Chapter 14
Differential Forms

When we introduced tensors in Chapter 12, we observed that there are two special classes of tensors whose values change predictably when their arguments are rearranged: *symmetric tensors* and *alternating tensors*. We saw a significant application of symmetric tensors in Chapter 13, in the form of Riemannian metrics. In this chapter, we begin to develop the theory of alternating tensors, and especially *differential forms*, which are alternating tensor fields on manifolds. It might come as a surprise, but these innocent-sounding objects turn out to be considerably more important in smooth manifold theory than symmetric tensor fields.

Much of the theory of differential forms can be viewed as a generalization of the theory of covector fields—which are, after all, the simplest examples of differential forms. Most fundamentally, covector fields are objects that can be integrated over curves in a coordinate invariant way; similarly, it turns out that differential forms are objects that can be integrated over higher-dimensional submanifolds. We will develop the theory of integration of differential forms in Chapter 16; the first section of that chapter gives a heuristic explanation of why alternating tensor fields make sense as objects to integrate.

In addition to their role in integration, differential forms provide a framework for generalizing such diverse concepts from multivariable calculus as the cross product, curl, divergence, and Jacobian determinant. They are also essential to the theories of orientations (Chapter 15), de Rham cohomology (Chapters 17–18), foliations (Chapter 19), and symplectic manifolds (Chapter 22).

Because of the many uses of differential forms, differential geometers have developed an array of technical tools for manipulating them. The purpose of this chapter is to describe those tools. If you have never been exposed to differential forms before, you might find much of the theory in this chapter unmotivated the first time through; but rest assured that your effort will be repaid many times over in the many applications of differential forms throughout the rest of the book.

We begin the chapter by examining the algebra of alternating tensors on a finite-dimensional vector space. After exploring the computational properties of these tensors in a linear-algebraic setting, we transfer everything to smooth manifolds, and begin to explore the properties of differential forms.

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The heart of the chapter is the introduction of the most important operation on differential forms, called the *exterior derivative*, which generalizes the differential of a smooth function that we introduced in Chapter 11, as well as the gradient, divergence, and curl operators of multivariable calculus. It acts on a differential form and yields another differential form of one higher degree. It is remarkable in that it is one of the very few differential operators that are naturally defined on every smooth manifold without any arbitrary choices.

At the end of the chapter, we will see how the exterior derivative can be used to simplify the computation of Lie derivatives of differential forms.

The Algebra of Alternating Tensors

Let V be a finite-dimensional (real) vector space. Recall that a covariant k-tensor on V is said to be *alternating* if its value changes sign whenever two arguments are interchanged, or equivalently if any permutation of the arguments causes its value to be multiplied by the sign of the permutation (see Exercise 12.17). Alternating covariant k-tensors are also called ***exterior forms***, ***multicovectors***, or ***k-covectors***. The vector space of all k-covectors on V is denoted by $\Lambda^k(V^*)$. As we noted in Chapter 12, all 0-tensors and 1-tensors are alternating.

The next lemma gives two more useful characterizations of alternating tensors.

Lemma 14.1. *Let α be a covariant k-tensor on a finite-dimensional vector space V. The following are equivalent:*

(a) *α is alternating.*
(b) *$\alpha(v_1,\dots,v_k)=0$ whenever the k-tuple $(v_1,\dots,v_k)$ is linearly dependent.*
(c) *α gives the value zero whenever two of its arguments are equal:*

$$\alpha(v_1,\dots,w,\dots,w,\dots,v_k)=0.$$

Proof. The implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c) are immediate. We complete the proof by showing that (c) implies both (a) and (b).

Assume that α satisfies (c). For any vectors $v_1,\dots,v_k$, the hypothesis implies

$$\begin{aligned}
0 &= \alpha(v_1,\dots,v_i+v_j,\dots,v_i+v_j,\dots,v_k)\\
&= \alpha(v_1,\dots,v_i,\dots,v_i,\dots,v_k)+\alpha(v_1,\dots,v_i,\dots,v_j,\dots,v_k)\\
&\quad +\alpha(v_1,\dots,v_j,\dots,v_i,\dots,v_k)+\alpha(v_1,\dots,v_j,\dots,v_j,\dots,v_k)\\
&= \alpha(v_1,\dots,v_i,\dots,v_j,\dots,v_k)+\alpha(v_1,\dots,v_j,\dots,v_i,\dots,v_k).
\end{aligned}$$

Thus α is alternating. On the other hand, if $(v_1,\dots,v_k)$ is a linearly dependent k-tuple, then one of the v_i's can be written as a linear combination of the others. For simplicity, let us assume that $v_k = \sum_{j=1}^{k-1} a^j v_j$. Then multilinearity of α implies

$$\alpha(v_1, \dots, v_k) = \sum_{j=1}^{k-1} a^j \alpha(v_1, \dots, v_{k-1}, v_j).$$

In each of these terms, α has two identical arguments, so every term is zero. □

In Chapter 12, we defined a projection Sym: $T^k(V^*) \to \Sigma^k(V^*)$ called *symmetrization*. We define a similar projection Alt: $T^k(V^*) \to \Lambda^k(V^*)$, called ***alternation***, as follows:

$$\operatorname{Alt} \alpha = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\left({}^{\sigma}\alpha\right),$$

where S_k is the symmetric group on k elements. More explicitly, this means

$$(\operatorname{Alt} \alpha)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\alpha\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right).$$

Example 14.2. If α is any 1-tensor, then $\operatorname{Alt} \alpha = \alpha$. If β is a 2-tensor, then

$$(\operatorname{Alt} \beta)(v, w) = \tfrac{1}{2}\big(\beta(v, w) - \beta(w, v)\big).$$

For a 3-tensor γ,

$$\begin{aligned}(\operatorname{Alt} \gamma)(v, w, x) = \tfrac{1}{6}\big(&\gamma(v, w, x) + \gamma(w, x, v) + \gamma(x, v, w) \\ &- \gamma(w, v, x) - \gamma(v, x, w) - \gamma(x, w, v)\big).\end{aligned}$$

//

The next proposition is the analogue for alternating tensors of Proposition 12.14.

Proposition 14.3 (Properties of Alternation). *Let α be a covariant tensor on a finite-dimensional vector space.*

(a) $\operatorname{Alt} \alpha$ *is alternating.*
(b) $\operatorname{Alt} \alpha = \alpha$ *if and only if* α *is alternating.*

► **Exercise 14.4.** Prove Proposition 14.3.

Elementary Alternating Tensors

For computations with alternating tensors, the following notation is exceedingly useful. Given a positive integer k, an ordered k-tuple $I = (i_1, \dots, i_k)$ of positive integers is called a ***multi-index of length k***. If I is such a multi-index and $\sigma \in S_k$ is a permutation of $\{1, \dots, k\}$, we write I_σ for the following multi-index:

$$I_\sigma = \left(i_{\sigma(1)}, \dots, i_{\sigma(k)}\right).$$

Note that $I_{\sigma\tau} = (I_\sigma)_\tau$ for $\sigma, \tau \in S_k$.

Let V be an n-dimensional vector space, and suppose $\left(\varepsilon^1, \dots, \varepsilon^n\right)$ is any basis for V^*. We now define a collection of k-covectors on V that generalize the determinant function on $\mathbb{R}^n$. For each multi-index $I = (i_1, \dots, i_k)$ of length k such that $1 \le i_1, \dots, i_k \le n$, define a covariant k-tensor $\varepsilon^I = \varepsilon^{i_1 \dots i_k}$ by

$$\varepsilon^I(v_1, \dots, v_k) = \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \dots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \dots & \varepsilon^{i_k}(v_k) \end{pmatrix} = \det \begin{pmatrix} v_1^{i_1} & \dots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \dots & v_k^{i_k} \end{pmatrix}. \tag{14.1}$$

In other words, if $\mathbb{v}$ denotes the $n \times k$ matrix whose columns are the components of the vectors $v_1, \dots, v_k$ with respect to the basis (E_i) dual to $\left(\varepsilon^i\right)$, then $\varepsilon^I(v_1, \dots, v_k)$ is the determinant of the $k \times k$ submatrix consisting of rows $i_1, \dots, i_k$ of $\mathbb{v}$. Because the determinant changes sign whenever two columns are interchanged, it is clear that ε^I is an alternating k-tensor. We call ε^I an ***elementary alternating tensor*** or ***elementary k-covector***.

