Chapter 19
Distributions and Foliations

Suppose V is a nonvanishing vector field on a smooth manifold M. The results of Chapter 9 imply that each integral curve of V is a smooth immersion, and that locally the images of the integral curves fit together nicely like parallel lines in Euclidean space. The fundamental theorem on flows tells us that these curves are determined by the knowledge of their velocity vectors.

In this chapter we explore an important generalization of this idea to higher-dimensional submanifolds. The general setup is this: suppose M is a smooth manifold, and we are given a k-dimensional subbundle of TM, called a *distribution* on M. Is there a k-dimensional submanifold (called an *integral manifold* of the distribution) whose tangent space at each point is the given subspace? The answer in this case is more complicated than in the case of vector fields: there is a nontrivial necessary condition, called *involutivity*, that must be satisfied by the distribution.

In the first section of the chapter, we define involutivity and give examples of both involutive and noninvolutive distributions. Next, we show how the involutivity condition can be rephrased in terms of differential forms.

The main theorem of this chapter, the *Frobenius theorem*, tells us that involutivity is also sufficient for the existence of an integral manifold through each point. We prove the Frobenius theorem in two forms. First, we prove a local form, which says that a neighborhood of every point is filled up with integral manifolds, fitting together nicely like parallel affine subspaces of $\mathbb{R}^n$. Then we prove a global form, which says that associated with each involutive distribution is a partition of the entire manifold into immersed integral manifolds fitting together nicely, called a *foliation*.

In the next section, we apply the theory of foliations to prove an important result in the theory of Lie groups. We already know that to each Lie subgroup of a Lie group G, there corresponds a Lie subalgebra of $\mathrm{Lie}(G)$. Using the theory of foliations, we prove that the correspondence also goes the other way: every Lie subalgebra of $\mathrm{Lie}(G)$ corresponds to some Lie subgroup of G.

At the end of the chapter, we give a few applications of the Frobenius theorem to partial differential equations.

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Distributions and Involutivity

Let M be a smooth manifold. A ***distribution on M of rank k*** is a rank-k subbundle of TM. It is called a ***smooth distribution*** if it is a smooth subbundle. Distributions are also sometimes called ***tangent distributions*** (especially if there is any opportunity for confusion with the use of the term *distribution* for generalized functions in analysis), ***k-plane fields***, or ***tangent subbundles***.

Often a rank-k distribution is described by specifying for each $p \in M$ a linear subspace $D_p \subseteq T_pM$ of dimension k, and letting $D = \bigcup_{p\in M} D_p$. It then follows from the local frame criterion for subbundles (Lemma 10.32) that D is a smooth distribution if and only if each point of M has a neighborhood U on which there are smooth vector fields $X_1, \dots, X_k \colon U \to TM$ such that $X_1|_q, \dots, X_k|_q$ form a basis for D_q at each $q \in U$. In this case, we say that D is the distribution **(*locally*) *spanned by the vector fields $X_1, \dots, X_k$***.

Integral Manifolds and Involutivity

Suppose $D \subseteq TM$ is a smooth distribution. A nonempty immersed submanifold $N \subseteq M$ is called an ***integral manifold of D*** if $T_pN = D_p$ at each point $p \in N$. The main question we want to address in this chapter is that of the existence of integral manifolds.

Before we proceed with the general theory, let us describe some examples of distributions and integral manifolds that you should keep in mind.

Example 19.1 (Distributions and Integral Manifolds).

(a) If V is a nowhere-vanishing smooth vector field on a manifold M, then V spans a smooth rank-1 distribution on M (see Example 10.33(a)). The image of any integral curve of V is an integral manifold of D.

(b) In $\mathbb{R}^n$, the vector fields $\partial/\partial x^1, \dots, \partial/\partial x^k$ span a smooth distribution of rank k. The k-dimensional affine subspaces parallel to $\mathbb{R}^k$ are integral manifolds.

(c) Let R be the distribution on $\mathbb{R}^n \smallsetminus \{0\}$ spanned by the unit radial vector field $x^i\,\partial/\partial x^i$, and let $R^\perp$ be its orthogonal complement bundle (see Lemma 10.35). Then $R^\perp$ is a smooth rank-$(n-1)$ distribution on $\mathbb{R}^n \smallsetminus \{0\}$. Through each point $x \in \mathbb{R}^n \smallsetminus \{0\}$, the sphere of radius $|x|$ around 0 is an integral manifold of $R^\perp$.

(d) Let D be the smooth distribution on $\mathbb{R}^3$ spanned by the following vector fields:

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y}.$$

(See Fig. 19.1.) It turns out that D has no integral manifolds. To get an idea why, suppose N is an integral manifold through the origin. Because X and Y are tangent to N, any integral curve of X or Y that starts in N has to stay in N, at least for a short time (Problem 9-2). Thus, N contains an open subset of the x-axis (which is an integral curve of X). It also contains, for each sufficiently small x, an open subset of the line parallel to the y-axis and passing through

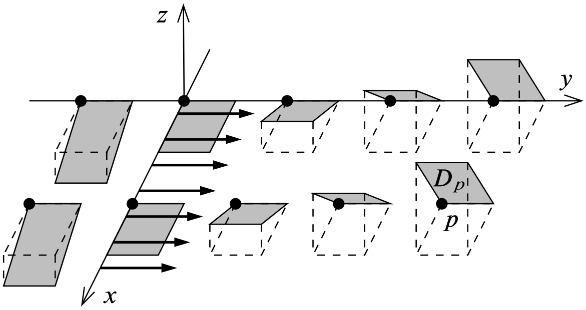


Fig. 19.1 A smooth distribution with no integral manifolds

$(x,0,0)$ (which is an integral curve of Y). Therefore, N contains an open subset of the (x,y)-plane. However, the tangent plane to the (x,y)-plane at any point p off of the x-axis is not equal to D_p. Therefore, no such integral manifold exists. //

The last example shows that in general, integral manifolds may fail to exist. Suppose D is a smooth distribution on M. We say that D is ***involutive*** if given any pair of smooth local sections of D (i.e., smooth vector fields X, Y defined on an open subset of M such that $X_p, Y_p \in D_p$ for each p), their Lie bracket is also a local section of D. The next proposition shows that the involutivity condition can be expressed concisely in terms of lie algebras.

Proposition 19.2. *Let $D \subseteq TM$ be a smooth distribution, and let $\Gamma(D) \subseteq \mathfrak{X}(M)$ denote the space of smooth global sections of D. Then D is involutive if and only if $\Gamma(D)$ is a Lie subalgebra of $\mathfrak{X}(M)$.*

Proof. If D is involutive, the definition implies that $\Gamma(D)$ is closed under Lie brackets. Because it is also a linear subspace of $\mathfrak{X}(M)$, it is a Lie subalgebra.

Conversely, suppose $\Gamma(D)$ is a Lie subalgebra of $\mathfrak{X}(M)$, and let X, Y be smooth local sections of D over an open subset $U \subseteq M$. Given $p \in M$, let $\psi \in C^\infty(M)$ be a bump function that is identically 1 on a neighborhood of p and supported in U. Then ψX and ψY are smooth global sections of D, so their Lie bracket is also a section of D by hypothesis. This Lie bracket is $[\psi X, \psi Y] = \psi^2[X,Y] + \psi(X\psi)Y - \psi(Y\psi)X$, which is equal to $[X,Y]$ in a neighborhood of p. Thus, $[X,Y]_p \in D_p$ for each $p \in U$, so D is involutive. □

A smooth distribution D on M is said to be ***integrable*** if each point of M is contained in an integral manifold of D.

Proposition 19.3. *Every integrable distribution is involutive.*

Proof. Let $D \subseteq TM$ be an integrable distribution. Suppose X and Y are smooth local sections of D defined on some open subset $U \subseteq M$. Let p be any point in U, and let N be an integral manifold of D containing p. The fact that X and Y are sections of D means that X and Y are tangent to N. By Corollary 8.32, $[X,Y]$ is also tangent to N, and therefore $[X,Y]_p \in D_p$. Since this is true at each $p \in U$, it follows that D is involutive. □

Note, for example, that the distribution D of Example 19.1(d) is not involutive, because $[X,Y] = -\partial/\partial z$, which is not a section of D.

The next lemma shows that the involutivity condition does not have to be checked for every pair of smooth vector fields, just those of a smooth local frame in a neighborhood of each point.

Lemma 19.4 (Local Frame Criterion for Involutivity). *Let $D \subseteq TM$ be a distribution. If in a neighborhood of every point of M there exists a smooth local frame $(V_1,\dots,V_k)$ for D such that $[V_i,V_j]$ is a section of D for each $i,j = 1,\dots,k$, then D is involutive.*

Proof. Suppose the hypothesis holds, and suppose X and Y are smooth local sections of D over some open subset $U \subseteq M$. Given $p \in U$, choose a smooth local frame $(V_1,\dots,V_k)$ satisfying the hypothesis in a neighborhood of p, and write $X = X^i V_i$ and $Y = Y^i V_i$ in that neighborhood. Then, using (8.11),

$$[X,Y] = \left[X^i V_i, Y^j V_j\right] = X^i Y^j [V_i, V_j] + X^i \left(V_i Y^j\right) V_j - Y^j \left(V_j X^i\right) V_i.$$

It follows from the hypothesis that this last expression is a section of D. □

Involutivity and Differential Forms

Differential forms yield an alternative way to describe distributions and involutivity.

