Chapter 9
Integral Curves and Flows

In this chapter we continue our study of vector fields. The primary geometric objects associated with smooth vector fields are their *integral curves*, which are smooth curves whose velocity at each point is equal to the value of the vector field there. The collection of all integral curves of a given vector field on a manifold determines a family of diffeomorphisms of (open subsets of) the manifold, called a *flow*. Any smooth $\mathbb{R}$-action is a flow, for example; but there are flows that are not $\mathbb{R}$-actions because the diffeomorphisms may not be defined on the whole manifold for every $t \in \mathbb{R}$.

The main theorem of the chapter, the *fundamental theorem on flows*, asserts that every smooth vector field determines a unique maximal integral curve starting at each point, and the collection of all such integral curves determines a unique maximal flow. The proof is an application of the existence, uniqueness, and smoothness theorem for solutions of ordinary differential equations (see Appendix D).

After proving the fundamental theorem, we explore some of the properties of vector fields and flows. First, we investigate conditions under which a vector field generates a global flow. Then we show how "flowing out" from initial submanifolds along vector fields can be used to create useful parametrizations of larger submanifolds. Next we examine the local behavior of flows, and find that the behavior at points where the vector field vanishes, which correspond to *equilibrium points* of the flow, is very different from the behavior at points where it does not vanish, where the flow looks locally like translation along parallel coordinate lines.

We then introduce the *Lie derivative*, which is a coordinate-independent way of computing the rate of change of one vector field along the flow of another. It leads to some deep connections among vector fields, their Lie brackets, and their flows. In particular, we will prove that two vector fields have commuting flows if and only if their Lie bracket is zero. Based on this fact, we can prove a necessary and sufficient condition for a smooth local frame to be expressible as a coordinate frame. We then discuss how some of the results of this chapter can be generalized to *time-dependent vector fields* on manifolds.

In the last section of the chapter, we describe an important application of flows to the study of first-order partial differential equations.

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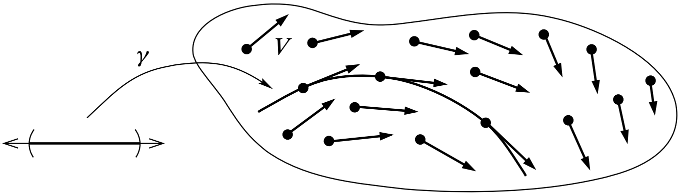


Fig. 9.1 An integral curve of a vector field

Integral Curves

Suppose M is a smooth manifold with or without boundary. If $\gamma\colon J \to M$ is a smooth curve, then for each $t \in J$, the velocity vector $\gamma'(t)$ is a vector in $T_{\gamma(t)}M$. In this section we describe a way to work backwards: given a tangent vector at each point, we seek a curve whose velocity at each point is equal to the given vector there.

If V is a vector field on M, an ***integral curve of V*** is a differentiable curve $\gamma\colon J \to M$ whose velocity at each point is equal to the value of V at that point:

$$\gamma'(t) = V_{\gamma(t)} \quad \text{for all } t \in J.$$

(See Fig. 9.1.) If $0 \in J$, the point $\gamma(0)$ is called the ***starting point of γ***. (The reason for the term "integral curve" will be explained shortly. Note that this is one definition that requires some differentiability hypothesis, because the definition of an integral curve would make no sense for a curve that is merely continuous.)

Example 9.1 (Integral Curves).

(a) Let (x, y) be standard coordinates on $\mathbb{R}^2$, and let $V = \partial/\partial x$ be the first coordinate vector field. It is easy to check that the integral curves of V are precisely the straight lines parallel to the x-axis, with parametrizations of the form $\gamma(t) = (a + t, b)$ for constants a and b (Fig. 9.2(a)). Thus, there is a unique integral curve starting at each point of the plane, and the images of different integral curves are either identical or disjoint.

(b) Let $W = x\,\partial/\partial y - y\,\partial/\partial x$ on $\mathbb{R}^2$ (Fig. 9.2(b)). If $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ is a smooth curve, written in standard coordinates as $\gamma(t) = (x(t), y(t))$, then the condition $\gamma'(t) = W_{\gamma(t)}$ for γ to be an integral curve translates to

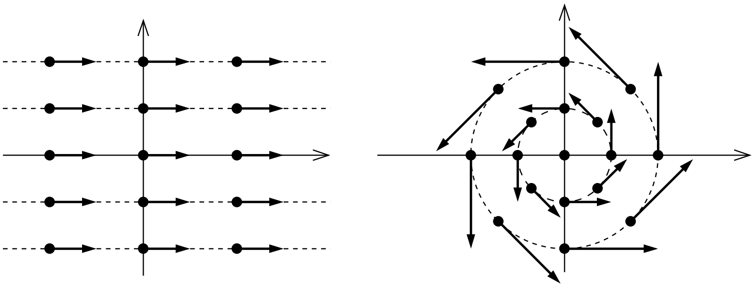
$$x'(t)\,\frac{\partial}{\partial x}\bigg|_{\gamma(t)} + y'(t)\,\frac{\partial}{\partial y}\bigg|_{\gamma(t)} = x(t)\,\frac{\partial}{\partial y}\bigg|_{\gamma(t)} - y(t)\,\frac{\partial}{\partial x}\bigg|_{\gamma(t)}.$$

Comparing the components of these vectors, we see that this is equivalent to the system of ordinary differential equations

$$x'(t) = -y(t),$$
$$y'(t) = x(t).$$

These equations have the solutions

$$x(t) = a\cos t - b\sin t, \qquad y(t) = a\sin t + b\cos t,$$



(a) $V = \partial/\partial x$. (b) $W = x\,\partial/\partial y - y\,\partial/\partial x$.

Fig. 9.2 Vector fields and their integral curves in the plane

for arbitrary constants a and b, and thus each curve of the form $\gamma(t) = (a\cos t - b\sin t, a\sin t + b\cos t)$ is an integral curve of W. When $(a,b) = (0,0)$, this is the constant curve $\gamma(t) \equiv (0,0)$; otherwise, it is a circle traversed counterclockwise. Since $\gamma(0) = (a,b)$, we see once again that there is a unique integral curve starting at each point $(a,b) \in \mathbb{R}^2$, and the images of the various integral curves are either identical or disjoint. //

As the second example above illustrates, finding integral curves boils down to solving a system of ordinary differential equations in a smooth chart. Suppose V is a smooth vector field on M and $\gamma\colon J \to M$ is a smooth curve. On a smooth coordinate domain $U \subseteq M$, we can write γ in local coordinates as $\gamma(t) = \big(\gamma^1(t), \dots, \gamma^n(t)\big)$. Then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V can be written

$$\dot\gamma^i(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} = V^i\big(\gamma(t)\big)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)},$$

which reduces to the following autonomous system of ordinary differential equations (ODEs):

$$\begin{aligned} \dot\gamma^1(t) &= V^1\left(\gamma^1(t), \dots, \gamma^n(t)\right), \\ &\ \ \vdots \\ \dot\gamma^n(t) &= V^n\left(\gamma^1(t), \dots, \gamma^n(t)\right). \end{aligned} \tag{9.1}$$

(We use a dot to denote an ordinary derivative with respect to t when there are superscripts that would make primes hard to read.) The fundamental fact about such systems is the existence, uniqueness, and smoothness theorem, Theorem D.1. (This is the reason for the terminology "integral curves," because solving a system of ODEs is often referred to as "integrating" the system.) We will derive detailed consequences of that theorem later; for now, we just note the following simple result.

Proposition 9.2. *Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exist $\varepsilon > 0$ and a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at p.*

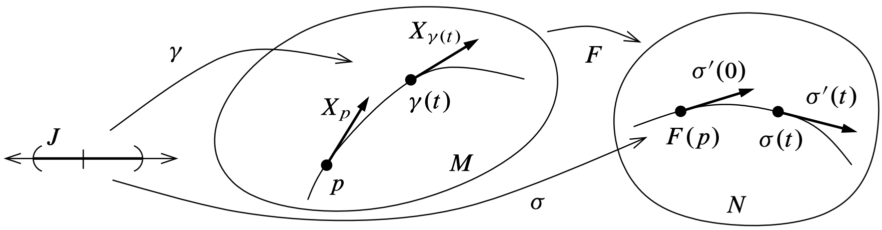


Fig. 9.3 Flows of F-related vector fields

Proof. This is just the existence statement of Theorem D.1 applied to the coordinate representation of V. □

The next two lemmas show how affine reparametrizations affect integral curves.

Lemma 9.3 (Rescaling Lemma). *Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma\colon J \to M$ be an integral curve of V. For any $a \in \mathbb{R}$, the curve $\widetilde{\gamma}\colon \widetilde{J} \to M$ defined by $\widetilde{\gamma}(t) = \gamma(at)$ is an integral curve of the vector field aV, where $\widetilde{J} = \{t : at \in J\}$.*

Proof. One way to see this is as a straightforward application of the chain rule in local coordinates. Somewhat more invariantly, we can examine the action of $\widetilde{\gamma}'(t)$ on a smooth real-valued function f defined in a neighborhood of a point $\widetilde{\gamma}(t_0)$. By the chain rule and the fact that γ is an integral curve of V,

$$\widetilde{\gamma}'(t_0)f = \frac{d}{dt}\bigg|_{t=t_0}(f \circ \widetilde{\gamma})(t) = \frac{d}{dt}\bigg|_{t=t_0}(f \circ \gamma)(at)$$
$$= a(f \circ \gamma)'(at_0) = a\gamma'(at_0)f = aV_{\widetilde{\gamma}(t_0)}f.$$ □

Lemma 9.4 (Translation Lemma). *Let V, M, J, and γ be as in the preceding lemma. For any $b \in \mathbb{R}$, the curve $\widehat{\gamma}\colon \widehat{J} \to M$ defined by $\widehat{\gamma}(t) = \gamma(t + b)$ is also an integral curve of V, where $\widehat{J} = \{t : t + b \in J\}$.*

► **Exercise 9.5.** Prove the translation lemma.

Proposition 9.6 (Naturality of Integral Curves). *Suppose M and N are smooth manifolds and $F\colon M \to N$ is a smooth map. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related if and only if F takes integral curves of X to integral curves of Y, meaning that for each integral curve γ of X, $F \circ \gamma$ is an integral curve of Y.*

Proof. Suppose first that X and Y are F-related, and $\gamma\colon J \to M$ is an integral curve of X. If we define $\sigma\colon J \to N$ by $\sigma = F \circ \gamma$ (see Fig. 9.3), then

$$\sigma'(t) = \big(F \circ \gamma\big)'(t) = dF_{\gamma(t)}\big(\gamma'(t)\big) = dF_{\gamma(t)}\big(X_{\gamma(t)}\big) = Y_{F(\gamma(t))} = Y_{\sigma(t)},$$

so σ is an integral curve of Y.

Conversely, suppose F takes integral curves of X to integral curves of Y. Given $p \in M$, let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be an integral curve of X starting at p. Since $F \circ \gamma$ is an integral curve of Y starting at $F(p)$, we have

$$Y_{F(p)} = (F \circ \gamma)'(0) = dF_p\big(\gamma'(0)\big) = dF_p(X_p),$$

which shows that X and Y are F-related. □

Flows

Here is another way to visualize the family of integral curves associated with a vector field. Let M be a smooth manifold and $V \in \mathfrak{X}(M)$, and suppose that for each point $p \in M$, V has a unique integral curve starting at p and defined for all $t \in \mathbb{R}$, which we denote by $\theta^{(p)}\colon \mathbb{R} \to M$. (It may not always be the case that every integral curve is defined for all t, but for purposes of illustration let us assume so for the time being.) For each $t \in \mathbb{R}$, we can define a map $\theta_t\colon M \to M$ by sending each $p \in M$ to the point obtained by following for time t the integral curve starting at p:

$$\theta_t(p) = \theta^{(p)}(t).$$

Each map θ_t "slides" the manifold along the integral curves for time t. The translation lemma implies that $t \mapsto \theta^{(p)}(t+s)$ is an integral curve of V starting at $q = \theta^{(p)}(s)$; since we are assuming uniqueness of integral curves, $\theta^{(q)}(t) = \theta^{(p)}(t+s)$. When we translate this into a statement about the maps θ_t, it becomes

$$\theta_t \circ \theta_s(p) = \theta_{t+s}(p).$$

Together with the equation $\theta_0(p) = \theta^{(p)}(0) = p$, which holds by definition, this implies that the map $\theta\colon \mathbb{R} \times M \to M$ is an action of the additive group $\mathbb{R}$ on M.