Example 14.5. In terms of the standard dual basis $\left(e^1, e^2, e^3\right)$ for $\left(\mathbb{R}^3\right)^*$, we have

$$e^{13}(v, w) = v^1 w^3 - w^1 v^3;$$
$$e^{123}(v, w, x) = \det(v, w, x). \qquad //$$

In order to streamline computations with the elementary k-covectors, it is useful to extend the Kronecker delta notation in the following way. If I and J are multi-indices of length k, we define

$$\delta^I_J = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \dots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \dots & \delta^{i_k}_{j_k} \end{pmatrix}.$$

► **Exercise 14.6.** Show that

$$\delta^I_J = \begin{cases} \operatorname{sgn}\sigma & \text{if neither } I \text{ nor } J \text{ has a repeated index and } J = I_\sigma \text{ for some } \sigma \in S_k, \\ 0 & \text{if } I \text{ or } J \text{ has a repeated index or } J \text{ is not a permutation of } I. \end{cases}$$

Lemma 14.7 (Properties of Elementary *k*-Covectors). *Let (E_i) be a basis for V, let $\left(\varepsilon^i\right)$ be the dual basis for V^*, and let ε^I be as defined above.*

(a) *If I has a repeated index, then $\varepsilon^I = 0$.*
(b) *If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^I = (\operatorname{sgn}\sigma)\varepsilon^J$.*
(c) *The result of evaluating ε^I on a sequence of basis vectors is*

$$\varepsilon^I\left(E_{j_1}, \dots, E_{j_k}\right) = \delta^I_J.$$

Proof. If I has a repeated index, then for any vectors $v_1, \dots, v_k$, the determinant in (14.1) has two identical rows and thus is equal to zero, which proves (a). On the other hand, if J is obtained from I by interchanging two indices, then the corresponding determinants have opposite signs; this implies (b). Finally, (c) follows immediately from the definition of ε^I. □

The significance of the elementary k-covectors is that they provide a convenient basis for $\Lambda^k(V^*)$. Of course, the ε^I's are not all linearly independent, because some of them are zero and the ones corresponding to different permutations of the same multi-index are scalar multiples of each other. But as the next proposition shows, we can get a basis by restricting attention to an appropriate subset of multi-indices. A multi-index $I = (i_1, \dots, i_k)$ is said to be ***increasing*** if $i_1 < \dots < i_k$. It is useful to use a primed summation sign to denote a sum over only increasing multi-indices, so that, for example,

$$\sum_I{}' \alpha_I \varepsilon^I = \sum_{\{I:\, i_1 < \dots < i_k\}} \alpha_I \varepsilon^I.$$

Proposition 14.8 (A Basis for $\Lambda^k(V^*)$). *Let V be an n-dimensional vector space. If $\left(\varepsilon^i\right)$ is any basis for V^*, then for each positive integer $k \le n$, the collection of k-covectors*

$$\mathcal{E} = \left\{\varepsilon^I : I \text{ is an increasing multi-index of length } k\right\}$$

is a basis for $\Lambda^k(V^*)$. Therefore,

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n$, then $\dim \Lambda^k(V^*) = 0$.

Proof. The fact that $\Lambda^k(V^*)$ is the trivial vector space when $k > n$ follows immediately from Lemma 14.1(b), since every k-tuple of vectors is linearly dependent in that case. For the case $k \le n$, we need to show that the set $\mathcal{E}$ spans $\Lambda^k(V^*)$ and is linearly independent. Let (E_i) be the basis for V dual to $\left(\varepsilon^i\right)$.

To show that $\mathcal{E}$ spans $\Lambda^k(V^*)$, let $\alpha \in \Lambda^k(V^*)$ be arbitrary. For each multi-index $I = (i_1, \dots, i_k)$ (not necessarily increasing), define a real number α_I by

$$\alpha_I = \alpha(E_{i_1}, \dots, E_{i_k}).$$

The fact that α is alternating implies that $\alpha_I = 0$ if I contains a repeated index, and $\alpha_J = (\operatorname{sgn}\sigma)\alpha_I$ if $J = I_\sigma$ for $\sigma \in S_k$. For any multi-index J, therefore, Lemma 14.7 gives

$$\sum_I{}' \alpha_I \varepsilon^I(E_{j_1}, \dots, E_{j_k}) = \sum_I{}' \alpha_I \delta^I_J = \alpha_J = \alpha(E_{j_1}, \dots, E_{j_k}).$$

Thus $\sum_I' \alpha_I \varepsilon^I = \alpha$, so $\mathcal{E}$ spans $\Lambda^k(V^*)$.

To show that $\mathcal{E}$ is a linearly independent set, suppose the identity $\sum_I' \alpha_I \varepsilon^I = 0$ holds for some coefficients α_I. Let J be any increasing multi-index. Applying both sides of the identity to the vectors $(E_{j_1}, \dots, E_{j_k})$ and using Lemma 14.7, we get

$$0 = \sum_I{}' \alpha_I \varepsilon^I(E_{j_1}, \dots, E_{j_k}) = \alpha_J.$$

Thus each coefficient α_J is zero. $\square$

In particular, for an n-dimensional vector space V, this proposition implies that $\Lambda^n(V^*)$ is 1-dimensional and is spanned by $\varepsilon^{1\dots n}$. By definition, this elementary n-covector acts on vectors $(v_1, \dots, v_n)$ by taking the determinant of their component matrix $\mathbb{v} = \left(v_j^i\right)$. For example, on $\mathbb{R}^n$ with the standard basis, $e^{1\dots n}$ is precisely the determinant function.

One consequence of this is the following useful description of the behavior of an n-covector on an n-dimensional space under linear maps. Recall that if $T: V \to V$ is a linear map, the determinant of T is defined to be the determinant of the matrix representation of T with respect to any basis (see Appendix B, p. 633).

Proposition 14.9. *Suppose V is an n-dimensional vector space and $\omega \in \Lambda^n(V^*)$. If $T: V \to V$ is any linear map and $v_1, \dots, v_n$ are arbitrary vectors in V, then*

$$\omega(Tv_1, \dots, Tv_n) = (\det T)\omega(v_1, \dots, v_n). \tag{14.2}$$

Proof. Let (E_i) be any basis for V, and let $\left(\varepsilon^i\right)$ be the dual basis. Let $\left(T_i^j\right)$ denote the matrix of T with respect to this basis, and let $T_i = TE_i = T_i^j E_j$. By Proposition 14.8, we can write $\omega = c\varepsilon^{1\dots n}$ for some real number c.

Since both sides of (14.2) are multilinear functions of $v_1, \dots, v_n$, it suffices to verify the identity when the v_i's are basis vectors. Furthermore, since both sides are alternating, we only need to check the case $(v_1, \dots, v_n) = (E_1, \dots, E_n)$. In this case, the right-hand side of (14.2) is

$$(\det T)c\varepsilon^{1\dots n}(E_1, \dots, E_n) = c \det T.$$

On the other hand, the left-hand side reduces to

$$\omega(TE_1, \dots, TE_n) = c\varepsilon^{1\dots n}(T_1, \dots, T_n) = c \det\left(\varepsilon^j(T_i)\right) = c \det\left(T_i^j\right),$$

which is equal to the right-hand side. $\square$

The Wedge Product

In Chapter 12, we defined the symmetric product, which takes a pair of symmetric tensors α, β and yields another symmetric tensor $\alpha\beta = \operatorname{Sym}(\alpha \otimes \beta)$ whose rank is the sum of the ranks of the original ones.

In this section we define a similar product operation for alternating tensors. One way to define it would be to mimic what we did in the symmetric case and define the product of alternating tensors ω and η to be $\operatorname{Alt}(\omega \otimes \eta)$. However, we will use a different definition that looks more complicated at first but turns out to be much better suited to computation.

We continue with the assumption that V is a finite-dimensional real vector space. Given $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, we define their ***wedge product*** or ***exterior product*** to be the following $(k+l)$-covector:

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!}\operatorname{Alt}(\omega \otimes \eta). \tag{14.3}$$

The mysterious coefficient is motivated by the simplicity of the statement of the following lemma.

Lemma 14.10. *Let V be an n-dimensional vector space and let $\left(\varepsilon^1,\dots,\varepsilon^n\right)$ be a basis for V^*. For any multi-indices $I=(i_1,\dots,i_k)$ and $J=(j_1,\dots,j_l)$,*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}, \tag{14.4}$$

where $IJ=(i_1,\dots,i_k,j_1,\dots,j_l)$ is obtained by concatenating I and J.

Proof. By multilinearity, it suffices to show that

$$\varepsilon^I \wedge \varepsilon^J \left(E_{p_1},\dots,E_{p_{k+l}}\right) = \varepsilon^{IJ}\left(E_{p_1},\dots,E_{p_{k+l}}\right) \tag{14.5}$$

for any sequence $\left(E_{p_1},\dots,E_{p_{k+l}}\right)$ of basis vectors. We consider several cases.

CASE 1: *$P=(p_1,\dots,p_{k+l})$ has a repeated index.* In this case, both sides of (14.5) are zero by Lemma 14.1(c).

CASE 2: *P contains an index that does not appear in either I or J.* In this case, the right-hand side is zero by Lemma 14.7(c). Similarly, each term in the expansion of the left-hand side involves either ε^I or ε^J evaluated on a sequence of basis vectors that is not a permutation of I or J, respectively, so the left-hand side is also zero.