Lemma 19.5 (1-Form Criterion for Smooth Distributions). *Suppose M is a smooth n-manifold and $D \subseteq TM$ is a distribution of rank k. Then D is smooth if and only if each point $p \in M$ has a neighborhood U on which there are smooth 1-forms $\omega^1,\dots,\omega^{n-k}$ such that for each $q \in U$,*

$$D_q = \operatorname{Ker} \omega^1\big|_q \cap \dots \cap \operatorname{Ker} \omega^{n-k}\big|_q. \tag{19.1}$$

Proof. First suppose that there exist such forms $\omega^1,\dots,\omega^{n-k}$ in a neighborhood of each point. The assumption (19.1) together with the fact that D has rank k implies that the forms $\omega^1,\dots,\omega^{n-k}$ are independent on U for dimensional reasons. By Proposition 10.15, we can complete them to a smooth coframe $\left(\omega^1,\dots,\omega^n\right)$ on a (possibly smaller) neighborhood of each point. If $(E_1,\dots,E_n)$ is the dual frame, it is easy to check that D is locally spanned by $E_{n-k+1},\dots,E_n$, so it is smooth by the local frame criterion.

Conversely, suppose D is smooth. In a neighborhood of any $p \in M$, there are smooth vector fields $Y_1,\dots,Y_k$ spanning D. By Proposition 10.15 again, we can complete these vector fields to a smooth local frame $(Y_1,\dots,Y_n)$ for M in a neighborhood of p. With the dual coframe denoted by $\left(\varepsilon^1,\dots,\varepsilon^n\right)$, it follows easily that D is characterized locally by

$$D_q = \operatorname{Ker} \varepsilon^{k+1}\big|_q \cap \dots \cap \operatorname{Ker} \varepsilon^n\big|_q.$$

□

If D is a rank-k distribution on a smooth n-manifold M, any $n-k$ linearly independent 1-forms $\omega^1,\dots,\omega^{n-k}$ defined on an open subset $U\subseteq M$ and satisfying (19.1) for each $q\in U$ are called ***local defining forms for D***. More generally, if $0\le p\le n$, we say that a p-form $\omega\in\Omega^p(M)$ ***annihilates D*** if $\omega(X_1,\dots,X_p)=0$ whenever $X_1,\dots,X_p$ are local sections of D. (In the case $p=0$, only the zero function annihilates D.)

Lemma 19.6. *Suppose M is a smooth n-manifold and D is a smooth rank-k distribution on M. Let $\omega^1,\dots,\omega^{n-k}$ be smooth local defining forms for D over an open subset $U\subseteq M$. A smooth p-form η defined on U annihilates D if and only if it can be expressed in the form*

$$\eta=\sum_{i=1}^{n-k}\omega^i\wedge\beta^i \tag{19.2}$$

for some smooth $(p-1)$-forms $\beta^1,\dots,\beta^{n-k}$ on U.

Remark. In the case $p=1$, the β^i's are smooth functions, and we interpret a wedge product with a smooth function to be ordinary multiplication. Thus, in this case, the lemma just says that η is a linear combination of the ω^i's with smooth coefficients.

Proof. It is easy to check that any form η that satisfies (19.2) in a neighborhood of each point annihilates D. Conversely, suppose η annihilates D on U. In a neighborhood of each point we can complete the $(n-k)$-tuple $(\omega^1,\dots,\omega^{n-k})$ to a smooth local coframe $(\omega^1,\dots,\omega^n)$ for M (Proposition 10.15). If $(E_1,\dots,E_n)$ is the dual frame, then D is locally spanned by $E_{n-k+1},\dots,E_n$. In terms of this coframe, any $\eta\in\Omega^p(M)$ can be written locally in a unique way as

$$\eta={\sum_I}'\eta_I\omega^{i_1}\wedge\cdots\wedge\omega^{i_p},$$

where the coefficients η_I are determined by $\eta_I=\eta(E_{i_1},\dots,E_{i_p})$. Thus, η annihilates D in U if and only if $\eta_I=0$ whenever $n-k+1\le i_1<\cdots<i_p\le n$, in which case η can be written locally as

$$\eta={\sum_{I:i_1\le n-k}}'\eta_I\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}=\sum_{i_1=1}^{n-k}\omega^{i_1}\wedge\left({\sum_{I'}}'\eta_{i_1I'}\omega^{i_2}\wedge\cdots\wedge\omega^{i_p}\right),$$

where we have written $I'=(i_2,\dots,i_p)$. This holds in a neighborhood of each point of U; patching together with a partition of unity, we obtain a similar expression on all of U. □

When expressed in terms of differential forms, the involutivity condition translates into a statement about exterior derivatives.

Theorem 19.7 (1-Form Criterion for Involutivity). *Suppose $D\subseteq TM$ is a smooth distribution. Then D is involutive if and only if the following condition is* *satisfied*:

> *If η is any smooth 1-form that annihilates D on an open subset $U \subseteq M$, then $d\eta$ also annihilates D on U.* (19.3)

Proof. First, assume that D is involutive, and suppose η is a smooth 1-form that annihilates D on $U \subseteq M$. Then for any smooth local sections X, Y of D, formula (14.28) for $d\eta$ gives

$$d\eta(X,Y) = X\big(\eta(Y)\big) - Y\big(\eta(X)\big) - \eta\big([X,Y]\big).$$

The hypothesis implies that each of the terms on the right-hand side is zero on U.

Conversely, suppose D satisfies (19.3), and suppose X and Y are smooth local sections of D. If $\omega^1, \dots, \omega^{n-k}$ are smooth local defining forms for D, then (14.28) shows that for each $i = 1, \dots, n-k$,

$$\omega^i\big([X,Y]\big) = X\big(\omega^i(Y)\big) - Y\big(\omega^i(X)\big) - d\omega^i(X,Y) = 0,$$

which implies that $[X,Y]$ takes its values in D. Thus D is involutive. □

Just like the Lie bracket condition for involutivity, the exterior derivative condition need only be checked for a particular set of smooth defining forms in a neighborhood of each point, as the next proposition shows.

Proposition 19.8 (Local Coframe Criterion for Involutivity). *Let D be a smooth distribution of rank k on a smooth n-manifold M, and let $\omega^1, \dots, \omega^{n-k}$ be smooth defining forms for D on an open subset $U \subseteq M$. The following are equivalent*:

(a) *D is involutive on U.*
(b) *$d\omega^1, \dots, d\omega^{n-k}$ annihilate D.*
(c) *There exist smooth 1-forms $\{\alpha_j^i : i, j = 1, \dots, n-k\}$ such that*

$$d\omega^i = \sum_{j=1}^{n-k} \omega^j \wedge \alpha_j^i, \quad \textit{for each } i = 1, \dots, n-k.$$

► **Exercise 19.9.** Prove the preceding proposition.

With a bit more algebraic terminology, there is an elegant way to express the involutivity condition in terms of differential forms. Recall that we have defined the graded algebra of smooth differential forms on a smooth n-manifold M as $\Omega^*(M) = \Omega^0(M) \oplus \cdots \oplus \Omega^n(M)$ (see p. 360). An ***ideal in $\Omega^*(M)$*** is a linear subspace $\mathcal{J} \subseteq \Omega^*(M)$ that is closed under wedge products with arbitrary elements of $\Omega^*(M)$; that is, $\omega \in \mathcal{J}$ implies $\eta \wedge \omega \in \mathcal{J}$ for every $\eta \in \Omega^*(M)$.

Now suppose D is a smooth distribution on a smooth n-manifold M. Let $\mathcal{J}^p(D) \subseteq \Omega^p(M)$ denote the space of smooth p-forms that annihilate D, and let $\mathcal{J}(D) = \mathcal{J}^0(D) \oplus \cdots \oplus \mathcal{J}^n(D) \subseteq \Omega^*(M)$.

► **Exercise 19.10.** For any smooth distribution $D \subseteq TM$, show that $\mathcal{J}(D)$ is an ideal in $\Omega^*(M)$.

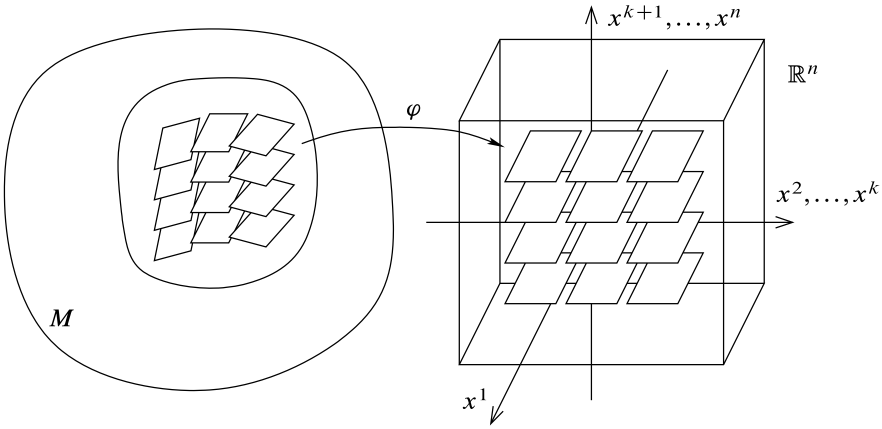


Fig. 19.2 A flat chart for a distribution

Any ideal of the form $\mathcal{I}(D)$ for some smooth distribution D is sometimes called a ***Pfaffian system***. An ideal $\mathcal{I} \subseteq \Omega^*(M)$ is said to be a ***differential ideal*** if $d(\mathcal{I}) \subseteq \mathcal{I}$, that is, if $\omega \in \mathcal{I}$ implies $d\omega \in \mathcal{I}$.