Motivated by these observations, we define a ***global flow*** on M (also called a ***one-parameter group action***) to be a continuous left $\mathbb{R}$-action on M; that is, a continuous map $\theta\colon \mathbb{R} \times M \to M$ satisfying the following properties for all $s, t \in \mathbb{R}$ and $p \in M$:

$$\theta\big(t, \theta(s,p)\big) = \theta(t+s, p), \qquad \theta(0,p) = p. \tag{9.2}$$

Given a global flow θ on M, we define two collections of maps as follows:

- For each $t \in \mathbb{R}$, define a continuous map $\theta_t\colon M \to M$ by

  $$\theta_t(p) = \theta(t,p).$$

  The defining properties (9.2) are equivalent to the ***group laws***

  $$\theta_t \circ \theta_s = \theta_{t+s}, \qquad \theta_0 = \mathrm{Id}_M. \tag{9.3}$$

  As is the case for any continuous group action, each map $\theta_t\colon M \to M$ is a homeomorphism, and if the flow is smooth, θ_t is a diffeomorphism.
- For each $p \in M$, define a curve $\theta^{(p)}\colon \mathbb{R} \to M$ by

  $$\theta^{(p)}(t) = \theta(t,p).$$

  The image of this curve is the orbit of p under the group action.

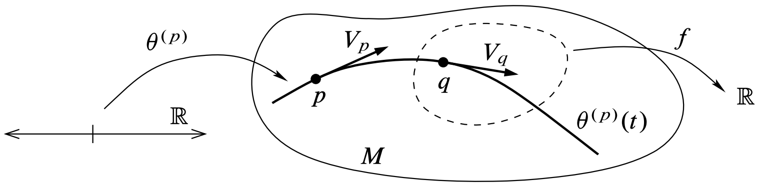


Fig. 9.4 The infinitesimal generator of a global flow

The next proposition shows that every smooth global flow is derived from the integral curves of some smooth vector field in precisely the way we described above. If $\theta\colon \mathbb{R} \times M \to M$ is a smooth global flow, for each $p \in M$ we define a tangent vector $V_p \in T_pM$ by

$$V_p = \theta^{(p)\prime}(0).$$

The assignment $p \mapsto V_p$ is a (rough) vector field on M, which is called the ***infinitesimal generator of*** $\boldsymbol{\theta}$, for reasons we will explain below.

Proposition 9.7. *Let $\theta\colon \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. The infinitesimal generator V of θ is a smooth vector field on M, and each curve $\theta^{(p)}$ is an integral curve of V.*

Proof. To show that V is smooth, it suffices by Proposition 8.14 to show that Vf is smooth for every smooth real-valued function f defined on an open subset $U \subseteq M$. For any such f and any $p \in U$, just note that

$$Vf(p) = V_p f = \theta^{(p)\prime}(0) f = \left.\frac{d}{dt}\right|_{t=0} f\left(\theta^{(p)}(t)\right) = \left.\frac{\partial}{\partial t}\right|_{(0,p)} f\left(\theta(t,p)\right).$$

Because $f(\theta(t,p))$ is a smooth function of (t,p) by composition, so is its partial derivative with respect to t. Thus, $Vf(p)$ depends smoothly on p, so V is smooth.

Next we need to show that $\theta^{(p)}$ is an integral curve of V, which means that $\theta^{(p)\prime}(t) = V_{\theta^{(p)}(t)}$ for all $p \in M$ and all $t \in \mathbb{R}$. Let $t_0 \in \mathbb{R}$ be arbitrary, and set $q = \theta^{(p)}(t_0) = \theta_{t_0}(p)$, so what we have to show is $\theta^{(p)\prime}(t_0) = V_q$ (see Fig. 9.4). By the group law, for all t,

$$\theta^{(q)}(t) = \theta_t(q) = \theta_t\left(\theta_{t_0}(p)\right) = \theta_{t+t_0}(p) = \theta^{(p)}(t+t_0). \tag{9.4}$$

Therefore, for any smooth real-valued function f defined in a neighborhood of q,

$$\begin{aligned} V_q f = \theta^{(q)\prime}(0) f &= \left.\frac{d}{dt}\right|_{t=0} f\left(\theta^{(q)}(t)\right) = \left.\frac{d}{dt}\right|_{t=0} f\left(\theta^{(p)}(t+t_0)\right) \\ &= \theta^{(p)\prime}(t_0) f, \end{aligned} \tag{9.5}$$

which was to be shown. □

Example 9.8 (Global Flows). The two vector fields on the plane described in Example 9.1 both had integral curves defined for all $t \in \mathbb{R}$, so they generate global flows. Using the results of that example, we can write down the flows explicitly.

(a) The flow of $V = \partial/\partial x$ in $\mathbb{R}^2$ is the map $\tau\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\tau_t(x, y) = (x + t, y).$$

For each nonzero $t \in \mathbb{R}$, τ_t translates the plane to the right ($t > 0$) or left ($t < 0$) by a distance $|t|$.

(b) The flow of $W = x\,\partial/\partial y - y\,\partial/\partial x$ is the map $\theta\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\theta_t(x, y) = (x \cos t - y \sin t,\ x \sin t + y \cos t).$$

For each $t \in \mathbb{R}$, θ_t rotates the plane through an angle t about the origin. //

The Fundamental Theorem on Flows

We have seen that every smooth global flow gives rise to a smooth vector field whose integral curves are precisely the curves defined by the flow. Conversely, we would like to be able to say that every smooth vector field is the infinitesimal generator of a smooth global flow. However, it is easy to see that this cannot be the case, because there are smooth vector fields whose integral curves are not defined for all $t \in \mathbb{R}$. Here are two examples.

Example 9.9. Let $M = \mathbb{R}^2 \smallsetminus \{0\}$ with standard coordinates (x, y), and let V be the vector field $\partial/\partial x$ on M. The unique integral curve of V starting at $(-1, 0) \in M$ is $\gamma(t) = (t - 1, 0)$. However, in this case, γ cannot be extended continuously past $t = 1$. This is intuitively evident because of the "hole" in M at the origin; to prove it rigorously, suppose $\tilde{\gamma}$ is any continuous extension of γ past $t = 1$. Then $\gamma(t) \to \tilde{\gamma}(1) \in \mathbb{R}^2 \smallsetminus \{0\}$ as $t \nearrow 1$. But we can also consider γ as a map into $\mathbb{R}^2$ by composing with the inclusion $M \hookrightarrow \mathbb{R}^2$, and it is obvious from the formula that $\gamma(t) \to (0, 0)$ as $t \nearrow 1$. Since limits in $\mathbb{R}^2$ are unique, this is a contradiction. //

Example 9.10. For a more subtle example, let M be all of $\mathbb{R}^2$ and let $W = x^2 \partial/\partial x$. You can check easily that the unique integral curve of W starting at $(1, 0)$ is

$$\gamma(t) = \left(\frac{1}{1 - t}, 0\right).$$

This curve also cannot be extended past $t = 1$, because its x-coordinate is unbounded as $t \nearrow 1$. //

For this reason, we make the following definitions. If M is a manifold, a ***flow domain*** for M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set $\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\}$ is an open interval containing 0 (Fig. 9.5). A ***flow*** on M is a continuous map $\theta\colon \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, that satisfies the following group laws: for all $p \in M$,

$$\theta(0, p) = p, \tag{9.6}$$

and for all $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{(\theta(s,p))}$ such that $s + t \in \mathcal{D}^{(p)}$,

$$\theta\big(t, \theta(s, p)\big) = \theta(t + s, p). \tag{9.7}$$

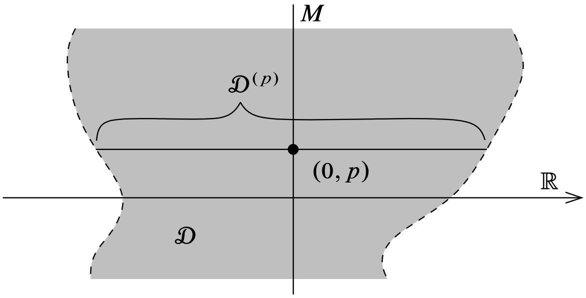


Fig. 9.5 A flow domain

We sometimes call θ a ***local flow*** to distinguish it from a global flow as defined earlier. The unwieldy term ***local one-parameter group action*** is also used.

If θ is a flow, we define $\theta_t(p) = \theta^{(p)}(t) = \theta(t, p)$ whenever $(t, p) \in \mathcal{D}$, just as for a global flow. For each $t \in \mathbb{R}$, we also define

$$M_t = \left\{p \in M : (t, p) \in \mathcal{D}\right\}, \tag{9.8}$$

so that

$$p \in M_t \Leftrightarrow t \in \mathcal{D}^{(p)} \Leftrightarrow (t, p) \in \mathcal{D}.$$

If θ is smooth, the ***infinitesimal generator of θ*** is defined by $V_p = \theta^{(p)\prime}(0)$.

Proposition 9.11. *If $\theta\colon \mathcal{D} \to M$ is a smooth flow, then the infinitesimal generator V of θ is a smooth vector field, and each curve $\theta^{(p)}$ is an integral curve of V.*

Proof. The proof is essentially identical to the analogous proof for global flows, Proposition 9.7. In the proof that V is smooth, we need only note that for any $p_0 \in M$, $\theta(t, p)$ is defined and smooth for all (t, p) sufficiently close to $(0, p_0)$ because $\mathcal{D}$ is open. In the proof that $\theta^{(p)}$ is an integral curve, we need to verify that all of the expressions in (9.4) and (9.5) make sense. Suppose $t_0 \in \mathcal{D}^{(p)}$. Because both $\mathcal{D}^{(p)}$ and $\mathcal{D}^{(\theta_{t_0}(p))}$ are open intervals containing 0, there is a positive number ε such that $t + t_0 \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{(\theta_{t_0}(p))}$ whenever $|t| < \varepsilon$, and then $\theta_t(\theta_{t_0}(p)) = \theta_{t+t_0}(p)$ by definition of a flow. The rest of the proof goes through just as before. □

The next theorem is the main result of this section. A ***maximal integral curve*** is one that cannot be extended to an integral curve on any larger open interval, and a ***maximal flow*** is a flow that admits no extension to a flow on a larger flow domain.

Theorem 9.12 (Fundamental Theorem on Flows). *Let V be a smooth vector field on a smooth manifold M. There is a unique smooth maximal flow $\theta\colon \mathcal{D} \to M$ whose infinitesimal generator is V. This flow has the following properties:*

(a) *For each $p \in M$, the curve $\theta^{(p)}\colon \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V starting at p.*
(b) *If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))}$ is the interval $\mathcal{D}^{(p)} - s = \left\{t - s : t \in \mathcal{D}^{(p)}\right\}$.*

(c) *For each $t \in \mathbb{R}$, the set M_t is open in M, and $\theta_t \colon M_t \to M_{-t}$ is a diffeomorphism with inverse θ_{-t}.*

Proof. Proposition 9.2 shows that there exists an integral curve starting at each point $p \in M$. Suppose $\gamma, \widetilde{\gamma} \colon J \to M$ are two integral curves of V defined on the same open interval J such that $\gamma(t_0) = \widetilde{\gamma}(t_0)$ for some $t_0 \in J$. Let $\mathcal{S}$ be the set of $t \in J$ such that $\gamma(t) = \widetilde{\gamma}(t)$. Clearly, $\mathcal{S} \neq \varnothing$, because $t_0 \in \mathcal{S}$ by hypothesis, and $\mathcal{S}$ is closed in J by continuity. On the other hand, suppose $t_1 \in \mathcal{S}$. Then in a smooth coordinate neighborhood around the point $p = \gamma(t_1)$, γ and $\widetilde{\gamma}$ are both solutions to same ODE with the same initial condition $\gamma(t_1) = \widetilde{\gamma}(t_1) = p$. By the uniqueness part of Theorem D.1, $\gamma \equiv \widetilde{\gamma}$ on an interval containing t_1, which implies that $\mathcal{S}$ is open in J. Since J is connected, $\mathcal{S} = J$, which implies that $\gamma = \widetilde{\gamma}$ on all of J. Thus, any two integral curves that agree at one point agree on their common domain.

