□

Remark The preceding proposition can be viewed as a generalization of Lemma 4.3 to stochastic integrals. However, in contrast with that lemma, it is essential in Proposition 5.9 to evaluate H at the **left end** of the interval $(t_i^n, t_{i+1}^n]$: The result will fail if we replace $H_{t_i^n}$ by $H_{t_{i+1}^n}$. Let us give a simple counterexample. We take $H_t = X_t$ and we assume that the sequence of subdivisions $(t_i^n)_{0 \leq i \leq p_n}$ is increasing. By the proposition, we have

$$\lim_{n\to\infty} \sum_{i=0}^{p_n-1} X_{t_i^n}(X_{t_{i+1}^n} - X_{t_i^n}) = \int_0^t X_s \, \mathrm{d}X_s,$$

in probability. On the other hand, writing

$$\sum_{i=0}^{p_n-1} X_{t^n_{i+1}}(X_{t^n_{i+1}} - X_{t^n_i}) = \sum_{i=0}^{p_n-1} X_{t^n_i}(X_{t^n_{i+1}} - X_{t^n_i}) + \sum_{i=0}^{p_n-1} (X_{t^n_{i+1}} - X_{t^n_i})^2,$$

and using Proposition 4.21, we get

$$\lim_{n\to\infty} \sum_{i=0}^{p_n-1} X_{t^n_{i+1}}(X_{t^n_{i+1}} - X_{t^n_i}) = \int_0^t X_s\,\mathrm{d}X_s + \langle X, X\rangle_t,$$

in probability. The resulting limit is different from $\int_0^t X_s \mathrm{d}X_s$ unless the martingale part of X is degenerate. Note that, if we add the previous two convergences, we arrive at the formula

$$(X_t)^2 - (X_0)^2 = 2\int_0^t X_s \mathrm{d}X_s + \langle X, X\rangle_t$$

which is a special case of Itô's formula of the next section.

5.2 Itô's Formula

Itô's formula is the cornerstone of stochastic calculus. It shows that, if we apply a twice continuously differentiable function to a p-tuple of continuous semimartingales, the resulting process is still a continuous semimartingale, and there is an explicit formula for the canonical decomposition of this semimartingale.

Theorem 5.10 (Itô's formula) *Let $X^1, \dots, X^p$ be p continuous semimartingales, and let F be a twice continuously differentiable real function on $\mathbb{R}^p$. Then, for every $t \geq 0$,*

$$F(X^1_t, \dots, X^p_t) = F(X^1_0, \dots, X^p_0) + \sum_{i=1}^{p} \int_0^t \frac{\partial F}{\partial x^i}(X^1_s, \dots, X^p_s)\,\mathrm{d}X^i_s$$
$$+\frac{1}{2}\sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 F}{\partial x^i \partial x^j}(X^1_s, \dots, X^p_s)\,\mathrm{d}\langle X^i, X^j\rangle_s.$$

Proof We first deal with the case $p = 1$ and we write $X = X^1$ for simplicity. Fix $t > 0$ and consider an increasing sequence $0 = t^n_0 < \cdots < t^n_{p_n} = t$ of subdivisions of $[0, t]$ whose mesh tends to 0. Then, for every n,

$$F(X_t) = F(X_0) + \sum_{i=0}^{p_n-1} (F(X_{t^n_{i+1}}) - F(X_{t^n_i})).$$

For every $i \in \{0, 1, \dots, p_n - 1\}$, we apply the Taylor–Lagrange formula to the function $[0, 1] \ni \theta \mapsto F(X_{t_i^n} + \theta(X_{t_{i+1}^n} - X_{t_i^n}))$, between $\theta = 0$ and $\theta = 1$, and we get that

$$F(X_{t_{i+1}^n}) - F(X_{t_i^n}) = F'(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n}) + \frac{1}{2} f_{n,i}\,(X_{t_{i+1}^n} - X_{t_i^n})^2,$$

where the quantity $f_{n,i}$ can be written as $F''(X_{t_i^n} + c(X_{t_{i+1}^n} - X_{t_i^n}))$ for some $c \in [0, 1]$. By Proposition 5.9 with $H_s = F'(X_s)$, we have

$$\lim_{n\to\infty} \sum_{i=0}^{p_n-1} F'(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n}) = \int_0^t F'(X_s)\, \mathrm{d}X_s,$$

in probability. To complete the proof of the case $p = 1$ of the theorem, it is therefore enough to verify that

$$\lim_{n\to\infty} \sum_{i=0}^{p_n-1} f_{n,i}(X_{t_{i+1}^n} - X_{t_i^n})^2 = \int_0^t F''(X_s)\, \mathrm{d}\langle X, X\rangle_s, \tag{5.15}$$

in probability. We observe that

$$\sup_{0\le i\le p_n-1} |f_{n,i} - F''(X_{t_i^n})| \le \sup_{0\le i\le p_n-1} \left(\sup_{x\in[X_{t_i^n}\wedge X_{t_{i+1}^n}, X_{t_i^n}\vee X_{t_{i+1}^n}]} |F''(x) - F''(X_{t_i^n})| \right).$$

The right-hand side of the preceding display tends to 0 a.s. as $n \to \infty$, as a simple consequence of the uniform continuity of F'' (and of the sample paths of X) over a compact interval.

Since $\sum_{i=0}^{p_n-1}(X_{t_{i+1}^n} - X_{t_i^n})^2$ converges in probability (Proposition 4.21), it follows from the last display that

$$\left| \sum_{i=0}^{p_n-1} f_{n,i}(X_{t_{i+1}^n} - X_{t_i^n})^2 - \sum_{i=0}^{p_n-1} F''(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n})^2 \right| \underset{n\to\infty}{\longrightarrow} 0$$

in probability. So the convergence (5.15) will follow if we can verify that

$$\lim_{n\to\infty} \sum_{i=0}^{p_n-1} F''(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n})^2 = \int_0^t F''(X_s)\, \mathrm{d}\langle X, X\rangle_s, \tag{5.16}$$

in probability. In fact, we will show that (5.16) holds a.s. along a suitable sequence of values of n (this suffices for our needs, because we can replace the initial sequence of subdivisions by a subsequence). To this end, we note that

$$\sum_{i=0}^{p_n-1} F''(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n})^2 = \int_{[0,t]} F''(X_s)\,\mu_n(\mathrm{d}s),$$

where μ_n is the random measure on $[0,t]$ defined by

$$\mu_n(\mathrm{d}r) := \sum_{i=0}^{p_n-1} (X_{t_{i+1}^n} - X_{t_i^n})^2\,\delta_{t_i^n}(\mathrm{d}r).$$

Write D for the dense subset of $[0,t]$ that consists of all t_i^n for $n \geq 1$ and $0 \leq i \leq p_n$. As a consequence of Proposition 4.21, we get for every $r \in D$,

$$\mu_n([0,r]) \underset{n\to\infty}{\longrightarrow} \langle X, X\rangle_r$$

in probability. Using a diagonal extraction, we can thus find a subsequence of values of n such that, along this subsequence, we have for every $r \in D$,

$$\mu_n([0,r]) \underset{n\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} \langle X, X\rangle_r,$$

which implies that the sequence μ_n converges a.s. to the measure $\mathbf{1}_{[0,t]}(r)\,\mathrm{d}\langle X,X\rangle_r$, in the sense of weak convergence of finite measures. We conclude that we have

$$\int_{[0,t]} F''(X_s)\,\mu_n(ds) \underset{n\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} \int_0^t F''(X_s)\,\mathrm{d}\langle X,X\rangle_s$$

along the chosen subsequence. This completes the proof of the case $p = 1$.

In the general case, the Taylor–Lagrange formula, applied for every $n \geq 1$ and every $i \in \{0,1,\ldots,p_n-1\}$ to the function

$$[0,1] \ni \theta \mapsto F(X^1_{t_i^n} + \theta(X^1_{t_{i+1}^n} - X^1_{t_i^n}), \ldots, X^p_{t_i^n} + \theta(X^p_{t_{i+1}^n} - X^p_{t_i^n}))\ ,$$

gives

$$\begin{aligned} F(X^1_{t_{i+1}^n}, \ldots, X^p_{t_{i+1}^n}) - F(X^1_{t_i^n}, \ldots, X^p_{t_i^n}) &= \sum_{k=1}^p \frac{\partial F}{\partial x^k}(X^1_{t_i^n}, \ldots, X^p_{t_i^n})\,(X^k_{t_{i+1}^n} - X^k_{t_i^n}) \\ &\quad + \sum_{k,l=1}^p \frac{f^{k,l}_{n,i}}{2}(X^k_{t_{i+1}^n} - X^k_{t_i^n})(X^l_{t_{i+1}^n} - X^l_{t_i^n}) \end{aligned}$$

where, for every $k, l \in \{1, \dots, p\}$,

$$f_{n,i}^{k,l} = \frac{\partial^2 F}{\partial x_k \partial x_l}(X_{t_i^n} + c(X_{t_{i+1}^n} - X_{t_i^n})),$$

for some $c \in [0, 1]$ (here we use the notation $X_t = (X_t^1, \dots, X_t^p)$).

Proposition 5.9 can again be used to handle the terms involving first derivatives. Moreover, a slight modification of the arguments of the case $p = 1$ shows that, at least along a suitable sequence of values of n, we have for every $k, l \in \{1, \dots, p\}$,

$$\lim_{n\to\infty} \sum_{i=0}^{p_n-1} f_{n,i}^{k,l}(X_{t_{i+1}^n}^k - X_{t_i^n}^k)(X_{t_{i+1}^n}^l - X_{t_i^n}^l) = \int_0^t \frac{\partial^2 F}{\partial x_k \partial x_l}(X_s^1, \dots, X_s^p)\, \mathrm{d}\langle X^k, X^l\rangle_s$$

in probability. This completes the proof of the theorem. □

An important special case of Itô's formula is the **formula of integration by parts**, which is obtained by taking $p = 2$ and $F(x, y) = xy$: if X and Y are two continuous semimartingales, we have

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, \mathrm{d}Y_s + \int_0^t Y_s\, \mathrm{d}X_s + \langle X, Y\rangle_t.$$

In particular, if $Y = X$,

$$X_t^2 = X_0^2 + 2\int_0^t X_s\, \mathrm{d}X_s + \langle X, X\rangle_t.$$

When $X = M$ is a continuous local martingale, we know from the definition of the quadratic variation that $M^2 - \langle M, M\rangle$ is a continuous local martingale. The previous formula shows that this continuous local martingale is

$$M_0^2 + 2\int_0^t M_s\, \mathrm{d}M_s.$$

We could have seen this directly from the construction of $\langle M, M\rangle$ in Chap. 4 (this construction involved approximations of the stochastic integral $\int_0^t M_s \mathrm{d}M_s$).

Let B be an $(\mathscr{F}_t)$-real Brownian motion (recall from Definition 3.11 that this means that B is a Brownian motion, which is adapted to the filtration $(\mathscr{F}_t)$ and such that, for every $0 \le s < t$, the variable $B_t - B_s$ is independent of the σ-field $\mathscr{F}_s$). An $(\mathscr{F}_t)$-Brownian motion is a continuous local martingale (a martingale if $B_0 \in L^1$) and we already noticed that its quadratic variation is $\langle B, B\rangle_t = t$.

In this particular case, Itô's formula reads

$$F(B_t) = F(B_0) + \int_0^t F'(B_s)\, \mathrm{d}B_s + \frac{1}{2}\int_0^t F''(B_s)\mathrm{d}s.$$

Taking $X_t^1 = t$, $X_t^2 = B_t$, we also get for every twice continuously differentiable function $F(t,x)$ on $\mathbb{R}_+ \times \mathbb{R}$,

$$F(t,B_t) = F(0,B_0) + \int_0^t \frac{\partial F}{\partial x}(s,B_s)\,\mathrm{d}B_s + \int_0^t \Big(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\Big)(s,B_s)\,\mathrm{d}s.$$

Let $B_t = (B_t^1,\ldots,B_t^d)$ be a d-dimensional $(\mathscr{F}_t)$-Brownian motion. Note that the components $B^1,\ldots,B^d$ are $(\mathscr{F}_t)$-Brownian motions. By Proposition 4.16, $\langle B^i, B^j\rangle = 0$ when $i \neq j$ (by subtracting the initial value, which does not change the bracket $\langle B^i, B^j\rangle$, we are reduced to the case where $B^1,\ldots,B^d$ are independent). Itô's formula then shows that, for every twice continuously differentiable function F on $\mathbb{R}^d$,

$$\begin{aligned} &F(B_t^1,\ldots,B_t^d) \\ &\quad = F(B_0^1,\ldots,B_0^d) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(B_s^1,\ldots,B_s^d)\,\mathrm{d}B_s^i + \frac{1}{2}\int_0^t \Delta F(B_s^1,\ldots,B_s^d)\,\mathrm{d}s. \end{aligned}$$

The latter formula is often written in the shorter form

$$F(B_t) = F(B_0) + \int_0^t \nabla F(B_s)\cdot \mathrm{d}B_s + \frac{1}{2}\int_0^t \Delta F(B_s)\,\mathrm{d}s,$$

where ∇F stands for the vector of first partial derivatives of F. There is again an analogous formula for $F(t,B_t)$.