Proposition 19.11 (Differential Ideal Criterion for Involutivity). *Let M be a smooth manifold. A smooth distribution $D \subseteq TM$ is involutive if and only if $\mathcal{I}(D)$ is a differential ideal in $\Omega^*(M)$.*

Proof. Problem 19-1. □

The Frobenius Theorem

In Example 19.1, all of the distributions we defined except the last one had the property that there was an integral manifold through each point. Moreover, locally these submanifolds all "fit together" nicely like parallel affine subspaces of $\mathbb{R}^n$. Given a rank-k distribution $D \subseteq TM$, let us say that a smooth coordinate chart (U, φ) on M is ***flat for D*** if $\varphi(U)$ is a cube in $\mathbb{R}^n$, and at points of U, D is spanned by the first k coordinate vector fields $\partial/\partial x^1, \dots, \partial/\partial x^k$ (Fig. 19.2). In any such chart, each slice of the form $x^{k+1} = c^{k+1}, \dots, x^n = c^n$ for constants $c^{k+1}, \dots, c^n$ is an integral manifold of D. This is the nicest possible local situation for integral manifolds. We say that a distribution $D \subseteq TM$ is ***completely integrable*** if there exists a flat chart for D in a neighborhood of each point of M. Obviously, every completely integrable distribution is integrable and therefore involutive. In summary,

$$\text{completely integrable} \Rightarrow \text{integrable} \Rightarrow \text{involutive}.$$

The next theorem is the main result of this chapter, and indeed one of the central theorems in smooth manifold theory. It says that the implications above are actually equivalences:

completely integrable $\Leftrightarrow$ integrable $\Leftrightarrow$ involutive.

Theorem 19.12 (Frobenius). *Every involutive distribution is completely integrable.*

Proof. The canonical form theorem for commuting vector fields (Theorem 9.46) implies that any distribution locally spanned by independent smooth *commuting* vector fields is completely integrable, because the coordinate chart whose existence is guaranteed by that theorem is flat (after shrinking the domain if necessary so the image is a cube). Thus, it suffices to show that every involutive distribution is locally spanned by independent smooth commuting vector fields.

Let D be an involutive distribution of rank k on an n-dimensional manifold M, and let $p \in M$. Since complete integrability is a local question, by passing to a smooth coordinate neighborhood of p, we may replace M by an open subset $U \subseteq \mathbb{R}^n$, and choose a smooth local frame $X_1, \dots, X_k$ for D. By reordering the coordinates if necessary, we may assume that D_p is complementary to the subspace of $T_p\mathbb{R}^n$ spanned by $\left(\partial/\partial x^{k+1}|_p, \dots, \partial/\partial x^n|_p\right)$.

Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first k coordinates, $\pi\left(x^1, \dots, x^n\right) = \left(x^1, \dots, x^k\right)$ (Fig. 19.3). This induces a smooth bundle homomorphism $d\pi\colon T\mathbb{R}^n \to T\mathbb{R}^k$, which can be written

$$d\pi\left(\sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_q\right) = \sum_{i=1}^{k} v^i \left.\frac{\partial}{\partial x^i}\right|_{\pi(q)}.$$

(Notice that the summation on the right-hand side is only over $i = 1, \dots, k$.) Because $d\pi|_D$ is the composition of the inclusion $D \hookrightarrow TU$ followed by $d\pi$, it is a smooth bundle homomorphism. Thus, the matrix entries of $d\pi|_{D_q}$ with respect to the frames $\left(X_i|_q\right)$ and $\left(\partial/\partial x^j|_{\pi(q)}\right)$ are smooth functions of q.

By our choice of coordinates, $D_p \subseteq T_p\mathbb{R}^n$ is complementary to the kernel of $d\pi_p$, so the restriction of $d\pi_p$ to D_p is bijective. By continuity, therefore, the same is true of $d\pi|_{D_q}$ for q in a neighborhood of p, and the matrix entries of $\left(d\pi|_{D_q}\right)^{-1}\colon T_{\pi(q)}\mathbb{R}^k \to D_q$ are also smooth functions of q. Define a new smooth local frame $V_1, \dots, V_k$ for D in a neighborhood of p by

$$V_i\big|_q = \left(d\pi\big|_{D_q}\right)^{-1} \left.\frac{\partial}{\partial x^i}\right|_{\pi(q)}. \tag{19.4}$$

The theorem will be proved if we can show that $[V_i, V_j] = 0$ for all i, j.

First observe that V_i and $\partial/\partial x^i$ are π-related, because (19.4) implies that

$$\left.\frac{\partial}{\partial x^i}\right|_{\pi(q)} = \left(d\pi\big|_{D_q}\right) V_i\big|_q = d\pi_q\left(V_i\big|_q\right).$$

Therefore, by the naturality of Lie brackets,

$$d\pi_q\left([V_i, V_j]_q\right) = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]_{\pi(q)} = 0.$$

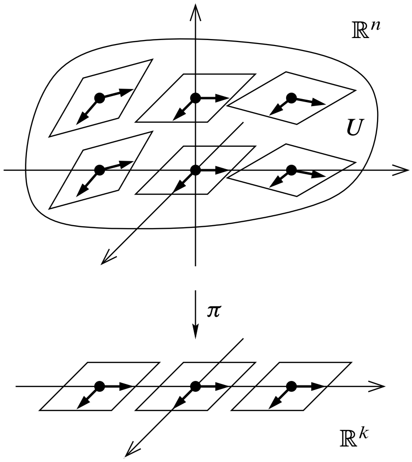


Fig. 19.3 Proof of the Frobenius theorem

Since involutivity of D implies that $[V_i, V_j]$ takes its values in D, and $d\pi$ is injective on each fiber of D, this implies that $[V_i, V_j]_q = 0$ for each q, thus completing the proof. □

For later use in our treatment of overdetermined partial differential equations, we note the following easy corollary to the proof.

Corollary 19.13. *Suppose M is a smooth manifold, D is an involutive rank-k distribution on M, and $S \subseteq M$ is a codimension-k embedded submanifold. If $p \in S$ is a point such that T_pS is complementary to D_p, then there is a flat chart $\left(U, \left(s^i\right)\right)$ for D centered at p in which $S \cap U$ is the slice $s^1 = \cdots = s^k = 0$.*

Proof. The proof of the theorem showed that locally D is spanned by k commuting vector fields, and then the corollary follows from Theorem 9.46. □

As is often the case, embedded in the proof of the Frobenius theorem is a technique for finding integral manifolds. The idea is to use a coordinate projection to find commuting vector fields spanning the same distribution, and then use the technique of Example 9.47 to find a flat chart. Here is an example.

Example 19.14. Let $D \subseteq T\mathbb{R}^3$ be the distribution spanned by the vector fields

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z},$$
$$Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}.$$

The computation of Example 8.27 showed that

$$[X, Y] = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z} = -Y,$$

so D is involutive. Let us try to find a flat chart in a neighborhood of the origin. Since D is complementary to the span of $\partial/\partial z$, the coordinate projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $\pi(x,y,z) = (x,y)$ induces an isomorphism $d\pi|_{D_{(x,y,z)}}\colon D_{(x,y,z)} \to T_{(x,y)}\mathbb{R}^2$ for each $(x,y,z) \in \mathbb{R}^3$. The proof of the Frobenius theorem shows that if we can find smooth local sections V, W of D that are π-related to $\partial/\partial x$ and $\partial/\partial y$, respectively, they will be commuting vector fields spanning D. It is easy to check that V, W have this property if and only if they take their values in D and are of the form

$$
\begin{aligned}
V &= \frac{\partial}{\partial x} + u(x,y,z)\frac{\partial}{\partial z},\\
W &= \frac{\partial}{\partial y} + v(x,y,z)\frac{\partial}{\partial z},
\end{aligned}
$$

for some smooth real-valued functions u, v. A bit of linear algebra shows that the vector fields

$$
\begin{aligned}
V = Y &= \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},\\
W = X - xY &= \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},
\end{aligned}
$$

do the trick. The flows of these vector fields are easily found by solving the two systems of ODEs. Sparing the details, we find that the flow of V is

$$\alpha_t(x,y,z) = (x+t, y, z+ty), \tag{19.5}$$

and that of W is

$$\beta_t(x,y,z) = (x, y+t, z+tx). \tag{19.6}$$

Thus, by the procedure of Example 9.47, we can define the inverse Φ of our coordinate map by starting on the z-axis and flowing out along these two flows in succession:

$$\Phi(u,v,w) = \alpha_u \circ \beta_v(0,0,w) = \alpha_u(0,v,w) = (u,v,w+uv).$$

The flat coordinates we seek are given by inverting the map $(x,y,z) = \Phi(u,v,w) = (u,v,w+uv)$, to yield

$$(u,v,w) = \Phi^{-1}(x,y,z) = (x,y,z-xy).$$

It follows that the integral manifolds of D are the level sets of $w(x,y,z) = z - xy$. (Since the flat chart we have constructed is actually a global chart in this case, this describes all of the integral manifolds, not just the ones near the origin.) //

► **Exercise 19.15.** Verify that the flows of V and W are given by (19.5) and (19.6), respectively, and that the level sets of $z - xy$ are integral manifolds of D.