For each $p \in M$, let $\mathcal{D}^{(p)}$ be the union of all open intervals $J \subseteq \mathbb{R}$ containing 0 on which an integral curve starting at p is defined. Define $\theta^{(p)} \colon \mathcal{D}^{(p)} \to M$ by letting $\theta^{(p)}(t) = \gamma(t)$, where γ is any integral curve starting at p and defined on an open interval containing 0 and t. Since all such integral curves agree at t by the argument above, $\theta^{(p)}$ is well defined, and is obviously the unique maximal integral curve starting at p.

Now let $\mathcal{D} = \{(t, p) \in \mathbb{R} \times M : t \in \mathcal{D}^{(p)}\}$, and define $\theta \colon \mathcal{D} \to M$ by $\theta(t, p) = \theta^{(p)}(t)$. As usual, we also write $\theta_t(p) = \theta(t, p)$. By definition, θ satisfies property (a) in the statement of the fundamental theorem: for each $p \in M$, $\theta^{(p)}$ is the unique maximal integral curve of V starting at p. To verify the group laws, fix any $p \in M$ and $s \in \mathcal{D}^{(p)}$, and write $q = \theta(s, p) = \theta^{(p)}(s)$. The curve $\gamma \colon \mathcal{D}^{(p)} - s \to M$ defined by $\gamma(t) = \theta^{(p)}(t + s)$ starts at q, and the translation lemma shows that γ is an integral curve of V. By uniqueness of ODE solutions, γ agrees with $\theta^{(q)}$ on their common domain, which is equivalent to the second group law (9.7), and the first group law (9.6) is immediate from the definition. By maximality of $\theta^{(q)}$, the domain of γ cannot be larger than $\mathcal{D}^{(q)}$, which means that $\mathcal{D}^{(p)} - s \subseteq \mathcal{D}^{(q)}$. Since $0 \in \mathcal{D}^{(p)}$, this implies that $-s \in \mathcal{D}^{(q)}$, and the group law implies that $\theta^{(q)}(-s) = p$. Applying the same argument with $(-s, q)$ in place of (s, p), we find that $\mathcal{D}^{(q)} + s \subseteq \mathcal{D}^{(p)}$, which is the same as $\mathcal{D}^{(q)} \subseteq \mathcal{D}^{(p)} - s$. This proves (b).

Next we show that $\mathcal{D}$ is open in $\mathbb{R} \times M$ (so it is a flow domain), and that $\theta \colon \mathcal{D} \to M$ is smooth. Define a subset $W \subseteq \mathcal{D}$ as the set of all $(t, p) \in \mathcal{D}$ such that θ is defined and smooth on a product neighborhood of (t, p) of the form $J \times U \subseteq \mathcal{D}$, where $J \subseteq \mathbb{R}$ is an open interval containing 0 and t and $U \subseteq M$ is a neighborhood of p. Then W is open in $\mathbb{R} \times M$, and the restriction of θ to W is smooth, so it suffices to show that $W = \mathcal{D}$. Suppose this is not the case. Then there exists some point $(\tau, p_0) \in \mathcal{D} \smallsetminus W$. For simplicity, assume $\tau > 0$; the argument for $\tau < 0$ is similar.

Let $t_0 = \inf\{t \in \mathbb{R} : (t, p_0) \notin W\}$ (Fig. 9.6). By the ODE theorem (applied in smooth coordinates around p_0), θ is defined and smooth in some product neighborhood of $(0, p_0)$, so $t_0 > 0$. Since $t_0 \leq \tau$ and $\mathcal{D}^{(p_0)}$ is an open interval containing 0 and τ, it follows that $t_0 \in \mathcal{D}^{(p_0)}$. Let $q_0 = \theta^{(p_0)}(t_0)$. By the ODE theorem again, there exist $\varepsilon > 0$ and a neighborhood U_0 of q_0 such that $(-\varepsilon, \varepsilon) \times U_0 \subseteq W$. We will

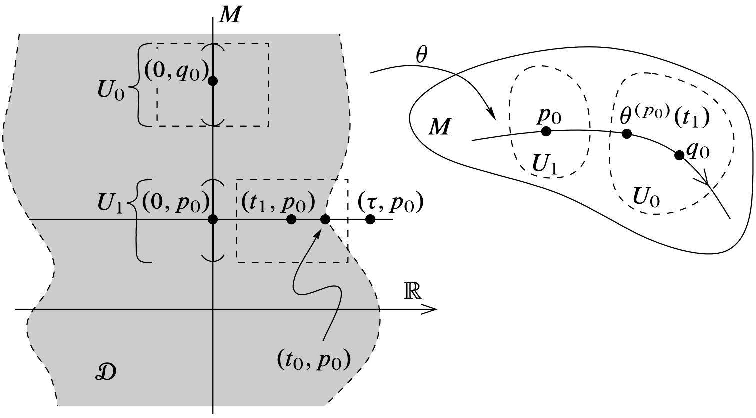


Fig. 9.6 Proof that $\mathcal{D}$ is open

use the group law to show that θ extends smoothly to a neighborhood of (t_0, p_0), which contradicts our choice of t_0.

Choose some $t_1 < t_0$ such that $t_1 + \varepsilon > t_0$ and $\theta^{(p_0)}(t_1) \in U_0$. Since $t_1 < t_0$, we have $(t_1, p_0) \in W$, and so there is a product neighborhood $(t_1 - \delta, t_1 + \delta) \times U_1 \subseteq W$. By definition of W, this implies that θ is defined and smooth on $[0, t_1 + \delta) \times U_1$. Because $\theta(t_1, p_0) \in U_0$, we can choose U_1 small enough that θ maps $\{t_1\} \times U_1$ into U_0. Define $\widetilde{\theta}\colon [0, t_1 + \varepsilon) \times U_1 \to M$ by

$$\widetilde{\theta}(t,p) = \begin{cases} \theta_t(p), & p \in U_1,\ 0 \le t < t_1, \\ \theta_{t-t_1} \circ \theta_{t_1}(p), & p \in U_1,\ t_1 - \varepsilon < t < t_1 + \varepsilon. \end{cases}$$

The group law for θ guarantees that these definitions agree where they overlap, and our choices of U_1, t_1, and ε ensure that this defines a smooth map. By the translation lemma, each map $t \mapsto \widetilde{\theta}(t,p)$ is an integral curve of V, so $\widetilde{\theta}$ is a smooth extension of θ to a neighborhood of (t_0, p_0), contradicting our choice of t_0. This completes the proof that $W = \mathcal{D}$.

Finally, we prove (c). The fact that M_t is open is an immediate consequence of the fact that $\mathcal{D}$ is open. From part (b) we deduce

$$\begin{aligned} p \in M_t &\Rightarrow t \in \mathcal{D}^{(p)} \Rightarrow \mathcal{D}^{(\theta_t(p))} = \mathcal{D}^{(p)} - t \\ &\Rightarrow -t \in \mathcal{D}^{(\theta_t(p))} \Rightarrow \theta_t(p) \in M_{-t}, \end{aligned}$$

which shows that θ_t maps M_t to M_{-t}. Moreover, the group laws then show that $\theta_{-t} \circ \theta_t$ is equal to the identity on M_t. Reversing the roles of t and $-t$ shows that $\theta_t \circ \theta_{-t}$ is the identity on M_{-t}, which completes the proof. □

The flow whose existence and uniqueness are asserted in the fundamental theorem is called the ***flow generated by*** V, or just the ***flow of*** V. The term "infinitesimal generator" comes from the following picture: in a smooth chart, a good approximation to an integral curve can be obtained by composing many small straight-line motions, with the direction and length of each motion determined by the value of the vector field at the point arrived at in the previous step. Intuitively, one can think of a flow as a sequence of infinitely many infinitesimally small linear steps.

The naturality of integral curves (Proposition 9.6) translates into the following naturality statement for flows.

Proposition 9.13 (Naturality of Flows). *Suppose M and N are smooth manifolds, $F\colon M \to N$ is a smooth map, $X \in \mathfrak{X}(M)$, and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η the flow of Y. If X and Y are F-related, then for each $t \in \mathbb{R}$, $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:*

$$\begin{CD} M_t @>F>> N_t \\ @V{\theta_t}VV @VV{\eta_t}V \\ M_{-t} @>>F> N_{-t}. \end{CD}$$

Proof. By Proposition 9.6, for any $p \in M$, the curve $F \circ \theta^{(p)}$ is an integral curve of Y starting at $F \circ \theta^{(p)}(0) = F(p)$. By uniqueness of integral curves, therefore, the maximal integral curve $\eta^{(F(p))}$ must be defined at least on the interval $\mathcal{D}^{(p)}$, and $F \circ \theta^{(p)} = \eta^{(F(p))}$ on that interval. This means that

$$p \in M_t \Rightarrow t \in \mathcal{D}^{(p)} \Rightarrow t \in \mathcal{D}^{(F(p))} \Rightarrow F(p) \in N_t,$$

which is equivalent to $F(M_t) \subseteq N_t$, and

$$F\big(\theta^{(p)}(t)\big) = \eta^{(F(p))}(t) \quad \text{for all } t \in \mathcal{D}^{(p)},$$

which is equivalent to $\eta_t \circ F(p) = F \circ \theta_t(p)$ for all $p \in M_t$. □

The next corollary is immediate.

Corollary 9.14 (Diffeomorphism Invariance of Flows). *Let $F\colon M \to N$ be a diffeomorphism. If $X \in \mathfrak{X}(M)$ and θ is the flow of X, then the flow of F_*X is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.* □

Complete Vector Fields

As we observed earlier in this chapter, not every smooth vector field generates a global flow. The ones that do are important enough to deserve a name. We say that a smooth vector field is ***complete*** if it generates a global flow, or equivalently if each of its maximal integral curves is defined for all $t \in \mathbb{R}$. For example, both of the vector fields on the plane whose flows we computed in Example 9.8 are complete, whereas those of Examples 9.9 and 9.10 are not.

It is not always easy to determine by looking at a vector field whether it is complete or not. If you can solve the ODE explicitly to find all of the integral curves, and they all exist for all time, then the vector field is complete. On the other hand, if you can find a single integral curve that cannot be extended to all of $\mathbb{R}$, as we did for the vector fields of Examples 9.9 and 9.10, then it is not complete. However, it is often impossible to solve the ODE explicitly, so it is useful to have some general criteria for determining when a vector field is complete.

We will show below that all compactly supported smooth vector fields, and therefore all smooth vector fields on a compact manifold, are complete. The proof will be based on the following lemma.