Important remark It frequently occurs that one needs to apply Itô's formula to a function F which is only defined (and twice continuously differentiable) on an open subset U of $\mathbb{R}^p$. In that case, we can argue in the following way. Suppose that there exists another open set V, such that $(X_0^1,\ldots,X_0^p) \in V$ a.s. and $\bar{V} \subset U$ (here $\bar{V}$ denotes the closure of V). Typically V will be the set of all points whose distance from U^c is strictly greater than ε, for some $\varepsilon > 0$. Set $T_V := \inf\{t \geq 0 : (X_t^1,\ldots,X_t^p) \notin V\}$, which is a stopping time by Proposition 3.9. Simple analytic arguments allow us to find a function G which is twice continuously differentiable on $\mathbb{R}^p$ and coincides with F on $\bar{V}$. We can now apply Itô's formula to obtain the canonical decomposition of the semimartingale $G(X_{t\wedge T_V}^1,\ldots,X_{t\wedge T_V}^p) = F(X_{t\wedge T_V}^1,\ldots,X_{t\wedge T_V}^p)$, and this decomposition only involves the first and second derivatives of F on V. If in addition we know that the process $(X_t^1,\ldots,X_t^p)$ a.s. does not exit U, we can let the open set V increase to U, and we get that Itô's formula for $F(X_t^1,\ldots,X_t^p)$ remains valid exactly in the same form as in Theorem 5.10. These considerations can be applied, for instance, to the function $F(x) = \log x$ and to a semimartingale X taking strictly positive values: see the proof of Proposition 5.21 below.

We now use Itô's formula to exhibit a remarkable class of (local) martingales, which extends the exponential martingales associated with processes with independent increments. A random process with values in the complex plane $\mathbb{C}$ is called a complex continuous local martingale if both its real part and its imaginary part are continuous local martingales.

Proposition 5.11 *Let M be a continuous local martingale and, for every $\lambda \in \mathbb{C}$, let*

$$\mathscr{E}(\lambda M)_t = \exp\Big(\lambda M_t - \frac{\lambda^2}{2}\langle M,M\rangle_t\Big).$$

The process $\mathscr{E}(\lambda M)$ is a complex continuous local martingale, which can be written in the form

$$\mathscr{E}(\lambda M)_t = \mathrm{e}^{\lambda M_0} + \lambda \int_0^t \mathscr{E}(\lambda M)_s \, \mathrm{d}M_s.$$

Remark The stochastic integral in the right-hand side of the last display is defined by dealing separately with the real and the imaginary part.

Proof If $F(r,x)$ is a twice continuously differentiable function on $\mathbb{R}^2$, Itô's formula gives

$$\begin{aligned} F(\langle M,M\rangle_t, M_t) = F(0,M_0) &+ \int_0^t \frac{\partial F}{\partial x}(\langle M,M\rangle_s, M_s)\, \mathrm{d}M_s \\ &+ \int_0^t \Big(\frac{\partial F}{\partial r} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\Big)(\langle M,M\rangle_s, M_s)\, \mathrm{d}\langle M,M\rangle_s. \end{aligned}$$

Hence, $F(\langle M,M\rangle_t, M_t)$ is a continuous local martingale as soon as F satisfies the equation

$$\frac{\partial F}{\partial r} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0.$$

This equation holds for $F(r,x) = \exp(\lambda x - \frac{\lambda^2}{2} r)$ (more precisely for both the real and the imaginary part of this function). Moreover, for this choice of F we have $\frac{\partial F}{\partial x} = \lambda F$, which leads to the formula of the statement. □

5.3 A Few Consequences of Itô's Formula

Itô's formula has a huge number of applications. In this section, we derive some of the most important ones.

5.3.1 Lévy's Characterization of Brownian Motion

We start with a striking characterization of real Brownian motion as the unique continuous local martingale M such that $\langle M, M\rangle_t = t$. In fact, we give a multidimensional version of this result, which is known as Lévy's theorem.

Theorem 5.12 *Let $X = (X^1, \dots, X^d)$ be an adapted process with continuous sample paths. The following are equivalent:*

(i) *X is a d-dimensional $(\mathscr{F}_t)$-Brownian motion.*
(ii) *The processes $X^1, \dots, X^d$ are continuous local martingales, and $\langle X^i, X^j\rangle_t = \delta_{ij}\, t$ for every $i, j \in \{1, \dots, d\}$ (here δ_{ij} is the Kronecker symbol, $\delta_{ij} = \mathbf{1}_{\{i=j\}}$).*

In particular, a continuous local martingale M is an $(\mathscr{F}_t)$-Brownian motion if and only if $\langle M, M\rangle_t = t$, for every $t \geq 0$, or equivalently if and only if $M_t^2 - t$ is a continuous local martingale.

Proof The fact that (i) $\Rightarrow$ (ii) has already been derived. Let us assume that (ii) holds. Let $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. Then, $\xi \cdot X_t = \sum_{j=1}^d \xi_j X_t^j$ is a continuous local martingale with quadratic variation

$$\sum_{j=1}^d \sum_{k=1}^d \xi_j \xi_k \langle X^j, X^k\rangle_t = |\xi|^2 t.$$

By Proposition 5.11, $\exp(\mathrm{i}\xi \cdot X_t + \frac{1}{2}|\xi|^2 t)$ is a complex continuous local martingale. This complex continuous local martingale is bounded on every interval $[0, a]$, $a > 0$, and is therefore a (true) martingale, in the sense that its real and imaginary parts are both martingales. Hence, for every $0 \leq s < t$,

$$E[\exp(\mathrm{i}\xi \cdot X_t + \frac{1}{2}|\xi|^2 t) \mid \mathscr{F}_s] = \exp(\mathrm{i}\xi \cdot X_s + \frac{1}{2}|\xi|^2 s),$$

and thus

$$E[\exp(\mathrm{i}\xi \cdot (X_t - X_s) \mid \mathscr{F}_s] = \exp(-\frac{1}{2}|\xi|^2 (t-s)).$$

It follows that, for every $A \in \mathscr{F}_s$,

$$E[\mathbf{1}_A \exp(\mathrm{i}\xi \cdot (X_t - X_s))] = P(A)\, \exp(-\frac{1}{2}|\xi|^2 (t-s)).$$

Taking $A = \Omega$, we get that $X_t - X_s$ is a centered Gaussian vector with covariance matrix $(t-s)\mathrm{Id}$ (in particular, the components $X_t^j - X_s^j$, $1 \leq j \leq d$ are independent). Furthermore, fix $A \in \mathscr{F}_s$ with $P(A) > 0$, and write P_A for the conditional probability measure $P_A(\cdot) = P(A)^{-1}P(\cdot \cap A)$. We also obtain that

$$P_A[\exp(\mathrm{i}\xi \cdot (X_t - X_s))] = \exp(-\frac{1}{2}|\xi|^2(t-s))$$

which means that the law of $X_t - X_s$ under P_A is the same as its law under P. Therefore, for any nonnegative measurable function f on $\mathbb{R}^d$, we have

$$P_A[f(X_t - X_s)] = E[f(X_t - X_s)],$$

or equivalently

$$E[\mathbf{1}_A f(X_t - X_s)] = P(A)\, E[f(X_t - X_s)].$$

This holds for any $A \in \mathscr{F}_s$ (when $P(A) = 0$ the equality is trivial), and thus $X_t - X_s$ is independent of $\mathscr{F}_s$.

It follows that, if $t_0 = 0 < t_1 < \ldots < t_p$, the vectors $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_p} - X_{t_{p-1}}$ are independent. Since the components of each of these vectors are independent random variables, we obtain that all variables $X^j_{t_k} - X^j_{t_{k-1}}, 1 \leq j \leq d, 1 \leq k \leq p$ are independent, and $X^j_{t_k} - X^j_{t_{k-1}}$ is distributed according to $\mathscr{N}(0, t_k - t_{k-1})$. This implies that $X - X_0$ is a d-dimensional Brownian motion started from 0. Since we also know that $X - X_0$ is independent of X_0 (as an easy consequence of the fact that $X_t - X_s$ is independent of $\mathscr{F}_s$, for every $0 \leq s < t$), we get that X is a d-dimensional Brownian motion. Finally, X is adapted and has independent increments with respect to the filtration $(\mathscr{F}_t)$ so that X is a d-dimensional $(\mathscr{F}_t)$-Brownian motion. □

5.3.2 Continuous Martingales as Time-Changed Brownian Motions

The next theorem shows that any continuous local martingale M can be written as a "time-changed" Brownian motion (in fact, we prove this only when $\langle M, M\rangle_\infty = \infty$, but see the remarks below). It follows that the sample paths of M are Brownian sample paths run at a different (varying) speed, and certain almost sure properties of sample paths of M can be deduced from the corresponding properties of Brownian sample paths. For instance, under the condition $\langle M, M\rangle_\infty = \infty$, the sample paths of M must oscillate between $+\infty$ and $-\infty$ as $t \to \infty$ (cf. the last assertion of Proposition 2.14).

Theorem 5.13 (Dambis–Dubins–Schwarz) *Let M be a continuous local martingale such that $\langle M,M\rangle_\infty = \infty$ a.s. There exists a Brownian motion $(\beta_s)_{s\geq 0}$ such that*

$$a.s.\ \forall t \geq 0, \quad M_t = \beta_{<M,M>_t}.$$

Remarks

(i) One can remove the assumption $\langle M,M\rangle_\infty = \infty$, at the cost of enlarging the underlying probability space, see [70, Chapter V].
(ii) The Brownian motion β is not adapted with respect to the filtration $(\mathscr{F}_t)$, but with respect to a "time-changed" filtration, as the following proof will show.

Proof We first assume that $M_0 = 0$. For every $r \geq 0$, we set

$$\tau_r = \inf\{t \geq 0 : \langle M,M\rangle_t \geq r\}.$$

Note that τ_r is a stopping time by Proposition 3.9. Furthermore, we have $\tau_r < \infty$ for every $r \geq 0$, on the event $\{\langle M,M\rangle_\infty = \infty\}$. It will be convenient to redefine the variables τ_r on the (negligible) event $\mathscr{N} = \{\langle M,M\rangle_\infty < \infty\}$ by taking $\tau_r(\omega) = 0$ for every $r \geq 0$ if $\omega \in \mathscr{N}$. Since the filtration is complete, τ_r remains a stopping time after this modification.

By construction, for every $\omega \in \Omega$, the function $r \mapsto \tau_r(\omega)$ is nondecreasing and left-continuous, and therefore has a right limit at every $r \geq 0$. This right limit is denoted by τ_{r+} and we have

$$\tau_{r+} = \inf\{t \geq 0 : \langle M,M\rangle_t > r\},$$

except of course on the negligible set $\mathscr{N}$, where $\tau_{r+} = 0$.

We set $\beta_r = M_{\tau_r}$ for every $r \geq 0$. By Theorem 3.7, the process $(\beta_r)_{r\geq 0}$ is adapted with respect to the filtration $(\mathscr{G}_r)$ defined by $\mathscr{G}_r = \mathscr{F}_{\tau_r}$ for every $r \geq 0$, and $\mathscr{G}_\infty = \mathscr{F}_\infty$. Note that the filtration $(\mathscr{G}_r)$ is complete since this property holds for $(\mathscr{F}_t)$.

The sample paths $r \mapsto \beta_r(\omega)$ are left-continuous and have right limits given for every $r \geq 0$ by

$$\beta_{r+} = \lim_{s \downarrow\downarrow r} \beta_s = M_{\tau_{r+}}.$$

In fact we have $\beta_{r+} = \beta_r$ for every $r \geq 0$, a.s., as a consequence of the following lemma.

Lemma 5.14 *We have a.s. for every $0 \leq a < b$,*

$$M_t = M_a,\ \forall t \in [a,b] \iff \langle M,M\rangle_b = \langle M,M\rangle_a.$$

Let us postpone the proof of the lemma. Since $\langle M,M\rangle_{\tau_r} = \langle M,M\rangle_{\tau_{r+}}$ for every $r \geq 0$, Lemma 5.14 implies that $M_{\tau_r} = M_{\tau_{r+}}$, for every $r \geq 0$, a.s. Hence the sample paths of β are continuous (to be precise, we should redefine $\beta_r = 0$, for every $r \geq 0$, on the zero probability set where the property of Lemma 5.14 fails).