The next proposition is one of the most important consequences of the Frobenius theorem.

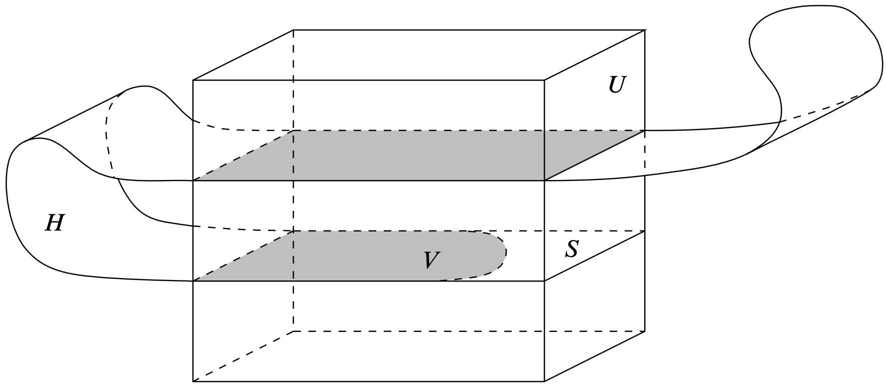


Fig. 19.4 The local structure of an integral manifold

Proposition 19.16 (Local Structure of Integral Manifolds). *Let D be an involutive distribution of rank k on a smooth manifold M, and let $\left(U,\left(x^i\right)\right)$ be a flat chart for D. If H is any integral manifold of D, then $H \cap U$ is a union of countably many disjoint open subsets of parallel k-dimensional slices of U, each of which is open in H and embedded in M.*

Proof. Let H be an integral manifold of D. Because the inclusion map $\iota\colon H \hookrightarrow M$ is continuous, $H \cap U = \iota^{-1}(U)$ is open in H, and thus consists of a countable disjoint union of connected components, each of which is open in H.

Let V be any component of $H \cap U$ (Fig. 19.4). We show first that V is contained in a single slice. Since $dx^{k+1},\dots,dx^n$ are local defining forms for D, it follows that the pullbacks of these 1-forms to V are identically zero. Because V is connected, this implies that $x^{k+1},\dots,x^n$ are all constant on V, so V lies in a single slice S.

Because S is embedded in M, the inclusion map $V \hookrightarrow M$ is also smooth as a map into S by Corollary 5.30. The inclusion $V \hookrightarrow S$ is thus an injective smooth immersion between manifolds of the same dimension, and therefore a local diffeomorphism, an open map, and a homeomorphism onto an open subset of S. The inclusion map $V \hookrightarrow M$ is a composition of the smooth embeddings $V \hookrightarrow S \hookrightarrow M$, so it is a smooth embedding. □

The preceding proposition implies the following important result about integral manifolds, which we will use in our study of Lie subgroups at the end of this chapter. Recall that a smooth submanifold $H \subseteq M$ is said to be *weakly embedded in M* if every smooth map $F\colon N \to M$ whose image lies in H is smooth as a map from N to H. (See Chapter 5.)

Theorem 19.17. *Every integral manifold of an involutive distribution is weakly embedded.*

Proof. Let M be a smooth n-manifold, let $H \subseteq M$ be an integral manifold of an involutive rank-k distribution D on M, and suppose $F\colon N \to M$ is a smooth map such that $F(N) \subseteq H$. Let $p \in N$ be arbitrary, and set $q = F(p) \in H$. Let $(y^1, \dots, y^n)$ be flat coordinates for D on a neighborhood U of q, and let (x^i) be smooth coordinates for N on a connected neighborhood B of p such that $F(B) \subseteq U$. With the coordinate representation of F written as

$$(y^1, \dots, y^n) = (F^1(x), \dots, F^n(x)),$$

the fact that $F(B) \subseteq H \cap U$ means that the coordinate functions $F^{k+1}, \dots, F^n$ take on only countably many values. Because B is connected, the intermediate value theorem implies that these coordinate functions are constant, and thus $F(B)$ lies in a single slice $S \subseteq U$. Because $S \cap H$ is an open subset of H that is embedded in M, it follows that $F|_B$ is smooth from B into $S \cap H$, and thus by composition, $F|_B \colon B \to (S \cap H) \hookrightarrow H$ is smooth into H. □

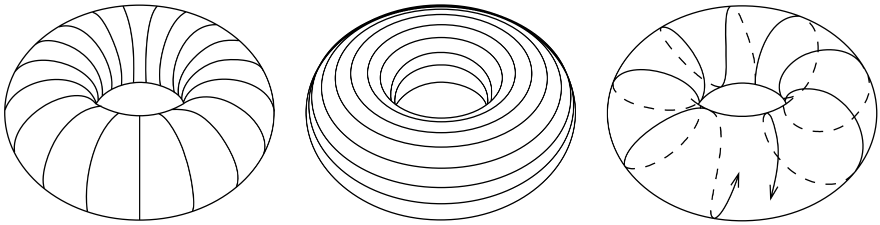
Foliations

When we put together all of the maximal integral manifolds of an involutive rank-k distribution on a smooth manifold M, we obtain a partition of M into k-dimensional submanifolds that "fit together" locally like the slices in a flat chart.

To express more precisely what we mean by "fitting together," we need to extend our notion of a flat chart slightly. Let M be a smooth n-manifold, and let $\mathcal{F}$ be any collection of k-dimensional submanifolds of M. A smooth chart (U, φ) for M is said to be ***flat for $\mathcal{F}$*** if $\varphi(U)$ is a cube in $\mathbb{R}^n$, and each submanifold in $\mathcal{F}$ intersects U in either the empty set or a countable union of k-dimensional slices of the form $x^{k+1} = c^{k+1}, \dots, x^n = c^n$. We define a ***foliation of dimension k on M*** to be a collection $\mathcal{F}$ of disjoint, connected, nonempty, immersed k-dimensional submanifolds of M (called the ***leaves of the foliation***), whose union is M, and such that in a neighborhood of each point $p \in M$ there exists a flat chart for $\mathcal{F}$.

Example 19.18 (Foliations).

(a) The collection of all k-dimensional affine subspaces of $\mathbb{R}^n$ parallel to $\mathbb{R}^k \times \{0\}$ is a k-dimensional foliation of $\mathbb{R}^n$.
(b) The collection of open rays of the form $\{\lambda x : \lambda > 0\}$ as x ranges over $\mathbb{R}^n \smallsetminus \{0\}$ is a 1-dimensional foliation of $\mathbb{R}^n \smallsetminus \{0\}$.
(c) The collection of all spheres centered at 0 is an $(n-1)$-dimensional foliation of $\mathbb{R}^n \smallsetminus \{0\}$.
(d) If M and N are connected smooth manifolds, the collection of subsets of the form $M \times \{q\}$ as q ranges over points in N forms a foliation of $M \times N$, each of whose leaves is diffeomorphic to M. For example, the collection of all circles of the form $\mathbb{S}^1 \times \{q\} \subseteq \mathbb{T}^2$ for $q \in \mathbb{S}^1$ yields a foliation of the torus $\mathbb{T}^2$ (Fig. 19.5(a)). A different foliation of $\mathbb{T}^2$ is given by the collection of circles of the form $\{p\} \times \mathbb{S}^1$ (Fig. 19.5(b)).



(a) (b) (c)

Fig. 19.5 Foliations of the torus

(e) If α is a fixed real number, the images of all curves of the form

$$\gamma_\theta(t) = \left(e^{it}, e^{i(\alpha t + \theta)}\right)$$

as θ ranges over $\mathbb{R}$ form a 1-dimensional foliation of the torus (Fig. 19.5(c)). If α is rational, each leaf is an embedded circle; whereas if α is irrational, each leaf is dense (see Example 4.20 and Problem 4-4).

(f) The collection of connected components of the curves in the (y,z)-plane defined by the following equations is a foliation of $\mathbb{R}^2$ (Fig. 19.6(a)):

$$z = \sec y + c, \quad c \in \mathbb{R};$$
$$y = \left(k + \tfrac{1}{2}\right)\pi, \quad k \in \mathbb{Z}.$$

(g) If we revolve the curves of the previous example around the z-axis, we obtain a 2-dimensional foliation of $\mathbb{R}^3$ in which some of the leaves are diffeomorphic to disks and some are diffeomorphic to cylinders (Fig. 19.6(b)). //

The main fact about foliations is that they are in one-to-one correspondence with involutive distributions. One direction, expressed in the next proposition, is an easy consequence of the definition.

Proposition 19.19. *Let $\mathcal{F}$ be a foliation on a smooth manifold M. The collection of tangent spaces to the leaves of $\mathcal{F}$ forms an involutive distribution on M.*

► **Exercise 19.20.** Prove Proposition 19.19.

