Chapter 17
De Rham Cohomology

In Chapter 14 we defined closed and exact forms: a smooth differential form ω is *closed* if $d\omega = 0$, and *exact* if it can be written $\omega = d\eta$. Because $d \circ d = 0$, every exact form is closed. In this chapter, we explore the implications of the converse question: Is every closed form exact? The answer, in general, is no: in Example 11.48, for instance, we saw a 1-form on the punctured plane that is closed but not exact; the failure of exactness seemed to be a consequence of the "hole" in the center of the domain. For higher-degree forms, the question of which closed forms are exact depends on subtle topological properties of the manifold, connected with the existence of "holes" of higher dimensions. Making this dependence quantitative leads to a new set of invariants of smooth manifolds, called the *de Rham cohomology groups*, which are the subject of this chapter.

Knowledge of which closed forms are exact has many important consequences. For example, Stokes's theorem implies that if ω is exact, then the integral of ω over any compact submanifold without boundary is zero. Proposition 11.42 showed that a smooth 1-form is conservative if and only if it is exact.

We begin by defining the de Rham cohomology groups and proving some of their basic properties, including diffeomorphism invariance. Then we prove an important generalization of this fact: the de Rham groups are in fact *homotopy invariants*, which implies in particular that they are topological invariants. Next, after computing the de Rham groups in some simple cases, we state a general theorem, called the *Mayer–Vietoris theorem*, that expresses the de Rham groups of a manifold in terms of those of its open subsets. Using this, we compute all of the de Rham groups of spheres and the top-degree groups of compact manifolds. Then we give an important application of these ideas to topology: there is a homotopically invariant integer associated with any continuous map between connected, compact, oriented, smooth manifolds of the same dimension, called the *degree* of the map.

At the end of the chapter, we prove the Mayer–Vietoris theorem.

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The de Rham Cohomology Groups

In Chapter 11, we studied the closed 1-form

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}, \tag{17.1}$$

and showed that it is not exact on $\mathbb{R}^2 \smallsetminus \{0\}$, but it is exact on some smaller domains such as the right half-plane $H = \{(x, y) : x > 0\}$, where it is equal to $d\theta$ (see Example 11.48).

As we will see in this chapter, that behavior is typical: closed forms are always *locally* exact, so whether a given closed form is exact depends on the global shape of the domain. To capture this dependence, we make the following definitions.

Let M be a smooth manifold with or without boundary, and let p be a nonnegative integer. Because $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ is linear, its kernel and image are linear subspaces. We define

$$\mathcal{Z}^p(M) = \operatorname{Ker}\Big(d\colon \Omega^p(M) \to \Omega^{p+1}(M)\Big) = \{\text{closed } p\text{-forms on } M\},$$

$$\mathcal{B}^p(M) = \operatorname{Im}\Big(d\colon \Omega^{p-1}(M) \to \Omega^p(M)\Big) = \{\text{exact } p\text{-forms on } M\}.$$

By convention, we consider $\Omega^p(M)$ to be the zero vector space when $p < 0$ or $p > n = \dim M$, so that, for example, $\mathcal{B}^0(M) = 0$ and $\mathcal{Z}^n(M) = \Omega^n(M)$.

The fact that every exact form is closed implies that $\mathcal{B}^p(M) \subseteq \mathcal{Z}^p(M)$. Thus, it makes sense to define the ***de Rham cohomology group in degree p*** (or the ***pth de Rham group***) ***of M*** to be the quotient vector space

$$H^p_{\mathrm{dR}}(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)}.$$

(It is a real vector space, and thus in particular a group under vector addition. Perhaps "de Rham cohomology space" would be a more appropriate term, but because most other cohomology theories produce only groups it is traditional to use the term *group* in this context as well, bearing in mind that these "groups" are actually real vector spaces.) It is clear that $H^p_{\mathrm{dR}}(M) = 0$ for $p < 0$ or $p > \dim M$, because $\Omega^p(M) = 0$ in those cases. For $0 \le p \le n$, the definition implies that $H^p_{\mathrm{dR}}(M) = 0$ if and only if every closed p-form on M is exact.

Example 17.1. The fact that there is a closed 1-form on $\mathbb{R}^2 \smallsetminus \{0\}$ that is not exact means that $H^1_{\mathrm{dR}}\big(\mathbb{R}^2 \smallsetminus \{0\}\big) \neq 0$ (see Example 11.48). On the other hand, the Poincaré lemma for 1-forms (Theorem 11.49) implies that $H^1_{\mathrm{dR}}(U) = 0$ for any star-shaped open subset $U \subseteq \mathbb{R}^n$. //

The first order of business is to show that the de Rham groups are diffeomorphism invariants. For any closed p-form ω on M, we let $[\omega]$ denote the equivalence class of ω in $H^p_{\mathrm{dR}}(M)$, called the ***cohomology class of ω***. If $[\omega] = [\omega']$ (that is, if ω and ω' differ by an exact form), we say that ω and ω' are ***cohomologous***.

Proposition 17.2 (Induced Cohomology Maps). *For any smooth map $F\colon M \to N$ between smooth manifolds with or without boundary, the pullback $F^*\colon \Omega^p(N) \to \Omega^p(M)$ carries $\mathcal{Z}^p(N)$ into $\mathcal{Z}^p(M)$ and $\mathcal{B}^p(N)$ into $\mathcal{B}^p(M)$. It thus descends to a linear map, still denoted by F^*, from $H^p_{\mathrm{dR}}(N)$ to $H^p_{\mathrm{dR}}(M)$, called the* ***induced cohomology map****. It has the following properties*:

(a) *If $G\colon N \to P$ is another smooth map, then*

$$(G \circ F)^* = F^* \circ G^*\colon H^p_{\mathrm{dR}}(P) \to H^p_{\mathrm{dR}}(M).$$

(b) *If* Id *denotes the identity map of M, then* Id* *is the identity map of $H^p_{\mathrm{dR}}(M)$.*

Proof. If ω is closed, then $d(F^*\omega) = F^*(d\omega) = 0$, so $F^*\omega$ is also closed. If $\omega = d\eta$ is exact, then $F^*\omega = F^*(d\eta) = d(F^*\eta)$, which is also exact. Therefore, F^* maps $\mathcal{Z}^p(N)$ into $\mathcal{Z}^p(M)$ and $\mathcal{B}^p(N)$ into $\mathcal{B}^p(M)$. The induced cohomology map $F^*\colon H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$ is defined in the obvious way: for a closed p-form ω, let

$$F^*[\omega] = [F^*\omega].$$

If $\omega' = \omega + d\eta$, then $[F^*\omega'] = [F^*\omega + d(F^*\eta)] = [F^*\omega]$, so this map is well defined. Properties (a) and (b) follow immediately from the analogous properties of the pullback map on forms. □

The next two corollaries are immediate.

Corollary 17.3 (Functoriality). *For any integer p, the assignment $M \mapsto H^p_{\mathrm{dR}}(M)$, $F \mapsto F^*$ is a contravariant functor from the category of smooth manifolds with boundary to the category of real vector spaces.* □

Corollary 17.4 (Diffeomorphism Invariance of de Rham Cohomology). *Diffeomorphic smooth manifolds (with or without boundary) have isomorphic de Rham cohomology groups.* □

Elementary Computations

The direct computation of the de Rham groups is not easy in general. However, there are a number of special cases that can be easily computed by various techniques. In this section, we describe a few of those cases. We begin with disjoint unions.

Proposition 17.5 (Cohomology of Disjoint Unions). *Let $\{M_j\}$ be a countable collection of smooth n-manifolds with or without boundary, and let $M = \coprod_j M_j$. For each p, the inclusion maps $\iota_j\colon M_j \hookrightarrow M$ induce an isomorphism from $H^p_{\mathrm{dR}}(M)$ to the direct product space $\prod_j H^p_{\mathrm{dR}}(M_j)$.*

Proof. The pullback maps $\iota_j^*\colon \Omega^p(M) \to \Omega^p(M_j)$ already induce an isomorphism from $\Omega^p(M)$ to $\prod_j \Omega^p(M_j)$, namely

$$\omega \mapsto \left(\iota_1^*\omega, \iota_2^*\omega, \dots\right) = \left(\omega|_{M_1}, \omega|_{M_2}, \dots\right).$$

This map is injective because any smooth p-form whose restriction to each M_j is zero must itself be zero, and it is surjective because giving an arbitrary smooth p-form on each M_j defines one on M. □

Because of this proposition, each de Rham group of a disconnected manifold is just the direct product of the corresponding groups of its components. Thus, we can concentrate henceforth on computing the de Rham groups of connected manifolds.

Next we give an explicit characterization of de Rham cohomology in degree zero.

Proposition 17.6 (Cohomology in Degree Zero). *If M is a connected smooth manifold with or without boundary, then $H^0_{\mathrm{dR}}(M)$ is equal to the space of constant functions and is therefore 1-dimensional.*

Proof. Because there are no (-1)-forms, $\mathcal{B}^0(M) = 0$. A closed 0-form is a smooth real-valued function f such that $df = 0$, and since M is connected, this is true if and only if f is constant. Therefore, $H^0_{\mathrm{dR}}(M) = \mathcal{Z}^0(M) = \{\text{constants}\}$. □

Corollary 17.7 (Cohomology of Zero-Manifolds). *Suppose M is a manifold of dimension 0. Then $H^0_{\mathrm{dR}}(M)$ is a direct product of 1-dimensional vector spaces, one for each point of M, and all other de Rham cohomology groups of M are zero.*

Proof. The statement about $H^0_{\mathrm{dR}}(M)$ follows from Propositions 17.5 and 17.6, and the cohomology groups in nonzero degrees vanish for dimensional reasons. □

Homotopy Invariance

In this section we present a profound generalization of Corollary 17.4, one surprising consequence of which is that the de Rham cohomology groups are actually *topological* invariants. In fact, they are something much more: they are ***homotopy invariants***, which means that homotopy equivalent manifolds have isomorphic de Rham groups. (See p. 614 for the definition of homotopy equivalence.)