Lemma 9.15 (Uniform Time Lemma). *Let V be a smooth vector field on a smooth manifold M, and let θ be its flow. Suppose there is a positive number ε such that for every $p \in M$, the domain of $\theta^{(p)}$ contains $(-\varepsilon, \varepsilon)$. Then V is complete.*

Proof. Suppose for the sake of contradiction that for some $p \in M$, the domain $\mathcal{D}^{(p)}$ of $\theta^{(p)}$ is bounded above. (A similar proof works if it is bounded below.) Let $b = \sup \mathcal{D}^{(p)}$, let t_0 be a positive number such that $b - \varepsilon < t_0 < b$, and let $q = \theta^{(p)}(t_0)$. The hypothesis implies that $\theta^{(q)}(t)$ is defined at least for $t \in (-\varepsilon, \varepsilon)$. Define a curve $\gamma \colon (-\varepsilon, t_0 + \varepsilon) \to M$ by

$$\gamma(t) = \begin{cases} \theta^{(p)}(t), & -\varepsilon < t < b, \\ \theta^{(q)}(t - t_0), & t_0 - \varepsilon < t < t_0 + \varepsilon. \end{cases}$$

These two definitions agree where they overlap, because $\theta^{(q)}(t - t_0) = \theta_{t-t_0}(q) = \theta_{t-t_0} \circ \theta_{t_0}(p) = \theta_t(p) = \theta^{(p)}(t)$ by the group law for θ. By the translation lemma, γ is an integral curve starting at p. Since $t_0 + \varepsilon > b$, this is a contradiction. □

Theorem 9.16. *Every compactly supported smooth vector field on a smooth manifold is complete.*

Proof. Suppose V is a compactly supported vector field on a smooth manifold M, and let $K = \operatorname{supp} V$. For each $p \in K$, there is a neighborhood U_p of p and a positive number ε_p such that the flow of V is defined at least on $\left(-\varepsilon_p, \varepsilon_p\right) \times U_p$. By compactness, finitely many such sets $U_{p_1}, \dots, U_{p_k}$ cover K. With $\varepsilon = \min\left\{\varepsilon_{p_1}, \dots, \varepsilon_{p_k}\right\}$, it follows that every maximal integral curve starting in K is defined at least on $(-\varepsilon, \varepsilon)$. Since $V \equiv 0$ outside of K, every integral curve starting in $M \smallsetminus K$ is constant and thus can be defined on all of $\mathbb{R}$. Thus the hypotheses of the uniform time lemma are satisfied, so V is complete. □

Corollary 9.17. *On a compact smooth manifold, every smooth vector field is complete.* □

Left-invariant vector fields on Lie groups form another class of vector fields that are always complete.

Theorem 9.18. *Every left-invariant vector field on a Lie group is complete.*

Proof. Let G be a Lie group, let $X \in \operatorname{Lie}(G)$, and let $\theta \colon \mathcal{D} \to G$ denote the flow of X. There is some $\varepsilon > 0$ such that $\theta^{(e)}$ is defined on $(-\varepsilon, \varepsilon)$.

Let $g \in G$ be arbitrary. Because X is L_g-related to itself, it follows from Proposition 9.6 that the curve $L_g \circ \theta^{(e)}$ is an integral curve of X starting at g and therefore

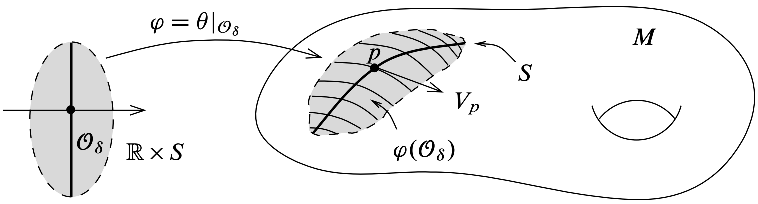


Fig. 9.7 A flowout

is equal to $\theta^{(g)}$. This shows that for each $g \in G$, the integral curve $\theta^{(g)}$ is defined at least on $(-\varepsilon, \varepsilon)$, so the uniform time lemma guarantees that X is complete. □

Here is another useful property of integral curves.

Lemma 9.19 (Escape Lemma). *Suppose M is a smooth manifold and $V \in \mathfrak{X}(M)$. If $\gamma\colon J \to M$ is a maximal integral curve of V whose domain J has a finite least upper bound b, then for any $t_0 \in J$, $\gamma\big([t_0, b)\big)$ is not contained in any compact subset of M.*

Proof. Problem 9-6. □

Flowouts

Flows provide the technical apparatus for many geometric constructions on manifolds. Most of those constructions are based on the following general theorem, which describes how flows behave in the vicinity of certain submanifolds.

Theorem 9.20 (Flowout Theorem). *Suppose M is a smooth manifold, $S \subseteq M$ is an embedded k-dimensional submanifold, and $V \in \mathfrak{X}(M)$ is a smooth vector field that is nowhere tangent to S. Let $\theta\colon \mathcal{D} \to M$ be the flow of V, let $\mathcal{O} = (\mathbb{R} \times S) \cap \mathcal{D}$, and let $\Phi = \theta|_{\mathcal{O}}$.*

(a) *$\Phi\colon \mathcal{O} \to M$ is an immersion.*
(b) *$\partial/\partial t \in \mathfrak{X}(\mathcal{O})$ is Φ-related to V.*
(c) *There exists a smooth positive function $\delta\colon S \to \mathbb{R}$ such that the restriction of Φ to $\mathcal{O}_\delta$ is injective, where $\mathcal{O}_\delta \subseteq \mathcal{O}$ is the flow domain*

$$\mathcal{O}_\delta = \big\{(t, p) \in \mathcal{O} : |t| < \delta(p)\big\}. \tag{9.9}$$

Thus, $\Phi(\mathcal{O}_\delta)$ is an immersed submanifold of M containing S, and V is tangent to this submanifold.

(d) *If S has codimension 1, then $\Phi|_{\mathcal{O}_\delta}$ a diffeomorphism onto an open submanifold of M.*

Remark. The submanifold $\Phi(\mathcal{O}_\delta) \subseteq M$ is called a ***flowout from S along V*** (see Fig. 9.7).

Proof. First we prove (b). Fix $p \in S$, and let $\sigma\colon \mathcal{D}^{(p)} \to \mathbb{R} \times S$ be the curve $\sigma(t) = (t, p)$. Then $\Phi \circ \sigma(t) = \theta(t, p)$ is an integral curve of V, so for any $t_0 \in \mathcal{D}^{(p)}$ it follows that

$$d\Phi_{(t_0,p)}\left(\left.\frac{\partial}{\partial t}\right|_{(t_0,p)}\right) = (\Phi \circ \sigma)'(t_0) = V_{\Phi(t_0,p)}. \tag{9.10}$$

Next we prove (a). The restriction of Φ to $\{0\} \times S$ is the composition of the diffeomorphism $\{0\} \times S \approx S$ with the embedding $S \hookrightarrow M$, so it is an embedding. Thus, the restriction of $d\Phi_{(0,p)}$ to T_pS (viewed as a subspace of $T_{(0,p)}\mathcal{O} \cong T_0\mathbb{R} \oplus T_pS$) is the inclusion $T_pS \hookrightarrow T_pM$. If $(E_1, \dots, E_k)$ is any basis for T_pS, it follows that $d\Phi_{(0,p)}$ maps the basis $\left(\partial/\partial t|_{(0,p)}, E_1, \dots, E_k\right)$ for $T_{(0,p)}\mathcal{O}$ to $(V_p, E_1, \dots, E_k)$. Since V_p is not tangent to S, this $(k+1)$-tuple is linearly independent and thus $d\Phi_{(0,p)}$ is injective.

To show $d\Phi$ is injective at other points, we argue as in the proof of the equivariant rank theorem. Given $(t_0, p_0) \in \mathcal{O}$, let $\tau_{t_0}\colon \mathcal{O} \to \mathbb{R} \times S$ be the translation $\tau_{t_0}(t, p) = (t + t_0, p)$. By the group law for θ, the following diagram commutes (where the horizontal maps might be defined only in open subsets containing $(0, p_0)$ and p_0, respectively):

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\tau_{t_0}} & \mathcal{O} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ M & \xrightarrow[\theta_{t_0}]{} & M. \end{array}$$

Both horizontal maps in the diagram above are local diffeomorphisms. Taking differentials, we obtain

$$\begin{array}{ccc} T_{(0,p_0)}\mathcal{O} & \xrightarrow{d(\tau_{t_0})_{(0,p_0)}} & T_{(t_0,p_0)}\mathcal{O} \\ {\scriptstyle d\Phi_{(0,p_0)}}\downarrow & & \downarrow{\scriptstyle d\Phi_{(t_0,p_0)}} \\ T_{p_0}M & \xrightarrow[d(\theta_{t_0})_{p_0}]{} & T_{\Phi(t_0,p_0)}M. \end{array}$$

Because the horizontal maps are isomorphisms, the two vertical maps have the same rank. Since we have already shown that $d\Phi_{(0,p_0)}$ has full rank, so does $d\Phi_{(t_0,p_0)}$. This completes the proof that Φ is an immersion.

Next we prove (c). Given a point $p_0 \in S$, choose a slice chart $\left(U, \left(x^i\right)\right)$ for S in M centered at p_0, so that $U \cap S$ is the set where $x^{k+1} = \dots = x^n = 0$ (where $n = \dim M$). Because V is not tangent to S, one of the last $n - k$ components of V_{p_0}, say $V^j(p_0)$, must be nonzero. Shrinking U if necessary, we may assume that there is a constant $c > 0$ such that

$$\left|V^j(p)\right| \geq c \quad \text{for all } p \in U. \tag{9.11}$$

Since $\Phi^{-1}(U)$ is open in $\mathbb{R} \times S$, we may choose a number $\varepsilon_{p_0} > 0$ and a neighborhood W_{p_0} of p_0 in S such that $\left(-\varepsilon_{p_0}, \varepsilon_{p_0}\right) \times W_{p_0} \subseteq \mathcal{O}$ and

$\Phi\big((-\varepsilon_{p_0},\varepsilon_{p_0})\times W_{p_0}\big)\subseteq U$. Write the component functions of Φ in these local coordinates as

$$\Phi(t,p)=\big(\Phi^1(t,p),\dots,\Phi^n(t,p)\big).$$

Because Φ is the restriction of the flow, the component function Φ^j satisfies

$$\frac{\partial\Phi^j}{\partial t}(t,p)=V^j\big(\Phi(t,p)\big),\qquad V^j(0,p)=0.$$

By (9.11) and the fundamental theorem of calculus, $\left|\Phi^j(t,p)\right|\geq c\,|t|$, and thus for $(t,p)\in(-\varepsilon_{p_0},\varepsilon_{p_0})\times W_{p_0}$ we conclude that $\Phi(t,p)\in S$ if and only if $t=0$.

Choose a smooth partition of unity $\{\psi_p : p\in S\}$ subordinate to the open cover $\{W_p : p\in S\}$ of S, and define $f\colon S\to\mathbb{R}$ by

$$f(q)=\sum_{p\in S}\varepsilon_p\psi_p(q). \tag{9.12}$$

Then f is smooth and positive. For each $q\in S$, there are finitely many $p\in S$ such that $\psi_p(q)>0$; if p_0 is one of these points such that ε_{p_0} is maximum among all such ε_p, then

$$f(q)\leq\varepsilon_{p_0}\sum_{p\in S}\psi_p(q)=\varepsilon_{p_0}.$$

It follows that if $(t,q)\in\mathcal{O}$ such that $|t|<f(q)$, then $(t,q)\in\big(-\varepsilon_{p_0},\varepsilon_{p_0}\big)\times W_{p_0}$, so $\Phi(t,q)\in S$ if and only if $t=0$.

Let $\delta=\frac{1}{2}f$. We will show that $\Phi|_{\mathcal{O}_\delta}$ is injective, where $\mathcal{O}_\delta$ is defined by (9.9). Suppose $\Phi(t,q)=\Phi(t',q')$ for some $(t,q),(t',q')\in\mathcal{O}_\delta$. By renaming the points if necessary, we may arrange that $f(q')\leq f(q)$. Our assumption means that $\theta_t(q)=\theta_{t'}(q')$, and the group law for θ then implies that $\theta_{t-t'}(q)=q'\in S$. The fact that (t,q) and (t',q') are in $\mathcal{O}_\delta$ implies that

$$|t-t'|\leq|t|+|t'|<\tfrac{1}{2}f(q)+\tfrac{1}{2}f(q')\leq f(q),$$

which forces $t=t'$ and thus $q=q'$.