The Frobenius theorem allows us to conclude the following converse, which is much more profound. By the way, it is worth noting that this result is one of the two primary reasons why the notion of immersed submanifold has been defined. (The other is for the study of Lie subgroups.)

Theorem 19.21 (Global Frobenius Theorem). *Let D be an involutive distribution on a smooth manifold M. The collection of all maximal connected integral manifolds of D forms a foliation of M.*

The theorem will be an easy consequence of the following lemma.

Lemma 19.22. *Suppose $D \subseteq TM$ is an involutive distribution, and let $\{N_\alpha\}_{\alpha\in A}$ be any collection of connected integral manifolds of D with a point in common. Then*

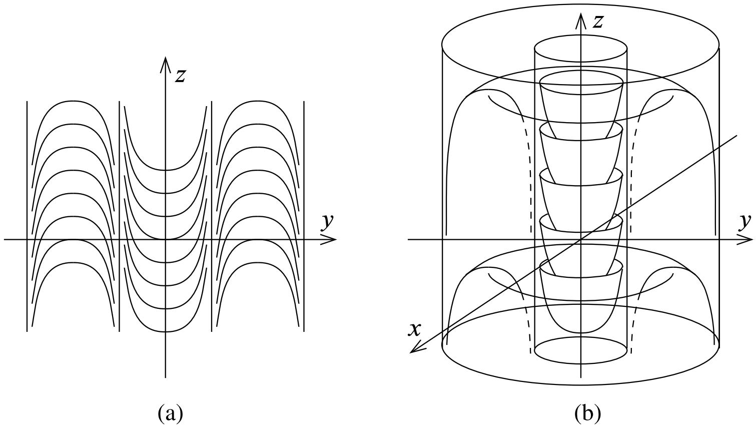


Fig. 19.6 Foliations of $\mathbb{R}^2$ and $\mathbb{R}^3$

$N = \bigcup_\alpha N_\alpha$ has a unique smooth manifold structure making it into a connected integral manifold of D.

Proof. If we can construct a topology and smooth manifold structure making N into an integral manifold of D, then Theorem 5.33 shows that the topology and smooth structure are uniquely determined, because integral manifolds are weakly embedded.

To construct the topology, first we need to show that $N_\alpha \cap N_\beta$ is open in N_α and in N_β for each $\alpha, \beta \in A$. Let $q \in N_\alpha \cap N_\beta$ be arbitrary, and choose a flat chart for D on a neighborhood W of q (Fig. 19.7). Let V_α, V_β denote the components of $N_\alpha \cap W$ and $N_\beta \cap W$, respectively, containing q. By Proposition 19.16, V_α and V_β are open subsets of single slices with the subspace topology, and since both contain q, they both must lie in the same slice S. Thus $V_\alpha \cap V_\beta$ is open in S and also in both N_α and N_β, so q has a neighborhood in N_α and a neighborhood in N_β contained in $N_\alpha \cap N_\beta$.

Define a topology on N by declaring a subset $U \subseteq N$ to be open if and only if $U \cap N_\alpha$ is open in N_α for each α. Using the result of the previous paragraph, it is easy to check that this is a topology and that each N_α is an open subspace of N. With this topology, N is locally Euclidean of dimension k, because each $q \in N$ has a coordinate neighborhood V in some N_α, and V is an open subset of N because N_α is open in N. Moreover, the inclusion map $N \hookrightarrow M$ is continuous: for any open subset $U \subseteq M$, $U \cap N$ is open in N because $U \cap N_\alpha$ is open in N_α for each α.

To see that N is Hausdorff, let q, q' be distinct points of N. There are disjoint open subsets $U, U' \subseteq M$ containing q and q', respectively, and because inclusion $N \hookrightarrow M$ is continuous, $N \cap U$ and $N \cap U'$ are disjoint open subsets of N containing q and q'.

Next we show that N is second-countable. We can cover M with countably many flat charts for D, say $\{W_i\}$. It suffices to show that $N \cap W_i$ is contained in a countable union of slices for each i, because any open subset of a single slice is second-

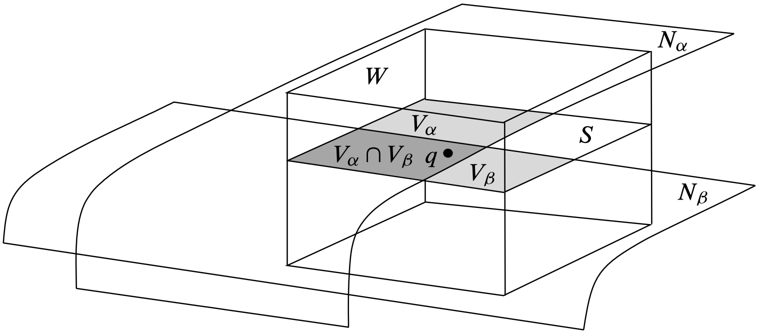


Fig. 19.7 A union of integral manifolds

countable, and thus N can be expressed as a union of countably many subsets, each of which is second-countable and open in N.

Let p_0 be a point contained in N_α for every α. Let us say that a slice S of some W_k is ***accessible from*** $\boldsymbol{p_0}$ if there is a finite sequence of indices $i_1, \dots, i_m$ and for each i_j a slice $S_{i_j} \subseteq W_{i_j}$ with the properties that $p \in S_{i_1}$, $S_{i_m} = S$, and $S_{i_j} \cap S_{i_{j+1}} \neq \varnothing$ for each $j = 1, \dots, m-1$ (Fig. 19.8).

Let W_k be one of our countable collection of flat charts, and suppose $S \subseteq W_k$ is a slice that contains a point $q \in N$. Then q is contained in one of the integral manifolds N_α. Because p_0 is also in N_α, there is a continuous path $\gamma\colon [0,1] \to N_\alpha$ connecting p_0 and q. Since $\gamma([0,1])$ is compact, there exist finitely many numbers $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for each $j = 1, \dots, m$, the set $\gamma([t_{j-1}, t_j])$ is contained in one of the flat charts W_{i_j}. Since $\gamma([t_{j-1}, t_j])$ is connected, it is contained in a single component of $W_{i_j} \cap N_\alpha$ and therefore in a single slice $S_{i_j} \subseteq W_{i_j}$. For each $j = 1, \dots, m-1$, the slices S_{i_j} and $S_{i_{j+1}}$ have the point $\gamma(t_j)$ in common, so it follows that the slice S is accessible from p_0.

This shows that every slice of some W_k containing a point of N is accessible from p_0. To complete the proof of second-countability, we just note that each S_{i_j} is itself an integral manifold, and therefore it meets at most countably many slices of $W_{i_{j+1}}$ by Proposition 19.16; thus, there are only countably many slices accessible from p_0. Therefore, N is a topological manifold of dimension k. It is connected because it is a union of connected subspaces with a point in common.

To construct a smooth structure on N, we define an atlas consisting of all charts of the form $(S \cap N, \psi)$, where S is a single slice of some flat chart, and $\psi\colon S \to \mathbb{R}^k$ is the map whose coordinate representation in the flat chart is projection onto the first k coordinates: $\psi\big(x^1, \dots, x^k, x^{k+1}, \dots, x^n\big) = \big(x^1, \dots, x^k\big)$. Because any slice is an embedded submanifold, its smooth structure is uniquely determined, and thus whenever two such slices S, S' overlap the transition map $\psi' \circ \psi^{-1}$ is smooth. With respect to this smooth structure, the inclusion map $N \hookrightarrow M$ is a smooth immersion (because it is a smooth embedding on each slice), and the tangent space to N at each point $q \in N$ is equal to D_q (because this is true for slices). $\square$

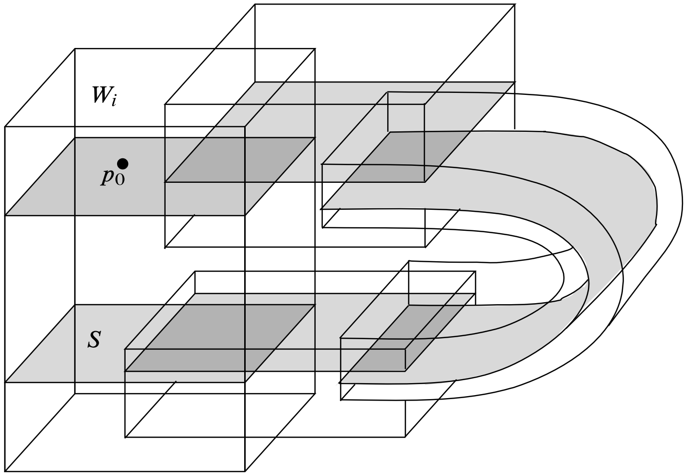


Fig. 19.8 A slice S accessible from p_0

Proof of the global Frobenius theorem. For each $p \in M$, let L_p be the union of all connected integral manifolds of D containing p. By the preceding lemma, L_p is a connected integral manifold of D containing p, and it is clearly maximal. If any two such maximal integral manifolds L_p and $L_{p'}$ intersect, their union $L_p \cup L_{p'}$ is an integral manifold containing both p and p', so by maximality $L_p = L_{p'}$. Thus, the various maximal connected integral manifolds are either disjoint or identical.