The underlying fact that allows us to prove the homotopy invariance of de Rham cohomology is that homotopic smooth maps induce the same cohomology map. To motivate the proof, suppose $F, G\colon M \to N$ are smooth maps, and let us think about what it means to prove that $F^* = G^*$. Given a closed p-form ω on N, we need somehow to produce a $(p-1)$-form η on M such that

$$G^*\omega - F^*\omega = d\eta,$$

from which it follows that $G^*[\omega] - F^*[\omega] = [d\eta] = 0$. One might hope to construct η in a systematic way, resulting in a map h from closed p-forms on N to $(p-1)$-forms on M that satisfies

$$d(h\omega) = G^*\omega - F^*\omega. \tag{17.2}$$

Instead of defining $h\omega$ only when ω is closed, it turns out to be far simpler to define a map h from the space of *all* smooth p-forms on N to the space of smooth

$(p-1)$-forms on M. Such a map cannot satisfy (17.2), but instead we will find a family of such maps, one for each p, satisfying

$$d(h\omega) + h(d\omega) = G^*\omega - F^*\omega. \tag{17.3}$$

This implies (17.2) when ω is closed.

In general, if $F, G\colon M \to N$ are smooth maps, a collection of linear maps $h\colon \Omega^p(N) \to \Omega^{p-1}(M)$ such that (17.3) is satisfied for all ω is called a ***homotopy operator between F^* and G^****. (The term ***cochain homotopy*** is used frequently in the algebraic topology literature.) The next proposition follows immediately from the argument in the preceding paragraph.

Proposition 17.8. *Suppose M and N are smooth manifolds with or without boundary. If $F, G\colon M \to N$ are smooth maps and there exists a homotopy operator between the pullback maps F^* and G^*, then the induced cohomology maps $F^*, G^*\colon H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$ are equal.* □

The key to our proof of homotopy invariance is to construct a homotopy operator first in the following special case. Let M be a smooth manifold with or without boundary, and for each $t \in I$, let $i_t\colon M \to M \times I$ be the map

$$i_t(x) = (x, t).$$

If M has empty boundary, then $M \times I$ is a smooth manifold with boundary, and all of the results above apply to it. But if $\partial M \neq \varnothing$, then $M \times I$ has to be considered as a smooth manifold with corners. It is straightforward to check that the definitions of the de Rham groups and induced homomorphisms make perfectly good sense on manifolds with corners, and Proposition 17.2 is valid in that context as well.

Lemma 17.9 (Existence of a Homotopy Operator). *For any smooth manifold M with or without boundary, there exists a homotopy operator between the two maps $i_0^*, i_1^*\colon \Omega^*(M \times I) \to \Omega^*(M)$.*

Proof. For each p, we need to define a linear map $h\colon \Omega^p(M \times I) \to \Omega^{p-1}(M)$ such that

$$h(d\omega) + d(h\omega) = i_1^*\omega - i_0^*\omega. \tag{17.4}$$

Let s denote the standard coordinate on $\mathbb{R}$, and let S be the vector field on $M \times \mathbb{R}$ given by $S_{(q,s)} = \big(0, \partial/\partial s|_s\big)$ under the usual identification $T_{(q,s)}M \leftrightarrow T_qM \times T_s\mathbb{R}$. Given a smooth p-form ω on $M \times I$, define $h\omega \in \Omega^{p-1}(M)$ by

$$h\omega = \int_0^1 i_t^*(S \lrcorner\, \omega)\, dt.$$

More specifically, for any $q \in M$, this means

$$(h\omega)_q = \int_0^1 i_t^*\left((S \lrcorner\, \omega)_{(q,t)}\right) dt,$$

where the integrand is interpreted as a function of t with values in the vector space $\Lambda^{p-1}(T_q^*M)$. On any smooth coordinate domain $U \subseteq M$, the components of the integrand are smooth functions of $(q,t) \in U \times I$, so the integral defines a smooth $(p-1)$-form on M. We can compute $d(h\omega)$ at any point by differentiating under the integral sign in local coordinates, which yields

$$d(h\omega) = \int_0^1 d\left(i_t^*(S \lrcorner \omega)\right) dt.$$

Therefore, using Cartan's magic formula, we obtain

$$\begin{aligned} h(d\omega) + d(h\omega) &= \int_0^1 \left(i_t^*(S \lrcorner d\omega) + d\left(i_t^*(S \lrcorner \omega)\right)\right) dt \\ &= \int_0^1 \left(i_t^*(S \lrcorner d\omega) + i_t^* d(S \lrcorner \omega)\right) dt \\ &= \int_0^1 i_t^*(\mathcal{L}_S \omega)\, dt. \end{aligned} \tag{17.5}$$

To simplify this last expression, we use the flow of S on $M \times \mathbb{R}$. (If M has nonempty boundary, note that S is tangent to $\partial(M \times \mathbb{R}) = \partial M \times \mathbb{R}$, so Theorem 9.34 applies.) The flow is given explicitly by $\theta_t(q,s) = (q, t+s)$, so S is complete. It follows that we can write $i_t = \theta_t \circ i_0$, and therefore by Proposition 12.36,

$$i_t^*(\mathcal{L}_S \omega) = i_0^*\left(\theta_t^*(\mathcal{L}_S \omega)\right) = i_0^*\left(\frac{d}{dt}(\theta_t^* \omega)\right) = \frac{d}{dt} i_0^*(\theta_t^* \omega) = \frac{d}{dt} i_t^* \omega.$$

Inserting this into (17.5) and applying the fundamental theorem of calculus, we obtain (17.4). □

Proposition 17.10. *Suppose M and N are smooth manifolds with or without boundary, and $F, G\colon M \to N$ are homotopic smooth maps. For every p, the induced cohomology maps $F^*, G^*\colon H^p_{dR}(N) \to H^p_{dR}(M)$ are equal.*

Proof. The preceding lemma implies that the two cohomology maps i_0^* and i_1^* from $H^p_{dR}(M \times I)$ to $H^p_{dR}(M)$ are equal. By Theorem 9.28, there is a smooth homotopy $H\colon M \times I \to N$ from F to G. Because $F = H \circ i_0$ and $G = H \circ i_1$ (see Fig. 17.1),Proposition 17.2 implies

$$F^* = (H \circ i_0)^* = i_0^* \circ H^* = i_1^* \circ H^* = (H \circ i_1)^* = G^*. \qquad \square$$

The next theorem is the main result of this section.

Theorem 17.11 (Homotopy Invariance of de Rham Cohomology). *If M and N are homotopy equivalent smooth manifolds with or without boundary, then $H^p_{dR}(M) \cong H^p_{dR}(N)$ for each p. The isomorphisms are induced by any smooth homotopy equivalence $F\colon M \to N$.*

Proof. Suppose $F\colon M \to N$ is a homotopy equivalence, with homotopy inverse $G\colon N \to M$. By the Whitney approximation theorem (Theorem 6.26 or 9.27), there are smooth maps $\widetilde{F}\colon M \to N$ homotopic to F and $\widetilde{G}\colon N \to M$ homotopic to G.

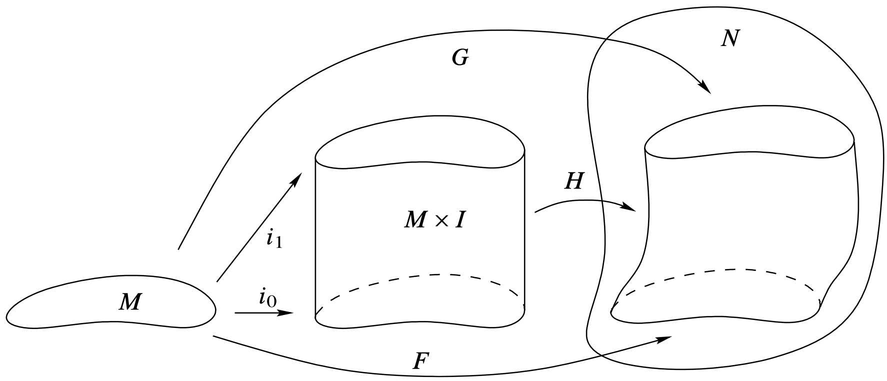

Fig. 17.1 Homotopic maps

Because homotopy is preserved by composition, it follows that $\widetilde{F} \circ \widetilde{G} \simeq F \circ G \simeq \mathrm{Id}_N$ and $\widetilde{G} \circ \widetilde{F} \simeq G \circ F \simeq \mathrm{Id}_M$, so $\widetilde{F}$ and $\widetilde{G}$ are homotopy inverses of each other.

Proposition 17.10 shows that, on cohomology,

$$\widetilde{F}^* \circ \widetilde{G}^* = \left(\widetilde{G} \circ \widetilde{F}\right)^* = (\mathrm{Id}_M)^* = \mathrm{Id}_{H^p_{\mathrm{dR}}(M)}.$$

The same argument shows that $\widetilde{G}^* \circ \widetilde{F}^*$ is also the identity, so $\widetilde{F}^*\colon H^p_{\mathrm{dR}}(N) \to H^p_{\mathrm{dR}}(M)$ is an isomorphism. □

Because every homeomorphism is a homotopy equivalence, the next corollary is immediate.

Corollary 17.12 (Topological Invariance of de Rham Cohomology). *The de Rham cohomology groups are topological invariants: if M and N are homeomorphic smooth manifolds with or without boundary, then their de Rham cohomology groups are isomorphic.* □

This result is remarkable, because the definition of the de Rham groups of M is intimately tied up with its smooth structure, and we had no reason to expect that different differentiable structures on the same topological manifold should give rise to the same de Rham groups.

Computations Using Homotopy Invariance

We can use homotopy invariance to compute a number of de Rham groups. We begin with the simplest case of homotopy equivalence. A topological space X is said to be ***contractible*** if the identity map of X is homotopic to a constant map.

Theorem 17.13 (Cohomology of Contractible Manifolds). *If M is a contractible smooth manifold with or without boundary, then $H^p_{\mathrm{dR}}(M) = 0$ for $p \geq 1$.*

Proof. The assumption means there is some point $q \in M$ such that the identity map of M is homotopic to the constant map $c_q\colon M \to M$ sending all of M to q.

If $\iota_q\colon \{q\} \hookrightarrow M$ denotes the inclusion map, it follows that $c_q \circ \iota_q = \mathrm{Id}_{\{q\}}$ and $\iota_q \circ c_q \simeq \mathrm{Id}_M$, so ι_q is a homotopy equivalence. The result then follows from the homotopy invariance of H^p_{dR} together with the obvious fact that $H^p_{\mathrm{dR}}(\{q\}) = 0$ for $p \geq 1$ because $\{q\}$ is a 0-manifold. □

In Theorem 11.49, we showed that every closed 1-form on a star-shaped open subset of $\mathbb{R}^n$ is exact. (Recall that a subset $U \subseteq \mathbb{R}^n$ is said to be *star-shaped* if there is a point $c \in U$ such that for every $x \in U$, the line segment from c to x is entirely contained in U.) The next theorem is a generalization of that result to forms of all degrees. Despite the apparent specialness of star-shaped domains, this theorem is one of the most important facts about de Rham cohomology.