Only (d) remains. If S has codimension 1, then $\Phi|_{\mathcal{O}_\delta}$ is an injective smooth immersion between manifolds of the same dimension, so it is an embedding (Proposition 4.22(d)) and a diffeomorphism onto an open submanifold (Proposition 5.1). □

Regular Points and Singular Points

If V is a vector field on M, a point $p\in M$ is said to be a ***singular point of V*** if $V_p=0$, and a ***regular point*** otherwise. The next proposition shows that the integral curves starting at regular and singular points behave very differently from each other.

Proposition 9.21. *Let V be a smooth vector field on a smooth manifold M, and let $\theta\colon\mathcal{D}\to M$ be the flow generated by V. If $p\in M$ is a singular point of V, then*

$\mathcal{D}^{(p)} = \mathbb{R}$ *and* $\theta^{(p)}$ *is the constant curve* $\theta^{(p)}(t) \equiv p$. *If* p *is a regular point, then* $\theta^{(p)}: \mathcal{D}^{(p)} \to M$ *is a smooth immersion.*

Proof. If $V_p = 0$, then the constant curve $\gamma: \mathbb{R} \to M$ given by $\gamma(t) \equiv p$ is clearly an integral curve of V, so by uniqueness and maximality it must be equal to $\theta^{(p)}$.

To verify the second statement, we prove its contrapositive: if $\theta^{(p)}$ is not an immersion, then p is a singular point. The assumption that $\theta^{(p)}$ is not an immersion means that $\theta^{(p)\prime}(s) = 0$ for some $s \in \mathcal{D}^{(p)}$. Write $q = \theta^{(p)}(s)$. Then the argument in the preceding paragraph implies that $\mathcal{D}^{(q)} = \mathbb{R}$ and $\theta^{(q)}(t) = q$ for all $t \in \mathbb{R}$. It follows from Theorem 9.12(b) that $\mathcal{D}^{(p)} = \mathbb{R}$ as well, and for all $t \in \mathbb{R}$ the group law gives

$$\theta^{(p)}(t) = \theta_t(p) = \theta_{t-s}\big(\theta_s(p)\big) = \theta_{t-s}(q) = q.$$

Setting $t = 0$ yields $p = q$, and thus $\theta^{(p)}(t) \equiv p$ and $V_p = \theta^{(p)\prime}(0) = 0$. □

If $\theta: \mathcal{D} \to M$ is a flow, a point $p \in M$ is called an ***equilibrium point of*** $\boldsymbol{\theta}$ if $\theta(t, p) = p$ for all $t \in \mathcal{D}^{(p)}$. Proposition 9.21 shows that the equilibrium points of a smooth flow are precisely the singular points of its infinitesimal generator.

The next theorem completely describes, up to diffeomorphism, exactly what a vector field looks like in a neighborhood of a regular point.

Theorem 9.22 (Canonical Form Near a Regular Point). *Let* V *be a smooth vector field on a smooth manifold* M, *and let* $p \in M$ *be a regular point of* V. *There exist smooth coordinates* (s^i) *on some neighborhood of* p *in which* V *has the coordinate representation* $\partial/\partial s^1$. *If* $S \subseteq M$ *is any embedded hypersurface with* $p \in S$ *and* $V_p \notin T_pS$, *then the coordinates can also be chosen so that* s^1 *is a local defining function for* S.

Proof. If no hypersurface S is given, choose any smooth coordinates $\big(U, (x^i)\big)$ centered at p, and let $S \subseteq U$ be the hypersurface defined by $x^j = 0$, where j is chosen so that $V^j(p) \neq 0$. (Recall that p is a regular point of V.)

Regardless of whether S was given or was constructed as above, since $V_p \notin T_pS$, we can shrink S if necessary so that V is nowhere tangent to S. The flowout theorem then says that there is a flow domain $\mathcal{O}_\delta \subseteq \mathbb{R} \times S$ such that the flow of V restricts to a diffeomorphism Φ from $\mathcal{O}_\delta$ onto an open subset $W \subseteq M$ containing S. There is a product neighborhood $(-\varepsilon, \varepsilon) \times W_0$ of $(0, p)$ in $\mathcal{O}_\delta$. Choose a smooth local parametrization $X: \Omega \to S$ whose image is contained in W_0, where Ω is an open subset of $\mathbb{R}^{n-1}$ with coordinates denoted by $\big(s^2, \dots, s^n\big)$. It follows that the map $\Psi: (-\varepsilon, \varepsilon) \times \Omega \to M$ given by

$$\Psi\big(t, s^2, \dots, s^n\big) = \Phi\big(t, X\big(s^2, \dots, s^n\big)\big)$$

is a diffeomorphism onto a neighborhood of p in M. Because the diffeomorphism $\big(t, s^1, \dots, s^n\big) \mapsto \big(t, X\big(s^2, \dots, s^n\big)\big)$ pushes $\partial/\partial t$ forward to itself and $\Phi_*\big(\partial/\partial t\big) = V$, it follows that $\Psi_*\big(\partial/\partial t\big) = V$. Thus Ψ^{-1} is a smooth coordinate chart in which V has the coordinate representation $\partial/\partial t$. Renaming t to s^1 completes the proof. □

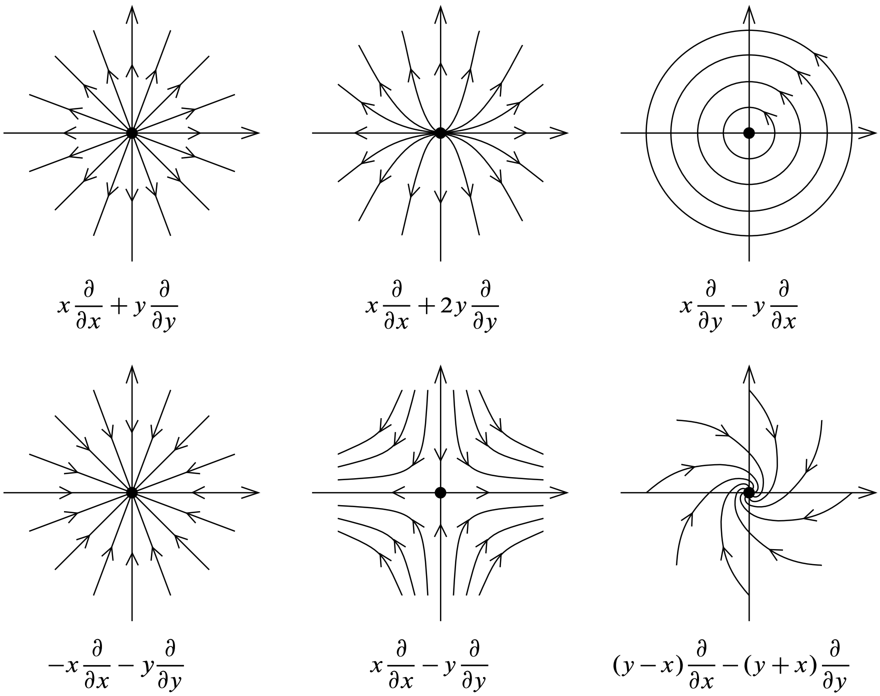


Fig. 9.8 Examples of flows near equilibrium points

The proof of the canonical form theorem actually provides a technique for finding coordinates that put a given vector field V in canonical form, at least when the corresponding system of ODEs can be explicitly solved: begin with a hypersurface S to which V is not tangent and a local parametrization $X\colon \Omega \to S$, and form the composite map $\Psi(t,s) = \theta_t\big(X(s)\big)$, where θ is the flow of V. The desired coordinate map is then the inverse of Ψ. The procedure is best illustrated by an example.

Example 9.23. Let $W = x\,\partial/\partial y - y\,\partial/\partial x$ on $\mathbb{R}^2$. We computed the flow of W in Example 9.8(b). The point $(1,0) \in \mathbb{R}^2$ is a regular point of W, because $W_{(1,0)} = \partial/\partial y|_{(1,0)} \neq 0$. Because W has nonzero y-coordinate there, we can take S to be the x-axis, parametrized by $X(s) = (s,0)$. We define $\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Psi(t,s) = \theta_t(s,0) = (s\cos t, s\sin t),$$

and then solve locally for (t,s) in terms of (x,y) to obtain the following coordinate map in a neighborhood of $(1,0)$:

$$(t,s) = \Psi^{-1}(x,y) = \left(\tan^{-1}(y/x), \sqrt{x^2+y^2}\right). \tag{9.13}$$

It is easy to check that $W = \partial/\partial t$ in these coordinates. (They are, as you might have noticed, just polar coordinates with different names.) //

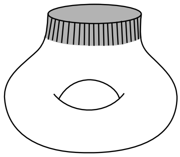


Fig. 9.9 A collar neighborhood of the boundary

The canonical form theorem shows that a flow in a neighborhood of a regular point behaves, up to diffeomorphism, just like translation along parallel coordinate lines in $\mathbb{R}^n$. Thus all of the interesting local behavior of the flow is concentrated near its equilibrium points. The flow around equilibrium points can exhibit a wide variety of behaviors, such as closed orbits surrounding the equilibrium point, orbits converging to the equilibrium point as $t \to +\infty$ or $-\infty$, and many more complicated phenomena. Some typical 2-dimensional examples are illustrated in Fig. 9.8. A systematic study of the local behavior of flows near equilibrium points in the plane can be found in many ODE texts, such as [BD09]. The study of global and long-time behaviors of flows on manifolds, called *smooth dynamical systems theory*, is a deep subject with many applications both inside and outside of mathematics.

Flows and Flowouts on Manifolds with Boundary

In general, a smooth vector field on a manifold with boundary need not generate a flow, because, for example, the integral curves starting at some boundary points might be defined only on half-open intervals. But there is a variant of the flowout theorem for manifolds with boundary, which has many important applications.

Suppose M is a smooth manifold with nonempty boundary. The next theorem describes a sort of "one-sided flowout" from ∂M, determined by a vector field that is inward pointing everywhere on ∂M.

Theorem 9.24 (Boundary Flowout Theorem). *Let M be a smooth manifold with nonempty boundary, and let N be a smooth vector field on M that is inward-pointing at each point of ∂M. There exist a smooth function $\delta\colon \partial M \to \mathbb{R}^+$ and a smooth embedding $\Phi\colon \mathcal{P}_\delta \to M$, where $\mathcal{P}_\delta = \{(t,p) : p \in \partial M,\ 0 \le t < \delta(p)\} \subseteq \mathbb{R} \times \partial M$, such that $\Phi(\mathcal{P}_\delta)$ is a neighborhood of ∂M, and for each $p \in \partial M$ the map $t \mapsto \Phi(t,p)$ is an integral curve of N starting at p.*

Proof. Problem 9-11. □

Let M be a smooth manifold with boundary. A neighborhood of ∂M is called a ***collar neighborhood*** if it is the image of a smooth embedding $[0,1) \times \partial M \to M$ that restricts to the obvious identification $\{0\} \times \partial M \to \partial M$. (See Fig. 9.9.)

Theorem 9.25 (Collar Neighborhood Theorem). *If M is a smooth manifold with nonempty boundary, then ∂M has a collar neighborhood.*

Proof. By the result of Problem 8-4, there exists a smooth vector field $N \in \mathfrak{X}(M)$ whose restriction to ∂M is everywhere inward-pointing. Let $\delta\colon M \to \mathbb{R}^+$ and $\Phi\colon \mathcal{P}_\delta \to M$ be as in Theorem 9.24, and define a map $\psi\colon [0,1) \times \partial M \to \mathcal{P}_\delta$ by $\psi(t,p) = \big(t\delta(p), p\big)$. Then ψ is a diffeomorphism that restricts to the identity on $\{0\} \times \partial M$, and therefore the map $\Phi \circ \psi\colon [0,1) \times \partial M \to M$ is a smooth embedding with open image that restricts to the usual identification $\{0\} \times \partial M \to \partial M$. The image of $\Phi \circ \psi$ is a collar neighborhood of ∂M. □

Our first application of the collar neighborhood theorem shows (among other things) that every smooth manifold with boundary is homotopy equivalent to its interior.