CASE 3: *$P=IJ$ and P has no repeated indices.* In this case, the right-hand side of (14.5) is equal to 1 by Lemma 14.7(c), so we need to show that the left-hand side is also equal to 1. By definition,

$$\begin{aligned}
&\varepsilon^I \wedge \varepsilon^J \left(E_{p_1},\dots,E_{p_{k+l}}\right)\\
&\quad= \frac{(k+l)!}{k!l!}\operatorname{Alt}\left(\varepsilon^I \otimes \varepsilon^J\right)\left(E_{p_1},\dots,E_{p_{k+l}}\right)\\
&\quad= \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)\varepsilon^I\left(E_{p_{\sigma(1)}},\dots,E_{p_{\sigma(k)}}\right)\varepsilon^J\left(E_{p_{\sigma(k+1)}},\dots,E_{p_{\sigma(k+l)}}\right).
\end{aligned}$$

By Lemma 14.7 again, the only terms in the sum above that give nonzero values are those in which σ permutes the first k indices and the last l indices of P separately. In other words, σ must be of the form $\sigma=\tau\eta$, where $\tau\in S_k$ acts by permuting $\{1,\dots,k\}$ and $\eta\in S_l$ acts by permuting $\{k+1,\dots,k+l\}$. Since

$\operatorname{sgn}(\tau\eta) = (\operatorname{sgn}\tau)(\operatorname{sgn}\eta)$, we have

$$
\begin{aligned}
&\varepsilon^I \wedge \varepsilon^J \left(E_{p_1}, \dots, E_{p_{k+l}}\right) \\
&\quad = \frac{1}{k!l!} \sum_{\substack{\tau \in S_k \\ \eta \in S_l}} (\operatorname{sgn}\tau)(\operatorname{sgn}\eta)\varepsilon^I \left(E_{p_{\tau(1)}}, \dots, E_{p_{\tau(k)}}\right) \varepsilon^J \left(E_{p_{k+\eta(1)}}, \dots, E_{p_{k+\eta(l)}}\right) \\
&\quad = \left(\frac{1}{k!} \sum_{\tau \in S_k} (\operatorname{sgn}\tau)\varepsilon^I \left(E_{p_{\tau(1)}}, \dots, E_{p_{\tau(k)}}\right) \right) \\
&\qquad \times \left(\frac{1}{l!} \sum_{\eta \in S_l} (\operatorname{sgn}\eta)\varepsilon^J \left(E_{p_{k+\eta(1)}}, \dots, E_{p_{k+\eta(l)}}\right) \right) \\
&\quad = \left(\operatorname{Alt} \varepsilon^I\right) \left(E_{p_1}, \dots, E_{p_k}\right) \left(\operatorname{Alt} \varepsilon^J\right) \left(E_{p_{k+1}}, \dots, E_{p_{k+l}}\right) \\
&\quad = \varepsilon^I \left(E_{p_1}, \dots, E_{p_k}\right) \varepsilon^J \left(E_{p_{k+1}}, \dots, E_{p_{k+l}}\right) = 1.
\end{aligned}
$$

CASE 4: *P is a permutation of IJ and has no repeated indices*. In this case, applying a permutation to P brings us back to Case . Since the effect of the permutation is to multiply both sides of (14.5) by the same sign, the result holds in this case as well. □

Proposition 14.11 (Properties of the Wedge Product). *Suppose ω, ω', η, η', and ξ are multicovectors on a finite-dimensional vector space V.*

(a) BILINEARITY: *For $a, a' \in \mathbb{R}$,*

$$
\begin{aligned}
(a\omega + a'\omega') \wedge \eta &= a(\omega \wedge \eta) + a'(\omega' \wedge \eta), \\
\eta \wedge (a\omega + a'\omega') &= a(\eta \wedge \omega) + a'(\eta \wedge \omega').
\end{aligned}
$$

(b) ASSOCIATIVITY:

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi.$$

(c) ANTICOMMUTATIVITY: *For $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,*

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega. \tag{14.6}$$

(d) *If $\left(\varepsilon^i\right)$ is any basis for V^* and $I = (i_1, \dots, i_k)$ is any multi-index, then*

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^I. \tag{14.7}$$

(e) *For any covectors $\omega^1, \dots, \omega^k$ and vectors $v_1, \dots, v_k$,*

$$\omega^1 \wedge \cdots \wedge \omega^k (v_1, \dots, v_k) = \det\left(\omega^j(v_i)\right). \tag{14.8}$$

Proof. Bilinearity follows immediately from the definition, because the tensor product is bilinear and Alt is linear. To prove associativity, note that Lemma 14.10 gives

$$\left(\varepsilon^I \wedge \varepsilon^J\right) \wedge \varepsilon^K = \varepsilon^{IJ} \wedge \varepsilon^K = \varepsilon^{IJK} = \varepsilon^I \wedge \varepsilon^{JK} = \varepsilon^I \wedge \left(\varepsilon^J \wedge \varepsilon^K\right).$$

The general case follows from bilinearity. Similarly, using Lemma 14.10 again, we get

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} = (\operatorname{sgn}\tau)\varepsilon^{JI} = (\operatorname{sgn}\tau)\varepsilon^J \wedge \varepsilon^I,$$

where τ is the permutation that sends IJ to JI. It is easy to check that $\operatorname{sgn}\tau = (-1)^{kl}$, because τ can be decomposed as a composition of kl transpositions (each index of I must be moved past each of the indices of J). Anticommutativity then follows from bilinearity.

Part (d) is an immediate consequence of Lemma 14.10 and induction. To prove (e), we note that the special case in which each ω^j is one of the basis covectors ε^{i_j} reduces to (14.7). Since both sides of (14.8) are multilinear in $\left(\omega^1,\dots,\omega^k\right)$, this suffices. □

Because of part (d) of this lemma, henceforth we generally use the notations ε^I and $\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$ interchangeably.

A k-covector η is said to be ***decomposable*** if it can be expressed in the form $\eta = \omega^1 \wedge \cdots \wedge \omega^k$, where $\omega^1,\dots,\omega^k$ are covectors. It is important to be aware that not every k-covector is decomposable when $k > 1$ (see Problem 14-2); however, it follows from Propositions 14.8 and 14.11(d) that every k-covector can be written as a linear combination of decomposable ones.

The definition and computational properties of the wedge product can seem daunting at first sight. However, the only properties that you need to remember for most practical purposes are the ones expressed in the preceding proposition. In fact, these properties are more than enough to determine the wedge product uniquely, as the following exercise shows.

► **Exercise 14.12.** Show that the wedge product is the unique associative, bilinear, and anticommutative map $\Lambda^k(V^*) \times \Lambda^l(V^*) \to \Lambda^{k+l}(V^*)$ satisfying (14.7).

For any n-dimensional vector space V, define a vector space $\Lambda(V^*)$ by

$$\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*).$$

It follows from Proposition 14.8 that $\dim \Lambda(V^*) = 2^n$. Proposition 14.11 shows that the wedge product turns $\Lambda(V^*)$ into an associative algebra, called the ***exterior algebra*** (or ***Grassmann algebra***) ***of*** V. This algebra is not commutative, but it has a closely related property. An algebra A is said to be ***graded*** if it has a direct sum decomposition $A = \bigoplus_{k\in\mathbb{Z}} A^k$ such that the product satisfies $\left(A^k\right)\left(A^l\right) \subseteq A^{k+l}$ for each k and l. A graded algebra is ***anticommutative*** if the product satisfies $ab = (-1)^{kl}ba$ for $a \in A^k$, $b \in A^l$. Proposition 14.11(c) shows that $\Lambda(V^*)$ is an anticommutative graded algebra (where we interpret $A^k = \Lambda^k(V^*)$ for $0 \le k \le n$, and $A^k = \{0\}$ otherwise).

As we observed at the beginning of this section, one could also define the wedge product without the unwieldy coefficient of (14.3). Some authors choose this alternative definition of the wedge product. To avoid confusion, we denote it by $\barwedge$:

$$\omega \barwedge \eta = \operatorname{Alt}(\omega \otimes \eta). \tag{14.9}$$

With this definition, (14.4) is replaced by

$$\varepsilon^I \mathbin{\overline{\wedge}} \varepsilon^J = \frac{k!l!}{(k+l)!}\varepsilon^{IJ},$$

and (14.8) is replaced by

$$\omega^1 \mathbin{\overline{\wedge}} \cdots \mathbin{\overline{\wedge}} \omega^k(v_1, \dots, v_k) = \frac{1}{k!}\det\left(\omega^i(v_j)\right) \tag{14.10}$$

whenever $\omega^1, \dots, \omega^k$ are covectors, as you can check.

Because of (14.8), we call definition (14.3) the ***determinant convention*** for the wedge product, and (14.9) the ***Alt convention***. The choice of which definition to use is largely a matter of taste. Although the definition of the Alt convention is perhaps a bit more natural, the computational advantages of the determinant convention make it preferable for most applications, and we use it exclusively in this book. (But see Problem 14-3 for an argument in favor of the Alt convention.) The determinant convention is most common in introductory differential geometry texts, and is used, for example, in [Boo86, Cha06, dC92, LeeJeff09, Pet06, Spi99]. The Alt convention is used in [KN69] and is more common in complex differential geometry.

Interior Multiplication

There is an important operation that relates vectors with alternating tensors. Let V be a finite-dimensional vector space. For each $v \in V$, we define a linear map $i_v \colon \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$, called ***interior multiplication by v***, as follows:

$$i_v\omega(w_1, \dots, w_{k-1}) = \omega(v, w_1, \dots, w_{k-1}).$$

In other words, $i_v\omega$ is obtained from ω by inserting v into the first slot. By convention, we interpret $i_v\omega$ to be zero when ω is a 0-covector (i.e., a number). Another common notation is

$$v \lrcorner\, \omega = i_v\omega.$$

This is often read "v into ω."

Lemma 14.13. *Let V be a finite-dimensional vector space and $v \in V$.*

(a) $i_v \circ i_v = 0$.
(b) *If $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,*

$$i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^k \omega \wedge (i_v\eta). \tag{14.11}$$

Proof. On k-covectors for $k \geq 2$, part (a) is immediate from the definition, because any alternating tensor gives zero when two of its arguments are identical. On 1-covectors and 0-covectors, it follows from the fact that $i_v \equiv 0$ on 0-covectors.