Theorem 17.14 (The Poincaré Lemma). *If U is a star-shaped open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, then $H^p_{\mathrm{dR}}(U) = 0$ for $p \geq 1$.*

Proof. If U is star-shaped with respect to c, then it is contractible by the following straight-line homotopy:

$$H(x,t) = c + t(x - c).$$ □

Corollary 17.15 (Local Exactness of Closed Forms). *Let M be a smooth manifold with or without boundary. Each point of M has a neighborhood on which every closed form is exact.*

Proof. Every point of M has a neighborhood diffeomorphic to an open ball in $\mathbb{R}^n$ or an open half-ball in $\mathbb{H}^n$, each of which is star-shaped. The result follows from the Poincaré lemma and the diffeomorphism invariance of de Rham cohomology. □

Corollary 17.16 (Cohomology of Euclidean Spaces and Half-Spaces). *For any integers $n \geq 0$ and $p \geq 1$, $H^p_{\mathrm{dR}}(\mathbb{R}^n) = 0$ and $H^p_{\mathrm{dR}}(\mathbb{H}^n) = 0$.*

Proof. Both $\mathbb{R}^n$ and $\mathbb{H}^n$ are star-shaped. □

Another case in which we can say quite a lot about de Rham cohomology is in degree 1. Suppose M is a connected smooth manifold and q is any point in M. Let $\mathrm{Hom}\big(\pi_1(M,q),\mathbb{R}\big)$ denote the set of group homomorphisms from $\pi_1(M,q)$ to the additive group $\mathbb{R}$; it is a vector space under pointwise addition of homomorphisms and multiplication by constants. We define a linear map $\Phi\colon H^1_{\mathrm{dR}}(M) \to \mathrm{Hom}\big(\pi_1(M,q),\mathbb{R}\big)$ as follows: given a cohomology class $[\omega] \in H^1_{\mathrm{dR}}(M)$, define $\Phi[\omega]\colon \pi_1(M,q) \to \mathbb{R}$ by

$$\Phi[\omega][\gamma] = \int_{\widetilde{\gamma}} \omega,$$

where $[\gamma]$ is any path homotopy class in $\pi_1(M,q)$, and $\widetilde{\gamma}$ is any piecewise smooth curve representing the same path class.

Theorem 17.17 (First Cohomology and the Fundamental Group). *Suppose M is a connected smooth manifold. For each $q \in M$, the linear map $\Phi\colon H^1_{\mathrm{dR}}(M) \to \mathrm{Hom}\big(\pi_1(M,q),\mathbb{R}\big)$ is well defined and injective.*

Remark. It is actually the case that Φ is an isomorphism, but we do not quite have the tools to prove this. See Problem 18-2.

Proof. Given $[\gamma] \in \pi_1(M,q)$, it follows from the Whitney approximation theorem that there is some smooth closed curve segment $\widetilde{\gamma}$ in the same path class as γ, and from Theorem 16.26 that $\int_{\widetilde{\gamma}} \omega$ gives the same result for every piecewise smooth curve $\widetilde{\gamma}$ in the given class. Moreover, if $\widetilde{\omega}$ is another smooth 1-form in the same cohomology class as ω, then $\widetilde{\omega} - \omega = df$ for some smooth function f, which implies

$$\int_{\widetilde{\gamma}} \widetilde{\omega} - \int_{\widetilde{\gamma}} \omega = \int_{\widetilde{\gamma}} df = f(q) - f(q) = 0.$$

Thus Φ is well defined. It follows from Proposition 11.34(c) that $\Phi[\omega]$ is a group homomorphism from $\pi_1(M,q)$ to $\mathbb{R}$, and from linearity of the line integral that Φ itself is a linear map.

To see that Φ is injective, suppose $\Phi[\omega]$ is the zero homomorphism. This means that $\int_{\widetilde{\gamma}} \omega = 0$ for every piecewise smooth closed curve $\widetilde{\gamma}$ starting at q. If σ is a piecewise smooth closed curve starting at some other point $q' \in M$, we can choose a piecewise smooth curve α from q to q', so that the path product $\alpha \cdot \sigma \cdot \overline{\alpha}$ is a closed curve based at q, where $\overline{\alpha}$ is a backward reparametrization of α. It then follows that

$$0 = \int_{\alpha \cdot \sigma \cdot \overline{\alpha}} \omega = \int_{\alpha} \omega + \int_{\sigma} \omega - \int_{\alpha} \omega = \int_{\sigma} \omega.$$

Thus, ω is conservative and therefore exact. □

It follows from Corollary 16.27 that $H^1_{\mathrm{dR}}(M) = 0$ when M is simply connected. The next corollary generalizes that result.

Corollary 17.18. *If M is a connected smooth manifold with finite fundamental group, then $H^1_{\mathrm{dR}}(M) = 0$.*

Proof. There are no nontrivial homomorphisms from a finite group to $\mathbb{R}$. □

► **Exercise 17.19.** A group Γ is called a ***torsion group*** if for each $g \in \Gamma$ there exists an integer k such that $g^k = 1$. Show that if M is a connected smooth manifold whose fundamental group is a torsion group, then $H^1_{\mathrm{dR}}(M) = 0$.

The Mayer–Vietoris Theorem

In this section we state a general theorem that can be used to compute the de Rham cohomology groups of many manifolds, by expressing them as unions of open submanifolds with simpler cohomology. We use the theorem here to compute all of the de Rham cohomology groups of spheres and of punctured Euclidean spaces, and the top-degree cohomology groups of compact manifolds. In the next chapter, we will use it again as an essential ingredient in the proof of the de Rham theorem. Because the proof of the Mayer–Vietoris theorem is fairly technical, we defer it to the end of the chapter.

Here is the setup for the theorem. Suppose M is a smooth manifold with or without boundary, and U, V are open subsets of M such that $M = U \cup V$. We have four inclusions,

$$\begin{array}{ccccc} & & U & & \\ & \overset{i}{\nearrow} & & \overset{k}{\searrow} & \\ U \cap V & & & & M, \\ & \underset{j}{\searrow} & & \underset{l}{\nearrow} & \\ & & V & & \end{array} \tag{17.6}$$

which induce pullback maps on differential forms,

$$\begin{array}{ccccc} & & \Omega^p(U) & & \\ & \overset{k^*}{\nearrow} & & \overset{i^*}{\searrow} & \\ \Omega^p(M) & & & & \Omega^p(U \cap V), \\ & \underset{l^*}{\searrow} & & \underset{j^*}{\nearrow} & \\ & & \Omega^p(V) & & \end{array}$$

as well as corresponding induced cohomology maps. Note that these pullback maps are really just restrictions: for example, $k^*\omega = \omega|_U$. Consider the following sequence of maps:

$$0 \to \Omega^p(M) \xrightarrow{k^* \oplus l^*} \Omega^p(U) \oplus \Omega^p(V) \xrightarrow{i^* - j^*} \Omega^p(U \cap V) \to 0, \tag{17.7}$$

where

$$\begin{aligned} (k^* \oplus l^*)\omega &= (k^*\omega, l^*\omega), \\ (i^* - j^*)(\omega, \eta) &= i^*\omega - j^*\eta. \end{aligned} \tag{17.8}$$

Because pullbacks commute with d, these maps descend to linear maps on the corresponding de Rham cohomology groups.

In the statement of the Mayer–Vietoris theorem, we will use the following standard algebraic terminology. Suppose we are given a sequence of vector spaces and linear maps:

$$\cdots \to V^{p-1} \xrightarrow{F_{p-1}} V^p \xrightarrow{F_p} V^{p+1} \xrightarrow{F_{p+1}} V^{p+2} \to \cdots. \tag{17.9}$$

Such a sequence is said to be ***exact*** if the image of each map is equal to the kernel of the next: for each p,

$$\operatorname{Im} F_{p-1} = \operatorname{Ker} F_p.$$

Theorem 17.20 (Mayer–Vietoris). *Let M be a smooth manifold with or without boundary, and let U, V be open subsets of M whose union is M. For each p, there is a linear map $\delta\colon H^p_{dR}(U \cap V) \to H^{p+1}_{dR}(M)$ such that the following sequence, called the* ***Mayer–Vietoris sequence*** *for the open cover $\{U, V\}$, is exact:*

$$\cdots \xrightarrow{\delta} H^p_{\mathrm{dR}}(M) \xrightarrow{k^*\oplus l^*} H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) \xrightarrow{i^*-j^*} H^p_{\mathrm{dR}}(U\cap V)$$
$$\xrightarrow{\delta} H^{p+1}_{\mathrm{dR}}(M) \xrightarrow{k^*\oplus l^*} \cdots. \quad (17.10)$$

Computations Using the Mayer–Vietoris Theorem

Using the Mayer–Vietoris theorem, it is a simple matter to compute all of the de Rham cohomology groups of spheres.

Theorem 17.21 (Cohomology of Spheres). *For $n \geq 1$, the de Rham cohomology groups of $\mathbb{S}^n$ are*

$$H^p_{\mathrm{dR}}(\mathbb{S}^n) \cong \begin{cases} \mathbb{R} & \text{if } p = 0 \text{ or } p = n, \\ 0 & \text{if } 0 < p < n. \end{cases} \quad (17.11)$$

The cohomology class of any smooth orientation form is a basis for $H^n_{\mathrm{dR}}(\mathbb{S}^n)$.

Proof. Proposition 17.6 shows that $H^0_{\mathrm{dR}}(\mathbb{S}^n) \cong \mathbb{R}$, so we need only prove (17.11) for $p \geq 1$. We do so by induction on n. For $n = 1$, note first that any orientation form on $\mathbb{S}^1$ has nonzero integral, so it is not exact by Corollary 16.13; thus $\dim H^1_{\mathrm{dR}}(\mathbb{S}^1) \geq 1$. On the other hand, Theorem 17.17 implies that there is an injective linear map from $H^1_{\mathrm{dR}}(\mathbb{S}^n)$ into $\mathrm{Hom}(\pi_1(\mathbb{S}^1, 1), \mathbb{R})$, which is 1-dimensional. Thus, $H^1_{\mathrm{dR}}(\mathbb{S}^1)$ has dimension exactly 1, and is spanned by the cohomology class of any orientation form.