Let us verify that β_s and $\beta_s^2 - s$ are martingales with respect to the filtration $(\mathscr{G}_s)$. For every integer $n \geq 1$, the stopped continuous local martingales M^{τ_n} and $(M^{\tau_n})^2 - \langle M, M\rangle^{\tau_n}$ are uniformly integrable martingales (by Theorem 4.13, recalling that $M_0 = 0$ and noting that $\langle M^{\tau_n}, M^{\tau_n}\rangle_\infty = \langle M, M\rangle_{\tau_n} = n$ a.s.). The optional stopping theorem (Theorem 3.22) then implies that, for every $0 \leq r \leq s \leq n$,

$$E[\beta_s \mid \mathscr{G}_r] = E[M_{\tau_s}^{\tau_n} \mid \mathscr{F}_{\tau_r}] = M_{\tau_r}^{\tau_n} = \beta_r$$

and similarly

$$E[\beta_s^2 - s \mid \mathscr{G}_r] = E[(M_{\tau_s}^{\tau_n})^2 - \langle M^{\tau_n}, M^{\tau_n}\rangle_{\tau_s} \mid \mathscr{F}_{\tau_r}] = (M_{\tau_r}^{\tau_n})^2 - \langle M^{\tau_n}, M^{\tau_n}\rangle_{\tau_r} = \beta_r^2 - r.$$

Then the case $d = 1$ of Theorem 5.12 shows that β is a $(\mathscr{G}_r)$-Brownian motion. Finally, by the definition of β, we have a.s. for every $t \geq 0$,

$$\beta_{<M,M>_t} = M_{\tau_{<M,M>_t}}.$$

But since $\tau_{<M,M>_t -} \leq t \leq \tau_{<M,M>_t +}$ and since $\langle M, M\rangle$ takes the same value at $\tau_{<M,M>_t}$ and at $\tau_{<M,M>_t +}$, Lemma 5.14 shows that $M_t = M_{\tau_{<M,M>_t}}$ for every $t \geq 0$, a.s. We conclude that we have $M_t = \beta_{<M,M>_t}$ for every $t \geq 0$, a.s. This completes the proof when $M_0 = 0$.

If $M_0 \neq 0$, we write $M_t = M_0 + M'_t$, and we apply the previous argument to M', in order to get a Brownian motion β' with $\beta'_0 = 0$, such that $M'_t = \beta'_{<M',M'>_t}$ for every $t \geq 0$ a.s. Since β' is a $(\mathscr{G}_r)$-Brownian motion, β' is independent of $\mathscr{G}_0 = \mathscr{F}_0$, hence of M_0. Therefore, $\beta_s = M_0 + \beta'_s$ is also a Brownian motion, and we get the desired representation for M. □

Proof of Lemma 5.14 Thanks to the continuity of sample paths of M and $\langle M, M\rangle$, it is enough to verify that for any fixed a and b such that $0 \leq a < b$, we have

$$\{M_t = M_a,\ \forall t \in [a, b]\} = \{\langle M, M\rangle_b = \langle M, M\rangle_a\}\ , \quad \text{a.s.}$$

The fact that the event in the left-hand side is (a.s.) contained in the event in the right-hand side is easy from the approximations of $\langle M, M\rangle$ in Theorem 4.9.

Let us prove the converse. Consider the continuous local martingale $N_t = M_t - M_{t\wedge a}$ and note that

$$\langle N, N\rangle_t = \langle M, M\rangle_t - \langle M, M\rangle_{t\wedge a}.$$

For every $\varepsilon > 0$, introduce the stopping time

$$T_\varepsilon = \inf\{t \geq 0 : \langle N, N\rangle_t \geq \varepsilon\}.$$

Then N^{T_ε} is a martingale in $\mathbb{H}^2$ (since $\langle N^{T_\varepsilon}, N^{T_\varepsilon}\rangle_\infty \le \varepsilon$). Fix $t \in [a,b]$. We have

$$E[N^2_{t\wedge T_\varepsilon}] = E[\langle N,N\rangle_{t\wedge T_\varepsilon}] \le \varepsilon.$$

Hence, considering the event $A := \{\langle M,M\rangle_b = \langle M,M\rangle_a\} \subset \{T_\varepsilon \ge b\}$,

$$E[\mathbf{1}_A N_t^2] = E[\mathbf{1}_A N^2_{t\wedge T_\varepsilon}] \le E[N^2_{t\wedge T_\varepsilon}] \le \varepsilon.$$

By letting ε go to 0, we get $E[\mathbf{1}_A N_t^2] = 0$ and thus $N_t = 0$ a.s. on A, which completes the proof. □

We can combine the arguments of the proof of Theorem 5.13 with Theorem 5.12 to get the following technical result, which will be useful when we consider the image of planar Brownian motion under holomorphic transformations in Chap. 7.

Proposition 5.15 *Let M and N be two continuous local martingales such that $M_0 = N_0 = 0$. Assume that*

(i) $\langle M,M\rangle_t = \langle N,N\rangle_t$ *for every* $t \ge 0$, *a.s.*
(ii) *M and N are orthogonal* ($\langle M,N\rangle_t = 0$ *for every* $t \ge 0$, *a.s.)*
(iii) $\langle M,M\rangle_\infty = \langle N,N\rangle_\infty = \infty$, *a.s.*

Let $\beta = (\beta_t)_{t\ge 0}$, resp. $\gamma = (\gamma_t)_{t\ge 0}$, be the real Brownian motion such $M_t = \beta_{\langle M,M\rangle_t}$, resp. $N_t = \gamma_{\langle N,N\rangle_t}$, for every $t \ge 0$, a.s. Then β and γ are independent.

Proof We use the notation of the proof of Theorem 5.13 and note that we have $\beta_r = M_{\tau_r}$ and $\gamma_r = N_{\tau_r}$, where

$$\tau_r = \inf\{t \ge 0 : \langle M,M\rangle_t \ge r\} = \inf\{t \ge 0 : \langle N,N\rangle_t \ge r\}.$$

We know that β and γ are $(\mathscr{G}_r)$-Brownian motions. Since M and N are orthogonal martingales, we also know that M_tN_t is a local martingale. As in the proof of Theorem 5.13, and using now Proposition 4.15 (v), we get that, for every $n \ge 1$, $M_t^{\tau_n}N_t^{\tau_n}$ is a uniformly integrable martingale, and by applying the optional stopping theorem, we obtain that for $r \le s \le n$,

$$E[\beta_s\gamma_s \mid \mathscr{G}_r] = E[M^{\tau_n}_{\tau_s}N^{\tau_n}_{\tau_s} \mid \mathscr{F}_{\tau_r}] = M^{\tau_n}_{\tau_r}N^{\tau_n}_{\tau_r} = \beta_s\gamma_r$$

so that $\beta_r\gamma_r$ is a $(\mathscr{G}_r)$-martingale and the bracket $\langle \beta,\gamma\rangle$ (evaluated in the filtration $(\mathscr{G}_r)$) is identically zero. By Theorem 5.12, it follows that (β,γ) is a two-dimensional Brownian motion and, since $\beta_0 = \gamma_0 = 0$, this implies that β and γ are independent. □

5.3.3 The Burkholder–Davis–Gundy Inequalities

We now state important inequalities connecting a continuous local martingale with its quadratic variation. If M is a continuous local martingale, we set

$$M_t^* = \sup_{s \le t} |M_s|$$

for every $t \ge 0$. Theorem 5.16 below shows that, under the condition $M_0 = 0$, for every $p > 0$, the p-th moment of M_t^* is bounded above and below (up to universal multiplicative constants) by the p-th moment of $\sqrt{\langle M, M \rangle_t}$. These bounds are very useful because, in particular when M is a stochastic integral, it is often easier to estimate the moments of $\sqrt{\langle M, M \rangle_t}$ than those of M_t^*. Such applications arise, for instance, in the study of stochastic differential equations (see e.g. the proof of Theorem 8.5 below).

Theorem 5.16 (Burkholder–Davis–Gundy inequalities) *For every real $p > 0$, there exist two constants c_p, $C_p > 0$ depending only on p such that, for every continuous local martingale M with $M_0 = 0$, and every stopping time T,*

$$c_p\, E[\langle M, M \rangle_T^{p/2}] \le E[(M_T^*)^p] \le C_p\, E[\langle M, M \rangle_T^{p/2}].$$

Remark It may happen that the quantities $E[\langle M, M \rangle_T^{p/2}]$ and $E[(M_T^*)^p]$ are infinite. The theorem says that these quantities are either both finite (then the stated bounds hold) or both infinite.

Proof Replacing M by the stopping martingale M^T, we see that it is enough to treat the special case $T = \infty$. We then observe that it suffices to consider the case when M is bounded: Assuming that the bounded case has been treated, we can replace M by M^{T_n}, where $T_n = \inf\{t \ge 0 : |M_t| = n\}$, and we get the general case by letting n tend to ∞.

The left-hand side inequality, in the case $p \ge 4$, follows from the result of question **4.** in Exercise 4.26. We prove below the right-hand side inequality for all values of p. This is the inequality we will use in the sequel (we refer to [70, Chapter IV] for the remaining case).

We first consider the case $p \ge 2$. We apply Itô's formula to the function $|x|^p$:

$$|M_t|^p = \int_0^t p|M_s|^{p-1} \mathrm{sgn}(M_s)\, \mathrm{d}M_s + \frac{1}{2} \int_0^t p(p-1)|M_s|^{p-2}\, \mathrm{d}\langle M, M \rangle_s.$$

Since M is bounded, hence in particular $M \in \mathbb{H}^2$, the process

$$\int_0^t p|M_s|^{p-1} \mathrm{sgn}(M_s)\, \mathrm{d}M_s$$

is a martingale in $\mathbb{H}^2$. We therefore get

$$\begin{aligned}E[|M_t|^p] &= \frac{p(p-1)}{2}E\Big[\int_0^t |M_s|^{p-2}\,\mathrm{d}\langle M,M\rangle_s\Big]\\&\le \frac{p(p-1)}{2}E[(M_t^*)^{p-2}\langle M,M\rangle_t]\\&\le \frac{p(p-1)}{2}(E[(M_t^*)^p])^{(p-2)/p}(E[\langle M,M\rangle_t^{p/2}])^{2/p},\end{aligned}$$

by Hölder's inequality. On the other hand, by Doob's inequality in L^p (Proposition 3.15),

$$E[(M_t^*)^p] \le \Big(\frac{p}{p-1}\Big)^p E[|M_t|^p]$$

and combining this bound with the previous one, we arrive at

$$E[(M_t^*)^p] \le \Big(\Big(\frac{p}{p-1}\Big)^p\ \frac{p(p-1)}{2}\Big)^{p/2} E[\langle M,M\rangle_t^{p/2}].$$

It now suffices to let t tend to ∞.

Consider then the case $p < 2$. Since $M \in \mathbb{H}^2$, $M^2 - \langle M,M\rangle$ is a uniformly integrable martingale and we have, for every stopping time T,

$$E[(M_T)^2] = E[\langle M,M\rangle_T].$$

Let $x > 0$ and consider the stopping time $T_x := \inf\{t \ge 0 : (M_t)^2 \ge x\}$. Then, if T is any bounded stopping time,

$$\begin{aligned}P((M_T^*)^2 \ge x) = P(T_x \le T) = P((M_{T_x\wedge T})^2 \ge x) &\le \frac{1}{x}E[(M_{T_x\wedge T})^2]\\&= \frac{1}{x}E[\langle M,M\rangle_{T_x\wedge T}]\\&\le \frac{1}{x}E[\langle M,M\rangle_T].\end{aligned}$$

Next consider the stopping time $S_x := \inf\{t \ge 0 : \langle M,M\rangle_t \ge x\}$. Observe that, for every $t \ge 0$, we have $\{(M_t^*)^2 \ge x\} \subset (\{(M_{S_x\wedge t}^*)^2 \ge x\} \cup \{S_x \le t\})$. Using the preceding bound with $T = S_x \wedge t$, we thus get

$$\begin{aligned}P((M_t^*)^2 \ge x) &\le P((M_{S_x\wedge t}^*)^2 \ge x) + P(S_x \le t)\\&\le \frac{1}{x}E[\langle M,M\rangle_{S_x\wedge t}] + P(\langle M,M\rangle_t \ge x)\end{aligned}$$

$$= \frac{1}{x} E[\langle M, M \rangle_t \wedge x] + P(\langle M, M \rangle_t \geq x)$$
$$= \frac{1}{x} E[\langle M, M \rangle_t \, \mathbf{1}_{\{\langle M,M \rangle_t < x\}}] + 2\, P(\langle M, M \rangle_t \geq x).$$

To complete the proof, set $q = p/2 \in (0, 1)$ and integrate each side of the last bound with respect to the measure $q\, x^{q-1}\, \mathrm{d}x$. We have first

$$\int_0^\infty P((M_t^*)^2 \geq x)\, q\, x^{q-1}\, \mathrm{d}x = E\Big[\int_0^{(M_t^*)^2} q\, x^{q-1}\, \mathrm{d}x\Big] = E[(M_t^*)^{2q}],$$

and similarly

$$\int_0^\infty P(\langle M, M \rangle_t \geq x)\, q\, x^{q-1}\, \mathrm{d}x = E[\langle M, M \rangle_t^q].$$

Furthermore,

$$\int_0^\infty \frac{1}{x} E[\langle M, M \rangle_t \, \mathbf{1}_{\{\langle M,M \rangle_t < x\}}]\, q\, x^{q-1}\, \mathrm{d}x$$
$$= E\Big[\langle M, M \rangle_t \int_{\langle M,M \rangle_t}^\infty q\, x^{q-2}\, \mathrm{d}x\Big] = \frac{q}{1-q} E[\langle M, M \rangle_t^q].$$

Summarizing, we have obtained the bound

$$E[(M_t^*)^{2q}] \leq \Big(2 + \frac{q}{1-q}\Big) E[\langle M, M \rangle_t^q],$$

and we just have to let $t \to \infty$ to get the desired result. □

Corollary 5.17 *Let M be a continuous local martingale such that $M_0 = 0$. The condition*

$$E[\langle M, M \rangle_\infty^{1/2}] < \infty$$

implies that M is a uniformly integrable martingale.