To prove (b), it suffices to consider the case in which both ω and η are decomposable, since every alternating tensor of positive rank can be written as a linear combination of decomposable ones. It is straightforward to verify that (b) follows in this special case from the following general formula for covectors $\omega^1, \dots, \omega^k$:

$$v \lrcorner \left(\omega^1 \wedge \cdots \wedge \omega^k\right) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v) \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k, \qquad (14.12)$$

where the hat indicates that ω^i is omitted.

To prove (14.12), let us write $v_1 = v$ and apply both sides to an arbitrary $(k-1)$-tuple of vectors $(v_2, \dots, v_k)$; then what we have to prove is

$$\begin{aligned} &\left(\omega^1 \wedge \cdots \wedge \omega^k\right)(v_1, \dots, v_k) \\ &\quad = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v_1) \left(\omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k\right)(v_2, \dots, v_k). \end{aligned} \qquad (14.13)$$

The left-hand side of (14.13) is the determinant of the matrix $\mathbb{v}$ whose (i, j)-entry is $\omega^i(v_j)$. To simplify the right-hand side, let $\mathbb{v}_j^i$ denote the $(k-1) \times (k-1)$ submatrix of $\mathbb{v}$ obtained by deleting the ith row and jth column. Then the right-hand side of (14.13) is

$$\sum_{i=1}^{k} (-1)^{i-1} \omega^i(v_1) \det \mathbb{v}_1^i.$$

This is just the expansion of $\det \mathbb{v}$ by minors along the first column, and therefore is equal to $\det \mathbb{v}$. □

When the wedge product is defined using the Alt convention, interior multiplication of a vector with a k-form has to be defined with an extra factor of k:

$$\bar{\imath}_v \omega(w_1, \dots, w_{k-1}) = k \omega(v, w_1, \dots, w_{k-1}).$$

This definition ensures that interior multiplication $\bar{\imath}_v$ still satisfies (14.11)—the factor of k compensates for the difference between the factors of $1/k!$ and $1/(k-1)!$ that occur when the left-hand and right-hand sides of (14.13) are evaluated using the Alt convention.

Differential Forms on Manifolds

Now we turn our attention to an n-dimensional smooth manifold M (with or without boundary). Recall that $T^k T^* M$ is the bundle of covariant k-tensors on M. The subset of $T^k T^* M$ consisting of alternating tensors is denoted by $\Lambda^k T^* M$:

$$\Lambda^k T^* M = \coprod_{p \in M} \Lambda^k (T_p^* M).$$

► **Exercise 14.14.** Show that $\Lambda^k T^* M$ is a smooth subbundle of $T^k T^* M$, and therefore is a smooth vector bundle of rank $\binom{n}{k}$ over M.

A section of $\Lambda^k T^*M$ is called a ***differential k-form***, or just a ***k-form***; this is a (continuous) tensor field whose value at each point is an alternating tensor. The integer k is called the ***degree*** of the form. We denote the vector space of smooth k-forms by

$$\Omega^k(M) = \Gamma\big(\Lambda^k T^*M\big).$$

The wedge product of two differential forms is defined pointwise: $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$. Thus, the wedge product of a k-form with an l-form is a $(k+l)$-form. If f is a 0-form and η is a k-form, we interpret the wedge product $f \wedge \eta$ to mean the ordinary product $f\eta$. If we define

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M), \tag{14.14}$$

then the wedge product turns $\Omega^*(M)$ into an associative, anticommutative graded algebra.

In any smooth chart, a k-form ω can be written locally as

$$\omega = {\sum_I}' \omega_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} = {\sum_I}' \omega_I \, dx^I,$$

where the coefficients ω_I are continuous functions defined on the coordinate domain, and we use dx^I as an abbreviation for $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ (not to be mistaken for the differential of a real-valued function x^I). Proposition 10.22 shows that ω is smooth on U if and only if the component functions ω_I are smooth. In terms of differential forms, the result of Lemma 14.7(c) translates to

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}\right) = \delta^I_J.$$

Thus the component functions ω_I of ω are determined by

$$\omega_I = \omega\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right).$$

Example 14.15. A 0-form is just a continuous real-valued function, and a 1-form is a covector field. On $\mathbb{R}^3$, some examples of smooth 2-forms are given by

$$\begin{aligned} \omega &= (\sin xy)\, dy \wedge dz; \\ \eta &= dx \wedge dy + dx \wedge dz + dy \wedge dz. \end{aligned}$$

Every 3-form on $\mathbb{R}^3$ is a continuous real-valued function times $dx \wedge dy \wedge dz$. //

If $F\colon M \to N$ is a smooth map and ω is a differential form on N, the pullback $F^*\omega$ is a differential form on M, defined as for any covariant tensor field:

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}\big(dF_p(v_1), \ldots, dF_p(v_k)\big).$$

Lemma 14.16. *Suppose $F\colon M \to N$ is smooth.*

(a) $F^*\colon \Omega^k(N) \to \Omega^k(M)$ *is linear over* $\mathbb{R}$.
(b) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.
(c) *In any smooth chart,*

$$F^*\left(\sum_I{}' \omega_I \, dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F)\, d\left(y^{i_1} \circ F\right) \wedge \cdots \wedge d\left(y^{i_k} \circ F\right).$$

► **Exercise 14.17.** Prove this lemma.

This lemma gives a computational rule for pullbacks of differential forms similar to the ones we developed for covector fields and arbitrary tensor fields earlier.

Example 14.18. Define $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ by $F(u,v) = \left(u, v, u^2 - v^2\right)$, and let ω be the 2-form $y\, dx \wedge dz + x\, dy \wedge dz$ on $\mathbb{R}^3$. The pullback $F^*\omega$ is computed as follows:

$$\begin{aligned} F^*(y\, dx \wedge dz + x\, dy \wedge dz) &= v\, du \wedge d\left(u^2 - v^2\right) + u\, dv \wedge d\left(u^2 - v^2\right) \\ &= v\, du \wedge (2u\, du - 2v\, dv) + u\, dv \wedge (2u\, du - 2v\, dv) \\ &= -2v^2\, du \wedge dv + 2u^2\, dv \wedge du, \end{aligned}$$

where we have used the fact $du \wedge du = dv \wedge dv = 0$ by anticommutativity. Because $dv \wedge du = -du \wedge dv$, this simplifies to

$$F^*\omega = -2\left(u^2 + v^2\right) du \wedge dv. \qquad //$$

The same technique can also be used to compute the expression for a differential form in another smooth chart.

Example 14.19. Let $\omega = dx \wedge dy$ on $\mathbb{R}^2$. Thinking of the transformation to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ as an expression for the identity map with respect to different coordinates on the domain and codomain, we obtain

$$\begin{aligned} dx \wedge dy &= d(r\cos\theta) \wedge d(r\sin\theta) \\ &= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= r\, dr \wedge d\theta. \end{aligned} \qquad //$$

The similarity between this formula and the formula for changing a double integral from Cartesian to polar coordinates is striking. The following proposition generalizes this.

Proposition 14.20 (Pullback Formula for Top-Degree Forms). *Let $F\colon M \to N$ be a smooth map between n-manifolds with or without boundary. If $\left(x^i\right)$ and $\left(y^j\right)$ are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:*

$$F^*\left(u\, dy^1 \wedge \cdots \wedge dy^n\right) = (u \circ F)(\det DF)\, dx^1 \wedge \cdots \wedge dx^n, \tag{14.15}$$

where DF represents the Jacobian matrix of F in these coordinates.

Proof. Because the fiber of $\Lambda^n T^*M$ is spanned by $dx^1 \wedge \cdots \wedge dx^n$ at each point, it suffices to show that both sides of (14.15) give the same result when evaluated on $\left(\partial/\partial x^1, \ldots, \partial/\partial x^n\right)$. From Lemma 14.16,

$$F^*\left(u\,dy^1 \wedge \cdots \wedge dy^n\right) = (u \circ F) dF^1 \wedge \cdots \wedge dF^n.$$

Proposition 14.11(e) shows that

$$dF^1 \wedge \cdots \wedge dF^n \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) = \det\left(dF^j\left(\frac{\partial}{\partial x^i}\right)\right) = \det\left(\frac{\partial F^j}{\partial x^i}\right).$$

Therefore, the left-hand side of (14.15) gives $(u \circ F)\det DF$ when applied to $\left(\partial/\partial x^1, \ldots, \partial/\partial x^n\right)$. On the other hand, the right-hand side gives the same thing, because $dx^1 \wedge \cdots \wedge dx^n\left(\partial/\partial x^1, \ldots, \partial/\partial x^n\right) = 1$. □

Corollary 14.21. *If $\left(U, \left(x^i\right)\right)$ and $\left(\widetilde{U}, \left(\widetilde{x}^j\right)\right)$ are overlapping smooth coordinate charts on M, then the following identity holds on $U \cap \widetilde{U}$:*

$$d\widetilde{x}^1 \wedge \cdots \wedge d\widetilde{x}^n = \det\left(\frac{\partial \widetilde{x}^j}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n. \tag{14.16}$$

Proof. Apply the previous proposition with F equal to the identity map of $U \cap \widetilde{U}$, but using coordinates $\left(x^i\right)$ in the domain and $\left(\widetilde{x}^j\right)$ in the codomain. □

Interior multiplication also extends naturally to vector fields and differential forms, simply by letting it act pointwise: if $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, define a $(k-1)$-form $X \lrcorner\, \omega = i_X \omega$ by

$$(X \lrcorner\, \omega)_p = X_p \lrcorner\, \omega_p.$$

► **Exercise 14.22.** Let X be a smooth vector field on M.