Next, suppose $n \geq 2$ and assume by induction that the theorem is true for $\mathbb{S}^{n-1}$. Because $\mathbb{S}^n$ is simply connected, $H^1_{\mathrm{dR}}(\mathbb{S}^n) = 0$ by Corollary 17.18. For $p > 1$, we use the Mayer–Vietoris theorem as follows. Let N and S be the north and south poles in $\mathbb{S}^n$, respectively, and let $U = \mathbb{S}^n \smallsetminus \{S\}$, $V = \mathbb{S}^n \smallsetminus \{N\}$. By stereographic projection (Problem 1-7), both U and V are diffeomorphic to $\mathbb{R}^n$ (Fig. 17.2), and thus $U \cap V$ is diffeomorphic to $\mathbb{R}^n \smallsetminus \{0\}$.

Part of the Mayer–Vietoris sequence for $\{U, V\}$ reads

$$H^{p-1}_{\mathrm{dR}}(U) \oplus H^{p-1}_{\mathrm{dR}}(V) \to H^{p-1}_{\mathrm{dR}}(U\cap V) \to H^p_{\mathrm{dR}}(\mathbb{S}^n) \to H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V).$$

Because U and V are diffeomorphic to $\mathbb{R}^n$, the groups on both ends are trivial when $p > 1$, which implies that $H^p_{\mathrm{dR}}(\mathbb{S}^n) \cong H^{p-1}_{\mathrm{dR}}(U \cap V)$. Moreover, $U \cap V$ is diffeomorphic to $\mathbb{R}^n \smallsetminus \{0\}$ and therefore homotopy equivalent to $\mathbb{S}^{n-1}$, so in the end we conclude that $H^p_{\mathrm{dR}}(\mathbb{S}^n) \cong H^{p-1}_{\mathrm{dR}}(\mathbb{S}^{n-1})$ for $p > 1$, and (17.11) follows by induction. As in the $n = 1$ case, any smooth orientation form on $\mathbb{S}^n$ determines a nonzero cohomology class, which therefore spans $H^n_{\mathrm{dR}}(\mathbb{S}^n)$. □

► **Exercise 17.22.** Show that $\eta \in \Omega^n(\mathbb{S}^n)$ is exact if and only if $\int_{\mathbb{S}^n} \eta = 0$.

Corollary 17.23 (Cohomology of Punctured Euclidean Space). *Suppose $n \geq 2$ and $x \in \mathbb{R}^n$, and let $M = \mathbb{R}^n \smallsetminus \{x\}$. The only nontrivial de Rham groups of M are*

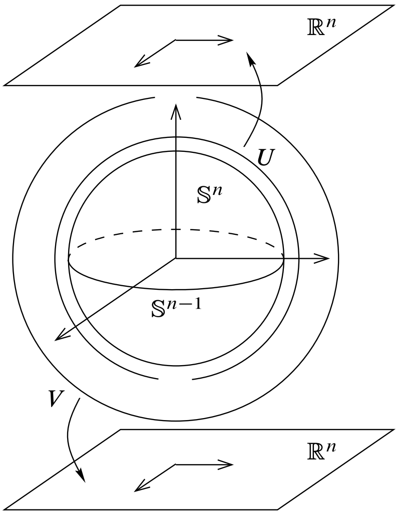

Fig. 17.2 Computing the de Rham cohomology of $\mathbb{S}^n$

$H^0_{\mathrm{dR}}(M)$ and $H^{n-1}_{\mathrm{dR}}(M)$, both of which are 1-dimensional. A closed $(n-1)$-form η on M is exact if and only if $\int_S \eta = 0$ for some (and hence every) $(n-1)$-dimensional sphere $S \subseteq M$ centered at x.

Proof. Let $S \subseteq M$ be any $(n-1)$-dimensional sphere centered at x. Because inclusion $\iota\colon S \hookrightarrow M$ is a homotopy equivalence, $\iota^*\colon H^p_{\mathrm{dR}}(M) \to H^p_{\mathrm{dR}}(S)$ is an isomorphism for each p, so the assertion about the dimension of $H^p_{\mathrm{dR}}(M)$ follows from Theorem 17.21. If η is a closed $(n-1)$-form on M, it follows that η is exact if and only if $\iota^*\eta$ is exact on S, which in turn is true if and only if $\int_S \eta = \int_S \iota^*\eta = 0$ by Exercise 17.22. □

► **Exercise 17.24.** Check that the statement and proof of Corollary 17.23 remain true if $\mathbb{R}^n \smallsetminus \{x\}$ is replaced by $\mathbb{R}^n \smallsetminus \bar{B}$ for some closed ball $\bar{B} \subseteq \mathbb{R}^n$.

Corollary 17.25. *Suppose $n \geq 2$, $U \subseteq \mathbb{R}^n$ is any open subset, and $x \in U$. Then $H^{n-1}_{\mathrm{dR}}(U \smallsetminus \{x\}) \neq 0$.*

Proof. Because U is open, there is an $(n-1)$-dimensional sphere S centered at x such that $S \subseteq U \smallsetminus \{x\}$. Let $\iota\colon S \hookrightarrow U \smallsetminus \{x\}$ be inclusion and $r\colon U \smallsetminus \{x\} \to S$ be the radial projection onto S. Then r and ι are smooth with $r \circ \iota = \mathrm{Id}_S$. This implies $\iota^* \circ r^* = \mathrm{Id}_{H^{n-1}_{\mathrm{dR}}}(S)$, and therefore $r^*\colon H^{n-1}_{\mathrm{dR}}(S) \to H^{n-1}_{\mathrm{dR}}(U \smallsetminus \{x\})$ is injective. Since $H^{n-1}_{\mathrm{dR}}(S) \neq 0$ by Theorem 17.21, the result follows. □

Here is an important application of the topological invariance of the de Rham cohomology groups. Recall the theorem on invariance of dimension (Theorem 1.2); it is a surprising fact that this purely topological theorem can be proved using de Rham cohomology. Before proving the theorem, we restate it here for convenience.

Theorem 17.26 (Topological Invariance of Dimension). *A nonempty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.*

Proof. If M is a topological n-manifold that is homeomorphic to an m-manifold, then M is itself both an n-manifold and an m-manifold. The case in which m or n is zero was already taken care of in Chapter 1, so assume that $m > n \geq 1$. Because M is an m-manifold, there is an open subset $V \subseteq M$ that is homeomorphic to $\mathbb{R}^m$. Because an open subset of an n-manifold is itself an n-manifold, any point $x \in V$ has a neighborhood $U \subseteq V$ that is homeomorphic to $\mathbb{R}^n$. On the one hand, because U is homeomorphic to $\mathbb{R}^n$, we can use the homeomorphism to define a smooth structure on U, and then $H^{m-1}_{dR}(U \smallsetminus \{x\}) = 0$ by Corollary 17.23. On the other hand, because U is homeomorphic to an open subset of $\mathbb{R}^m$, we can use that homeomorphism to define another smooth structure on U, and then Corollary 17.25 implies that $H^{m-1}_{dR}(U \smallsetminus \{x\}) \neq 0$. This contradicts the topological invariance of de Rham cohomology. □

As another application of Corollary 17.23, we prove a generalization of the Poincaré lemma for compactly supported forms. We will use it below to compute top-degree cohomology groups.

Lemma 17.27 (Poincaré Lemma with Compact Support). *Let $n \geq p \geq 1$, and suppose ω is a compactly supported closed p-form on $\mathbb{R}^n$. If $p = n$, suppose in addition that $\int_{\mathbb{R}^n} \omega = 0$. Then there exists a compactly supported smooth $(p-1)$-form η on $\mathbb{R}^n$ such that $d\eta = \omega$.*

Remark. Of course, we know that ω is exact by the Poincaré lemma, so the novelty here is the claim that it is the exterior derivative of a *compactly supported* form.

Proof. When $n = p = 1$, we can write $\omega = f\,dx$ for some smooth, compactly supported function $f \in C^\infty(\mathbb{R})$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

By the fundamental theorem of calculus, $dF = F'\,dx = f\,dx = \omega$. Choose $R > 0$ such that $\operatorname{supp} f \subseteq [-R, R]$. When $x < -R$, $F(x) = 0$ by our choice of R. When $x > R$, the fact that $\int_{\mathbb{R}} \omega = 0$ translates to

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{\infty} f(t)\,dt = 0,$$

so, in fact, $\operatorname{supp} F \subseteq [-R, R]$. This completes the proof for the case $n = p = 1$.

Now assume $n \geq 2$, and let $B, B' \subseteq \mathbb{R}^n$ be open balls centered at the origin such that $\operatorname{supp} \omega \subseteq B \subseteq \overline{B} \subseteq B'$. By the ordinary Poincaré lemma, there exists a smooth (but not necessarily compactly supported) $(p-1)$-form η_0 on $\mathbb{R}^n$ such that $d\eta_0 = \omega$. This implies, in particular, that $d\eta_0 = 0$ on $\mathbb{R}^n \smallsetminus \overline{B}$. To complete the proof, we consider three cases.

CASE 1: $p = 1$. In this case η_0 is a smooth function. Because $\mathbb{R}^n \smallsetminus \bar{B}$ is connected when $n \geq 2$, it follows that η_0 is equal to a constant c there. Letting $\eta = \eta_0 - c$, we find that η is compactly supported and satisfies $d\eta = \omega$ as claimed.

CASE 2: $1 < p < n$. Now the restriction of η_0 to $\mathbb{R}^n \smallsetminus \bar{B}$ is a closed $(p-1)$-form. Because $H^{p-1}_{\mathrm{dR}}\left(\mathbb{R}^n \smallsetminus \bar{B}\right) = 0$ by Exercise 17.24, there is a smooth $(p-2)$-form γ on $\mathbb{R}^n \smallsetminus \bar{B}$ such that $d\gamma = \eta_0$ there. If we let ψ be a smooth bump function that is supported in $\mathbb{R}^n \smallsetminus \bar{B}$ and equal to 1 on $\mathbb{R}^n \smallsetminus B'$, then $\eta = \eta_0 - d(\psi\gamma)$ is smooth on all of $\mathbb{R}^n$ and satisfies $d\eta = d\eta_0 = \omega$. Because $d(\psi\gamma) = d\gamma = \eta_0$ on $\mathbb{R}^n \smallsetminus B'$, η is compactly supported.