Proof By the case $p = 1$ of Theorem 5.16, the condition $E[\langle M, M \rangle_\infty^{1/2}] < \infty$ implies that $E[M_\infty^*] < \infty$. Proposition 4.7 (ii) then shows that the continuous local martingale M, which is dominated by the variable M_∞^*, is a uniformly integrable martingale. □

The condition $E[\langle M, M \rangle_\infty^{1/2}] < \infty$ is weaker than the condition $E[\langle M, M \rangle_\infty] < \infty$, which ensures that $M \in \mathbb{H}^2$. The corollary can be applied to stochastic integrals. If M is a continuous local martingale and H is a progressive process such that, for every $t \geq 0$,

$$E\Big[\Big(\int_0^t H_s^2 \,\mathrm{d}\langle M, M\rangle_s\Big)^{1/2}\Big] < \infty\,,$$

then $\int_0^t H_s \mathrm{d}M_s$ is a martingale, and formulas (5.6) and (5.9) for the first moment and the conditional expectations of $\int_0^t H_s \mathrm{d}M_s$ hold (of course with $t < \infty$).

5.4 The Representation of Martingales as Stochastic Integrals

In the special setting where the filtration on Ω is the completed canonical filtration of a Brownian motion, we will now show that all martingales can be represented as stochastic integrals with respect to that Brownian motion. For the sake of simplicity, we first consider a one-dimensional Brownian motion, but we will discuss the extension to Brownian motion in higher dimensions at the end of this section.

Theorem 5.18 *Assume that the filtration $(\mathscr{F}_t)$ on Ω is the completed canonical filtration of a real Brownian motion B started from 0. Then, for every random variable $Z \in L^2(\Omega, \mathscr{F}_\infty, P)$, there exists a unique progressive process $h \in L^2(B)$ (i.e. $E[\int_0^\infty h_s^2 \mathrm{d}s] < \infty$) such that*

$$Z = E[Z] + \int_0^\infty h_s \,\mathrm{d}B_s.$$

Consequently, for every martingale M that is bounded in L^2 (respectively, for every continuous local martingale M), there exists a unique process $h \in L^2(B)$ (resp. $h \in L^2_{\mathrm{loc}}(B)$) and a constant $C \in \mathbb{R}$ such that

$$M_t = C + \int_0^t h_s \,\mathrm{d}B_s.$$

Remark As the proof will show, the second part of the statement applies to a martingale M that is bounded in L^2, **without any assumption** on the continuity of sample paths of M. This observation will be useful later when we discuss consequences of the representation theorem. Note that continuous local martingales have continuous sample paths by definition.

Lemma 5.19 *Under the assumptions of the theorem, the vector space generated by the random variables*

$$\exp\Big(\mathrm{i}\sum_{j=1}^n \lambda_j (B_{t_j} - B_{t_{j-1}})\Big),$$

for any choice of $0 = t_0 < t_1 < \cdots < t_n$ *and* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$*, is dense in the space* $L^2_{\mathbb{C}}(\Omega, \mathscr{F}_\infty, P)$ *of all square-integrable complex-valued* $\mathscr{F}_\infty$*-measurable random variables.*

Proof It is enough to prove that, if $Z \in L^2_{\mathbb{C}}(\Omega, \mathscr{F}_\infty, P)$ is such that

$$E\Big[Z \exp\Big(\mathrm{i}\sum_{j=1}^{n} \lambda_j (B_{t_j} - B_{t_{j-1}})\Big)\Big] = 0 \tag{5.17}$$

for any choice of $0 = t_0 < t_1 < \cdots < t_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then $Z = 0$.

Fix $0 = t_0 < t_1 < \cdots < t_n$, and consider the complex measure μ on $\mathbb{R}^n$ defined by

$$\mu(F) = E\Big[Z\, 1_F(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})\Big]$$

for any Borel subset F of $\mathbb{R}^n$. Then (5.17) exactly shows that the Fourier transform of μ is identically zero. By the injectivity of the Fourier transform on complex measures on $\mathbb{R}^d$, it follows that $\mu = 0$. We have thus $E[Z\,\mathbf{1}_A] = 0$ for every $A \in \sigma(B_{t_1}, \ldots, B_{t_n})$.

A monotone class argument then shows that the identity $E[Z\,\mathbf{1}_A] = 0$ remains valid for any $A \in \sigma(B_t, t \geq 0)$, and then by completion for any $A \in \mathscr{F}_\infty$. It follows that $Z = 0$. □

Proof of Theorem 5.18 We start with the first assertion. We first observe that the uniqueness of h is easy since, if the representation of a given variable Z holds with two processes h and h' in $L^2(B)$, we have

$$E\Big[\int_0^\infty (h_s - h'_s)^2 \mathrm{d}s\Big] = E\Big[\Big(\int_0^\infty h_s\,\mathrm{d}B_s - \int_0^\infty h'_s\,\mathrm{d}B_s\Big)^2\Big] = 0,$$

hence $h = h'$ in $L^2(B)$.

Let us turn to the existence part. Let $\mathscr{H}$ stand for the vector space of all variables $Z \in L^2(\Omega, \mathscr{F}_\infty, P)$ for which the property of the statement holds. We note that if $Z \in \mathscr{H}$ and h is the associated process in $L^2(B)$, we have

$$E[Z^2] = (E[Z])^2 + E\Big[\int_0^\infty (h_s)^2\,\mathrm{d}s\Big].$$

It follows that $\mathscr{H}$ is a closed subspace of $L^2(\Omega, \mathscr{F}_\infty, P)$. Indeed, if (Z_n) is a sequence in $\mathscr{H}$ that converges to Z in $L^2(\Omega, \mathscr{F}_\infty, P)$, the processes $h^{(n)}$ corresponding respectively to the variables Z_n form a Cauchy sequence in $L^2(B)$, hence converge in $L^2(B)$ to a certain process $h \in L^2(B)$ – here we use the Hilbert space structure of $L^2(B)$ – and it immediately follows that $Z = E[Z] + \int_0^\infty h_s\,\mathrm{d}B_s$.

Since $\mathscr{H}$ is closed, in order to prove that $\mathscr{H} = L^2(\Omega, \mathscr{F}_\infty, P)$, we just have to verify that $\mathscr{H}$ contains a dense subset of $L^2(\Omega, \mathscr{F}_\infty, P)$. Let $0 = t_0 < t_1 < \cdots < t_n$

and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, and set $f(s) = \sum_{j=1}^n \lambda_j \mathbf{1}_{(t_{j-1},t_j]}(s)$. Write $\mathscr{E}_t^f$ for the exponential martingale $\mathscr{E}(i\int_0^\cdot f(s)\,\mathrm{d}B_s)$ (cf. Proposition 5.11). Proposition 5.11 shows that

$$\exp\Big(\mathrm{i}\sum_{j=1}^n \lambda_j(B_{t_j} - B_{t_{j-1}}) + \frac{1}{2}\sum_{j=1}^n \lambda_j^2(t_j - t_{j-1})\Big) = \mathscr{E}_\infty^f = 1 + \mathrm{i}\int_0^\infty \mathscr{E}_s^f f(s)\,\mathrm{d}B_s$$

and it follows that both the real part and the imaginary part of variables of the form $\exp\Big(\mathrm{i}\sum_{j=1}^n \lambda_j(B_{t_j} - B_{t_{j-1}})\Big)$ are in $\mathscr{H}$. By Lemma 5.19, linear combinations of such random variables are dense in $L^2(\Omega, \mathscr{F}_\infty, P)$. This completes the proof of the first assertion of the theorem.

Let us turn to the second assertion. If M is a martingale that is bounded in L^2, then $M_\infty \in L^2(\Omega, \mathscr{F}_\infty, P)$, and thus can be written in the form

$$M_\infty = E[M_\infty] + \int_0^\infty h_s\,\mathrm{d}B_s,$$

where $h \in L^2(B)$. Thanks to (5.9), it follows that

$$M_t = E[M_\infty \mid \mathscr{F}_t] = E[M_\infty] + \int_0^t h_s\,\mathrm{d}B_s$$

and the uniqueness of h is also immediate from the uniqueness in the first assertion.

Finally, if M is a continuous local martingale, we have first $M_0 = C \in \mathbb{R}$ because the σ-field $\mathscr{F}_0$ contains only events of probability zero or one. If $T_n = \inf\{t \geq 0 : |M_t| \geq n\}$ we can apply the case of martingales bounded in L^2 to M^{T_n} and we get a process $h^{(n)} \in L^2(B)$ such that

$$M_t^{T_n} = C + \int_0^t h_s^{(n)}\,\mathrm{d}B_s.$$

Using the uniqueness of the progressive process in the representation, we get that $h_s^{(m)} = \mathbf{1}_{[0,T_m]}(s)\,h_s^{(n)}$ if $m < n$, $\mathrm{d}s$ a.e., a.s. It is now easy to construct a process $h \in L^2_{\mathrm{loc}}(B)$ such that, for every m, $h_s^{(m)} = \mathbf{1}_{[0,T_m]}(s)\,h_s$, $\mathrm{d}s$ a.e., a.s. The representation formula of the theorem follows, and the uniqueness of h is also straightforward. □

Consequences Let us give two important consequences of the representation theorem. Under the assumptions of the theorem:

(1) **The filtration $(\mathscr{F}_t)_{t\geq 0}$ is right-continuous**. Indeed, let $t \geq 0$ and let Z be $\mathscr{F}_{t+}$-measurable and bounded. We can find $h \in L^2(B)$ such that

$$Z = E[Z] + \int_0^\infty h_s \mathrm{d}B_s.$$

If $\varepsilon > 0$, Z is $\mathscr{F}_{t+\varepsilon}$-measurable, and thus, using (5.9),

$$Z = E[Z \mid \mathscr{F}_{t+\varepsilon}] = E[Z] + \int_0^{t+\varepsilon} h_s \mathrm{d}B_s.$$

When $\varepsilon \to 0$ the right-hand side converges in L^2 to

$$E[Z] + \int_0^t h_s \mathrm{d}B_s.$$

Thus Z is equal a.s. to an $\mathscr{F}_t$-measurable random variable, and, since the filtration is complete, Z is $\mathscr{F}_t$-measurable.

A similar argument shows that the filtration $(\mathscr{F}_t)_{t\geq 0}$ is also left-continuous: If, for $t > 0$, we let

$$\mathscr{F}_{t-} = \bigvee_{s\in[0,t)} \mathscr{F}_s$$

be the smallest σ-field that contains all σ-fields $\mathscr{F}_s$ for $s \in [0,t)$, we have $\mathscr{F}_{t-} = \mathscr{F}_t$.

(2) All martingales of the filtration $(\mathscr{F}_t)_{t\geq 0}$ have a modification with continuous sample paths. For a martingale that is bounded in L^2, this follows from the representation formula (see the remark after the statement of the theorem). Then consider a uniformly integrable martingale M (if M is not uniformly integrable, we just replace M by $M_{t\wedge a}$ for every $a \geq 0$). In that case, we have, for every $t \geq 0$,

$$M_t = E[M_\infty \mid \mathscr{F}_t].$$

By Theorem 3.18 (whose application is justified as we know that the filtration is right-continuous), the process M_t has a modification with càdlàg sample paths, and we consider this modification. Let $M_\infty^{(n)}$ be a sequence of bounded random variables such that $M_\infty^{(n)} \longrightarrow M_\infty$ in L^1 as $n \to \infty$. Introduce the martingales

$$M_t^{(n)} = E[M_\infty^{(n)} \mid \mathscr{F}_t],$$

which are bounded in L^2. By the beginning of the argument, we can assume that, for every n, the sample paths of $M^{(n)}$ are continuous. On the other hand, Doob's maximal inequality (Proposition 3.15) implies that, for every $\lambda > 0$,

$$P\Big[\sup_{t\geq 0} |M_t^{(n)} - M_t| > \lambda\Big] \leq \frac{3}{\lambda} E[|M_\infty^{(n)} - M_\infty|].$$

It follows that we can find a sequence $n_k \uparrow \infty$ such that, for every $k \geq 1$,

$$P\Big[\sup_{t\geq 0} |M_t^{(n_k)} - M_t| > 2^{-k}\Big] \leq 2^{-k}.$$

An application of the Borel–Cantelli lemma now shows that

$$\sup_{t\geq 0} |M_t^{(n_k)} - M_t| \underset{k\to\infty}{\overset{\text{a.s.}}{\longrightarrow}} 0$$

and we get that the sample paths of M are continuous as uniform limits of continuous functions.