If (U, φ) is any flat chart for D, then $L_p \cap U$ is a countable union of open subsets of slices by Proposition 19.16. For any such slice S, if $L_p \cap S$ is neither empty nor all of S, then $L_p \cup S$ is a connected integral manifold properly containing L_p, which contradicts the maximality of L_p. Therefore, $L_p \cap U$ is precisely a countable union of slices, so the collection $\{L_p : p \in M\}$ is the desired foliation. □

Suppose M is a smooth manifold and $\Phi\colon M \to M$ is a diffeomorphism. A distribution D on M is said to be ***Φ-invariant*** if $d\Phi(D) = D$; or more precisely if for each $x \in M$, $d\Phi_x(D_x) = D_{\Phi(x)}$. Similarly, a foliation $\mathcal{F}$ on M is said to be ***Φ-invariant*** if for each leaf L of $\mathcal{F}$, the submanifold $\Phi(L)$ is also a leaf of $\mathcal{F}$.

Proposition 19.23. *Let M be a smooth manifold and $\Phi\colon M \to M$ be a diffeomorphism. Suppose D is an involutive distribution on M and $\mathcal{F}$ is the foliation it determines. Then D is Φ-invariant if and only if $\mathcal{F}$ is Φ-invariant.*

Proof. Problem 19-9. □

Lie Subalgebras and Lie Subgroups

Foliations have profound applications to the theory of Lie groups. Here we present two such applications; we will see many more in the next two chapters. Both rely on the following simple relationship between Lie subalgebras and distributions. A distribution D on a Lie group G is said to be ***left-invariant*** if it is invariant under every left translation. (Recall that this means $d(L_g)(D) = D$ for each $g \in G$.)

Lemma 19.24. *Let G be a Lie group. If $\mathfrak{h}$ is a Lie subalgebra of* $\operatorname{Lie}(G)$, *then the subset $D = \bigcup_{g \in G} D_g \subseteq TG$, where*

$$D_g = \{X_g : X \in \mathfrak{h}\} \subseteq T_g G, \tag{19.7}$$

is a left-invariant involutive distribution on G.

Proof. Each $X \in \mathfrak{h}$ is a left-invariant vector field on G. Thus, for any $g, g' \in G$, the differential $d\big(L_{g'g^{-1}}\big)$ restricts to an isomorphism from D_g to $D_{g'}$. It follows that D_g has the same dimension for each g, and D is left-invariant. Any basis $(X_1, \dots, X_k)$ for $\mathfrak{h}$ is a global smooth frame for D, so D is smooth. Moreover, because $[X_i, X_j] \in \mathfrak{h}$ for all $i, j \in \{1, \dots, k\}$, it follows from Lemma 19.4 that D is involutive. □

Theorem 19.25 (Lie Subgroups Are Weakly Embedded). *Every Lie subgroup is an integral manifold of an involutive distribution, and therefore is a weakly embedded submanifold.*

Proof. Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. Theorem 8.46 shows that the Lie algebra of H is canonically isomorphic to the Lie subalgebra $\mathfrak{h} = \iota_*\big(\operatorname{Lie}(H)\big) \subseteq \operatorname{Lie}(G)$, where $\iota\colon H \hookrightarrow G$ is inclusion. Let $D \subseteq TG$ be the involutive distribution determined by $\mathfrak{h}$ as in Lemma 19.24. It follows from the definitions that at each point $h \in H$, the tangent space $T_h H$ is equal to D_h, so H is an integral manifold of D. It then follows from Theorem 19.17 that H is weakly embedded in G. □

Theorem 19.26 (The Lie Subgroup Associated with a Lie Subalgebra). *Suppose G is a Lie group and $\mathfrak{g}$ is its Lie algebra. If $\mathfrak{h}$ is any Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup of G whose Lie algebra is $\mathfrak{h}$.*

Proof. Suppose $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Let $D \subseteq TG$ be the involutive distribution defined by (19.7). Let $\mathcal{H}$ denote the foliation determined by D, and for any $g \in G$, let $\mathcal{H}_g$ denote the leaf of $\mathcal{H}$ containing g (Fig. 19.9). Because D is left-invariant, it follows from Proposition 19.23 that each left translation takes leaves to leaves: for any $g, g' \in G$, we have $L_g(\mathcal{H}_{g'}) = \mathcal{H}_{gg'}$.

Define $H = \mathcal{H}_e$, the leaf containing the identity. We will show that H is the desired Lie subgroup.

First, to see that H is a subgroup, observe that for any $h, h' \in H$,

$$hh' = L_h(h') \in L_h(H) = L_h(\mathcal{H}_e) = \mathcal{H}_h = H.$$

Similarly,

$$h^{-1} = h^{-1}e \in L_{h^{-1}}(\mathcal{H}_e) = L_{h^{-1}}(\mathcal{H}_h) = \mathcal{H}_{h^{-1}h} = H.$$

To show that H is a Lie group, we need to show that the map $\mu\colon (h, h') \mapsto hh'^{-1}$ is smooth as a map from $H \times H$ to H. Because $H \times H$ is a submanifold of $G \times G$,

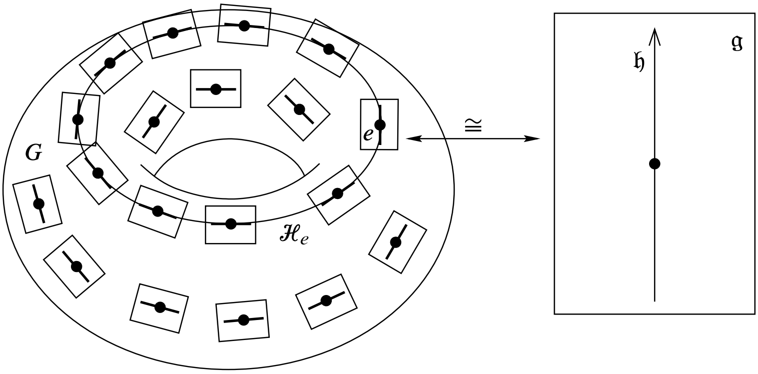


Fig. 19.9 Finding a subgroup whose Lie algebra is $\mathfrak{h}$

it is immediate that $\mu\colon H \times H \to G$ is smooth. Since H is an integral manifold of an involutive distribution, Theorem 19.17 shows that it is weakly embedded, so μ is also smooth as a map into H.

The fact that H is a leaf of $\mathscr{H}$ implies that the Lie algebra of H is $\mathfrak{h}$, because the tangent space to H at the identity is $D_e = \{X_e : X \in \mathfrak{h}\}$. To see that H is the unique connected subgroup with Lie algebra $\mathfrak{h}$, suppose $\widetilde{H}$ is any other connected subgroup with the same Lie algebra. Any such Lie subgroup is easily seen to be an integral manifold of D, so by maximality of $H = \mathscr{H}_e$, we must have $\widetilde{H} \subseteq H$. On the other hand, if U is the domain of a flat chart for D containing the identity, then by Proposition 19.16, $\widetilde{H} \cap U$ is a union of open subsets of slices. Since the slice containing e is an open subset of H, this implies that $\widetilde{H}$ contains a neighborhood of the identity in H. Since any neighborhood of the identity generates H (Proposition 7.14), this implies that $\widetilde{H} = H$. □

Overdetermined Systems of Partial Differential Equations

The partial differential equations we considered in Chapter 9 were all single equations for one unknown function. In some applications, it is necessary to consider systems of PDEs that are ***overdetermined***, which means that there are more equations than unknown functions. In general, overdetermined systems have solutions only if they satisfy certain compatibility conditions. For some first-order systems, the compatibility condition can be interpreted as a statement about involutivity of a distribution, and the Frobenius theorem can be used to prove local existence and uniqueness of solutions.

First, we consider certain linear systems. Suppose W is an open subset of $\mathbb{R}^n$ and m is a positive integer less than or equal to n. Consider the following system of

m linear partial differential equations for a single unknown function $u \in C^\infty(W)$:

$$a_1^1(x)\frac{\partial u}{\partial x^1}(x) + \cdots + a_1^n(x)\frac{\partial u}{\partial x^n}(x) = f_1(x),$$
$$\vdots \tag{19.8}$$
$$a_m^1(x)\frac{\partial u}{\partial x^1}(x) + \cdots + a_m^n(x)\frac{\partial u}{\partial x^n}(x) = f_m(x),$$

where $\left(a_i^j\right)$ is an $n \times m$ matrix of smooth real-valued functions and $f_1, \dots, f_m$ are smooth real-valued functions on W. The case $m = 1$ is covered by Theorem 9.51, so this discussion is useful primarily when $m > 1$.

If we let A_i denote the vector field $a_i^j \partial/\partial x^j$, the system (19.8) can be written more succinctly as $A_1 u = f_1, \dots, A_m u = f_m$. To avoid redundant or degenerate systems of equations, we assume that the matrix $\left(a_i^j\right)$ has rank m at each point of W, or equivalently that the vector fields $A_1, \dots, A_m$ are linearly independent. The following theorem is an analogue of Theorem 9.51 for the overdetermined case.