CASE 3: $p = n$. In this case, we cannot use the same argument as in Case 2 because $H^{n-1}_{\mathrm{dR}}\left(\mathbb{R}^n \smallsetminus \bar{B}\right) \neq 0$. However, it follows from Corollary 17.23 and Exercise 17.24 that the restriction of η_0 to $\mathbb{R}^n \smallsetminus \bar{B}$ is exact provided its integral is zero over some sphere centered at the origin and contained in $\mathbb{R}^n \smallsetminus \bar{B}$. Stokes's theorem implies that

$$0 = \int_{\mathbb{R}^n} \omega = \int_{\bar{B}'} \omega = \int_{\bar{B}'} d\eta_0 = \int_{\partial B'} \eta_0.$$

Thus η_0 is exact on $\mathbb{R}^n \smallsetminus \bar{B}$, and the proof proceeds exactly as in Case 2. □

For some purposes it is useful to define a generalization of the de Rham cohomology groups using only compactly supported forms. Let M be a smooth manifold with or without boundary and let $\Omega^p_c(M)$ denote the vector space of compactly supported smooth p-forms on M. The ***pth compactly supported de Rham cohomology group of M*** is the quotient space

$$H^p_c(M) = \frac{\operatorname{Ker}\big(d\colon \Omega^p_c(M) \to \Omega^{p+1}_c(M)\big)}{\operatorname{Im}\big(d\colon \Omega^{p-1}_c(M) \to \Omega^p_c(M)\big)}.$$

Of course, when M is compact, this just reduces to ordinary de Rham cohomology. But for noncompact manifolds the two groups can be different, as the next theorem illustrates.

Theorem 17.28 (Compactly Supported Cohomology of $\mathbb{R}^n$). *For $n \geq 1$, the compactly supported de Rham cohomology groups of $\mathbb{R}^n$ are*

$$H^p_c(\mathbb{R}^n) \cong \begin{cases} 0 & \text{if } 0 \leq p < n, \\ \mathbb{R} & \text{if } p = n. \end{cases}$$

► **Exercise 17.29.** Prove this theorem.

In general, a smooth map need not pull back compactly supported forms to compactly supported ones, so it does not induce a map on compactly supported cohomology. However, a *proper* map does pull back compactly supported forms to compactly supported ones, so for a proper smooth map $F\colon M \to N$ there is an induced cohomology map $F^*\colon H^p_c(N) \to H^p_c(M)$ for each p.

Compactly supported cohomology has a number of important applications in algebraic topology. One important application is the Poincaré duality theorem, which is outlined in Problem 18-7.

Another application is to facilitate the computation of de Rham cohomology in the top degree. Suppose first that M is an oriented smooth n-manifold. There is a natural linear map $I\colon \Omega_c^n(M) \to \mathbb{R}$ given by integration over M:

$$I(\omega) = \int_M \omega.$$

Because the integral of the exterior derivative of a compactly supported $(n-1)$-form is zero, I descends to a linear map, still denoted by the same symbol, from $H_c^n(M)$ to $\mathbb{R}$. (Note that every smooth n-form on an n-manifold is closed.)

Theorem 17.30 (Top Cohomology, Orientable Compact Support Case). *If M is a connected oriented smooth n-manifold, then the integration map $I\colon H_c^n(M) \to \mathbb{R}$ is an isomorphism, so $H_c^n(M)$ is 1-dimensional.*

Proof. Because a connected 0-manifold is a single point, the 0-dimensional case is an immediate consequence of Corollary 17.7, so we may assume $n \geq 1$. Let $\left(U, \left(x^i\right)\right)$ be an oriented smooth coordinate chart on M, and let f be a smooth bump function with compact support in U. Then the n-form defined by $\theta_0 = f\, dx^1 \wedge \cdots \wedge dx^n$ in U and 0 outside U is smooth and compactly supported on M, and satisfies $I(\theta_0) > 0$. Thus, I is surjective, so we need only show that it is injective. In other words, we have to show the following: if ω is a smooth, compactly supported n-form on M satisfying $\int_M \omega = 0$, then there is a smooth, compactly supported $(n-1)$-form η such that $\omega = d\eta$.

Let $\{U_i\}$ be a countable cover of M by open subsets that are diffeomorphic to $\mathbb{R}^n$, and let $M_k = U_1 \cup \cdots \cup U_k$ for each k. Because M is connected, by renumbering the sequence if necessary, we can arrange the $M_k \cap U_{k+1} \neq \varnothing$ for each k. Since every compactly supported n-form is supported in M_k for some finite k, it suffices to prove that if $\omega \in \Omega_c^n(M_k)$ has zero integral, then $\omega = d\eta$ for some $\eta \in \Omega_c^{n-1}(M_k)$, and then we can extend η by zero to a compactly supported form on all of M. We will prove this claim by induction on k.

For $k = 1$, since $M_1 = U_1$ is diffeomorphic to $\mathbb{R}^n$, the claim reduces to Lemma 17.27. So assume that the claim is true for some $k \geq 1$, and suppose ω is a compactly supported smooth n-form on $M_{k+1} = M_k \cup U_{k+1}$ that satisfies $\int_{M_{k+1}} \omega = 0$.

Let $\theta \in \Omega_c^n(M_{k+1})$ be an auxiliary form that is supported in $M_k \cap U_{k+1}$ and satisfies $\int_{M_{k+1}} \theta = 1$. (Such a form is easily constructed by using a bump function in coordinates as above.) Let $\{\varphi, \psi\}$ be a smooth partition of unity for M_{k+1} subordinate to the cover $\{M_k, U_{k+1}\}$ (Fig. 17.3),and let $c = \int_{M_{k+1}} \varphi\omega$. Observe that $\varphi\omega - c\theta$ is compactly supported in M_k, and its integral is equal to zero by our choice of c. Therefore, by the induction hypothesis, there is a compactly supported smooth $(n-1)$-form α on M_k such that $d\alpha = \varphi\omega - c\theta$. Similarly, $\psi\omega + c\theta$ is compactly

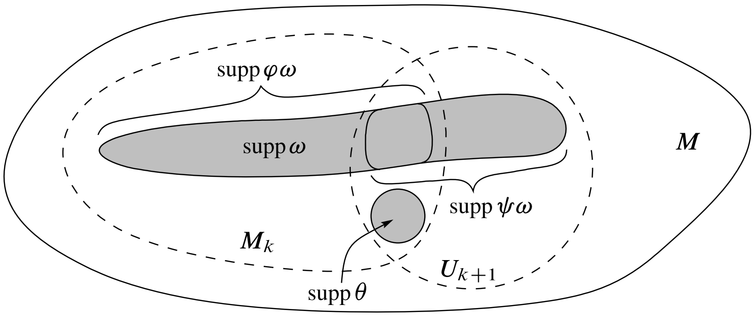

Fig. 17.3 Computing the top-degree cohomology

supported in U_{k+1}, and its integral is

$$\begin{aligned}\int_{U_{k+1}} (\psi\omega + c\theta) &= \int_{M_{k+1}} (1-\varphi)\omega + c\int_{M_{k+1}} \theta \\ &= \int_{M_{k+1}} \omega - \int_{M_{k+1}} \varphi\omega + c = 0.\end{aligned}$$

Thus by Lemma 17.27, there exists another smooth $(n-1)$-form β, compactly supported in U_{k+1}, such that $d\beta = \psi\omega + c\theta$. Both α and β can be extended by zero to smooth compactly supported forms on M_{k+1}. We compute

$$d(\alpha+\beta) = (\varphi\omega - c\theta) + (\psi\omega + c\theta) = (\varphi + \psi)\omega = \omega,$$

which completes the inductive step. □

Theorem 17.31 (Top Cohomology, Orientable Compact Case). *If M is a compact connected orientable smooth n-manifold, then $H^n_{\mathrm{dR}}(M)$ is 1-dimensional, and is spanned by the cohomology class of any smooth orientation form.*

Proof. This follows from the preceding theorem, because $H^p_{\mathrm{dR}}(M) = H^p_c(M)$ in that case, and the integral of any orientation form is nonzero. □

Theorem 17.32 (Top Cohomology, Orientable Noncompact Case). *If M is a noncompact connected orientable smooth n-manifold, then $H^n_{\mathrm{dR}}(M) = 0$.*

Proof. Choose an orientation on M. Let $f \in C^\infty(M)$ be a smooth exhaustion function. By adding a constant, we can arrange that $\inf_M f = 0$, and then connectedness and noncompactness of M imply that $f(M) = [0,\infty)$. For each positive integer i, let $V_i = f^{-1}\big((i-2,i)\big)$. Thus, $\{V_i\}_{i=1}^\infty$ is a cover of M by nonempty precompact open sets, with $V_i \cap V_j \neq \varnothing$ if and only if $j = i-1$, i, or $i+1$. Let $\{\psi_i\}$ be a smooth partition of unity subordinate to this cover, and for each i, let $\theta_i \in \Omega^n_c(M)$ be a smooth n-form compactly supported in $V_i \cap V_{i+1}$ with $\int_M \theta_i = 1$.

Suppose ω is any smooth n-form on M, and let $\omega_i = \psi_i\omega$ for each i, so $\omega_i \in \Omega^n_c(V_i)$. Let $c_1 = \int_{V_1} \omega_1$, so that $\omega_1 - c_1\theta_1$ is compactly supported in V_1 and has zero integral. It follows from Theorem 17.30 that there exists $\eta_1 \in \Omega^n_c(V_1)$ such that $d\eta_1 = \omega_1 - c_1\theta_1$. Next, choose $c_2 \in \mathbb{R}$ such that $\int_{V_2}(\omega_2 + c_1\theta_1 - c_2\theta_2) = 0$, so there exists $\eta_2 \in \Omega^n_c(V_2)$ with $d\eta_2 = \omega_2 + c_1\theta_1 - c_2\theta_2$. Continuing by induction, we can choose $c_j \in \mathbb{R}$ and $\eta_j \in \Omega^n_c(V_j)$ such that $d\eta_j = \omega_j + c_{j-1}\theta_{j-1} - c_j\theta_j$. Set $\eta = \sum_{j=1}^{\infty} \eta_j$, with each η_j extended to be zero on $M \smallsetminus V_j$. Because at most three terms in this sum are nonzero on each V_i, this is a smooth n-form on M. When we take its exterior derivative, the $c_j\theta_j$ terms all cancel, so $d\eta = \sum_j \omega_j = \omega$. □

Next we consider the nonorientable case. If M is a nonorientable smooth manifold, the key to analyzing its cohomology groups is the orientation covering $\widehat{\pi}\colon \widehat{M} \to M$ (see Theorem 15.41). Because a finite-sheeted covering map is a proper map by Exercise A.75, $\widehat{\pi}$ induces cohomology maps on both compactly supported and ordinary de Rham cohomology. The next lemma shows that these maps are all injective.