(a) Show that if ω is a smooth differential form, then $i_X\omega$ is smooth.
(b) Verify that $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is linear over $C^\infty(M)$ and therefore corresponds to a smooth bundle homomorphism $i_X : \Lambda^k T^*M \to \Lambda^{k-1} T^*M$.

Exterior Derivatives

In this section we define a natural differential operator on smooth forms, called the *exterior derivative*. It is a generalization of the differential of a function.

To give some idea of where the motivation for the exterior derivative comes from, let us look back at a question we addressed in Chapter 11. Recall that not all 1-forms are differentials of functions: given a smooth 1-form ω, a necessary condition for the existence of a smooth function f such that $\omega = df$ is that ω be ***closed***, which means that it satisfies

$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0 \tag{14.17}$$

in every smooth coordinate chart. Since this is a coordinate-independent property by Proposition 11.45, one might hope that the expression on the left side of (14.17) would have a meaning of its own. The key is that it is antisymmetric in the indices i and j, so it can be interpreted as the ij-component of an alternating tensor field, i.e., a 2-form. We can define a 2-form $d\omega$ locally in each smooth chart by

$$d\omega = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j, \tag{14.18}$$

so it follows that ω is closed if and only if $d\omega = 0$ in each chart.

It turns out that $d\omega$ is actually well defined globally, independently of the choice of coordinate chart, and this definition has a significant generalization to differential forms of all degrees. For each smooth manifold M with or without boundary, we will show that there is a differential operator $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying $d(d\omega) = 0$ for all ω. Thus, it will follow that a necessary condition for a smooth k-form ω to be equal to $d\eta$ for some $(k-1)$-form η is that $d\omega = 0$.

The definition of d on Euclidean space is straightforward: if $\omega = \sum'_J \omega_J\, dx^J$ is a smooth k-form on an open subset $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, we define its ***exterior derivative*** $d\omega$ to be the following $(k+1)$-form:

$$d\left({\sum_J}' \omega_J\, dx^J \right) = {\sum_J}' d\omega_J \wedge dx^J, \tag{14.19}$$

where $d\omega_J$ is the differential of the function ω_J. In somewhat more detail, this is

$$d\left({\sum_J}' \omega_J\, dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right) = {\sum_J}' \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}. \tag{14.20}$$

Observe that when ω is a 1-form, this becomes

$$\begin{aligned} d\left(\omega_j\, dx^j\right) &= \sum_{i,j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \sum_{i>j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j \end{aligned}$$

after we interchange i and j in the second sum and use the fact that $dx^j \wedge dx^i = -dx^i \wedge dx^j$, so this is consistent with our earlier definition. For a smooth 0-form f (a real-valued function), (14.20) reduces to

$$df = \frac{\partial f}{\partial x^i} dx^i,$$

which is just the differential of f.

In order to transfer this definition to manifolds, we need to check that it satisfies the following properties.

Proposition 14.23 (Properties of the Exterior Derivative on $\mathbb{R}^n$).

(a) *d is linear over $\mathbb{R}$.*

(b) *If ω is a smooth k-form and η is a smooth l-form on an open subset $U \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) *$d \circ d \equiv 0$.*

(d) *d commutes with pullbacks: if U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, V is an open subset of $\mathbb{R}^m$ or $\mathbb{H}^m$, $F\colon U \to V$ is a smooth map, and $\omega \in \Omega^k(V)$, then*

$$F^*(d\omega) = d\big(F^*\omega\big). \tag{14.21}$$

Proof. Linearity of d is an immediate consequence of the definition. To prove (b), by linearity it suffices to consider terms of the form $\omega = u\, dx^I \in \Omega^k(U)$ and $\eta = v\, dx^J \in \Omega^l(U)$ for smooth real-valued functions u and v. First, though, we need to know that d satisfies $d(u\, dx^I) = du \wedge dx^I$ for *any* multi-index I, not just increasing ones. If I has repeated indices, then clearly $d\big(u\, dx^I\big) = 0 = du \wedge dx^I$. If not, let σ be the permutation sending I to an increasing multi-index J. Then

$$d\big(u\, dx^I\big) = (\operatorname{sgn}\sigma) d\big(u\, dx^J\big) = (\operatorname{sgn}\sigma)\, du \wedge dx^J = du \wedge dx^I.$$

Using this, we compute

$$\begin{aligned}
d(\omega \wedge \eta) &= d\left(\big(u\, dx^I\big) \wedge \big(v\, dx^J\big)\right) \\
&= d\big(uv\, dx^I \wedge dx^J\big) \\
&= (v\, du + u\, dv) \wedge dx^I \wedge dx^J \\
&= \big(du \wedge dx^I\big) \wedge \big(v\, dx^J\big) + (-1)^k \big(u\, dx^I\big) \wedge \big(dv \wedge dx^J\big) \\
&= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta,
\end{aligned}$$

where the $(-1)^k$ comes from the fact that $dv \wedge dx^I = (-1)^k dx^I \wedge dv$ because dv is a 1-form and dx^I is a k-form.

We prove (c) first for the special case of a 0-form, which is just a real-valued function. In this case,

$$\begin{aligned}
d(du) &= d\left(\frac{\partial u}{\partial x^j}\, dx^j\right) = \frac{\partial^2 u}{\partial x^i \partial x^j}\, dx^i \wedge dx^j \\
&= \sum_{i<j} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j = 0.
\end{aligned}$$

For the general case, we use the $k = 0$ case together with (b) to compute

$$\begin{aligned} d(d\omega) &= d\left(\sum_J{}' d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) \\ &= \sum_J{}' d(d\omega_J) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \\ &\quad + \sum_J{}' \sum_{i=1}^{k} (-1)^i d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge d\left(dx^{j_i}\right) \wedge \cdots \wedge dx^{j_k} = 0. \end{aligned}$$

Finally, to prove (d), again it suffices to consider $\omega = u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. For such a form, the left-hand side of (14.21) is

$$\begin{aligned} F^*\left(d\left(u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right)\right) &= F^*\left(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) \\ &= d(u \circ F) \wedge d\left(x^{i_1} \circ F\right) \wedge \cdots \wedge d\left(x^{i_k} \circ F\right), \end{aligned}$$

and the right-hand side is

$$\begin{aligned} d\left(F^*\left(u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right)\right) &= d\left((u \circ F)\, d\left(x^{i_1} \circ F\right) \wedge \cdots \wedge d\left(x^{i_k} \circ F\right)\right) \\ &= d(u \circ F) \wedge d\left(x^{i_1} \circ F\right) \wedge \cdots \wedge d\left(x^{i_k} \circ F\right), \end{aligned}$$

so they are equal. □

These results allow us to transplant the definition of the exterior derivative to manifolds.

Theorem 14.24 (Existence and Uniqueness of Exterior Differentiation). *Suppose M is a smooth manifold with or without boundary. There are unique operators $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ for all k, called **exterior differentiation**, satisfying the following four properties*:

(i) *d is linear over $\mathbb{R}$.*
(ii) *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(iii) $d \circ d \equiv 0$.
(iv) *For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.*

In any smooth coordinate chart, d is given by (14.19).

Proof. First, we prove existence. Suppose $\omega \in \Omega^k(M)$. We wish to define $d\omega$ by means of the coordinate formula (14.19) in each chart; more precisely, this means that for each smooth chart (U, φ) for M, we wish to set

$$d\omega = \varphi^* d\left(\varphi^{-1*}\omega\right). \tag{14.22}$$

To see that this is well defined, we just note that for any other smooth chart (V,ψ), the map $\varphi\circ\psi^{-1}$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, so Proposition 14.23(d) implies

$$\left(\varphi\circ\psi^{-1}\right)^* d\left(\varphi^{-1*}\omega\right) = d\left(\left(\varphi\circ\psi^{-1}\right)^*\varphi^{-1*}\omega\right).$$

Together with the fact that $\left(\varphi\circ\psi^{-1}\right)^* = \psi^{-1*}\circ\varphi^*$, this implies $\varphi^* d\left(\varphi^{-1*}\omega\right) = \psi^* d\left(\psi^{-1*}\omega\right)$, so $d\omega$ is well defined. It satisfies (i)–(iv) by virtue of Proposition 14.23.

To prove uniqueness, suppose that d is any operator satisfying (i)–(iv). First we need to show that $d\omega$ is determined locally: if ω_1 and ω_2 are k-forms that agree on an open subset $U\subseteq M$, then $d\omega_1 = d\omega_2$ on U. To see this, let $p\in U$ be arbitrary, let $\eta=\omega_1-\omega_2$, and let $\psi\in C^\infty(M)$ be a bump function that is identically 1 on some neighborhood of p and supported in U. Then $\psi\eta$ is identically zero, so (i)–(iv) imply $0 = d(\psi\eta) = d\psi\wedge\eta + \psi\, d\eta$. Evaluating this at p and using the facts that $\psi(p)=1$ and $d\psi_p = 0$, we conclude that $d\omega_1|_p - d\omega_2|_p = d\eta_p = 0$.