Theorem 9.26. *Let M be a smooth manifold with nonempty boundary, and let $\iota\colon \operatorname{Int} M \hookrightarrow M$ denote inclusion. There exists a proper smooth embedding $R\colon M \to \operatorname{Int} M$ such that both $\iota \circ R\colon M \to M$ and $R \circ \iota\colon \operatorname{Int} M \to \operatorname{Int} M$ are smoothly homotopic to identity maps. Therefore, ι is a homotopy equivalence.*

Proof. Theorem 9.25 shows that ∂M has a collar neighborhood C in M, which is the image of a smooth embedding $E\colon [0,1) \times \partial M \to M$ satisfying $E(0,x) = x$ for all $x \in \partial M$. To simplify notation, we will use this embedding to identify C with $[0,1) \times \partial M$, and denote a point in C as an ordered pair (s,x), with $s \in [0,1)$ and $x \in \partial M$; thus $(s,x) \in \partial M$ if and only if $s = 0$. For any $a \in (0,1)$, let $C(a) = \{(s,x) \in C : 0 \le t < a\}$ and $M(a) = M \smallsetminus C(a)$, which is a regular domain in $\operatorname{Int} M$.

Let $\psi\colon [0,1) \to \left[\frac{1}{3}, 1\right)$ be an increasing diffeomorphism that satisfies $\psi(s) = s$ for $\frac{2}{3} \le s < 1$, and define $R\colon M \to \operatorname{Int} M$ by

$$R(p) = \begin{cases} p, & p \in \operatorname{Int} M\left(\frac{2}{3}\right), \\ (\psi(s), x), & p = (s,x) \in C. \end{cases}$$

These definitions both give the identity map on the set $C \smallsetminus C\left(\frac{2}{3}\right)$ where they overlap, so R is smooth by the gluing lemma. It is a diffeomorphism onto the closed subset $M\left(\frac{1}{3}\right)$, so it is a proper smooth embedding of M into $\operatorname{Int} M$.

Define $H\colon M \times I \to M$ by

$$H(p,t) = \begin{cases} p, & p \in \operatorname{Int} M\left(\frac{2}{3}\right), \\ (ts + (1-t)\psi(s), x), & p = (s,x) \in C. \end{cases}$$

As before, H is smooth, and a straightforward verification shows that it is a homotopy from $\iota \circ R$ to Id_M. If $p \in \operatorname{Int} M$, then $H(p,t) \in \operatorname{Int} M$ for all $t \in I$, so the restriction of H to $(\operatorname{Int} M) \times I$ is a smooth homotopy from $R \circ \iota$ to $\mathrm{Id}_{\operatorname{Int} M}$. □

Theorem 9.26 is the main ingredient in the following generalization of the Whitney approximation theorem.

Theorem 9.27 (Whitney Approximation for Manifolds with Boundary). *If M and N are smooth manifolds with boundary, then every continuous map from M to N is homotopic to a smooth map.*

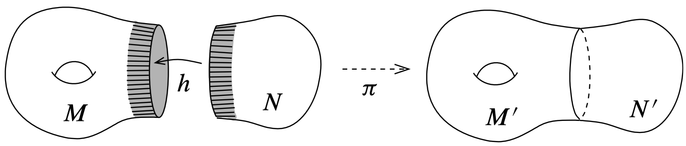


Fig. 9.10 Attaching manifolds along their boundaries

Proof. Theorem 6.26 takes care of the case in which $\partial N = \varnothing$, so we may assume that $\partial N \neq \varnothing$. Let $F: M \to N$ be a continuous map, let $\iota: \operatorname{Int} N \hookrightarrow N$ be inclusion, and let $R: N \to \operatorname{Int} N$ be the map constructed in Theorem 9.26, so that $\iota \circ R: N \to N$ is smoothly homotopic to Id_N. Theorem 6.26 shows that $R \circ F: M \to \operatorname{Int} N$ is homotopic to a smooth map G. It follows that $\iota \circ G \simeq \iota \circ R \circ F \simeq F$, so $\iota \circ G: M \to N$ is a smooth map homotopic to F. □

The next theorem generalizes the main result of Theorem 6.29 to the case of maps into a manifold with boundary.

Theorem 9.28. *Suppose M and N are smooth manifolds with or without boundary. If $F, G: M \to N$ are homotopic smooth maps, then they are smoothly homotopic.*

Proof. Theorem 6.29 takes care of the case $\partial N = \varnothing$, so we may assume that N has nonempty boundary. Let $\iota: \operatorname{Int} N \hookrightarrow N$ and $R: N \to \operatorname{Int} N$ be as in Theorem 9.26. Then $R \circ F$ and $R \circ G$ are homotopic smooth maps from M to $\operatorname{Int} N$, so Theorem 6.29 shows that they are smoothly homotopic to each other. Thus we have smooth homotopies $F \simeq \iota \circ R \circ F \simeq \iota \circ R \circ G \simeq G$. By transitivity of smooth homotopy (Lemma 6.28), it follows that F is smoothly homotopic to G. □

The following theorem is probably the most important application of the collar neighborhood theorem.

Theorem 9.29 (Attaching Smooth Manifolds Along Their Boundaries). *Let M and N be smooth n-manifolds with nonempty boundaries, and suppose $h: \partial N \to \partial M$ is a diffeomorphism (Fig. 9.10). Let $M \cup_h N$ be the adjunction space formed by identifying each $x \in \partial N$ with $h(x) \in \partial M$. Then $M \cup_h N$ is a topological manifold (without boundary), and has a smooth structure such that there are regular domains $M', N' \subseteq M \cup_h N$ diffeomorphic to M and N, respectively, and satisfying*

$$M' \cup N' = M \cup_h N, \qquad M' \cap N' = \partial M' = \partial N'. \tag{9.14}$$

If M and N are both compact, then $M \cup_h N$ is compact, and if they are both connected, then $M \cup_h N$ is connected.

Proof. For simplicity, let $X = M \cup_h N$ denote the quotient space and $\pi: M \amalg N \to X$ the quotient map. Let $V \subseteq M$ and $W \subseteq N$ be collar neighborhoods of ∂M and ∂N, respectively, and denote the corresponding diffeomorphisms by $\alpha: [0,1) \times \partial M \to V$ and $\beta: [0,1) \times \partial N \to W$. Define a continuous map

$\Phi\colon V \amalg W \to (-1,1) \times \partial M$ by

$$\Phi(x) = \begin{cases} (-t, p), & x = \alpha(t,p) \in V, \\ (t, h(q)), & x = \beta(t,q) \in W. \end{cases}$$

Then the restriction of Φ to V or W is a topological embedding with closed image, from which it follows easily that Φ is a closed map. Because Φ is constant on the fibers of π, it descends to a continuous map $\widetilde{\Phi}\colon \pi(V \amalg W) \to (-1,1) \times \partial M$. This map is bijective, and it is a homeomorphism because it too is a closed map: if $K \subseteq \pi(V \amalg W)$ is closed, then $\pi^{-1}(K)$ is closed in $V \amalg W$, and therefore $\widetilde{\Phi}(K) = \Phi\big(\pi^{-1}(K)\big)$ is closed. Thus, $\pi(V \amalg W)$ is a topological n-manifold. On the other hand, the restriction of π to the saturated open subset $\operatorname{Int} M \amalg \operatorname{Int} N$ is an injective quotient map and thus a homeomorphism onto its image; this shows that X is locally Euclidean of dimension n. Since X is the union of the second-countable open subsets $\pi(\operatorname{Int} M \amalg \operatorname{Int} N)$ and $\pi(V \amalg W)$, it is second-countable. Any two fibers in $M \amalg N$ can be separated by saturated open subsets, so X is Hausdorff. Thus it is a topological n-manifold.

We define a collection of charts on X as follows:

$\big(\pi(U),\ \varphi \circ \pi^{-1}|_{\pi(U)}\big)$, for each smooth chart (U,φ) for $\operatorname{Int} M$ or $\operatorname{Int} N$;

$\big(\widetilde{\Phi}^{-1}(U),\ \varphi \circ \widetilde{\Phi}|_{\widetilde{\Phi}^{-1}(U)}\big)$, for each smooth chart (U,φ) for $(-1,1) \times \partial M$.

These maps are compositions of homeomorphisms, so they define coordinate charts on X, and it is straightforward to check that they are all smoothly compatible and thus define a smooth structure on X. The restriction of π to M is continuous, closed, and injective, and thus it is a proper embedding. In terms of any of the smooth charts constructed above and corresponding charts on M, π has a coordinate representation that is either an identity map or an inclusion map, so it is a smooth embedding, and its image M' is therefore a regular domain in X. Similar considerations apply to N; and the relations (9.14) follow immediately from the definitions.

If M and N are compact, then X is the union of the compact sets M' and N', so it is compact; and if they are connected, then X is the union of the connected sets M' and N' with points of $\partial M' = \partial N'$ in common, so it is connected. □

► **Exercise 9.30.** Suppose M and N are smooth n-manifolds with boundary, $A \subseteq \partial M$ and $B \subseteq \partial N$ are nonempty subsets that are unions of components of the respective boundaries, and $h\colon B \to A$ is a diffeomorphism. Verify that the proof of Theorem 9.29 goes through with only trivial changes to show that $M \cup_h N$ is a topological manifold with boundary, and can be given a smooth structure such that M and N are diffeomorphic to regular domains in $M \cup_h N$.

Example 9.31 (Connected Sums). Let M_1, M_2 be connected smooth manifolds of dimension n. For $i = 1,2$, let U_i be a regular coordinate ball centered at some point $p_i \in M_i$, and let $M_i' = M_i \smallsetminus U_i$ (Fig. 9.11). Problem 5-8 shows that each M_i' is a smooth manifold with boundary whose boundary is diffeomorphic to $\mathbb{S}^{n-1}$. A ***smooth connected sum of M_1 and M_2***, denoted by $M_1 \# M_2$, is a smooth manifold formed by choosing a diffeomorphism from ∂M_1 to ∂M_2 and attaching M_1' and M_2' along their boundaries. //

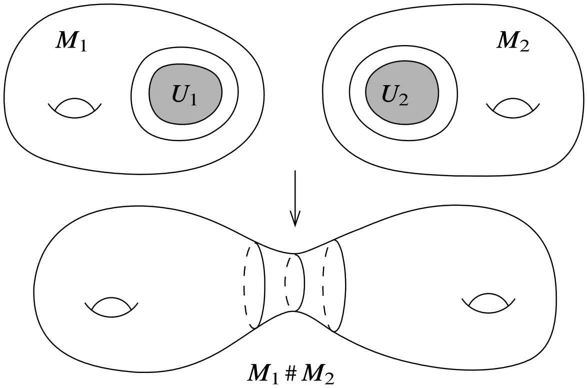


Fig. 9.11 A smooth connected sum

If M is any smooth manifold with boundary, Theorem 9.26 shows that M can be properly embedded into a smooth manifold without boundary (namely, a copy of $\operatorname{Int} M$). The next example shows a different way that M can be so embedded; this construction has the advantage of embedding M into a compact manifold when M itself is compact.

Example 9.32 (The Double of a Smooth Manifold with Boundary). Let M be a smooth manifold with boundary. The ***double of M*** is the manifold $D(M) = M \cup_{\mathrm{Id}} M$, where $\mathrm{Id}\colon \partial M \to \partial M$ is the identity map of ∂M; it is obtained from $M \amalg M$ by identifying each boundary point in one copy of M with the same boundary point in the other. It is a smooth manifold without boundary, and contains two regular domains diffeomorphic to M. It is easy to check that $D(M)$ is compact if and only if M is compact, and connected if and only if M is connected. (It is useful to extend the definition to manifolds without boundary by defining $D(M) = M \amalg M$ when $\partial M = \varnothing$.) //

Although vector fields on manifolds with boundary do not always generate flows, there is one circumstance in which they do: when the vector field is everywhere tangent to the boundary. To prove this, we begin with the following special case.