Lemma 17.33. *Suppose M is a connected nonorientable smooth manifold and $\widehat{\pi}\colon \widehat{M} \to M$ is its orientation covering. For each p, the induced cohomology maps $\widehat{\pi}^*\colon H^p_{\mathrm{dR}}(M) \to H^p_{\mathrm{dR}}\big(\widehat{M}\big)$ and $\widehat{\pi}^*\colon H^p_c(M) \to H^p_c\big(\widehat{M}\big)$ are injective.*

Proof. First, we prove the lemma for compactly supported cohomology. Suppose ω is a closed, compactly supported p-form on M such that $\widehat{\pi}^*[\omega] = 0 \in H^p_c\big(\widehat{M}\big)$. Then there exists $\eta \in \Omega^p_c\big(\widehat{M}\big)$ such that $d\eta = \widehat{\pi}^*\omega$. Let $\alpha\colon \widehat{M} \to \widehat{M}$ be the unique nontrivial covering automorphism of $\widehat{M}$ (see Fig. 15.9), and let $\widetilde{\eta} = \frac{1}{2}(\eta + \alpha^*\eta)$, which is also compactly supported. Using the fact that $\alpha \circ \alpha = \mathrm{Id}_{\widehat{M}}$, we compute

$$\alpha^*\widetilde{\eta} = \tfrac{1}{2}\big(\alpha^*\eta + (\alpha \circ \alpha)^*\eta\big) = \widetilde{\eta}.$$

Because $\widehat{\pi} \circ \alpha = \widehat{\pi}$, this implies

$$d\widetilde{\eta} = \tfrac{1}{2}(d\eta + d\alpha^*\eta) = \tfrac{1}{2}(d\eta + \alpha^* d\eta) = \tfrac{1}{2}(\widehat{\pi}^*\omega + \alpha^*\widehat{\pi}^*\omega) = \widehat{\pi}^*\omega.$$

Let $U \subseteq M$ be any evenly covered open subset. There are exactly two smooth local sections $\sigma_1, \sigma_2\colon U \to \widehat{M}$ over U, which are related by $\sigma_2 = \alpha \circ \sigma_1$. Observe that

$$\sigma_2^*\widetilde{\eta} = (\alpha \circ \sigma_1)^*\widetilde{\eta} = \sigma_1^*\alpha^*\widetilde{\eta} = \sigma_1^*\widetilde{\eta}.$$

Therefore, we can define a smooth global $(p-1)$-form β on M by setting $\beta|_U = \sigma^*\widetilde{\eta}$ for any smooth local section $\sigma\colon U \to \widehat{M}$; the argument above guarantees that the various definitions agree where they overlap. Because $\operatorname{supp}\beta = \widehat{\pi}\big(\operatorname{supp}\widetilde{\eta}\big)$, it follows that β is compactly supported. To determine the exterior derivative of β, given $p \in M$, choose a smooth local section σ defined on a neighborhood U of p, and compute

$$d\beta = d\sigma^*\widetilde{\eta} = \sigma^* d\widetilde{\eta} = \sigma^*\widehat{\pi}^*\omega = (\widehat{\pi} \circ \sigma)^*\,\omega = \omega,$$

because $\widehat{\pi} \circ \sigma = \mathrm{Id}_U$.

The argument for ordinary de Rham cohomology is the same, but with all references to compact support deleted. □

Theorem 17.34 (Top Cohomology, Nonorientable Case). *If M is a connected nonorientable smooth n-manifold, then $H^n_c(M) = 0$ and $H^n_{dR}(M) = 0$.*

Proof. First consider the case of compactly supported cohomology. By the preceding lemma, it suffices to show that $\widehat{\pi}^*\colon H^n_c(M) \to H^n_c\big(\widehat{M}\big)$ is the zero map, where $\widehat{\pi}\colon \widehat{M} \to M$ is the orientation covering of M. Let $\alpha\colon \widehat{M} \to \widehat{M}$ be the nontrivial covering automorphism as in the preceding proof. Now, α cannot be orientation-preserving: if it were, the entire covering automorphism group $\big\{\mathrm{Id}_{\widehat{M}}, \alpha\big\}$ would be orientation-preserving, and then M would be orientable by Theorem 15.36. By connectedness of $\widehat{M}$ and the fact that α is a diffeomorphism, it follows that α is orientation-reversing.

Suppose ω is any compactly supported smooth n-form on M, and let $\widehat{\omega} = \widehat{\pi}^*\omega$. Because $\widehat{\pi}$ is proper, $\widehat{\omega}$ is compactly supported, and $\widehat{\pi} \circ \alpha = \widehat{\pi}$ implies

$$\alpha^*\widehat{\omega} = \alpha^*\widehat{\pi}^*\omega = (\widehat{\pi} \circ \alpha)^*\,\omega = \widehat{\pi}^*\omega = \widehat{\omega}.$$

Because α is orientation-reversing, we conclude from Proposition 16.6(d) that

$$\int_{\widehat{M}} \widehat{\omega} = -\int_{\widehat{M}} \alpha^*\widehat{\omega} = -\int_{\widehat{M}} \widehat{\omega}.$$

This implies that $\int_{\widehat{M}} \widehat{\omega} = 0$, so $[\widehat{\omega}] = 0 \in H^n_c\big(\widehat{M}\big)$ by Theorem 17.31. This completes the proof that $H^n_c(M) = 0$.

It remains only to handle ordinary cohomology. If M is compact, it follows from the argument above that $H^n_{dR}(M) = H^n_c(M) = 0$. On the other hand, if M is noncompact, then so is $\widehat{M}$, and Theorem 17.32 shows that $H^n_{dR}\big(\widehat{M}\big) = 0$. It follows from Lemma 17.33 that $H^n_{dR}(M) = 0$ as well. □

Degree Theory

Now that we know the top-degree cohomology groups of all compact smooth manifolds, we can use them to draw a number of significant conclusions about smooth maps between certain compact manifolds of the same dimension. They all follow from the fact that we can associate an integer to each such map, called its *degree*, in such a way that homotopic maps have the same degree.

Theorem 17.35 (Degree of a Smooth Map). *Suppose M and N are compact, connected, oriented, smooth manifolds of dimension n, and $F\colon M \to N$ is a smooth map. There exists a unique integer k, called the* ***degree of F***, *that satisfies both of the following conditions.*

(a) *For every smooth n-form ω on N,*

$$\int_M F^*\omega = k \int_N \omega.$$

(b) *If $q \in N$ is a regular value of F, then*

$$k = \sum_{x \in F^{-1}(q)} \operatorname{sgn}(x),$$

where $\operatorname{sgn}(x) = +1$ *if dF_x is orientation-preserving, and* -1 *if it is orientation-reversing.*

Proof. By Theorem 17.31, two smooth n-forms on either M or N are cohomologous if and only if they have the same integral. Let θ be any smooth n-form on N such that $\int_N \theta = 1$, and let $k = \int_M F^*\theta$. If $\omega \in \Omega^n(N)$ is arbitrary, then ω is cohomologous to $a\theta$, where $a = \int_N \omega$, and therefore $F^*\omega$ is cohomologous to $aF^*\theta$. It follows that

$$\int_M F^*\omega = a \int_M F^*\theta = ak = k \int_N \omega.$$

Thus k satisfies (a), and is clearly the only number that does so.

Next we show that k also has the characterization given in part (b), from which it follows that it is an integer. Let $q \in N$ be an arbitrary regular value of F. Because $F^{-1}(q)$ is a properly embedded 0-dimensional submanifold of M, it is finite. Suppose first that $F^{-1}(q)$ is not empty—say, $F^{-1}(q) = \{x_1, \dots, x_m\}$. By the inverse function theorem, for each i there is a neighborhood U_i of x_i such that F is a diffeomorphism from U_i to a neighborhood W_i of q, and by shrinking the U_i's if necessary, we may assume that they are pairwise disjoint. Then $K = M \smallsetminus (U_1 \cup \dots \cup U_m)$ is closed in M and thus compact, so $F(K)$ is closed in N and disjoint from q. Let W be the connected component of $W_1 \cap \dots \cap W_m \cap \big(N \smallsetminus F(K)\big)$ containing q, and let $V_i = F^{-1}(W) \cap U_i$. It follows that W is a connected neighborhood of q whose preimage under F is the disjoint union $V_1 \amalg \dots \amalg V_m$, and F restricts to a diffeomorphism from each V_i to W. Since each V_i is connected, the restriction of F to V_i must be either orientation-preserving or orientation-reversing.

Let ω be a smooth n-form on N that is compactly supported in W and satisfies $\int_N \omega = \int_W \omega = 1$. It follows from part (a) that $\int_M F^*\omega = k$. Since $F^*\omega$ is compactly supported in $F^{-1}(W)$, we have $\int_M F^*\omega = \sum_{i=1}^m \int_{V_i} F^*\omega$. From Proposition 16.6(d) we conclude that for each i, $\int_{V_i} F^*\omega = \pm \int_W \omega = \pm 1$, with the positive sign if F is orientation-preserving on V_i and the negative sign otherwise. This proves (b) when $F^{-1}(q) \neq \varnothing$.

On the other hand, suppose $F^{-1}(q) = \varnothing$. Then q has a neighborhood W contained in $N \smallsetminus F(M)$ (because $F(M)$ is compact and thus closed). If ω is any smooth n-form on N that is compactly supported in W, then $\int_M F^*\omega = 0$, so $k = 0$. This proves (b). □

Much of the power of degree theory arises from the fact that the two different characterizations of the degree can be played off against each other. For example, it is often easy to compute the degree of a particular map simply by counting the points in the preimage of a regular value, with appropriate signs. On the other hand, the characterization in terms of differential forms makes it easy to prove many important properties, such as the ones given in the next proposition.

Proposition 17.36 (Properties of the Degree). *Suppose M, N, and P are compact, connected, oriented, smooth n-manifolds.*

(a) *If $F: M \to N$ and $G: N \to P$ are both smooth maps, then* $\deg(G \circ F) = (\deg G)(\deg F)$.
(b) *If $F: M \to N$ is a diffeomorphism, then* $\deg F = +1$ *if F is orientation-preserving and -1 if it is orientation-reversing.*
(c) *If two smooth maps $F_0, F_1: M \to N$ are homotopic, then they have the same degree.*

► **Exercise 17.37.** Prove the preceding proposition.

This proposition allows us to define the ***degree of a continuous map*** $F: M \to N$ between compact, connected, oriented, smooth n-manifolds, by letting $\deg F$ be the degree of any smooth map that is homotopic to F. The Whitney approximation theorem guarantees that there is such a map, and the preceding proposition guarantees that the degree is the same for every map homotopic to F.

Here are some applications of degree theory.

Theorem 17.38. *Suppose N is a compact, connected, oriented, smooth n-manifold, and X is a compact, oriented, smooth $(n+1)$-manifold with connected boundary. If $f: \partial X \to N$ is a continuous map that has a continuous extension to X, then* $\deg f = 0$.