Now let $\omega\in\Omega^k(M)$ be arbitrary, and let (U,φ) be any smooth coordinate chart on M. We can write ω in coordinates as $\sum_I' \omega_I\, dx^I$ on U. For any $p\in U$, by means of a bump function we can construct global smooth functions $\widetilde{\omega}_I$ and $\widetilde{x}^i$ on M that agree with ω_I and x^i in a neighborhood of p. By virtue of (i)–(iv) together with the observation in the preceding paragraph, it follows that (14.19) holds at p. Since p was arbitrary, this d must be equal to the one we defined above. □

If $A=\bigoplus_k A^k$ is a graded algebra, a linear map $T\colon A\to A$ is said to be a ***map of degree m*** if $T\left(A^k\right)\subseteq A^{k+m}$ for each k. It is said to be an ***antiderivation*** if it satisfies $T(xy) = (Tx)y + (-1)^k x(Ty)$ whenever $x\in A^k$ and $y\in A^l$. The preceding theorem can be summarized by saying that the differential on functions extends uniquely to an antiderivation of $\Omega^*(M)$ of degree $+1$ whose square is zero.

► **Exercise 14.25.** Suppose M is a smooth manifold and $X\in\mathfrak{X}(M)$. Show that interior multiplication $i_X\colon \Omega^*(M)\to\Omega^*(M)$ is an antiderivation of degree -1 whose square is zero.

Another important feature of the exterior derivative is that it commutes with all pullbacks.

Proposition 14.26 (Naturality of the Exterior Derivative). *If $F\colon M\to N$ is a smooth map, then for each k the pullback map $F^*\colon\Omega^k(N)\to\Omega^k(M)$ commutes with d: for all $\omega\in\Omega^k(N)$,*

$$F^*(d\omega) = d(F^*\omega). \tag{14.23}$$

Proof. If (U,φ) and (V,ψ) are smooth charts for M and N, respectively, we can apply Proposition 14.23(d) to the coordinate representation $\psi\circ F\circ\varphi^{-1}$. Using (14.22) twice, we compute as follows on $U\cap F^{-1}(V)$:

$$\begin{aligned} F^*(d\omega) &= F^*\psi^* d\left(\psi^{-1*}\omega\right) \\ &= \varphi^*\circ\left(\psi\circ F\circ\varphi^{-1}\right)^* d\left(\psi^{-1*}\omega\right) \end{aligned}$$

$$
\begin{aligned}
&= \varphi^* d\left(\left(\psi \circ F \circ \varphi^{-1}\right)^* \psi^{-1*}\omega\right) \\
&= \varphi^* d\left(\varphi^{-1*} F^*\omega\right) \\
&= d(F^*\omega). \qquad \square
\end{aligned}
$$

Extending the terminology we introduced for covector fields in Chapter 11, we say that a smooth differential form $\omega \in \Omega^k(M)$ is ***closed*** if $d\omega = 0$, and ***exact*** if there exists a smooth $(k-1)$-form η on M such that $\omega = d\eta$. Because the exterior derivative of a 1-form satisfies (14.18), this definition of closed 1-forms agrees with the one we gave in Chapter 11.

The fact that $d \circ d = 0$ implies that every exact form is closed. In Chapter 11, we saw that the converse might not be true: the 1-form ω of Example 11.36 is closed but not exact on $\mathbb{R}^2 \smallsetminus \{0\}$. On the other hand, we showed there that every closed 1-form is locally exact. We will return to these questions in Chapter 17, where we will show that this behavior is typical: closed forms are always locally exact but not necessarily globally, so the question of whether a given closed form is exact depends on global properties of the manifold.

Exterior Derivatives and Vector Calculus in $\mathbb{R}^3$

Example 14.27. Let us work out the exterior derivatives of arbitrary 1-forms and 2-forms on $\mathbb{R}^3$. Any smooth 1-form can be written

$$\omega = P\,dx + Q\,dy + R\,dz$$

for some smooth functions P, Q, R. Using (14.19) and the fact that the wedge product of any 1-form with itself is zero, we compute

$$
\begin{aligned}
d\omega &= dP \wedge dx + dQ \wedge dy + dR \wedge dz \\
&= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right) \wedge dy \\
&\quad + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right) \wedge dz \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dx \wedge dz \\
&\quad + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz.
\end{aligned}
$$

An arbitrary 2-form on $\mathbb{R}^3$ can be written

$$\eta = u\,dx \wedge dy + v\,dx \wedge dz + w\,dy \wedge dz.$$

A similar computation shows that

$$d\eta = \left(\frac{\partial u}{\partial z} - \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\right) dx \wedge dy \wedge dz. \qquad //$$

Recall the classical vector calculus operators on $\mathbb{R}^n$: the (Euclidean) gradient of a function $f \in C^\infty(\mathbb{R}^n)$ and the ***divergence*** of a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ are defined by

$$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}, \qquad \operatorname{div} X = \sum_{i=1}^{n} \frac{\partial X^i}{\partial x^i}. \tag{14.24}$$

In addition, in the case $n = 3$, the curl of a vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ is defined by (11.26). It is interesting to note that the components of the 2-form $d\omega$ in the preceding example are exactly the components of the curl of the vector field with components (P, Q, R) (except perhaps in a different order and with different signs). Similarly, but for signs and ordering of terms, there is a strong analogy between the formula for $d\eta$ and the divergence of a vector field. These analogies can be made precise in the following way.

The Euclidean metric on $\mathbb{R}^3$ yields an index-lowering isomorphism $\flat\colon \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$. Interior multiplication yields another map $\beta\colon \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ as follows:

$$\beta(X) = X \lrcorner (dx \wedge dy \wedge dz). \tag{14.25}$$

It is easy to check that β is linear over $C^\infty(\mathbb{R}^3)$, so it corresponds to a smooth bundle homomorphism from TM to $\Lambda^2 T^*\mathbb{R}^3$. It is a bundle isomorphism because it is injective and both TM and $\Lambda^2 T^*\mathbb{R}^3$ are bundles of rank 3. Similarly, we define a smooth bundle isomorphism $*\colon C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$ by

$$*(f) = f\, dx \wedge dy \wedge dz. \tag{14.26}$$

The relationships among all of these operators are summarized in the following diagram:

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\operatorname{grad}"] \arrow[d, "\mathrm{Id}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\operatorname{curl}"] \arrow[d, "\flat"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\operatorname{div}"] \arrow[d, "\beta"] & C^\infty(\mathbb{R}^3) \arrow[d, "*"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^3(\mathbb{R}^3).
\end{tikzcd} \tag{14.27}$$

► **Exercise 14.28.** Prove that diagram (14.27) commutes, and use it to give a quick proof that $\operatorname{curl} \circ \operatorname{grad} \equiv 0$ and $\operatorname{div} \circ \operatorname{curl} \equiv 0$ on $\mathbb{R}^3$. Prove also that the analogues of the left-hand and right-hand squares commute when $\mathbb{R}^3$ is replaced by $\mathbb{R}^n$ for any n.

The desire to generalize these vector calculus operators from $\mathbb{R}^3$ to higher dimensions was one of the main motivations for developing the theory of differential forms. The curl, in particular, makes sense as an operator on vector fields only in dimension 3, whereas the exterior derivative expresses the same information but makes sense in all dimensions.

An Invariant Formula for the Exterior Derivative

In addition to the coordinate formula (14.19) that we used in the definition of d, there is another formula for d that is often useful, not least because it is manifestly coordinate-independent. The formula for 1-forms is by far the most important, and is the easiest to state and prove, so we begin with that. Note the similarity between this and the formula of Proposition 11.45.

Proposition 14.29 (Exterior Derivative of a 1-Form). *For any smooth* 1*-form* ω *and smooth vector fields* X *and* Y,

$$d\omega(X,Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X,Y]\big). \tag{14.28}$$

Proof. Since any smooth 1-form can be expressed locally as a sum of terms of the form $u\,dv$ for smooth functions u and v, it suffices to consider that case. Suppose $\omega = u\,dv$, and X,Y are smooth vector fields. Then the left-hand side of (14.28) is

$$\begin{aligned} d(u\,dv)(X,Y) &= du\wedge dv(X,Y) = du(X)\,dv(Y) - dv(X)\,du(Y)\\ &= Xu\,Yv - Xv\,Yu. \end{aligned}$$

The right-hand side is

$$\begin{aligned} &X\big(u\,dv(Y)\big) - Y\big(u\,dv(X)\big) - u\,dv\big([X,Y]\big)\\ &\quad = X(u\,Yv) - Y(u\,Xv) - u\,[X,Y]v\\ &\quad = (Xu\,Yv + u\,XYv) - (Yu\,Xv + u\,YXv) - u(XYv - YXv). \end{aligned}$$

After the two $u\,XYv$ terms and the two $u\,YXv$ terms are canceled, this is equal to the left-hand side. □

We will see some applications of (14.28) in later chapters. Here is our first one. It shows that the exterior derivative is in a certain sense dual to the Lie bracket. In particular, it shows that if we know all the Lie brackets of basis vector fields in a smooth local frame, we can compute the exterior derivatives of the dual covector fields, and vice versa.