Theorem 19.27. *Let $W \subseteq \mathbb{R}^n$ be an open subset and let m be an integer such that $1 \le m \le n$. Suppose we are given an embedded codimension-m submanifold $S \subseteq W$, a linearly independent m-tuple of smooth vector fields $(A_1, \dots, A_m)$ on W whose span is complementary to T_pS at each $p \in S$, and functions $f_1, \dots, f_m \in C^\infty(W)$. Suppose also that there are smooth functions $c_{ij}^k \in C^\infty(W)$ for $i, j, k = 1, \dots, m$ such that the following compatibility conditions are satisfied*:

$$[A_i, A_j] = c_{ij}^k A_k, \tag{19.9}$$
$$A_i f_j - A_j f_i = c_{ij}^k f_k. \tag{19.10}$$

(*In these expressions*, k *is implicitly summed from* 1 *to* m.) *Then for each $p \in S$, there is a neighborhood U of p such that for every $\varphi \in C^\infty(S \cap U)$, there exists a unique solution $u \in C^\infty(U)$ to the following overdetermined Cauchy problem*:

$$A_i u = f_i \quad \text{for } i = 1, \dots, m, \tag{19.11}$$
$$u\big|_{S \cap U} = \varphi. \tag{19.12}$$

Proof. Let D be the distribution on W spanned by $A_1, \dots, A_m$, and let $p \in S$ be arbitrary. It follows from (19.9) that D is involutive, so by Corollary 19.13, on some neighborhood U of p there is a flat chart for D centered at p that is also a slice chart for S. Label the coordinates in this chart as $(v, w) = \left(v^1, \dots, v^m, w^1, \dots, w^{n-m}\right)$, so that $S \cap U$ is the slice where $v^1 = \cdots = v^m = 0$, and each $w =$ constant slice is an integral manifold of D in U, which we denote by H_w. Because (19.11)–(19.12) is a coordinate-independent statement, we can replace A_i and f_i by their coordinate representations in U, solve the equation there, and then use the inverse coordinate transformation to convert the solution back to the original coordinates.

Because $\operatorname{span}\left(A_1|_q, \dots, A_m|_q\right) = \operatorname{span}\left(\partial/\partial v^1|_q, \dots, \partial/\partial v^m|_q\right)$ for each $q \in U$, the n-tuple $\left(A_1, \dots, A_m, \partial/\partial w^1, \dots, \partial/\partial w^{n-m}\right)$ is a smooth local frame for U. Let

$(\alpha^1, \ldots, \alpha^m, \beta^1, \ldots, \beta^{n-m})$ denote the dual coframe, and define a smooth 1-form $\omega \in \Omega^1(U)$ by $\omega = f_k \alpha^k$ (with the implied summation from 1 to m). The system of equations (19.11) is satisfied if and only if $du(A_i) = \omega(A_i)$ for $i = 1, \ldots, m$, which is equivalent to saying that the pullback of $du - \omega$ to each H_w is equal to zero.

Using formula (14.28) for the exterior derivative together with (19.9), we obtain

$$d\alpha^k(A_i, A_j) = A_i\big(\alpha^k(A_j)\big) - A_j\big(\alpha^k(A_i)\big) - \alpha^k\big([A_i, A_j]\big) = -c_{ij}^k$$

for each $i, j, k = 1, \ldots, m$. It then follows from (19.10) that

$$\begin{aligned} d\omega(A_i, A_j) &= \big(df_k \wedge \alpha^k + f_k d\alpha^k\big)(A_i, A_j) \\ &= (A_i f_k)\delta_j^k - (A_j f_k)\delta_i^k - f_k c_{ij}^k = A_i f_j - A_j f_i - f_k c_{ij}^k = 0. \end{aligned}$$

Since $(A_1, \ldots, A_m)$ restricts to a frame on each integral manifold H_w, this shows that the pullback of ω to each H_w is closed.

Given $\varphi \in C^\infty(U \cap S)$, let $u = u_0 + u_1$, where $u_0, u_1 \in C^\infty(U)$ are defined by

$$\begin{aligned} u_0(v, w) &= \varphi(0, w), \\ u_1(v, w) &= \int_0^1 \omega_k(tv, w) v^k \, dt, \end{aligned}$$

and $\omega_k dv^k$ is the coordinate expression for ω.

Recall that a flat chart is cubical by definition, and thus star-shaped, so the integral is well defined for all $(v, w) \in U$, and differentiation under the integral sign shows that u_1 is a smooth function of (v, w). Because $u_0|_{S \cap U} = \varphi$ and $u_1|_{S \cap U} = 0$, it follows that u satisfies the initial condition (19.12).

The function u_0 satisfies $A_1 u_0 = \cdots = A_m u_0 = 0$ because it is independent of the v-coordinates. On the other hand, for each fixed w, u_1 is the potential function on H_w for $\iota_{H_w}^* \omega$ given by formula (11.24). The proof of Theorem 11.49 shows that $\iota_{H_w}^* du = \iota_{H_w}^* \omega$ for each w. It follows that $A_k u = A_k(u_1) = f_k$ for each $k = 1, \ldots, m$, so u is a solution to (19.11) as well.

To prove uniqueness, suppose $\widetilde{u}$ is any other solution to (19.11)–(19.12) on U, and let $\psi = u - \widetilde{u}$. Then $A_k \psi = 0$ for each k, so ψ is independent of v. It follows that $\psi(v, w) = \psi(0, w)$, which is zero because u and $\widetilde{u}$ satisfy (19.12). □

Next we apply the Frobenius theorem to a class of nonlinear overdetermined PDEs. These are equations for a vector-valued function $u = (u^1, \ldots, u^m)$ that express all first partial derivatives of u in terms of the independent variables and the values of u. We explain it first in the case of a single real-valued function u of two independent variables (x, y), in which case the notation is considerably simpler.

Suppose we seek a solution u to the system

$$\begin{aligned} \frac{\partial u}{\partial x}(x, y) &= \alpha\big(x, y, u(x, y)\big), \\ \frac{\partial u}{\partial y}(x, y) &= \beta\big(x, y, u(x, y)\big), \end{aligned} \tag{19.13}$$

where α and β are smooth real-valued functions defined on some open subset $W \subseteq \mathbb{R}^3$. This is an overdetermined system of (possibly nonlinear) first-order PDEs. (In fact, almost any pair of smooth first-order partial differential equations for one unknown function of two variables can be put into this form, at least locally, simply by solving the two equations for $\partial u/\partial x$ and $\partial u/\partial y$. Whether this can be done in principle is a question that is completely answered by the implicit function theorem; whether it can be done in practice depends on the specific equations and how clever you are.)

To determine the compatibility conditions that α and β must satisfy for solvability of (19.13), assume u is a smooth solution on some open subset of $\mathbb{R}^2$. Because $\partial^2 u/\partial x \partial y = \partial^2 u/\partial y \partial x$, (19.13) implies

$$\frac{\partial}{\partial y}\big(\alpha\big(x, y, u(x, y)\big)\big) = \frac{\partial}{\partial x}\big(\beta\big(x, y, u(x, y)\big)\big)$$

and therefore by the chain rule

$$\frac{\partial \alpha}{\partial y} + \beta \frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial x} + \alpha \frac{\partial \beta}{\partial z}. \tag{19.14}$$

This is true at a point $(x, y, z) \in W$ provided there is a smooth solution u with $u(x, y) = z$. In particular, (19.14) is a necessary condition for (19.13) to have a solution in a neighborhood of each point (x_0, y_0) with freely specified initial value $u(x_0, y_0) = z_0$. Using the Frobenius theorem, we can show that this condition is sufficient.

Proposition 19.28. *Suppose α and β are smooth real-valued functions defined on some open subset $W \subseteq \mathbb{R}^3$ and satisfying (19.14) there. For each $(x_0, y_0, z_0) \in W$, there exist a neighborhood U of (x_0, y_0) in $\mathbb{R}^2$ and a unique smooth function $u\colon U \to \mathbb{R}$ satisfying (19.13) and $u(x_0, y_0) = z_0$.*

Proof. The idea of the proof is that the system (19.13) determines the partial derivatives of u in terms of its values, and therefore determines the tangent plane to the graph of u at each point in terms of the coordinates of the point on the graph. This collection of tangent planes defines a smooth rank-2 distribution on W (Fig. 19.10), and (19.14) is equivalent to the involutivity condition for this distribution.