Proof. Suppose f has an extension to a continuous map $F: X \to N$. By the Whitney approximation theorem, there is a smooth map $\widetilde{F}: X \to N$ that is homotopic to F. Replacing F by $\widetilde{F}$ and f by $\widetilde{F}|_{\partial X}$, we may assume that both f and F are smooth.

Let ω be any smooth n-form on N. Then $d\omega = 0$ because it is an $(n+1)$-form on an n-manifold. From Stokes's theorem, we obtain

$$\int_{\partial X} f^*\omega = \int_{\partial X} F^*\omega = \int_X d(F^*\omega) = \int_X F^* d\omega = 0.$$

It follows from Theorem 17.35 that f has degree zero. □

Theorem 17.39 (Brouwer Fixed-Point Theorem). *Every continuous map from $\overline{\mathbb{B}}^n$ to itself has a fixed point.*

Proof. Suppose for the sake of contradiction that $F: \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ is continuous and has no fixed points. We can define a continuous map $G: \overline{\mathbb{B}}^n \to \mathbb{S}^{n-1}$ by

$$G(x) = \frac{x - F(x)}{|x - F(x)|},$$

and let $g = G|_{\mathbb{S}^{n-1}}: \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. On the one hand, the previous theorem implies that g has degree zero. On the other hand, consider the map $H: \mathbb{S}^{n-1} \times I \to \mathbb{S}^{n-1}$ defined by

$$H(x,t) = \frac{x - tF(x)}{|x - tF(x)|}.$$

The denominator never vanishes when $t = 1$ because F has no fixed points, and when $t < 1$ it cannot vanish because $|x| = 1$ while $|tF(x)| \le t < 1$. Thus H is continuous, so it is a homotopy from the identity to g. It follows from Proposition 17.36 that g has degree 1, which is a contradiction. □

With a little more machinery from algebraic topology, it is possible to give many more applications of degree theory. For example, it turns out that continuous maps from $\mathbb{S}^n$ to itself ($n \ge 1$) are classified up to homotopy by degree (see [Hat02, Cor. 4.25]). This is not true for other compact orientable manifolds, however; Problem 17-13 describes a counterexample.

Proof of the Mayer–Vietoris Theorem

In this section we give the proof of the Mayer–Vietoris theorem. For this purpose we need to introduce some simple algebraic concepts. More details about the ideas introduced here can be found in [LeeTM, Chap. 13] or in any textbook on algebraic topology.

Let $\mathcal{R}$ be a commutative ring, and suppose we are given a sequence of $\mathcal{R}$-modules and $\mathcal{R}$-linear maps:

$$\cdots \to A^{p-1} \xrightarrow{d} A^p \xrightarrow{d} A^{p+1} \to \cdots . \tag{17.12}$$

(In all of our applications, the ring will be either $\mathbb{Z}$, in which case we are looking at abelian groups and homomorphisms, or $\mathbb{R}$, in which case we have vector spaces and linear maps. The terminology of modules is just a convenient way to combine the two cases.) Such a sequence is said to be a ***complex*** if the composition of any two successive applications of d is the zero map:

$$d \circ d = 0\colon A^p \to A^{p+2} \quad \text{for each } p.$$

Just as in the case of vector spaces, such a sequence of modules is called an ***exact sequence*** if the image of each d is equal to the kernel of the next. Clearly, every exact sequence is a complex, but the converse need not be true.

Let us denote the sequence (17.12) by A^*. If it is a complex, then the image of each map d is contained in the kernel of the next, so we define the ***pth cohomology group of A^**** to be the quotient module

$$H^p(A^*) = \frac{\operatorname{Ker}(d\colon A^p \to A^{p+1})}{\operatorname{Im}(d\colon A^{p-1} \to A^p)}.$$

It can be thought of as a quantitative measure of the failure of exactness at A^p. The obvious example is the ***de Rham complex*** of a smooth n-manifold M:

$$0 \to \Omega^0(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0,$$

whose cohomology groups are the de Rham groups of M. (In algebraic topology, a complex as we have defined it is usually called a ***cochain complex***, while a ***chain***

complex is defined similarly except that the maps go in the direction of decreasing indices:

$$\cdots \to A_{p+1} \xrightarrow{\partial} A_p \xrightarrow{\partial} A_{p-1} \to \cdots.$$

In that case, the term ***homology*** is used in place of cohomology.)

If A^* and B^* are complexes, a ***cochain map from A^* to B^****, denoted by $F\colon A^* \to B^*$, is a collection of linear maps $F\colon A^p \to B^p$ (it is easiest to use the same symbol for all of the maps) such that the following diagram commutes for each p:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^p & \xrightarrow{d} & A^{p+1} & \longrightarrow & \cdots \\ & & {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} & & \\ \cdots & \longrightarrow & B^p & \xrightarrow[d]{} & B^{p+1} & \longrightarrow & \cdots. \end{array}$$

The fact that $F \circ d = d \circ F$ means that any cochain map induces a linear map on cohomology $F^*\colon H^p(A^*) \to H^p(B^*)$ for each p, just as in the case of de Rham cohomology. (A map between chain complexes satisfying the analogous relations is called a ***chain map***; the same argument shows that a chain map induces a linear map on homology.)

A ***short exact sequence of complexes*** consists of three complexes A^*, B^*, C^*, together with cochain maps

$$0 \to A^* \xrightarrow{F} B^* \xrightarrow{G} C^* \to 0$$

such that each sequence

$$0 \to A^p \xrightarrow{F} B^p \xrightarrow{G} C^p \to 0$$

is exact. This means that F is injective, G is surjective, and $\operatorname{Im} F = \operatorname{Ker} G$.

Lemma 17.40 (The Zigzag Lemma). *Given a short exact sequence of complexes as above, for each p there is a linear map*

$$\delta\colon H^p(C^*) \to H^{p+1}(A^*),$$

called the **connecting homomorphism**, *such that the following sequence is exact*:

$$\cdots \xrightarrow{\delta} H^p(A^*) \xrightarrow{F^*} H^p(B^*) \xrightarrow{G^*} H^p(C^*) \xrightarrow{\delta} H^{p+1}(A^*) \xrightarrow{F^*} \cdots. \qquad (17.13)$$

Proof. We sketch only the main idea; you can either carry out the details yourself or look them up.

The hypothesis means that the following diagram commutes and has exact horizontal rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^p & \xrightarrow{F} & B^p & \xrightarrow{G} & C^p & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & A^{p+1} & \xrightarrow{F} & B^{p+1} & \xrightarrow{G} & C^{p+1} & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & A^{p+2} & \xrightarrow{F} & B^{p+2} & \xrightarrow{G} & C^{p+2} & \longrightarrow & 0.
\end{array}$$

Suppose $c^p \in C^p$ represents a cohomology class; this means that $dc^p = 0$. Since $G\colon B^p \to C^p$ is surjective, there is some element $b^p \in B^p$ such that $Gb^p = c^p$. Because the diagram commutes, $Gdb^p = dGb^p = dc^p = 0$, and therefore $db^p \in \operatorname{Ker} G = \operatorname{Im} F$. Thus, there exists $a^{p+1} \in A^{p+1}$ satisfying $Fa^{p+1} = db^p$. By commutativity of the diagram again, $Fda^{p+1} = dFa^{p+1} = ddb^p = 0$. Since F is injective, this implies $da^{p+1} = 0$, so a^{p+1} represents a cohomology class in $H^{p+1}(A^*)$. The connecting homomorphism δ is defined by setting $\delta[c^p] = [a^{p+1}]$ for any such $a^{p+1} \in A^{p+1}$, that is, provided there exists $b^p \in B^p$ such that

$$Gb^p = c^p, \qquad Fa^{p+1} = db^p.$$

A number of facts have to be verified: that the cohomology class $[a^{p+1}]$ is well defined, independently of the choices made along the way; that the resulting map δ is linear; and that the resulting sequence (17.13) is exact. Each of these verifications is a routine "diagram chase" like the one we used to define δ; the details are left as an exercise. □

► **Exercise 17.41.** Complete (or look up) the proof of the zigzag lemma.

Proof of the Mayer–Vietoris Theorem. Suppose M is a smooth manifold with or without boundary, and U, V are open subsets of M whose union is M. The heart of the proof is to show that the sequence (17.7) is exact for each p. Because pullback maps commute with the exterior derivative, (17.7) therefore defines a short exact sequence of cochain maps, and the Mayer–Vietoris theorem follows immediately from the zigzag lemma.

We begin by proving exactness at $\Omega^p(M)$, which just means showing that $k^* \oplus l^*$ is injective. Suppose that $\sigma \in \Omega^p(M)$ satisfies $(k^* \oplus l^*)\sigma = (\sigma|_U, \sigma|_V) = (0,0)$. This means that the restrictions of σ to U and V are both zero. Since $\{U, V\}$ is an open cover of M, this implies that σ is zero.

To prove exactness at $\Omega^p(U) \oplus \Omega^p(V)$, first observe that

$$(i^* - j^*) \circ (k^* \oplus l^*)(\sigma) = (i^* - j^*)(\sigma|_U, \sigma|_V) = \sigma|_{U \cap V} - \sigma|_{U \cap V} = 0,$$

which shows that $\operatorname{Im}(k^* \oplus l^*) \subseteq \operatorname{Ker}(i^* - j^*)$. Conversely, suppose we are given $(\eta, \eta') \in \Omega^p(U) \oplus \Omega^p(V)$ such that $(i^* - j^*)(\eta, \eta') = 0$. This means that $\eta|_{U \cap V} =$

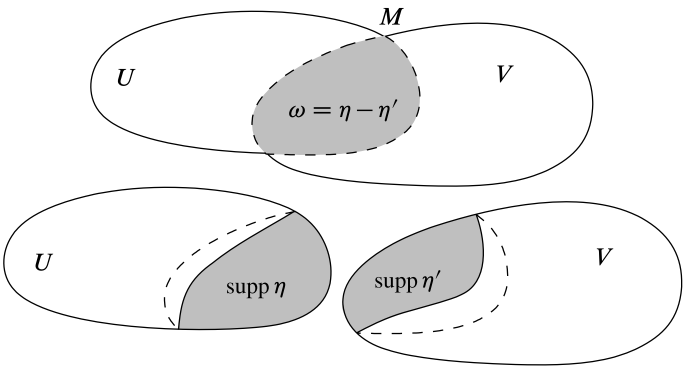

Fig. 17.4 Surjectivity of $i^* - j^*$

$\eta'|_{U\cap V}$, so there is a global smooth p-form σ on M defined by

$$\sigma = \begin{cases} \eta & \text{on } U, \\ \eta' & \text{on } V. \end{cases}$$

Clearly, $(\eta, \eta') = (k^* \oplus l^*)\sigma$, so $\operatorname{Ker}(i^* - j^*) \subseteq \operatorname{Im}(k^* \oplus l^*)$.