Multidimensional extension Let us briefly describe the multidimensional extension of the preceding results. We now assume that the filtration $(\mathscr{F}_t)$ on Ω is the completed canonical filtration of a d-dimensional Brownian motion $B = (B^1, \dots, B^d)$ started from 0. Then, for every random variable $Z \in L^2(\Omega, \mathscr{F}_\infty, P)$, there exists a unique d-tuple $(h^1, \dots, h^d)$ of progressive processes, satisfying

$$E\Big[\int_0^\infty (h_s^i)^2\,\mathrm{d}s\Big] < \infty\,, \qquad \forall i \in \{1, \dots, d\},$$

such that

$$Z = E[Z] + \sum_{i=1}^d \int_0^\infty h_s^i\,\mathrm{d}B_s^i.$$

Similarly, if M is a continuous local martingale, there exist a constant C and a unique d-tuple $(h^1, \dots, h^d)$ of progressive processes, satisfying

$$\int_0^t (h_s^i)^2\,\mathrm{d}s < \infty\,,\ \text{a.s.} \qquad \forall t \geq 0,\ \forall i \in \{1, \dots, d\},$$

such that

$$M_t = C + \sum_{i=1}^d \int_0^t h_s^i\,\mathrm{d}B_s^i.$$

The proofs are exactly the same as in the case $d = 1$ (Theorem 5.18). Consequences (1) and (2) above remain valid.

5.5 Girsanov's Theorem

Throughout this section, we assume that the filtration $(\mathscr{F}_t)$ is both complete and right-continuous. Our goal is to study how the notions of a martingale and of a semimartingale are affected when the underlying probability measure P is replaced by another probability measure Q. Most of the time we will assume that P and Q are mutually absolutely continuous, and then the fact that the filtration $(\mathscr{F}_t)$ is complete with respect to P implies that it is complete with respect to Q. When there is a risk of confusion, we will write E_P for the expectation under the probability measure P, and similarly E_Q for the expectation under Q. Unless otherwise specified, the notions of a (local) martingale or of a semimartingale refer to the underlying probability measure P (when we consider these notions under Q we will say so explicitly). Note that, in contrast with the notion of a martingale, the notion of a finite variation process does not depend on the underlying probability measure.

Proposition 5.20 *Assume that Q is a probability measure on $(\Omega, \mathscr{F})$, which is absolutely continuous with respect to P on the σ-field $\mathscr{F}_\infty$. For every $t \in [0, \infty]$, let*

$$D_t = \frac{\mathrm{d}Q}{\mathrm{d}P}_{|\mathscr{F}_t}$$

be the Radon–Nikodym derivative of Q with respect to P on the σ-field $\mathscr{F}_t$. The process $(D_t)_{t\geq 0}$ is a uniformly integrable martingale. Consequently $(D_t)_{t\geq 0}$ has a càdlàg modification. Keeping the same notation $(D_t)_{t\geq 0}$ for this modification, we have, for every stopping time T,

$$D_T = \frac{\mathrm{d}Q}{\mathrm{d}P}_{|\mathscr{F}_T}.$$

Finally, if we assume furthermore that P and Q are mutually absolutely continuous on $\mathscr{F}_\infty$, we have

$$\inf_{t\geq 0} D_t > 0\,, \quad P\ a.s.$$

Proof If $A \in \mathscr{F}_t$, we have

$$Q(A) = E_Q[\mathbf{1}_A] = E_P[\mathbf{1}_A D_\infty] = E_P[\mathbf{1}_A E_P[D_\infty \mid \mathscr{F}_t]]$$

and, by the uniqueness of the Radon–Nikodym derivative on $\mathscr{F}_t$, it follows that

$$D_t = E_P[D_\infty \mid \mathscr{F}_t], \qquad \text{a.s.}$$

Hence D is a uniformly integrable martingale, which is closed by D_∞. Theorem 3.18 (using the fact that $(\mathscr{F}_t)$ is both complete and right-continuous) then allows us to find a càdlàg modification of $(D_t)_{t\geq 0}$, which we consider from now on.

Then, if T is a stopping time, the optional stopping theorem (Theorem 3.22) gives for every $A \in \mathscr{F}_T$,

$$Q(A) = E_Q[\mathbf{1}_A] = E_P[\mathbf{1}_A D_\infty] = E_P[\mathbf{1}_A\, E_P[D_\infty \mid \mathscr{F}_T]] = E_P[\mathbf{1}_A D_T],$$

and, since D_T is $\mathscr{F}_T$-measurable, it follows that

$$D_T = \frac{\mathrm{d}Q}{\mathrm{d}P_{\,|\mathscr{F}_T}}.$$

Let us prove the last assertion. For every $\varepsilon > 0$, set

$$T_\varepsilon = \inf\{t \geq 0 : D_t < \varepsilon\}$$

and note that T_ε is a stopping time as the first hitting time of an open set by a càdlàg process (recall Proposition 3.9 and the fact that the filtration is right-continuous). Then, noting that the event $\{T_\varepsilon < \infty\}$ is $\mathscr{F}_{T_\varepsilon}$-measurable,

$$Q(T_\varepsilon < \infty) = E_P[\mathbf{1}_{\{T_\varepsilon < \infty\}}\, D_{T_\varepsilon}] \leq \varepsilon$$

since $D_{T_\varepsilon} \leq \varepsilon$ on $\{T_\varepsilon < \infty\}$ by the right-continuity of sample paths. It immediately follows that

$$Q\Big(\bigcap_{n=1}^{\infty} \{T_{1/n} < \infty\}\Big) = 0$$

and since P is absolutely continuous with respect to Q we have also

$$P\Big(\bigcap_{n=1}^{\infty} \{T_{1/n} < \infty\}\Big) = 0.$$

But this exactly means that, P a.s., there exists an integer $n \geq 1$ such that $T_{1/n} = \infty$, giving the last assertion of the proposition. □

Proposition 5.21 *Let D be a continuous local martingale taking (strictly) positive values. There exists a unique continuous local martingale L such that*

$$D_t = \exp\Big(L_t - \frac{1}{2}\langle L, L\rangle_t\Big) = \mathscr{E}(L)_t.$$

Moreover, L is given by the formula

$$L_t = \log D_0 + \int_0^t D_s^{-1}\, \mathrm{d}D_s.$$

Proof Uniqueness is an easy consequence of Theorem 4.8. Then, since D takes positive values, we can apply Itô's formula to $\log D_t$ (see the remark before Proposition 5.11), and we get

$$\log D_t = \log D_0 + \int_0^t \frac{\mathrm{d}D_s}{D_s} - \frac{1}{2}\int_0^t \frac{\mathrm{d}\langle D, D\rangle_s}{D_s^2} = L_t - \frac{1}{2}\langle L, L\rangle_t,$$

where L is as in the statement. □

We now state the main theorem of this section, which explains the relation between continuous local martingales under P and continuous local martingales under Q.

Theorem 5.22 (Girsanov) *Assume that the probability measures P and Q are mutually absolutely continuous on $\mathscr{F}_\infty$. Let $(D_t)_{t\geq 0}$ be the martingale with càdlàg sample paths such that, for every $t \geq 0$,*

$$D_t = \frac{\mathrm{d}Q}{\mathrm{d}P_{\,|\mathscr{F}_t}}.$$

Assume that D has continuous sample paths, and let L be the unique continuous local martingale such that $D_t = \mathscr{E}(L)_t$. Then, if M is a continuous local martingale under P, the process

$$\tilde{M} = M - \langle M, L\rangle$$

is a continuous local martingale under Q.

Remark By consequences of the martingale representation theorem explained at the end of the previous section, the continuity assumption for the sample paths of D always holds when $(\mathscr{F}_t)$ is the (completed) canonical filtration of a Brownian motion. In applications of Theorem 5.22, one often starts from the martingale (D_t) to define the probability measure Q, so that the continuity assumption is satisfied by construction (see the examples in the next section).

Proof The fact that D_t can be written in the form $D_t = \mathscr{E}(L)_t$ follows from Proposition 5.21 (we are assuming that D has continuous sample paths, and we also know from Proposition 5.20 that D takes positive values). Then, let T be a stopping time and let X be an adapted process with continuous sample paths. We claim that, if $(XD)^T$ is a martingale under P, then X^T is a martingale under Q. Let us verify the claim. By Proposition 5.20, $E_Q[|X_{T\wedge t}|] = E_P[|X_{T\wedge t}D_{T\wedge t}|] < \infty$, and it follows that $X_t^T \in L^1(Q)$. Then, let $A \in \mathscr{F}_s$ and $s < t$. Since $A \cap \{T > s\} \in \mathscr{F}_s$, we have, using the fact that $(XD)^T$ is a martingale under P,

$$E_P[\mathbf{1}_{A\cap\{T>s\}}X_{T\wedge t}D_{T\wedge t}] = E_P[\mathbf{1}_{A\cap\{T>s\}}X_{T\wedge s}D_{T\wedge s}].$$

By Proposition 5.20,

$$D_{T\wedge t} = \frac{\mathrm{d}Q}{\mathrm{d}P}_{|\mathscr{F}_{T\wedge t}}, \quad D_{T\wedge s} = \frac{\mathrm{d}Q}{\mathrm{d}P}_{|\mathscr{F}_{T\wedge s}},$$

and thus, since $A \cap \{T > s\} \in \mathscr{F}_{T\wedge s} \subset \mathscr{F}_{T\wedge t}$, it follows that

$$E_Q[\mathbf{1}_{A\cap\{T>s\}} X_{T\wedge t}] = E_Q[\mathbf{1}_{A\cap\{T>s\}} X_{T\wedge s}].$$

On the other hand, it is immediate that

$$E_Q[\mathbf{1}_{A\cap\{T\leq s\}} X_{T\wedge t}] = E_Q[\mathbf{1}_{A\cap\{T\leq s\}} X_{T\wedge s}].$$

By combining with the previous display, we have $E_Q[\mathbf{1}_A X_{T\wedge t}] = E_Q[\mathbf{1}_A X_{T\wedge s}]$, giving our claim. As a consequence of the claim, we get that, if XD is a continuous local martingale under P, then X is a continuous local martingale under Q.

Next let M be a continuous local martingale under P, and let $\tilde{M}$ be as in the statement of the theorem. We apply the preceding observation to $X = \tilde{M}$, noting that, by Itô's formula,

$$\begin{aligned}\tilde{M}_t D_t &= M_0 D_0 + \int_0^t \tilde{M}_s \,\mathrm{d}D_s + \int_0^t D_s \,\mathrm{d}M_s - \int_0^t D_s \,\mathrm{d}\langle M, L\rangle_s + \langle M, D\rangle_t \\ &= M_0 D_0 + \int_0^t \tilde{M}_s \,\mathrm{d}D_s + \int_0^t D_s \,\mathrm{d}M_s\end{aligned}$$

since $\mathrm{d}\langle M, L\rangle_s = D_s^{-1}\mathrm{d}\langle M, D\rangle_s$ by Proposition 5.21. We get that $\tilde{M}D$ is a continuous local martingale under P, and thus $\tilde{M}$ is a continuous local martingale under Q. □

Consequences

(a) A process M which is a continuous local martingale under P remains a semimartingale under Q, and its canonical decomposition under Q is $M = \tilde{M} + \langle M, L\rangle$ (recall that the notion of a finite variation process does not depend on the underlying probability measure). It follows that the class of semimartingales under P is contained in the class of semimartingales under Q.

In fact these two classes are equal. Indeed, under the assumptions of Theorem 5.22, P and Q play symmetric roles, since the Radon–Nikodym derivative of P with respect to Q on the σ-field $\mathscr{F}_t$ is D_t^{-1}, which has continuous sample paths if D does.

We may furthermore notice that

$$D_t^{-1} = \exp\Big(-L_t + \langle L, L\rangle_t - \frac{1}{2}\langle L, L\rangle_t\Big) = \exp\Big(-\tilde{L}_t - \frac{1}{2}\langle \tilde{L}, \tilde{L}\rangle_t\Big) = \mathscr{E}(-\tilde{L})_t,$$

where $\tilde{L} = L - \langle L, L\rangle$ is a continuous local martingale under Q, and $\langle \tilde{L}, \tilde{L}\rangle = \langle L, L\rangle$. So, under the assumptions of Theorem 5.22, the roles of P and Q can be interchanged provided D is replaced by D^{-1} and L is replaced by $-\tilde{L}$.