Proposition 14.30. *Let* M *be a smooth* n*-manifold with or without boundary, let* (E_i) *be a smooth local frame for* M, *and let* $\big(\varepsilon^i\big)$ *be the dual coframe. For each* i, *let* b^i_{jk} *denote the component functions of the exterior derivative of* ε^i *in this frame, and for each* j,k, *let* c^i_{jk} *be the component functions of the Lie bracket* $[E_j,E_k]$:

$$d\varepsilon^i = \sum_{j<k} b^i_{jk}\varepsilon^j\wedge\varepsilon^k; \qquad [E_j,E_k] = c^i_{jk}E_i.$$

Then $b^i_{jk} = -c^i_{jk}$.

► **Exercise 14.31.** Use (14.28) to prove the preceding proposition.

The generalization of (14.28) to higher-degree forms is more complicated.

Proposition 14.32 (Invariant Formula for the Exterior Derivative). *Let M be a smooth manifold with or without boundary, and $\omega \in \Omega^k(M)$. For any smooth vector fields $X_1, \dots, X_{k+1}$ on M,*

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= \sum_{1 \le i \le k+1} (-1)^{i-1} X_i \big(\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\big) \\ &\quad + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}\big), \end{aligned} \tag{14.29}$$

where the hats indicate omitted arguments.

Proof. For this proof, let us denote the entire expression on the right-hand side of (14.29) by $D\omega(X_1, \dots, X_{k+1})$, and the two sums on the right-hand side by $I(X_1, \dots, X_{k+1})$ and $II(X_1, \dots, X_{k+1})$, respectively. Note that $D\omega$ is obviously multilinear over $\mathbb{R}$. We begin by showing that, like $d\omega$, it is actually multilinear over $C^\infty(M)$, which is to say that for $1 \le p \le k+1$ and $f \in C^\infty(M)$,

$$D\omega(X_1, \dots, fX_p, \dots, X_{k+1}) = fD\omega(X_1, \dots, X_p, \dots, X_{k+1}).$$

In the expansion of $I(X_1, \dots, fX_p, \dots, X_{k+1})$, f obviously factors out of the $i = p$ term. The other terms expand as follows:

$$\begin{aligned} &\sum_{i \ne p} (-1)^{i-1} X_i \big(f\omega\big(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}\big)\big) \\ &\quad = \sum_{i \ne p} (-1)^{i-1} \Big(fX_i \big(\omega\big(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}\big)\big) \\ &\qquad + (X_i f)\omega\big(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}\big)\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} &I(X_1, \dots, fX_p, \dots, X_{k+1}) \\ &\quad = fI(X_1, \dots, X_p, \dots, X_{k+1}) + \sum_{i \ne p} (-1)^{i-1} (X_i f)\omega\big(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}\big). \end{aligned} \tag{14.30}$$

Consider next the expansion of II. Again, f factors out of all the terms in which $i \ne p$ and $j \ne p$. To expand the other terms, we use (8.11), which implies

$$\begin{aligned} [fX_p, X_j] &= f[X_p, X_j] - (X_j f)X_p, \\ [X_i, fX_p] &= f[X_i, X_p] + (X_i f)X_p. \end{aligned}$$

Inserting these formulas into the $i = p$ and $j = p$ terms, we obtain

$$\begin{aligned}
&\mathit{II}(X_1, \dots, fX_p, \dots, X_{k+1}) \\
&\quad = f\,\mathit{II}(X_1, \dots, X_p, \dots, X_{k+1}) \\
&\qquad + \sum_{p<j} (-1)^{p+j+1} (X_j f)\omega\big(X_p, X_1, \dots, \widehat{X_p}, \dots, \widehat{X_j}, \dots, X_{k+1}\big) \\
&\qquad + \sum_{i<p} (-1)^{i+p} (X_i f)\omega\big(X_p, X_1, \dots, \widehat{X_i}, \dots, \widehat{X_p}, \dots, X_{k+1}\big).
\end{aligned}$$

Rearranging the arguments in these two sums so as to put X_p into its original position, we see that they exactly cancel the sum in (14.30). This completes the proof that $D\omega$ is multilinear over $C^\infty(M)$, so it defines a smooth $(k+1)$-tensor field.

Since both $D\omega$ and $d\omega$ are smooth tensor fields, we can verify the equation $D\omega = d\omega$ in any frame that is convenient. By multilinearity, it suffices to show that both sides give the same result when applied to an arbitrary sequence of basis vectors in some chosen local frame in a neighborhood of each point. The computations are greatly simplified by working in a coordinate frame, for which all the Lie brackets vanish. Thus, let $\big(U, \big(x^i\big)\big)$ be an arbitrary smooth chart on M. Because both $d\omega$ and $D\omega$ depend linearly on ω, we may assume that $\omega = u\,dx^I$ for some smooth function u and some increasing multi-index $I = (i_1, \dots, i_k)$, so

$$d\omega = du \wedge dx^I = \sum_m \frac{\partial u}{\partial x^m}\, dx^m \wedge dx^I.$$

If $J = (j_1, \dots, j_{k+1})$ is any multi-index of length $k+1$, it follows that

$$d\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) = \sum_m \frac{\partial u}{\partial x^m} \delta^{mI}_J.$$

The only terms in this sum that can possibly be nonzero are those for which m is equal to one of the indices in J, say $m = j_p$. In this case, it is easy to check that $\delta^{mI}_J = (-1)^{p-1}\delta^I_{\widehat{J}_p}$, where $\widehat{J}_p = \big(j_1, \dots, \widehat{j_p}, \dots, j_{k+1}\big)$, so

$$d\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) = \sum_{1 \le p \le k+1} (-1)^{p-1} \frac{\partial u}{\partial x^{j_p}} \delta^I_{\widehat{J}_p}. \tag{14.31}$$

On the other hand, because all the Lie brackets are zero, we have

$$\begin{aligned}
&D\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right) \\
&\quad = \sum_{1 \le p \le k+1} (-1)^{p-1} \frac{\partial}{\partial x^{j_p}}\left(u\,dx^I\left(\frac{\partial}{\partial x^{j_1}}, \dots, \widehat{\frac{\partial}{\partial x^{j_p}}}, \dots, \frac{\partial}{\partial x^{j_{k+1}}}\right)\right) \\
&\quad = \sum_{1 \le p \le k+1} (-1)^{p-1} \frac{\partial u}{\partial x^{j_p}} \delta^I_{\widehat{J}_p},
\end{aligned}$$

which agrees with (14.31). □

It is worth remarking that formula (14.29) can be used to give an invariant definition of d, as well as an alternative proof of Theorem 14.24 on the existence, uniqueness, and properties of d. As the proof of Proposition 14.32 showed, the right-hand side of (14.29) is multilinear over $C^\infty(M)$ as a function of $(X_1, \dots, X_{k+1})$. By the tensor characterization lemma (Lemma 12.24), therefore, it defines a smooth covariant $(k+1)$-tensor field. A straightforward (if slightly tedious) verification shows that it changes sign whenever two of its arguments are interchanged, so in fact it defines a smooth $(k+1)$-form, which we could have used as a definition of $d\omega$. The rest of the proof of Proposition 14.32 then shows that $d\omega$ is actually given locally by the coordinate formula (14.19), and so the properties asserted in Theorem 14.24 follow just as before. We have chosen to define d by means of its coordinate formula because that formula is generally much easier to remember and to work with. Except in the $k = 1$ case, the invariant formula (14.29) is too complicated to be of much use for computation; in addition, it has the serious flaw that in order to compute the action of $d\omega$ on vectors $(v_1, \dots, v_k)$ at a point $p \in M$, one must first extend them to vector *fields* in a neighborhood of p. Nonetheless, it does have some important theoretical consequences, so it is useful to know that it exists.

Lie Derivatives of Differential Forms

In Chapter 12, we derived some formulas for computing Lie derivatives of smooth tensor fields (see Corollary 12.33 and Example 12.35), which apply equally well to differential forms. However, in the case of differential forms, the exterior derivative yields a much more powerful formula for computing Lie derivatives, which also has significant theoretical consequences. As we did in Chapter 12, we restrict attention to the case of manifolds without boundary for simplicity; but these results extend easily to manifolds with boundary and vector fields tangent to the boundary.

First, we note a simple fact that will be useful in both proofs and computations: Lie differentiation satisfies a product rule with respect to wedge products.

Proposition 14.33. *Suppose M is a smooth manifold, $V \in \mathfrak{X}(M)$, and $\omega, \eta \in \Omega^*(M)$. Then*

$$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V \omega) \wedge \eta + \omega \wedge (\mathcal{L}_V \eta).$$

► **Exercise 14.34.** Prove the preceding proposition.

The next theorem is the main result of this section. It gives a remarkable formula for Lie derivatives of differential forms, which dates back to Élie Cartan (1869–1951), the French mathematician who invented the theory of differential forms.

Theorem 14.35 (Cartan's Magic Formula). *On a smooth manifold M, for any smooth vector field V and any smooth differential form ω,*

$$\mathcal{L}_V \omega = V \lrcorner (d\omega) + d(V \lrcorner \omega). \tag{14.32}$$

Proof. We prove that (14.32) holds for smooth k-forms by induction on k. We begin with a smooth 0-form f, in which case

$$V \lrcorner (df) + d(V \lrcorner f) = V \lrcorner df = df(V) = Vf = \mathscr{L}_V f,$$

which is (14.32).