Lemma 9.33. *Suppose M is a smooth manifold and $D \subseteq M$ is a regular domain. If V is a smooth vector field on M that is tangent to ∂D, then every integral curve of V that starts in D remains in D as long as it is defined.*

Proof. Suppose $\gamma\colon J \to M$ is an integral curve of V with $\gamma(0) \in D$. Define $\mathcal{T} \subseteq J$ by $\mathcal{T} = \{t \in J : \gamma(t) \in D\}$. We will show that $\mathcal{T}$ is both open and closed in J; since J is an interval, this implies $\mathcal{T} = J$ and proves the lemma.

Since D is closed in M (by definition of a regular domain), $\mathcal{T}$ is closed in J by continuity. To prove it is open, suppose $t_0 \in \mathcal{T}$. If $\gamma(t_0) \in \operatorname{Int} D$, then a neighborhood of t_0 is contained in $\mathcal{T}$ by continuity, so we can assume $\gamma(t_0) \in \partial D$. Because V is tangent to ∂D, Proposition 8.23 shows that there is a smooth vector field $W = V|_{\partial D}$ that is ι-related to V, where $\iota\colon \partial D \hookrightarrow M$ is inclusion. Let $\widetilde{\gamma}\colon (t_0 - \varepsilon, t_0 + \varepsilon) \to$

∂D be an integral curve of W with $\widetilde{\gamma}(t_0) = \gamma(t_0)$. By naturality of integral curves (Proposition 9.6), $\iota \circ \widetilde{\gamma}$ is an integral curve of V with the same initial condition, so by uniqueness it must be equal to γ where both are defined. This shows that $\gamma(t) \in \partial D \subseteq D$ for t in some neighborhood of t_0, so $\mathcal{T}$ is open in J as claimed. □

Theorem 9.34 (Flows on Manifolds with Boundary). *The conclusions of Theorem 9.12 remain true if M is a smooth manifold with boundary and V is a smooth vector field on M that is tangent to ∂M.*

Proof. Example 9.32 shows that we can consider M as a regular domain in its double $D(M)$. By the extension lemma for vector fields, we can extend V to a smooth vector field $\widetilde{V}$ on $D(M)$. Let $\widetilde{\theta}\colon \widetilde{\mathcal{D}} \to D(M)$ be the flow of $\widetilde{V}$, and let $\mathcal{D} = \widetilde{\mathcal{D}} \cap (\mathbb{R} \times M)$ and $\theta = \widetilde{\theta}|_{\mathcal{D}}$. Then Lemma 9.33 guarantees that θ maps $\mathcal{D}$ into M, and the rest of the conclusions follow from Theorem 9.12 applied to $\widetilde{V}$. □

For manifolds with boundary, the canonical form theorem has the following variant.

Theorem 9.35 (Canonical Form Near a Regular Point on the Boundary). *Let M be a smooth manifold with boundary and let V be a smooth vector field on M that is tangent to ∂M. If $p \in \partial M$ is a regular point of V, there exist smooth boundary coordinates $\left(s^i\right)$ on some neighborhood of p in which V has the coordinate representation $\partial/\partial s^1$.*

Proof. Problem 9-15. □

Lie Derivatives

We know how to make sense of directional derivatives of real-valued functions on a manifold. Indeed, a tangent vector $v \in T_pM$ is by definition an operator that acts on a smooth function f to give a number vf that we interpret as a directional derivative of f at p. In Chapter 3 we showed that this number can be interpreted as the ordinary derivative of f along any curve whose initial velocity is v.

What about the directional derivative of a vector field? In Euclidean space, it makes perfectly good sense to define the directional derivative of a smooth vector field W in the direction of a vector $v \in T_p\mathbb{R}^n$. It is the vector

$$D_vW(p) = \left.\frac{d}{dt}\right|_{t=0} W_{p+tv} = \lim_{t\to 0} \frac{W_{p+tv} - W_p}{t}. \tag{9.15}$$

An easy calculation shows that $D_vW(p)$ can be evaluated by applying D_v to each component of W separately:

$$D_vW(p) = D_vW^i(p) \left.\frac{\partial}{\partial x^i}\right|_p.$$

Unfortunately, this definition is heavily dependent upon the fact that $\mathbb{R}^n$ is a vector space, so that the tangent vectors W_{p+tv} and W_p can both be viewed as elements

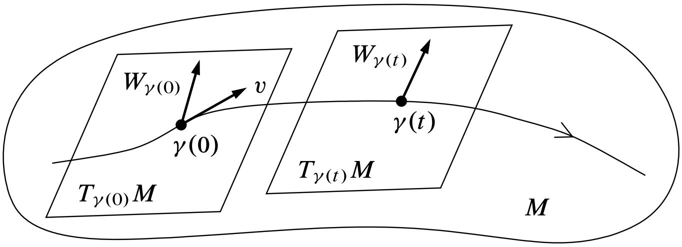


Fig. 9.12 The problem with directional derivatives of vector fields

of $\mathbb{R}^n$. If we search for a way to make invariant sense of (9.15) on a manifold, we see very quickly what the problem is. To begin with, we can replace $p + tv$ by a curve $\gamma(t)$ that starts at p and whose initial velocity is v. But even with this substitution, the difference quotient still makes no sense because $W_{\gamma(t)}$ and $W_{\gamma(0)}$ are elements of the two different vector spaces $T_{\gamma(t)}M$ and $T_{\gamma(0)}M$, respectively (see Fig. 9.12). We got away with it in Euclidean space because there is a canonical identification of each tangent space with $\mathbb{R}^n$ itself; but on a manifold there is no such identification. Thus there is no coordinate-independent way to make sense of the directional derivative of W in the direction of a vector v.

This problem can be circumvented if we replace the vector $v \in T_pM$ with a *vector field* $V \in \mathfrak{X}(M)$, so we can use the flow of V to push values of W back to p and then differentiate. Thus we make the following definition. Suppose M is a smooth manifold, V is a smooth vector field on M, and θ is the flow of V. For any smooth vector field W on M, define a rough vector field on M, denoted by $\mathcal{L}_V W$ and called the ***Lie derivative of W with respect to V***, by

$$(\mathcal{L}_V W)_p = \left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big)$$
$$= \lim_{t\to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) - W_p}{t}, \tag{9.16}$$

provided the derivative exists. For small $t \neq 0$, at least the difference quotient makes sense: θ_t is defined in a neighborhood of p, and θ_{-t} is the inverse of θ_t, so both $d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big)$ and W_p are elements of T_pM (Fig. 9.13).

If M has nonempty boundary, this definition of $\mathcal{L}_V W$ makes sense as long as V is tangent to ∂M so that its flow exists by Theorem 9.34. (We will define Lie derivatives on more general manifolds with boundary below; see the remark after the proof of Theorem 9.38.)

Lemma 9.36. *Suppose M is a smooth manifold with or without boundary, and $V, W \in \mathfrak{X}(M)$. If $\partial M \neq \varnothing$, assume in addition that V is tangent to ∂M. Then $(\mathcal{L}_V W)_p$ exists for every $p \in M$, and $\mathcal{L}_V W$ is a smooth vector field.*

Proof. Let θ be the flow of V. For arbitrary $p \in M$, let $\big(U, \big(x^i\big)\big)$ be a smooth chart containing p. Choose an open interval J_0 containing 0 and an open subset $U_0 \subseteq U$

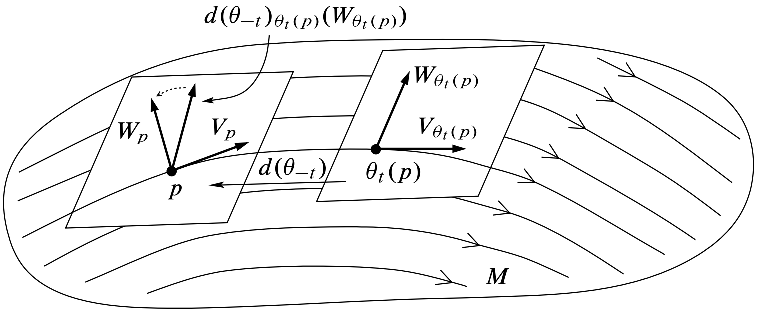


Fig. 9.13 The Lie derivative of a vector field

containing p such that θ maps $J_0 \times U_0$ into U. For $(t,x) \in J_0 \times U_0$, write the component functions of θ as $\big(\theta^1(t,x),\dots,\theta^n(t,x)\big)$. Then for any $(t,x) \in J_0 \times U_0$, the matrix of $d(\theta_{-t})_{\theta_t(x)} \colon T_{\theta_t(x)}M \to T_xM$ is

$$\left(\frac{\partial\theta^i}{\partial x^j}\big(-t,\theta(t,x)\big)\right).$$

Therefore,

$$d(\theta_{-t})_{\theta_t(x)}\big(W_{\theta_t(x)}\big) = \frac{\partial\theta^i}{\partial x^j}\big(-t,\theta(t,x)\big)W^j\big(\theta(t,x)\big)\left.\frac{\partial}{\partial x^i}\right|_x.$$

Because θ^i and W^j are smooth functions, the coefficient of $\partial/\partial x^i|_x$ depends smoothly on (t,x). It follows that $(\mathcal{L}_V W)_x$, which is obtained by taking the derivative of this expression with respect to t and setting $t = 0$, exists for each $x \in U_0$ and depends smoothly on x. □

► **Exercise 9.37.** Suppose $v \in \mathbb{R}^n$ and W is a smooth vector field on an open subset of $\mathbb{R}^n$. Show that the directional derivative $D_vW(p)$ defined by (9.15) is equal to $(\mathcal{L}_V W)_p$, where V is the vector field $V = v^i\partial/\partial x^i$ with constant coefficients in standard coordinates.

The definition of $\mathcal{L}_V W$ is not very useful for computations, because typically the flow is difficult or impossible to write down explicitly. Fortunately, there is a simple formula for computing the Lie derivative without explicitly finding the flow.

Theorem 9.38. *If M is a smooth manifold and $V, W \in \mathfrak{X}(M)$, then $\mathcal{L}_V W = [V,W]$.*

Proof. Suppose $V, W \in \mathfrak{X}(M)$, and let $\mathcal{R}(V) \subseteq M$ be the set of regular points of V (the set of points $p \in M$ such that $V_p \neq 0$). Note that $\mathcal{R}(V)$ is open in M by continuity, and its closure is the support of V. We will show that $(\mathcal{L}_V W)_p = [V,W]_p$ for all $p \in M$, by considering three cases.

CASE 1: $p \in \mathcal{R}(V)$. In this case, we can choose smooth coordinates $\big(u^i\big)$ on a neighborhood of p in which V has the coordinate representation $V = \partial/\partial u^1$ (Theorem 9.22). In these coordinates, the flow of V is $\theta_t(u) = \big(u^1 + t, u^2, \dots, u^n\big)$. For each fixed t, the matrix of $d(\theta_{-t})_{\theta_t(x)}$ in these coordinates (the Jacobian matrix of θ_{-t}) is the identity at every point. Consequently, for any $u \in U$,

$$\begin{aligned} d(\theta_{-t})_{\theta_t(u)}(W_{\theta_t(u)}) &= d(\theta_{-t})_{\theta_t(x)}\left(W^j\left(u^1+t,u^2,\dots,u^n\right) \left.\frac{\partial}{\partial u^j}\right|_{\theta_t(u)} \right) \\ &= W^j\left(u^1+t,u^2,\dots,u^n\right) \left.\frac{\partial}{\partial u^j}\right|_{u}. \end{aligned}$$

Using the definition of the Lie derivative, we obtain

$$(\mathcal{L}_V W)_u = \left.\frac{d}{dt}\right|_{t=0} W^j\left(u^1+t,u^2,\dots,u^n\right) \left.\frac{\partial}{\partial u^j}\right|_{u} = \frac{\partial W^j}{\partial u^1}\left(u^1,\dots,u^n\right) \left.\frac{\partial}{\partial u^j}\right|_{u}.$$

On the other hand, by virtue of formula (8.8) for the Lie bracket in coordinates, $[V,W]_u$ is easily seen to be equal to the same expression.