(b) Let X and Y be two semimartingales (under P or under Q). The bracket $\langle X, Y\rangle$ is the same under P and under Q. In fact this bracket is given in both cases by the approximation of Proposition 4.21 (this observation was used implicitly in (a) above).

Similarly, if H is a locally bounded progressive process, the stochastic integral $H \cdot X$ is the same under P and under Q. To see this it is enough to consider the case when $X = M$ is a continuous local martingale (under P). Write $(H \cdot M)_P$ for the stochastic integral under P and $(H \cdot M)_Q$ for the one under Q. By linearity,

$$(H \cdot \tilde{M})_P = (H \cdot M)_P - H \cdot \langle M, L\rangle = (H \cdot M)_P - \langle (H \cdot M)_P, L\rangle,$$

and Theorem 5.22 shows that $(H \cdot \tilde{M})_P$ is a continuous local martingale under Q. Furthermore the bracket of this continuous local martingale with any continuous local martingale N under Q is equal to $H \cdot \langle M, N\rangle = H \cdot \langle \tilde{M}, N\rangle$, and it follows from Theorem 5.6 that $(H \cdot \tilde{M})_P = (H \cdot \tilde{M})_Q$ hence also $(H \cdot M)_P = (H \cdot M)_Q$.

With the notation of Theorem 5.22, set $\tilde{M} = \mathscr{G}_Q^P(M)$. Then $\mathscr{G}_Q^P$ maps the set of all P-continuous local martingales onto the set of all Q-continuous local martingales. One easily verifies, using the remarks in (a) above, that $\mathscr{G}_P^Q \circ \mathscr{G}_Q^P = \mathrm{Id}$. Furthermore, the mapping $\mathscr{G}_Q^P$ commutes with the stochastic integral: if H is a locally bounded progressive process, $H \cdot \mathscr{G}_Q^P(M) = \mathscr{G}_Q^P(H \cdot M)$.

(c) Suppose that $M = B$ is an $(\mathscr{F}_t)$-Brownian motion under P, then $\tilde{B} = B - \langle B, L\rangle$ is a continuous local martingale under Q, with quadratic variation $\langle \tilde{B}, \tilde{B}\rangle_t = \langle B, B\rangle_t = t$. By Theorem 5.12, it follows that $\tilde{B}$ is an $(\mathscr{F}_t)$-Brownian motion under Q.

In most applications of Girsanov's theorem, one constructs the probability measure Q in the following way. Start from a continuous local martingale L such that $L_0 = 0$ and $\langle L, L\rangle_\infty < \infty$ a.s. The latter condition implies that the limit $L_\infty := \lim_{t\to\infty} L_t$ exists a.s. (see Exercise 4.24). Then $\mathscr{E}(L)_t$ is a nonnegative continuous local martingale hence a supermartingale (Proposition 4.7), which converges a.s. to $\mathscr{E}(L)_\infty = \exp(L_\infty - \frac{1}{2}\langle L, L\rangle_\infty)$, and $E[\mathscr{E}(L)_\infty] \leq 1$ by Fatou's lemma. If the property

$$E[\mathscr{E}(L)_\infty] = 1 \tag{5.18}$$

holds, then $\mathscr{E}(L)$ is a uniformly integrable martingale (by Fatou's lemma again, one has $\mathscr{E}(L)_t \geq E[\mathscr{E}(L)_\infty \mid \mathscr{F}_t]$, but (5.18) implies that $E[\mathscr{E}(L)_\infty] = E[\mathscr{E}(L)_0] = E[\mathscr{E}(L)_t]$ for every $t \geq 0$). If we let Q be the probability measure with density $\mathscr{E}(L)_\infty$ with respect to P, we are in the setting of Theorem 5.22, with $D_t = \mathscr{E}(L)_t$. It is therefore very important to give conditions that ensure that (5.18) holds.

Theorem 5.23 *Let L be a continuous local martingale such that* $L_0 = 0$. *Consider the following properties:*

(i) $E[\exp \frac{1}{2}\langle L, L\rangle_\infty] < \infty$ *(Novikov's criterion);*
(ii) *L is a uniformly integrable martingale, and* $E[\exp \frac{1}{2} L_\infty] < \infty$ *(Kazamaki's criterion);*
(iii) $\mathscr{E}(L)$ *is a uniformly integrable martingale.*

Then, (i) ⇒ (ii) ⇒ (iii).

Proof (i) ⇒ (ii) Property (i) implies that $E[\langle L, L\rangle_\infty] < \infty$ hence also that L is a continuous martingale bounded in L^2 (Theorem 4.13). Then,

$$\exp \frac{1}{2} L_\infty = (\mathscr{E}(L)_\infty)^{1/2} \, (\exp(\frac{1}{2}\langle L, L\rangle_\infty))^{1/2}$$

so that, by the Cauchy–Schwarz inequality,

$$\begin{aligned} E[\exp \frac{1}{2} L_\infty] &\leq (E[\mathscr{E}(L)_\infty])^{1/2} (E[\exp(\frac{1}{2}\langle L, L\rangle_\infty)])^{1/2} \\ &\leq (E[\exp(\frac{1}{2}\langle L, L\rangle_\infty)])^{1/2} < \infty. \end{aligned}$$

(ii) ⇒ (iii) Since L is a uniformly integrable martingale, Theorem 3.22 shows that, for any stopping time T, we have $L_T = E[L_\infty \mid \mathscr{F}_T]$. Jensen's inequality then gives

$$\exp \frac{1}{2} L_T \leq E[\exp \frac{1}{2} L_\infty \mid \mathscr{F}_T].$$

By assumption, $E[\exp \frac{1}{2} L_\infty] < \infty$, which implies that the collection of all variables of the form $E[\exp \frac{1}{2} L_\infty \mid \mathscr{F}_T]$, for any stopping time T, is uniformly integrable. The preceding bound then shows that the collection of all variables $\exp \frac{1}{2} L_T$, for any stopping time T, is also uniformly integrable.

For $0 < a < 1$, set $Z_t^{(a)} = \exp(\frac{aL_t}{1+a})$. Then, one easily verifies that

$$\mathscr{E}(aL)_t = (\mathscr{E}(L)_t)^{a^2} \, (Z_t^{(a)})^{1-a^2}.$$

If $\Gamma \in \mathscr{F}$ and T is a stopping time, Hölder's inequality gives

$$E[\mathbf{1}_\Gamma \mathscr{E}(aL)_T] \leq E[\mathscr{E}(L)_T]^{a^2} E[\mathbf{1}_\Gamma Z_T^{(a)}]^{1-a^2} \leq E[\mathbf{1}_\Gamma Z_T^{(a)}]^{1-a^2} \leq E[\mathbf{1}_\Gamma \exp \frac{1}{2} L_T]^{2a(1-a)}.$$

In the second inequality, we used the property $E[\mathscr{E}(L)_T] \leq 1$, which holds by Proposition 3.25 because $\mathscr{E}(L)$ is a nonnegative supermartingale and $\mathscr{E}(L)_0 = 1$. In the third inequality, we use Jensen's inequality, noting that $\frac{1+a}{2a} > 1$. Since the collection of all variables of the form $\exp \frac{1}{2}L_T$, for any stopping time T, is uniformly integrable, the preceding display shows that so is the collection of all variables $\mathscr{E}(aL)_T$ for any stopping time T. By the definition of a continuous local martingale, there is an increasing sequence $T_n \uparrow \infty$ of stopping times, such that, for every n, $\mathscr{E}(aL)_{t\wedge T_n}$ is a martingale. If $0 \le s \le t$, we can use uniform integrability to pass to the limit $n \to \infty$ in the equality $E[\mathscr{E}(aL)_{t\wedge T_n} \mid \mathscr{F}_s] = \mathscr{E}(aL)_{s\wedge T_n}$ and we get that $\mathscr{E}(aL)$ is a uniformly integrable martingale. It follows that

$$1 = E[\mathscr{E}(aL)_\infty] \le E[\mathscr{E}(L)_\infty]^{a^2} E[Z_\infty^{(a)}]^{1-a^2} \le E[\mathscr{E}(L)_\infty]^{a^2} E[\exp \frac{1}{2}L_\infty]^{2a(1-a)},$$

using again Jensen's inequality as above. When $a \to 1$, this gives $E[\mathscr{E}(L)_\infty] \ge 1$ hence $E[\mathscr{E}(L)_\infty] = 1$. □

5.6 A Few Applications of Girsanov's Theorem

In this section, we describe a few applications of Girsanov's theorem, which illustrate the strength of the previous results.

Constructing solutions of stochastic differential equations Let b be a bounded measurable function on $\mathbb{R}_+ \times \mathbb{R}$. We assume that there exists a function $g \in L^2(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+), \mathrm{d}t)$ such that $|b(t,x)| \le g(t)$ for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. This holds in particular if there exists an $A > 0$ such that $|b|$ is bounded on $[0,A] \times \mathbb{R}_+$ and vanishes on $(A,\infty) \times \mathbb{R}_+$.

Let B be an $(\mathscr{F}_t)$-Brownian motion. Consider the continuous local martingale

$$L_t = \int_0^t b(s,B_s)\,\mathrm{d}B_s$$

and the associated exponential martingale

$$D_t = \mathscr{E}(L)_t = \exp\Big(\int_0^t b(s,B_s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t b(s,B_s)^2 \mathrm{d}s\Big).$$

Our assumption on b ensures that condition (i) of Theorem 5.23 holds, and thus D is a uniformly integrable martingale. We set $Q = D_\infty \cdot P$. Girsanov's theorem, and remark (c) following the statement of this theorem, show that the process

$$\beta_t := B_t - \int_0^t b(s,B_s)\,\mathrm{d}s$$

is an $(\mathscr{F}_t)$-Brownian motion under Q.

We can restate the latter property by saying that, under the probability measure Q, there exists an $(\mathscr{F}_t)$-Brownian motion β such that the process $X = B$ solves the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}\beta_t + b(t, X_t)\,\mathrm{d}t.$$

This equation is of the type that will be considered in Chap. 7 below, but in contrast with the statements of this chapter, we are not making any regularity assumption on the function b. It is remarkable that Girsanov's theorem allows one to construct solutions of stochastic differential equations without regularity conditions on the coefficients.

The Cameron–Martin formula We now specialize the preceding discussion to the case where $b(t,x)$ does not depend on x. We assume that $b(t,x) = g(t)$, where $g \in L^2(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+), \mathrm{d}t)$, and we also set, for every $t \geq 0$,

$$h(t) = \int_0^t g(s)\,\mathrm{d}s.$$

The set $\mathscr{H}$ of all functions h that can be written in this form is called the Cameron–Martin space. If $h \in \mathscr{H}$, we sometimes write $\dot{h} = g$ for the associated function in $L^2(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+), \mathrm{d}t)$ (this is the derivative of h in the sense of distributions).

As a special case of the previous discussion, under the probability measure

$$Q := D_\infty \cdot P = \exp\Big(\int_0^\infty g(s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^\infty g(s)^2 \mathrm{d}s\Big) \cdot P,$$

the process $\beta_t := B_t - h(t)$ is a Brownian motion. Hence, for every nonnegative measurable function Φ on $C(\mathbb{R}_+, \mathbb{R})$,

$$\begin{aligned} E_P[D_\infty\, \Phi((B_t)_{t\geq 0})] = E_Q[\Phi((B_t)_{t\geq 0})] &= E_Q[\Phi((\beta_t + h(t))_{t\geq 0})] \\ &= E_P[\Phi((B_t + h(t))_{t\geq 0})]. \end{aligned}$$

The equality between the two ends of the last display is the Cameron–Martin formula. In the next proposition, we write this formula in the special case of the canonical construction of Brownian motion on the Wiener space (see the end of Sect. 2.2).

Proposition 5.24 (Cameron–Martin formula) *Let $W(\mathrm{dw})$ be the Wiener measure on $C(\mathbb{R}_+, \mathbb{R})$, and let h be a function in the Cameron–Martin space $\mathscr{H}$. Then, for every nonnegative measurable function Φ on $C(\mathbb{R}_+, \mathbb{R})$,*

$$\int W(\mathrm{dw})\, \Phi(\mathrm{w} + h) = \int W(\mathrm{dw})\, \exp\Big(\int_0^\infty \dot{h}(s)\,\mathrm{dw}(s) - \frac{1}{2}\int_0^\infty \dot{h}(s)^2\,\mathrm{d}s\Big)\Phi(\mathrm{w}).$$

Remark The integral $\int_0^\infty \dot{h}(s)\,\mathrm{dw}(s)$ is a stochastic integral with respect to $\mathrm{w}(s)$ (which is a Brownian motion under $W(d\mathrm{w})$), but it can also be viewed as a Wiener integral since the function $\dot{h}(s)$ is deterministic. The Cameron–Martin formula can be established by Gaussian calculations that do not involve stochastic integrals or Girsanov's theorem (see e.g. Chapter 1 of [62]). Still it is instructive to derive this formula as a special case of Girsanov's theorem.