If there is a solution u on an open subset $U \subseteq \mathbb{R}^2$, the map $F\colon U \to W$ given by

$$F(x, y) = \big(x, y, u(x, y)\big)$$

is a smooth global parametrization of the graph $\Gamma(u) \subseteq U \times \mathbb{R}$. At any point $p = F(x, y)$, the tangent space $T_p\Gamma(u)$ is spanned by the vectors

$$dF\left(\left.\frac{\partial}{\partial x}\right|_{(x,y)}\right) = \left.\frac{\partial}{\partial x}\right|_p + \frac{\partial u}{\partial x}(x, y) \left.\frac{\partial}{\partial z}\right|_p,$$

$$dF\left(\left.\frac{\partial}{\partial y}\right|_{(x,y)}\right) = \left.\frac{\partial}{\partial y}\right|_p + \frac{\partial u}{\partial y}(x, y) \left.\frac{\partial}{\partial z}\right|_p.$$

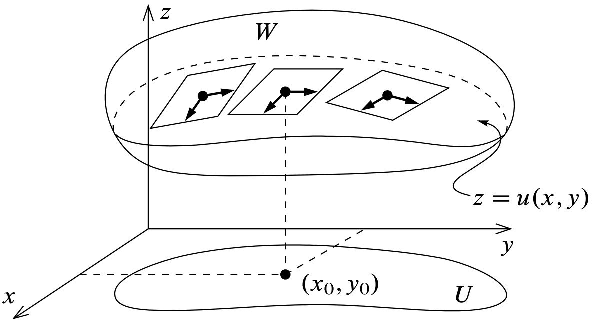


Fig. 19.10 Solving for u by finding its graph

The system (19.13) is satisfied if and only if

$$\begin{aligned} dF\left(\left.\frac{\partial}{\partial x}\right|_{(x,y)}\right) &= \left.\frac{\partial}{\partial x}\right|_p + \alpha\big(x, y, u(x,y)\big)\left.\frac{\partial}{\partial z}\right|_p, \\ dF\left(\left.\frac{\partial}{\partial y}\right|_{(x,y)}\right) &= \left.\frac{\partial}{\partial y}\right|_p + \beta\big(x, y, u(x,y)\big)\left.\frac{\partial}{\partial z}\right|_p. \end{aligned} \tag{19.15}$$

Let X and Y be the vector fields

$$\begin{aligned} X &= \frac{\partial}{\partial x} + \alpha(x,y,z)\frac{\partial}{\partial z}, \\ Y &= \frac{\partial}{\partial y} + \beta(x,y,z)\frac{\partial}{\partial z} \end{aligned} \tag{19.16}$$

on W, and let D be the distribution on W spanned by X and Y. Because (19.15) says that $T_p\Gamma(u)$ is spanned by X_p and Y_p, a necessary condition for the system (19.13) to be satisfied is that $\Gamma(u)$ be an integral manifold of D. On the other hand, this condition is also sufficient: if $\Gamma(u)$ is an integral manifold, then $dF(\partial/\partial x)$ and $dF(\partial/\partial y)$ must both be linear combinations of X and Y, and comparing $\partial/\partial x$ and $\partial/\partial y$ components shows that this can happen only if (19.15) holds.

A straightforward computation using (19.14) shows that $[X,Y] \equiv 0$, so given any point $p = (x_0, y_0, z_0) \in W$, there is an integral manifold N of D containing p. Let $\Phi\colon V \to \mathbb{R}$ be a defining function for N on some neighborhood V of p; for example, we could take Φ to be the third coordinate function in a flat chart. The tangent space to N at each point $p \in N$ (namely D_p) is equal to the kernel of $d\Phi_p$. Since $\partial/\partial z|_p \notin D_p$ at any point p, this implies that $\partial\Phi/\partial z \neq 0$ at p, so by the implicit function theorem N is the graph of a smooth function $z = u(x,y)$ in some neighborhood of p. You can verify easily that u is a solution to the problem. Uniqueness follows immediately from Proposition 19.16. □

As in several cases we have seen before, the proof of Proposition 19.28 actually contains a procedure for finding solutions to (19.13): find flat coordinates (u, v, w) for the distribution spanned by the vector fields X and Y defined by (19.16), and solve the equation $w = \text{constant}$ for $z = u(x, y)$. Some examples are given in Problem 19-13.

There is a straightforward generalization of this result to higher dimensions. The general statement of the theorem is a bit complicated, but verifying the necessary conditions in specific examples usually just amounts to computing mixed partial derivatives and applying the chain rule.

Proposition 19.29. *Suppose W is an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, and $\alpha = \left(\alpha^i_j\right) \colon W \to \mathrm{M}(m \times n, \mathbb{R})$ is a smooth matrix-valued function satisfying*

$$\frac{\partial \alpha^i_j}{\partial x^k} + \alpha^l_k \frac{\partial \alpha^i_j}{\partial z^l} = \frac{\partial \alpha^i_k}{\partial x^j} + \alpha^l_j \frac{\partial \alpha^i_k}{\partial z^l} \quad \textit{for all } i, j, k,$$

where we denote a point in $\mathbb{R}^n \times \mathbb{R}^m$ by $(x, z) = \left(x^1, \dots, x^n, z^1, \dots, z^m\right)$. For any $(x_0, z_0) \in W$, there is a neighborhood U of x_0 in $\mathbb{R}^n$ and a unique smooth function $u \colon U \to \mathbb{R}^m$ such that $u(x_0) = z_0$ and the Jacobian of u satisfies

$$\frac{\partial u^i}{\partial x^j}\left(x^1, \dots, x^n\right) = \alpha^i_j\left(x^1, \dots, x^n, u^1(x), \dots, u^m(x)\right).$$

► **Exercise 19.30.** Prove Proposition 19.29.

Problems

19-1. Prove Proposition 19.11 (a smooth distribution is involutive if and only if it determines a differential ideal).

19-2. Let D be a smooth distribution of rank k on a smooth n-manifold M, and suppose $\omega^1, \dots, \omega^{n-k}$ are smooth local defining forms for D on an open subset $U \subseteq M$. Show that D is involutive on U if and only if the following identity holds for each $i = 1, \dots, n-k$:

$$d\omega^i \wedge \omega^1 \wedge \cdots \wedge \omega^{n-k} = 0.$$

(*Used on p. 582.*)

19-3. Let ω be a smooth 1-form on a smooth manifold M. A smooth positive function μ on some open subset $U \subseteq M$ is called an ***integrating factor*** for ω if $\mu\omega$ is exact on U. Prove the following statements:

(a) If ω is nowhere-vanishing, then ω admits an integrating factor in a neighborhood of each point if and only if $d\omega \wedge \omega \equiv 0$.

(b) If $\dim M = 2$, then every nonvanishing smooth 1-form admits an integrating factor in a neighborhood of each point.

19-4. Let $U \subseteq \mathbb{R}^3$ be the subset where all three coordinates are positive, and let D be the distribution on U spanned by the vector fields

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}.$$

Find an explicit global flat chart for D on U.

19-5. Let D be the distribution on $\mathbb{R}^3$ spanned by

$$X = \frac{\partial}{\partial x} + yz\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y}.$$

(a) Find an integral submanifold of D passing through the origin.
(b) Is D involutive? Explain your answer in light of part (a).

19-6. Let D be an involutive distribution on a smooth manifold M, and let $\gamma\colon J \to M$ be a smooth curve. Prove the following statements.
(a) If H is an integral manifold of D, and the image of γ is contained in H, then $\gamma'(t)$ is in $T_{\gamma(t)}H \subseteq T_{\gamma(t)}M$ for all $t \in J$.
(b) Conversely, if $\gamma'(t)$ lies in D for all t, then the image of γ is contained in a single leaf of the foliation determined by D.
[Remark: compare this to the result of Problem 5-19.]

19-7. Let D be an involutive distribution on a smooth manifold M, and let N be a connected integral manifold of D. Show that if N is a closed subset of M, then it is a maximal connected integral manifold and is therefore a leaf of the foliation determined by D. (*Used on p. 545.*)

19-8. Suppose M and N are smooth manifolds and $F\colon M \to N$ is a smooth submersion. Show that the connected components of the nonempty level sets of F form a foliation of M.

19-9. Prove Proposition 19.23 (invariant distributions vs. invariant foliations).

19-10. Let M and N be smooth manifolds. Suppose $\mathcal{F}$ is a foliation on M of codimension k, and $\varphi\colon N \to M$ is a smooth map. Show that if φ is transverse to each leaf of $\mathcal{F}$, then there is a unique codimension-k foliation $\varphi^*\mathcal{F}$ on N, called the ***pullback of $\mathcal{F}$***, such that φ maps each leaf of $\varphi^*\mathcal{F}$ into a single leaf of $\mathcal{F}$.

19-11. Consider the following system of PDEs for $u \in C^\infty\left(\mathbb{R}^3\right)$:

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + x(y+1)\frac{\partial u}{\partial z} = xy,$$
$$\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial z} = y - 1.$$

Find a solution u to this system satisfying $u(0,0,z) = z$. [Hint: look at Example 19.14.]

19-12. Consider the following system of PDEs for $u \in C^\infty\left(\mathbb{R}^4\right)$:

$$\frac{\partial u}{\partial w} + x\frac{\partial u}{\partial y} + 2(w + xy)\frac{\partial u}{\partial z} = 0,$$
$$\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial y} + 2(x - wy)\frac{\partial u}{\partial z} = 0.$$

(a) Show that there do not exist two solutions u^1, u^2 with linearly independent differentials on any open subset of $\mathbb{R}^4$.

(b) Show that in a neighborhood of each point, there exists a solution with nonvanishing differential.

19-13. Of the systems of partial differential equations below, determine which ones have solutions $z(x,y)$ (or, for part (c), $z(x,y)$ and $w(x,y)$) in a neighborhood of the origin for arbitrary positive values of $z(0,0)$ (respectively, $z(0,0)$ and $w(0,0)$).

(a) $\dfrac{\partial z}{\partial x} = z\cos y; \qquad \dfrac{\partial z}{\partial y} = -z\log z\tan y.$

(b) $\dfrac{\partial z}{\partial x} = e^{xz}; \qquad \dfrac{\partial z}{\partial y} = xe^{yz}.$

(c) $\dfrac{\partial z}{\partial x} = z; \qquad \dfrac{\partial z}{\partial y} = w; \qquad \dfrac{\partial w}{\partial x} = w; \qquad \dfrac{\partial w}{\partial y} = z.$