Now let $k \geq 1$, and suppose (14.32) has been proved for forms of degree less than k. Let ω be an arbitrary smooth k-form, written in smooth local coordinates as

$$\omega = \sum_I{}' \omega_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Writing $u = x^{i_1}$ and $\beta = \omega_I \, dx^{i_2} \wedge \cdots \wedge dx^{i_k}$, we see that each term in this sum can be written in the form $du \wedge \beta$, where u is a smooth function and β is a smooth $(k-1)$-form. Corollary 12.34 showed that $\mathscr{L}_V du = d(\mathscr{L}_V u) = d(Vu)$. Thus Proposition 14.33 and the induction hypothesis imply

$$\begin{aligned} \mathscr{L}_V(du \wedge \beta) &= (\mathscr{L}_V \, du) \wedge \beta + du \wedge (\mathscr{L}_V \beta) \\ &= d(Vu) \wedge \beta + du \wedge \big(V \lrcorner \, d\beta + d(V \lrcorner \beta)\big). \end{aligned} \tag{14.33}$$

On the other hand, using the facts that both d and interior multiplication by V are antiderivations, and $V \lrcorner du = du(V) = Vu$, we compute

$$\begin{aligned} &V \lrcorner d(du \wedge \beta) + d\big(V \lrcorner (du \wedge \beta)\big) \\ &\quad = V \lrcorner (-du \wedge d\beta) + d\big((Vu)\beta - du \wedge (V \lrcorner \beta)\big) \\ &\quad = -(Vu)d\beta + du \wedge (V \lrcorner d\beta) + d(Vu) \wedge \beta + (Vu)\, d\beta + du \wedge d(V \lrcorner \beta). \end{aligned}$$

After the $(Vu)\, d\beta$ terms are canceled, this is equal to (14.33). □

Corollary 14.36 (The Lie Derivative Commutes with d). *If V is a smooth vector field and ω is a smooth differential form, then*

$$\mathscr{L}_V(d\omega) = d(\mathscr{L}_V \omega).$$

Proof. This follows from Cartan's formula and the fact that $d \circ d = 0$:

$$\begin{aligned} \mathscr{L}_V d\omega &= V \lrcorner d(d\omega) + d(V \lrcorner d\omega) = d(V \lrcorner d\omega); \\ d\,\mathscr{L}_V \omega &= d(V \lrcorner d\omega) + d\big(d(V \lrcorner \omega)\big) = d(V \lrcorner d\omega). \end{aligned}$$ □

Problems

14-1. Show that covectors $\omega^1, \ldots, \omega^k$ on a finite-dimensional vector space are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

14-2. For what values of k and n is it true that every k-covector on $\mathbb{R}^n$ is decomposable? [Suggestion: first do the cases $n = 3$ and $n = 4$, and then see if you can figure out how to generalize your results to other dimensions.]

14-3. We have two ways to think about covariant k-tensors on a finite-dimensional vector space V: concretely, as k-multilinear functionals on V, and abstractly, as elements of the abstract tensor product space $V^* \otimes \cdots \otimes V^*$. However, we have defined alternating and symmetric tensors only in terms of the concrete definition. This problem outlines an abstract approach to alternating tensors. (Symmetric tensors can be handled similarly.) Suppose V is a finite-dimensional real vector space. Let $\mathcal{A}$ denote the subspace of the k-fold abstract tensor product $V^* \otimes \cdots \otimes V^*$ spanned by all elements of the form $\omega^1 \otimes \cdots \otimes \omega^k$ with $\omega^i = \omega^j$ for some $i \neq j$. (Thus $\mathcal{A}$ is the trivial subspace if $k < 2$.) Let $\mathrm{A}^k(V^*)$ denote the quotient vector space $(V^* \otimes \cdots \otimes V^*)/\mathcal{A}$.

(a) Show that there is a unique isomorphism $F\colon \mathrm{A}^k(V^*) \to \Lambda^k(V^*)$ such that the following diagram commutes:

$$\begin{tikzcd} V^* \otimes \cdots \otimes V^* \arrow[r, "\cong"] \arrow[d, "\pi"'] & T^k(V^*) \arrow[d, "\mathrm{Alt}"] \\ \mathrm{A}^k(V^*) \arrow[r, dashed, "F"'] & \Lambda^k(V^*), \end{tikzcd}$$

where $\pi\colon V^* \otimes \cdots \otimes V^* \to \mathrm{A}^k(V^*)$ is the projection.

(b) Define a wedge product on $\bigoplus_k \mathrm{A}^k(V^*)$ by $\omega \wedge \eta = \pi\big(\widetilde{\omega} \otimes \widetilde{\eta}\big)$, where $\widetilde{\omega}, \widetilde{\eta}$ are arbitrary tensors such that $\pi\big(\widetilde{\omega}\big) = \omega$, $\pi\big(\widetilde{\eta}\big) = \eta$. Show that this wedge product is well defined, and that F takes this wedge product to the Alt convention wedge product on $\Lambda(V^*)$.

[Remark: this is one reason why some authors consider the Alt convention for the wedge product to be more natural than the determinant convention.]

14-4. This chapter focused on alternating covariant tensors because of their many important applications in differential geometry. Alternating *contravariant* tensors have a few applications as well (one is described in Problem 21-14). If V is a finite-dimensional vector space, we can define alternating contravariant k-tensors either as multilinear functionals from $V^* \times \cdots \times V^*$ to $\mathbb{R}$ that change sign whenever two arguments are interchanged, or as elements of a quotient space $(V \otimes \cdots \otimes V)/\mathcal{A}$ analogous to the one defined in Problem 14-3. In the first case, the wedge product is defined just as in (14.3), but with the roles of vectors and covectors interchanged. In the second case, the wedge product is defined as the image of the tensor product in the quotient space as in Problem 14-3. Whichever definition is used, alternating contravariant k-tensors are called ***multivectors*** or ***k-vectors***. For this problem, choose whichever of these definitions you prefer.

(a) Show that an ordered k-tuple $(v_1, \ldots, v_k)$ of elements of V is linearly dependent if and only if $v_1 \wedge \cdots \wedge v_k = 0$.

(b) Show that two linearly independent ordered k-tuples $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ have the same span if and only if

$$v_1 \wedge \cdots \wedge v_k = c\, w_1 \wedge \cdots \wedge w_k$$

for some nonzero real number c.

14-5. CARTAN'S LEMMA: Let M be a smooth n-manifold with or without boundary, and let $\left(\omega^1, \ldots, \omega^k\right)$ be an ordered k-tuple of smooth 1-forms on an open subset $U \subseteq M$ such that $\left(\omega^1|_p, \ldots, \omega^k|_p\right)$ is linearly independent for each $p \in U$. Given smooth 1-forms $\alpha^1, \ldots, \alpha^k$ on U such that

$$\sum_{i=1}^{k} \alpha^i \wedge \omega^i = 0,$$

show that each α^i can be written as a linear combination of $\omega^1, \ldots, \omega^k$ with smooth coefficients.

14-6. Define a 2-form ω on $\mathbb{R}^3$ by

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

(a) Compute ω in spherical coordinates (ρ, φ, θ) defined by $(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)$.

(b) Compute $d\omega$ in both Cartesian and spherical coordinates and verify that both expressions represent the same 3-form.

(c) Compute the pullback $\iota_{\mathbb{S}^2}^* \omega$ to $\mathbb{S}^2$, using coordinates (φ, θ) on the open subset where these coordinates are defined.

(d) Show that $\iota_{\mathbb{S}^2}^* \omega$ is nowhere zero.

14-7. In each of the following cases, M and N are smooth manifolds; $F\colon M \to N$ is a smooth map; and ω is a smooth differential form on N. In each case, compute $d\omega$ and $F^*\omega$, and verify by direct computation that $F^*(d\omega) = d(F^*\omega)$. (Cf. Problem 11-7.)

(a) $M = N = \mathbb{R}^2$;
$F(s,t) = (st, e^t)$;
$\omega = x\, dy$.

(b) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$;
$F(\theta, \varphi) = \big((\cos\varphi + 2)\cos\theta,\ (\cos\varphi + 2)\sin\theta,\ \sin\varphi\big)$;
$\omega = y\, dz \wedge dx$.

(c) $M = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$ and $N = \mathbb{R}^3 \smallsetminus \{0\}$;
$F(u,v) = \left(u, v, \sqrt{1 - u^2 - v^2}\right)$;
$$\omega = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

14-8. For each nonnegative integer k, show that there is a contravariant functor $\Omega^k \colon \mathsf{Diff} \to \mathsf{Vec}_{\mathbb{R}}$, which to each smooth manifold M assigns the vector space $\Omega^k(M)$ and to each smooth map F the pullback F^*. Show that the exterior derivative is a natural transformation from Ω^k to Ω^{k+1}. (See Problem 11-18.)

14-9. Let M, N be smooth manifolds, and suppose $\pi \colon M \to N$ is a surjective smooth submersion with connected fibers. We say that a tangent vector $v \in T_pM$ is ***vertical*** if $d\pi_p(v) = 0$. Suppose $\omega \in \Omega^k(M)$. Show that there exists $\eta \in \Omega^k(N)$ such that $\omega = \pi^*\eta$ if and only if $v \lrcorner \omega_p = 0$ and $v \lrcorner d\omega_p = 0$ for every $p \in M$ and every vertical vector $v \in T_pM$. [Hint: first, do the case in which $\pi \colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ is projection onto the first n coordinates.]