The Cameron–Martin formula gives a "quasi-invariance" property of Wiener measure under the translations by functions of the Cameron–Martin space: The image of Wiener measure $W(\mathrm{dw})$ under the mapping $\mathrm{w} \mapsto \mathrm{w} + h$ has a density with respect to $W(\mathrm{dw})$ and this density is the terminal value of the exponential martingale associated with the martingale $\int_0^t \dot{h}(s)\mathrm{dw}(s)$.

Law of hitting times for Brownian motion with drift Let B be a real Brownian motion with $B_0 = 0$, and for every $a > 0$, let $T_a := \inf\{t \geq 0 : B_t = a\}$. If $c \in \mathbb{R}$ is given, we aim at computing the law of the stopping time

$$U_a := \inf\{t \geq 0 : B_t + ct = a\}.$$

Of course, if $c = 0$, we have $U_a = T_a$, and the desired distribution is given by Corollary 2.22. Girsanov's theorem (or rather the Cameron–Martin formula) will allow us to derive the case where c is arbitrary from the special case $c = 0$.

Fix $t > 0$ and apply the Cameron–Martin formula with

$$\dot{h}(s) = c\,\mathbf{1}_{\{s \leq t\}} \quad , \quad h(s) = c(s \wedge t)\ ,$$

and, for every $\mathrm{w} \in C(\mathbb{R}_+, \mathbb{R})$,

$$\Phi(\mathrm{w}) = \mathbf{1}_{\{\max_{[0,t]} \mathrm{w}(s) \geq a\}}.$$

It follows that

$$\begin{aligned}
P(U_a \leq t) &= E[\Phi(B+h)] \\
&= E\Big[\Phi(B)\ \exp\Big(\int_0^\infty \dot{h}(s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^\infty \dot{h}(s)^2\,\mathrm{d}s\Big)\Big] \\
&= E[\mathbf{1}_{\{T_a \leq t\}}\ \exp(cB_t - \frac{c^2}{2}t))] \\
&= E[\mathbf{1}_{\{T_a \leq t\}}\ \exp(cB_{t \wedge T_a} - \frac{c^2}{2}(t \wedge T_a))] \\
&= E[\mathbf{1}_{\{T_a \leq t\}}\ \exp(ca - \frac{c^2}{2}T_a)]
\end{aligned}$$

$$= \int_0^t \mathrm{d}s \, \frac{a}{\sqrt{2\pi s^3}} \mathrm{e}^{-\frac{a^2}{2s}} \, \mathrm{e}^{ca - \frac{c^2}{2} s}$$
$$= \int_0^t \mathrm{d}s \, \frac{a}{\sqrt{2\pi s^3}} \mathrm{e}^{-\frac{1}{2s}(a-cs)^2},$$

where, in the fourth equality, we used the optional stopping theorem (Corollary 3.23) to write

$$E[\exp(cB_t - \frac{c^2}{2}t) \mid \mathscr{F}_{t\wedge T_a}] = \exp(cB_{t\wedge T_a} - \frac{c^2}{2}(t \wedge T_a)),$$

and we also made use of the explicit density of T_a given in Corollary 2.22. This calculation shows that the variable U_a has a density on $\mathbb{R}_+$ given by

$$\psi(s) = \frac{a}{\sqrt{2\pi s^3}} \mathrm{e}^{-\frac{1}{2s}(a-cs)^2}.$$

By integrating this density, we can verify that

$$P(U_a < \infty) = \begin{cases} 1 & \text{if } c \geq 0, \\ \mathrm{e}^{2ca} & \text{if } c \leq 0, \end{cases}$$

which may also be checked more easily by applying the optional stopping theorem to the continuous martingale $\exp(-2c(B_t + ct))$.

Exercises

In the following exercises, processes are defined on a probability space $(\Omega, \mathscr{F}, P)$ *equipped with a complete filtration* $(\mathscr{F}_t)_{t\in[0,\infty]}$.

Exercise 5.25 Let B be an $(\mathscr{F}_t)$-Brownian motion with $B_0 = 0$, and let H be an adapted process with continuous sample paths. Show that $\frac{1}{B_t}\int_0^t H_s \mathrm{d}B_s$ converges in probability when $t \to 0$ and determine the limit.

Exercise 5.26

1. Let B be a one-dimensional $(\mathscr{F}_t)$-Brownian motion with $B_0 = 0$. Let f be a twice continuously differentiable function on $\mathbb{R}$, and let g be a continuous function on $\mathbb{R}$. Verify that the process

$$X_t = f(B_t) \, \exp\Big(- \int_0^t g(B_s) \, \mathrm{d}s \Big)$$

is a semimartingale, and give its decomposition as the sum of a continuous local martingale and a finite variation process.

2. Prove that X is a continuous local martingale if and only if the function f satisfies the differential equation

$$f'' = 2gf.$$

3. From now on, we suppose in addition that g is nonnegative and vanishes outside a compact subinterval of $(0, \infty)$. Justify the existence and uniqueness of a solution f_1 of the equation $f'' = 2gf$ such that $f_1(0) = 1$ and $f_1'(0) = 0$. Let $a > 0$ and $T_a = \inf\{t \geq 0 : B_t = a\}$. Prove that

$$E\Big[\exp\Big(-\int_0^{T_a} g(B_s)\,\mathrm{d}s\Big)\Big] = \frac{1}{f_1(a)}.$$

Exercise 5.27 (*Stochastic calculus with the supremum*) *Preliminary question*. Let $m : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a continuous function such that $m(0) = 0$, and let $s : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be the monotone increasing function defined by

$$s(t) = \sup_{0 \leq r \leq t} m(r).$$

Show that, for every bounded Borel function h on $\mathbb{R}$ and every $t > 0$,

$$\int_0^t (s(r) - m(r))\, h(r)\,\mathrm{d}s(r) = 0.$$

(One may first observe that $\int \mathbf{1}_I(r)\,\mathrm{d}s(r) = 0$ for every open interval I that does not intersect $\{r \geq 0 : s(r) = m(r)\}$.)

1. Let M be a continuous local martingale such that $M_0 = 0$, and for every $t \geq 0$, let

$$S_t = \sup_{0 \leq r \leq t} M_r.$$

Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a twice continuously differentiable function. Justify the equality

$$\varphi(S_t) = \varphi(0) + \int_0^t \varphi'(S_s)\,\mathrm{d}S_s.$$

2. Show that

$$(S_t - M_t)\,\varphi(S_t) = \Phi(S_t) - \int_0^t \varphi(S_s)\,\mathrm{d}M_s$$

where $\Phi(x) = \int_0^x \varphi(y)\,dy$ for every $x \in \mathbb{R}$.

3. Infer that, for every $\lambda > 0$,

$$e^{-\lambda S_t} + \lambda(S_t - M_t)e^{-\lambda S_t}$$

is a continuous local martingale.

4. Let $a > 0$ and $T = \inf\{t \geq 0 : S_t - M_t = a\}$. We assume that $\langle M, M\rangle_\infty = \infty$ a.s. Show that $T < \infty$ a.s. and S_T is exponentially distributed with parameter $1/a$.

Exercise 5.28 Let B be an $(\mathscr{F}_t)$-Brownian motion started from 1. We fix $\varepsilon \in (0, 1)$ and set $T_\varepsilon = \inf\{t \geq 0 : B_t = \varepsilon\}$. We also let $\lambda > 0$ and $\alpha \in \mathbb{R}\backslash\{0\}$.

1. Show that $Z_t = (B_{t\wedge T_\varepsilon})^\alpha$ is a semimartingale and give its canonical decomposition as the sum of a continuous local martingale and a finite variation process.

2. Show that the process

$$Z_t = (B_{t\wedge T_\varepsilon})^\alpha \,\exp\Big(-\lambda \int_0^{t\wedge T_\varepsilon} \frac{\mathrm{d}s}{B_s^2}\Big)$$

is a continuous local martingale if α and λ satisfy a polynomial equation to be determined.

3. Compute

$$E\Big[\exp\Big(-\lambda \int_0^{T_\varepsilon} \frac{\mathrm{d}s}{B_s^2}\Big)\Big].$$

Exercise 5.29 Let $(X_t)_{t\geq 0}$ be a semimartingale. We assume that there exists an $(\mathscr{F}_t)$-Brownian motion $(B_t)_{t\geq 0}$ started from 0 and a continuous function $b : \mathbb{R} \longrightarrow \mathbb{R}$, such that

$$X_t = B_t + \int_0^t b(X_s)\,\mathrm{d}s.$$

1. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a twice continuously differentiable function on $\mathbb{R}$. Show that, for $F(X_t)$ to be a continuous local martingale, it suffices that F satisfies a second-order differential equation to be determined.

2. Give the solution of this differential equation which is such that $F(0) = 0$ and $F'(0) = 1$. In what follows, F stands for this particular solution, which can be written in the form $F(x) = \int_0^x \exp(-2\beta(y))\,\mathrm{d}y$, with a function β that will be determined in terms of b.

3. In this question only, we assume that b is integrable, i.e. $\int_{\mathbb{R}} |b(x)|\mathrm{d}x < \infty$.

(a) Show that the continuous local martingale $M_t = F(X_t)$ is a martingale.
(b) Show that $\langle M, M\rangle_\infty = \infty$ a.s.
(c) Infer that

$$\limsup_{t\to\infty} X_t = +\infty\ ,\ \liminf_{t\to\infty} X_t = -\infty\ ,\ \text{a.s.}$$

4. We come back to the general case. Let $c < 0$ and $d > 0$, and

$$T_c = \inf\{t \geq 0 : X_t \leq c\}\ ,\ T_d = \inf\{t \geq 0 : X_t \geq d\}\ .$$

Show that, on the event $\{T_c \wedge T_d = \infty\}$, the random variables $|B_{n+1} - B_n|$, for integers $n \geq 0$, are bounded above by a (deterministic) constant which does not depend on n. Infer that $P(T_c \wedge T_d = \infty) = 0$.

5. Compute $P(T_c < T_d)$ in terms of $F(c)$ of $F(d)$.

6. We assume that b vanishes on $(-\infty, 0]$ and that there exists a constant $\alpha > 1/2$ such that $b(x) \geq \alpha/x$ for every $x \geq 1$. Show that, for every $\varepsilon > 0$, one can choose $c < 0$ such that

$$P(T_n < T_c,\ \text{for every } n \geq 1) \geq 1 - \varepsilon.$$

Infer that $X_t \longrightarrow +\infty$ as $t \to \infty$, a.s. (*Hint*: Observe that the continuous local martingale $M_{t\wedge T_c}$ is bounded.)

7. Suppose now $b(x) = 1/(2x)$ for every $x \geq 1$. Show that

$$\liminf_{t\to\infty}\ X_t = -\infty\ ,\ \text{a.s.}$$

Exercise 5.30 (*Lévy area*) Let $(X_t, Y_t)_{t\geq 0}$ be a two-dimensional $(\mathscr{F}_t)$-Brownian motion started from 0. We set, for every $t \geq 0$:

$$\mathscr{A}_t = \int_0^t X_s\,\mathrm{d}Y_s - \int_0^t Y_s\,\mathrm{d}X_s \qquad (\textit{Lévy's area}).$$

1. Compute $\langle \mathscr{A}, \mathscr{A}\rangle_t$ and infer that $(\mathscr{A}_t)_{t\geq 0}$ is a square-integrable (true) martingale.

2. Let $\lambda > 0$. Justify the equality

$$E[\mathrm{e}^{\mathrm{i}\lambda\mathscr{A}_t}] = E[\cos(\lambda\mathscr{A}_t)].$$

3. Let f be a twice continuously differentiable function on $\mathbb{R}_+$. Give the canonical decomposition of the semimartingales

$$Z_t = \cos(\lambda \mathscr{A}_t),$$
$$W_t = -\frac{f'(t)}{2}(X_t^2 + Y_t^2) + f(t).$$

Verify that $\langle Z, W \rangle_t = 0$.

4. Show that, for the process $Z_t\, e^{W_t}$ to be a continuous local martingale, it suffices that f solves the differential equation

$$f''(t) = f'(t)^2 - \lambda^2 .$$

5. Let $r > 0$. Verify that the function

$$f(t) = -\log \cosh(\lambda(r-t))$$

solves the differential equation of question **4.** and derive the formula

$$E[\mathrm{e}^{\mathrm{i}\lambda \mathscr{A}_r}] = \frac{1}{\cosh(\lambda r)}.$$

Exercise 5.31 (*Squared Bessel processes*) Let B be an $(\mathscr{F}_t)$-Brownian motion started from 0, and let X be a continuous semimartingale. We assume that X takes values in $\mathbb{R}_+$, and is such that, for every $t \geq 0$,

$$X_t = x + 2\int_0^t \sqrt{X_s}\, \mathrm{d}B_s + \alpha\, t$$

where x and α are nonnegative real numbers.

1. Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a continuous function, and let φ be a twice continuously differentiable function on $\mathbb{R}_+$, taking **strictly positive** values, which solves the differential equation

$$\varphi'' = 2f\,\varphi$$

and satisfies $\varphi(0) = 1$ and $\varphi'(1) = 0$. Observe that the function φ must then be decreasing over the interval $[0, 1]$.

We set

$$u(t) = \frac{\varphi'(t)}{2\varphi(t)}$$

for every $t \geq 0$. Verify that we have, for every $t \geq 0$,

$$u'(t) + 2\,u(t)^2 = f(t),$$

then show that, for every $t \geq 0$,

$$u(t)X_t - \int_0^t f(s)\,X_s\,\mathrm{d}s = u(0)x + \int_0^t u(s)\,\mathrm{d}X_s - 2\int_0^t u(s)^2 X_s\,\mathrm{d}s.$$

We set

$$Y_t = u(t)X_t - \int_0^t f(s)\,X_s\,\mathrm{d}s.$$

2. Show that, for every $t \geq 0$,

$$\varphi(t)^{-\alpha/2}\,\mathrm{e}^{Y_t} = \mathscr{E}(N)_t$$

where $\mathscr{E}(N)_t = \exp(N_t - \frac{1}{2}\langle N, N\rangle_t)$ denotes the exponential martingale associated with the continuous local martingale

$$N_t = u(0)x + 2\int_0^t u(s)\,\sqrt{X_s}\,\mathrm{d}B_s.$$

3. Infer from the previous question that

$$E\Big[\exp\Big(-\int_0^1 f(s)\,X_s\,\mathrm{d}s\Big)\Big] = \varphi(1)^{\alpha/2}\,\exp(\frac{x}{2}\varphi'(0)).$$

4. Let $\lambda > 0$. Show that

$$E\Big[\exp\Big(-\lambda\int_0^1 X_s\,\mathrm{d}s\Big)\Big] = (\cosh(\sqrt{2\lambda}))^{-\alpha/2}\,\exp(-\frac{x}{2}\,\sqrt{2\lambda}\,\tanh(\sqrt{2\lambda})).$$

5. Show that, if $\beta = (\beta_t)_{t\geq 0}$ is a real Brownian motion started from y, one has, for every $\lambda > 0$,

$$E\Big[\exp\Big(-\lambda\int_0^1 \beta_s^2\,\mathrm{d}s\Big)\Big] = (\cosh(\sqrt{2\lambda}))^{-1/2}\,\exp(-\frac{y^2}{2}\,\sqrt{2\lambda}\,\tanh(\sqrt{2\lambda})).$$

Exercise 5.32 (*Tanaka's formula and local time*) Let B be an $(\mathscr{F}_t)$-Brownian motion started from 0. For every $\varepsilon > 0$, we define a function $g_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ by setting $g_\varepsilon(x) = \sqrt{\varepsilon + x^2}$.

1. Show that

$$g_\varepsilon(B_t) = g_\varepsilon(0) + M_t^\varepsilon + A_t^\varepsilon$$

where M^ε is a square integrable continuous martingale that will be identified in the form of a stochastic integral, and A^ε is an increasing process.

2. We set $\text{sgn}(x) = \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x<0\}}$ for every $x \in \mathbb{R}$. Show that, for every $t \geq 0$,

$$M_t^\varepsilon \xrightarrow[\varepsilon\to 0]{L^2} \int_0^t \text{sgn}(B_s)\,\mathrm{d}B_s.$$

Infer that there exists an increasing process L such that, for every $t \geq 0$,

$$|B_t| = \int_0^t \text{sgn}(B_s)\,\mathrm{d}B_s + L_t\ .$$

3. Observing that $A_t^\varepsilon \longrightarrow L_t$ when $\varepsilon \to 0$, show that, for every $\delta > 0$, for every choice of $0 < u < v$, the condition ($|B_t| \geq \delta$ for every $t \in [u, v]$) a.s. implies that $L_v = L_u$. Infer that the function $t \mapsto L_t$ is a.s. constant on every connected component of the open set $\{t \geq 0 : B_t \neq 0\}$.

4. We set $\beta_t = \int_0^t \text{sgn}(B_s)\,\mathrm{d}B_s$ for every $t \geq 0$. Show that $(\beta_t)_{t\geq 0}$ is an $(\mathscr{F}_t)$-Brownian motion started from 0.

5. Show that $L_t = \sup_{s\leq t}(-\beta_s)$, a.s. (In order to derive the bound $L_t \leq \sup_{s\leq t}(-\beta_s)$, one may consider the last zero of B before time t, and use question **3.**) Give the law of L_t.

6. For every $\varepsilon > 0$, we define two sequences of stopping times $(S_n^\varepsilon)_{n\geq 1}$ and $(T_n^\varepsilon)_{n\geq 1}$, by setting

$$S_1^\varepsilon = 0\ , \quad T_1^\varepsilon = \inf\{t \geq 0 : |B_t| = \varepsilon\}$$

and then, by induction,

$$S_{n+1}^\varepsilon = \inf\{t \geq T_n^\varepsilon : B_t = 0\}\ , \quad T_{n+1}^\varepsilon = \inf\{t \geq S_{n+1}^\varepsilon : |B_t| = \varepsilon\}.$$

For every $t \geq 0$, we set $N_t^\varepsilon = \sup\{n \geq 1 : T_n^\varepsilon \leq t\}$, where $\sup \varnothing = 0$. Show that

$$\varepsilon N_t^\varepsilon \xrightarrow[\varepsilon\to 0]{L^2} L_t.$$

(One may observe that

$$L_t + \int_0^t \Big(\sum_{n=1}^\infty \mathbf{1}_{[S_n^\varepsilon, T_n^\varepsilon]}(s)\Big)\,\text{sgn}(B_s)\,\mathrm{d}B_s = \varepsilon\, N_t^\varepsilon + r_t^\varepsilon$$

where the "remainder" r_t^ε satisfies $|r_t^\varepsilon| \leq \varepsilon$.)

7. Show that $N_t^1/\sqrt{t}$ converges in law as $t \to \infty$ to $|U|$, where U is $\mathscr{N}(0,1)$-distributed.

(*Many results of Exercise 5.32 are reproved and generalized in Chap. 8.*)

Exercise 5.33 (*Study of multidimensional Brownian motion*) Let $B_t = (B_t^1, B_t^2, \ldots, B_t^N)$ be an N-dimensional $(\mathscr{F}_t)$-Brownian motion started from $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. We suppose that $N \geq 2$.

1. Verify that $|B_t|^2$ is a continuous semimartingale, and that the martingale part of $|B_t|^2$ is a true martingale.

2. We set

$$\beta_t = \sum_{i=1}^{N} \int_0^t \frac{B_s^i}{|B_s|} \, \mathrm{d}B_s^i$$

with the convention that $\frac{B_s^i}{|B_s|} = 0$ if $|B_s| = 0$. Justify the definition of the stochastic integrals appearing in the definition of β_t, then show that the process $(\beta_t)_{t\geq 0}$ is an $(\mathscr{F}_t)$-Brownian motion started from 0.

3. Show that

$$|B_t|^2 = |x|^2 + 2\int_0^t |B_s| \, \mathrm{d}\beta_s + N\,t.$$

4. From now on, we assume that $x \neq 0$. Let $\varepsilon \in (0, |x|)$ and $T_\varepsilon = \inf\{t \geq 0 : |B_t| \leq \varepsilon\}$. We set $f(a) = \log a$ if $N = 2$, and $f(a) = a^{2-N}$ if $N \geq 3$, for every $a > 0$. Verify that $f(|B_{t\wedge T_\varepsilon}|)$ is a continuous local martingale.

5. Let $R > |x|$ and $S_R = \inf\{t \geq 0 : |B_t| \geq R\}$. Show that

$$P(T_\varepsilon < S_R) = \frac{f(R) - f(|x|)}{f(R) - f(\varepsilon)}.$$

Observing that $P(T_\varepsilon < S_R) \longrightarrow 0$ when $\varepsilon \to 0$, show that $B_t \neq 0$ for every $t \geq 0$, a.s.

6. Show that, a.s., for every $t \geq 0$,

$$|B_t| = |x| + \beta_t + \frac{N-1}{2} \int_0^t \frac{\mathrm{d}s}{|B_s|}.$$

7. We assume that $N \geq 3$. Show that $|B_t| \longrightarrow \infty$ when $t \to \infty$, a.s. (*Hint*: Observe that $|B_t|^{2-N}$ is a nonnegative supermartingale.)

8. We assume $N = 3$. Using the form of the Gaussian density, verify that the collection of random variables $(|B_t|^{-1})_{t\geq 0}$ is bounded in L^2. Show that $(|B_t|^{-1})_{t\geq 0}$ is a continuous local martingale but is not a (true) martingale.

(*Chapter 7 presents a slightly different approach to the results of this exercise, see in particular Proposition 7.16.*)

Exercise 5.34 (*Application of the Cameron–Martin formula*) Let B be an $(\mathscr{F}_t)$-Brownian motion started from 0. We set $B^*_t = \sup\{|B_s| : s \le t\}$ for every $t \ge 0$.

1. Set $U_1 = \inf\{t \ge 0 : |B_t| = 1\}$ and $V_1 = \inf\{t \ge U_1 : B_t = 0\}$. Justify the equality $P(B^*_{V_1} < 2) = 1/2$, and then show that one can find two constants $\alpha > 0$ and $\gamma > 0$ such that

$$P(V_1 \ge \alpha,\, B^*_{V_1} < 2) = \gamma > 0.$$

2. Show that, for every integer $n \ge 1$, $P(B^*_{n\alpha} < 2) \ge \gamma^n$. *Hint*: Construct a suitable sequence $V_1, V_2, \ldots$ of stopping times such that, for every $n \ge 2$,

$$P(V_n \ge n\alpha\ , B^*_{V_n} < 2) \ge \gamma\, P(V_{n-1} \ge (n-1)\alpha\ , B^*_{V_{n-1}} < 2).$$

Conclude that, for every $\varepsilon > 0$ and $t \ge 0$, $P(B^*_t \le \varepsilon) > 0$.

3. Let h be a twice continuously differentiable function on $\mathbb{R}_+$ such that $h(0) = 0$, and let $K > 0$. Via a suitable application of Itô's formula, show that there exists a constant A such that, for every $\varepsilon > 0$,

$$\Big|\int_0^K h'(s)\,\mathrm{d}B_s\Big| \le A\,\varepsilon \quad \text{a.s. on the event } \{B^*_K \le \varepsilon\}.$$

4. We set $X_t = B_t - h(t)$ and $X^*_t = \sup\{|X_s| : s \le t\}$. Infer from question **3.** that

$$\lim_{\varepsilon \downarrow 0} \frac{P(X^*_K \le \varepsilon)}{P(B^*_K \le \varepsilon)} = \exp\Big(-\frac{1}{2}\int_0^K h'(s)^2\,\mathrm{d}s\Big).$$

Notes and Comments

The reader who wishes to learn more about the topics of this chapter is strongly advised to look at the excellent books by Karatzas and Shreve [49], Revuz and Yor [70] and Rogers and Williams [72]. A more concise introduction to stochastic integration can also be found in Chung and Williams [10].

Stochastic integrals with respect to Brownian motion were introduced by Itô [36] in 1944. His motivation was to give a rigorous approach to the stochastic differential equations that govern diffusion processes. Doob [15] suggested to study stochastic integrals as martingales. Several authors then contributed to the theory, including Kunita and Watanabe [50] and Meyer [60]. We have chosen to restrict our attention to stochastic integration with respect to continuous semimartingales. The reader interested in the more general case of semimartingales with jumps can consult the treatise of Dellacherie and Meyer [14] and the more recent books of Protter [63] or Jacod and Shiryaev [44]. Itô's formula was derived in [40] for processes that are stochastic integrals with respect to Brownian motions, and in our general context it appeared in the work of Kunita and Watanabe [50]. Theorem 5.12, at least in the case $d = 1$, is usually attributed to Lévy, although it seems difficult to find this statement in Lévy's work (see however [54, Chapitre III]). Theorem 5.13 showing that any continuous martingale can be written as a time-changed Brownian motion is due to Dambis [11] and Dubins–Schwarz [17]. The Burkholder–Davis–Gundy inequalities appear in [7], see also the expository article of Burkholder [6] for the history of these famous inequalities. Theorem 5.18 goes back to Itô [39] – in the different form of the chaos decomposition of Wiener functionals – and was a great success of the theory of stochastic integration. This theorem and its numerous extensions have found many applications in the area of mathematical finance. Girsanov's theorem appears in [29] in 1960, whereas the Cameron–Martin formula goes back to [8] in 1944. Applications of Girsanov's theorem to stochastic differential equations are developed in the book [77] of Stroock and Varadhan. Exercise 5.30 is concerned with the so-called Lévy area of planar Brownian motion, which was studied by Lévy [53, 54] with a different definition. Exercise 5.31 is inspired by Pitman and Yor [67].