Exactness at $\Omega^p(U \cap V)$ means that $i^* - j^*$ is surjective. This is the only nontrivial part of the proof, and the only part that really uses any properties of smooth manifolds and differential forms.

Let $\omega \in \Omega^p(U \cap V)$ be arbitrary. We need to show that there exist $\eta \in \Omega^p(U)$ and $\eta' \in \Omega^p(V)$ such that

$$\omega = (i^* - j^*)(\eta, \eta') = i^*\eta - j^*\eta' = \eta|_{U\cap V} - \eta'|_{U\cap V}.$$

(See Fig. 17.4.) Let $\{\varphi, \psi\}$ be a smooth partition of unity subordinate to the open cover $\{U, V\}$ of M, and define $\eta \in \Omega^p(U)$ by

$$\eta = \begin{cases} \psi\omega & \text{on } U \cap V, \\ 0 & \text{on } U \smallsetminus \operatorname{supp}\psi. \end{cases} \tag{17.14}$$

On the set $(U \cap V) \smallsetminus \operatorname{supp}\psi$ where these definitions overlap, they both give zero, so this defines η as a smooth p-form on U.Similarly, define $\eta' \in \Omega^p(V)$ by

$$\eta' = \begin{cases} -\varphi\omega & \text{on } U \cap V, \\ 0 & \text{on } V \smallsetminus \operatorname{supp}\varphi. \end{cases} \tag{17.15}$$

Then we have

$$\eta|_{U\cap V} - \eta'|_{U\cap V} = \psi\omega - (-\varphi\omega) = (\psi + \varphi)\omega = \omega,$$

which was to be proved. □

For use in the next chapter, we record the following corollary to the proof, which explicitly characterizes the connecting homomorphism δ.

Corollary 17.42. *The connecting homomorphism in the Mayer–Vietoris sequence, $\delta\colon H^p_{dR}(U\cap V)\to H^{p+1}_{dR}(M)$, is defined as follows. For each $\omega\in\mathcal{Z}^p(U\cap V)$, there are p-forms $\eta\in\Omega^p(U)$ and $\eta'\in\Omega^p(V)$ such that $\omega=\eta|_{U\cap V}-\eta'|_{U\cap V}$; and then $\delta[\omega]=[\sigma]$, where σ is the $(p+1)$-form on M that is equal to $d\eta$ on U and to $d\eta'$ on V. If $\{\varphi,\psi\}$ is a smooth partition of unity subordinate to $\{U,V\}$, we can take $\eta=\psi\omega$ and $\eta'=-\varphi\omega$, both extended by zero outside the supports of ψ and φ.*

Proof. A characterization of the connecting homomorphism was given in the proof of the zigzag lemma. Specializing this characterization to the situation of the short exact sequence (17.7), we find that $\delta[\omega]=[\sigma]$, provided there exists $(\eta,\eta')\in\Omega^p(U)\oplus\Omega^p(V)$ such that

$$i^*\eta-j^*\eta'=\omega,\qquad (k^*\sigma,l^*\sigma)=(d\eta,d\eta'). \tag{17.16}$$

Just as in the proof of the Mayer–Vietoris theorem, if $\{\varphi,\psi\}$ is a smooth partition of unity subordinate to $\{U,V\}$, then formulas (17.14) and (17.15) define smooth forms $\eta\in\Omega^p(U)$ and $\eta'\in\Omega^p(V)$ satisfying the first equation of (17.16). Given such forms η, η', the fact that ω is closed implies that $d\eta=d\eta'$ on $U\cap V$. Thus there is a smooth $(p+1)$-form σ on M that is equal to $d\eta$ on U and $d\eta'$ on V, and it satisfies the second equation of (17.16). □

Problems

17-1. Let M be a smooth manifold with or without boundary, and let $\omega\in\Omega^p(M)$, $\eta\in\Omega^q(M)$ be closed forms. Show that the de Rham cohomology class of $\omega\wedge\eta$ depends only on the cohomology classes of ω and η, and thus there is a well-defined bilinear map $\cup\colon H^p_{dR}(M)\times H^q_{dR}(M)\to H^{p+q}_{dR}(M)$, called the ***cup product***, given by $[\omega]\cup[\eta]=[\omega\wedge\eta]$.

17-2. Let (M,g) be an oriented compact Riemannian n-manifold. For each $0\le p\le n$, the ***Laplace–Beltrami operator*** $\Delta\colon\Omega^p(M)\to\Omega^p(M)$ is the linear map defined by

$$\Delta\omega=d\,d^*\omega+d^*\,d\omega,$$

where d^* is the operator defined in Problem 16-22. A smooth form $\omega\in\Omega^p(M)$ is said to be ***harmonic*** if $\Delta\omega=0$. Show that the following are equivalent for any $\omega\in\Omega^p(M)$.

(a) ω is harmonic.
(b) $d\omega=0$ and $d^*\omega=0$.
(c) $d\omega=0$ and ω is the unique smooth p-form in its de Rham cohomology class with minimum norm $\|\omega\|=(\omega,\omega)^{1/2}$. (Here $(\cdot,\cdot)$ is the inner product on $\Omega^p(M)$ defined in Problem 16-22.)

[Hint: for (c), consider $f(t)=\|\omega+d(td^*\omega)\|^2$.] [Remark: there is a deep theorem called the *Hodge theorem*, which says that on every compact, oriented Riemannian manifold, there is a unique harmonic form in every de Rham cohomology class. See [Gil95] or [War83] for a proof.]

17-3. Let (M, g) be an oriented Riemannian manifold, and let $\Delta = dd^* + d^*d$ be the Laplace–Beltrami operator on p-forms as in Problem 17-2. When $p = 0$, show that Δ agrees with the geometric Laplacian $\Delta u = -\operatorname{div}(\operatorname{grad} u)$ defined on real-valued functions in Problem 16-13.

17-4. Suppose $U \subseteq \mathbb{R}^n$ is open and star-shaped with respect to 0, and $\omega = \sum' \omega_I \, dx^I$ is a closed p-form on U. Show either directly or by tracing through the proof of the Poincaré lemma that the $(p-1)$-form η given explicitly by the formula

$$\eta = \sum_I{}' \sum_{q=1}^{p} (-1)^{q-1} \left(\int_0^1 t^{p-1} \omega_I(tx) \, dt \right) x^{i_q} \, dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_q}} \wedge \cdots \wedge dx^{i_p}$$

satisfies $d\eta = \omega$. In the case that ω is a smooth closed 1-form, show that η is equal to the potential function f defined in Theorem 11.49.

17-5. For each $n \geq 1$, compute the de Rham cohomology groups of $\mathbb{R}^n \smallsetminus \{e_1, -e_1\}$; and for each nonzero cohomology group, give specific differential forms whose cohomology classes form a basis.

17-6. Let M be a connected smooth manifold of dimension $n \geq 3$. For any $x \in M$ and $0 \leq p \leq n-2$, prove that the map $H^p_{\mathrm{dR}}(M) \to H^p_{\mathrm{dR}}(M \smallsetminus \{x\})$ induced by inclusion $M \smallsetminus \{x\} \hookrightarrow M$ is an isomorphism. Prove that the same is true for $p = n-1$ if M is compact and orientable. [Hint: use the Mayer–Vietoris theorem. The cases $p = 0$, $p = 1$, and $p = n-1$ require special handling.]

17-7. Let M_1, M_2 be connected smooth manifolds of dimension $n \geq 3$, and let $M_1 \# M_2$ denote their smooth connected sum (Example 9.31). Prove that $H^p_{\mathrm{dR}}(M_1 \# M_2) \cong H^p_{\mathrm{dR}}(M_1) \oplus H^p_{\mathrm{dR}}(M_2)$ for $0 < p < n-1$. Prove that the same is true for $p = n-1$ if M_1 and M_2 are both compact and orientable. [Hint: use Problems 9-12 and 17-6.]

17-8. Suppose M is a compact, connected, orientable, smooth n-manifold.
(a) Show that there is a one-to-one correspondence between orientations of M and orientations of the vector space $H^n_{\mathrm{dR}}(M)$, under which the cohomology class of a smooth orientation form is an oriented basis for $H^n_{\mathrm{dR}}(M)$.
(b) Now suppose M and N are smooth n-manifolds with given orientations. Show that a diffeomorphism $F\colon M \to N$ is orientation preserving if and only if $F^*\colon H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M)$ is orientation preserving.

17-9. Prove Theorem 1.37 (topological invariance of the boundary).

17-10. Let p be a nonzero polynomial in one variable with complex coefficients, and let $\widetilde{p}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ be the smooth map defined in Problem 2-9. Prove that the degree of $\widetilde{p}$ (as a smooth map between manifolds) is equal to the degree of the polynomial p in the usual sense.

17-11. This problem shows that some parts of degree theory can be extended to proper maps between noncompact manifolds. Suppose M and N are noncompact, connected, oriented, smooth n-manifolds.

(a) Suppose $F\colon M \to N$ is a proper smooth map. Prove that there is a unique integer k called the ***degree of F*** such that for each smooth, compactly supported n-form ω on N,

$$\int_M F^*\omega = k \int_N \omega,$$

and for each regular value q of F,

$$k = \sum_{x\in F^{-1}(q)} \operatorname{sgn}(dF_x),$$

where $\operatorname{sgn}(dF_x)$ is defined in Theorem 17.35.

(b) By considering the maps $F, G\colon \mathbb{C} \to \mathbb{C}$ given by $F(z) = z$ and $G(z) = z^2$, show that the degree of a proper map is not a homotopy invariant.

17-12. Suppose M and N are compact, connected, oriented, smooth n-manifolds, and $F\colon M \to N$ is a smooth map. Prove that if $\int_M F^*\eta \neq 0$ for some $\eta \in \Omega^n(N)$, then F is surjective. Give an example to show that F can be surjective even if $\int_M F^*\eta = 0$ for every $\eta \in \Omega^n(N)$.

17-13. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the 2-torus. Consider the two maps $f, g\colon \mathbb{T}^2 \to \mathbb{T}^2$ given by $f(w,z) = (w,z)$ and $g(w,z) = (z,\bar{w})$. Show that f and g have the same degree, but are not homotopic. [Suggestion: consider the induced homomorphisms on the first cohomology group or the fundamental group.]