CASE 2: $p \in \operatorname{supp} V$. Because $\operatorname{supp} V$ is the closure of $\mathcal{R}(V)$, it follows by continuity from Case that $(\mathcal{L}_V W)_p = [V,W]_p$ for $p \in \operatorname{supp} V$.

CASE 3: $p \in M \smallsetminus \operatorname{supp} V$. In this case, $V \equiv 0$ on a neighborhood of p. On the one hand, this implies that θ_t is equal to the identity map in a neighborhood of p for all t, so $d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = W_p$, which implies $(\mathcal{L}_V W)_p = 0$. On the other hand, $[V,W]_p = 0$ by formula (8.8). □

This theorem allows us to extend the definition of the Lie derivative to arbitrary smooth vector fields on a smooth manifold M with boundary. Given $V,W \in \mathfrak{X}(M)$, we define $(\mathcal{L}_V W)_p$ for $p \in \partial M$ by embedding M in a smooth manifold $\widetilde{M}$ without boundary (such as the double of M), extending V and W to smooth vector fields on $\widetilde{M}$, and computing the Lie derivative there. By virtue of the preceding theorem, $(\mathcal{L}_V W)_p = [X,Y]_p$ is independent of the choice of extension.

Theorem 9.38 also gives us a geometric interpretation of the Lie bracket of two vector fields: it is the directional derivative of the second vector field along the flow of the first. A number of nonobvious properties of the Lie derivative follow immediately from things we already know about Lie brackets.

Corollary 9.39. *Suppose M is a smooth manifold with or without boundary, and $V,W,X \in \mathfrak{X}(M)$.*

(a) $\mathcal{L}_V W = -\mathcal{L}_W V$.
(b) $\mathcal{L}_V [W,X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.
(c) $\mathcal{L}_{[V,W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$.
(d) *If $g \in C^\infty(M)$, then $\mathcal{L}_V(gW) = (Vg)W + g\mathcal{L}_V W$.*
(e) *If $F\colon M \to N$ is a diffeomorphism, then $F_*(\mathcal{L}_V X) = \mathcal{L}_{F_*V} F_* X$.*

► **Exercise 9.40.** Prove this corollary.

Part (d) of this corollary gives a meaning to the mysterious formula (8.11) for Lie brackets of vector fields multiplied by functions: because the Lie bracket $[fV, gW]$ can be thought of as the Lie derivative $\mathcal{L}_{fV}(gW)$, it satisfies a product rule in g

and W; and because it can also be thought of as $-\mathcal{L}_{gW}(fV)$, it satisfies a product rule in f and V as well. Expanding out these two product rules yields (8.11).

If V and W are vector fields on M and θ is the flow of V, the Lie derivative $(\mathcal{L}_V W)_p$, by definition, expresses the t-derivative of the time-dependent vector $d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) \in T_pM$ at $t = 0$. The next proposition shows how it can also be used to compute the derivative of this expression at other times. We will use this result in the proof of Theorem 9.42 below.

Proposition 9.41. *Suppose M is a smooth manifold with or without boundary and $V, W \in \mathfrak{X}(M)$. If $\partial M \neq \varnothing$, assume also that V is tangent to ∂M. Let θ be the flow of V. For any (t_0, p) in the domain of θ,*

$$\left.\frac{d}{dt}\right|_{t=t_0} d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) = d\left(\theta_{-t_0}\right)\left((\mathcal{L}_V W)_{\theta_{t_0}(p)}\right). \tag{9.17}$$

Proof. Let $p \in M$ be arbitrary, let $\mathcal{D}^{(p)} \subseteq \mathbb{R}$ denote the domain of the integral curve $\theta^{(p)}$, and consider the map $X\colon \mathcal{D}^{(p)} \to T_pM$ given by $X(t) = d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right)$. The argument in the proof of Lemma 9.36 shows that X is a smooth curve in the vector space T_pM. Making the change of variables $t = t_0 + s$, we obtain

$$\begin{aligned} X'(t_0) &= \left.\frac{d}{ds}\right|_{s=0} X(t_0+s) = \left.\frac{d}{ds}\right|_{s=0} d\left(\theta_{-t_0-s}\right)\left(W_{\theta_{s+t_0}(p)}\right) \\ &= \left.\frac{d}{ds}\right|_{s=0} d\left(\theta_{-t_0}\right) \circ d(\theta_{-s})\left(W_{\theta_s(\theta_{t_0}(p))}\right) \\ &= d\left(\theta_{-t_0}\right)\left(\left.\frac{d}{ds}\right|_{s=0} d(\theta_{-s})\left(W_{\theta_s(\theta_{t_0}(p))}\right)\right). \end{aligned}$$

(The last equality follows because $d\left(\theta_{-t_0}\right)\colon T_{\theta_{t_0}(p)}M \to T_pM$ is a linear map that is independent of s. See Fig. 9.14.) By definition of the Lie derivative, this last expression is equal to the right-hand side of (9.17). □

Commuting Vector Fields

Let M be a smooth manifold and $V, W \in \mathfrak{X}(M)$. We say that ***V and W commute*** if $VWf = WVf$ for every smooth function f, or equivalently if $[V, W] \equiv 0$. If θ is a smooth flow, a vector field W is said to be ***invariant under θ*** if W is θ_t-related to itself for each t; more precisely, this means that $W|_{M_t}$ is θ_t-related to $W|_{M_{-t}}$ for each t, or equivalently that $d(\theta_t)_p(W_p) = W_{\theta_t(p)}$ for all (t, p) in the domain of θ. The next proposition shows that these two concepts are intimately related.

Theorem 9.42. *For smooth vector fields V and W on a smooth manifold M, the following are equivalent:*

(a) *V and W commute.*

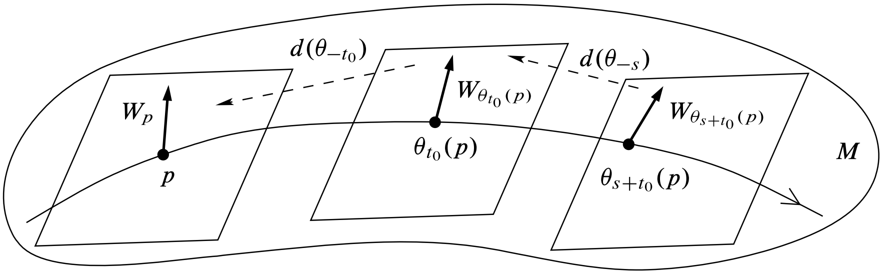


Fig. 9.14 Proof of Proposition 9.41

(b) *W is invariant under the flow of V.*
(c) *V is invariant under the flow of W.*

Proof. Suppose $V, W \in \mathfrak{X}(M)$, and let θ denote the flow of V. If (b) holds, then $W_{\theta_t(p)} = d(\theta_t)_p(W_p)$ whenever (t, p) is in the domain of θ. Applying $d(\theta_{-t})_{\theta_t(p)}$ to both sides, we conclude that $d(\theta_{-t})_{\theta_t(p)}\big(W_{\theta_t(p)}\big) = W_p$, which obviously implies $[V, W] = \mathcal{L}_V W = 0$ directly from the definition of the Lie derivative. The same argument shows that (c) implies (a).

To prove that (a) implies (b), assume that $[V, W] = \mathcal{L}_V W = 0$. Let $p \in M$ be arbitrary, and let $X(t) = d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right)$ for $t \in \mathcal{D}^{(p)}$. Proposition 9.41 shows that $X'(t) \equiv 0$. Since $X(0) = W_p$, this implies that $X(t) = W_p$ for all $t \in \mathcal{D}^{(p)}$, and applying $d(\theta_t)_p$ to both sides yields the identity that says W is invariant under θ. The same proof also shows that (a) implies (c). □

Corollary 9.43. *Every smooth vector field is invariant under its own flow.*

Proof. Use the preceding proposition together with the fact that $[V, V] \equiv 0$. □

The deepest characterization of commuting vector fields is in terms of the relationship between their respective flows. The next theorem says that two vector fields commute if and only if their flows commute. But before we state the theorem formally, we need to examine exactly what this means. Suppose V and W are smooth vector fields on M, and let θ and ψ denote their respective flows. If V and W are complete, it is clear what we should mean by saying their flows commute: simply that $\theta_t \circ \psi_s = \psi_s \circ \theta_t$ for all $s, t \in \mathbb{R}$. However, if either V or W is not complete, the most we can hope for is that this equation holds for all s and t such that both sides are defined. Unfortunately, even when the vector fields commute, their flows might not commute in this naive sense, because there are examples of commuting vector fields V and W and particular choices of t, s, and p for which both $\theta_t \circ \psi_s(p)$ and $\psi_s \circ \theta_t(p)$ are defined, but they are not equal (see Problem 9-19 for one such example). Here is the problem: if $\theta_t \circ \psi_s(p)$ is defined for $t = t_0$ and $s = s_0$, then by the properties of flow domains, it must be defined for all t in some open interval containing 0 and t_0, but the analogous statement need not be true of s—there might be values of s between 0 and s_0 for which the integral curve of V starting at $\psi_s(p)$ does not extend all the way to $t = t_0$.

Thus we make the following definition. If θ and ψ are flows on M, we say that ***θ and ψ commute*** if the following condition holds for every $p \in M$: whenever J and K are open intervals containing 0 such that one of the expressions $\theta_t \circ \psi_s(p)$ or $\psi_s \circ \theta_t(p)$ is defined for all $(s,t) \in J \times K$, both are defined and they are equal. For global flows, this is the same as saying that $\theta_t \circ \psi_s = \psi_s \circ \theta_t$ for all s and t.

Theorem 9.44. *Smooth vector fields commute if and only if their flows commute.*

Proof. Let V and W be smooth vector fields on a smooth manifold M, and let θ and ψ denote their respective flows. Assume first that V and W commute. Suppose that $p \in M$, and J and K are open intervals containing 0 such that $\psi_s \circ \theta_t(p)$ is defined for all $(s,t) \in J \times K$. (The same proof with V and W reversed works under the assumption that the other expression is defined on such a rectangle.) By Theorem 9.42, the hypothesis implies that V is invariant under ψ. Fix any $s \in J$, and consider the curve $\gamma\colon K \to M$ defined by $\gamma(t) = \psi_s \circ \theta_t(p) = \psi_s\big(\theta^{(p)}(t)\big)$. This curve satisfies $\gamma(0) = \psi_s(p)$, and its velocity at $t \in K$ is

$$\gamma'(t) = \frac{d}{dt}\Big(\psi_s\big(\theta^{(p)}(t)\big)\Big) = d(\psi_s)\big(\theta^{(p)\prime}(t)\big) = d(\psi_s)\big(V_{\theta^{(p)}(t)}\big) = V_{\gamma(t)}.$$

Thus, γ is an integral curve of V starting at $\psi_s(p)$. By uniqueness, therefore,

$$\gamma(t) = \theta^{\psi_s(p)}(t) = \theta_t\big(\psi_s(p)\big).$$

This proves that θ and ψ commute.

Conversely, assume that the flows commute, and let $p \in M$. If $\varepsilon > 0$ is chosen small enough that $\psi_s \circ \theta_t(p)$ is defined whenever $|s| < \varepsilon$ and $|t| < \varepsilon$, then the hypothesis guarantees that $\psi_s \circ \theta_t(p) = \theta_t \circ \psi_s(p)$ for all such s and t. This can be rewritten in the form

$$\psi^{\theta_t(p)}(s) = \theta_t\big(\psi^{(p)}(s)\big).$$

Differentiating this relation with respect to s, we get

$$W_{\theta_t(p)} = \frac{d}{ds}\bigg|_{s=0} \psi^{\theta_t(p)}(s) = \frac{d}{ds}\bigg|_{s=0} \theta_t\big(\psi^{(p)}(s)\big) = d(\theta_t)_p(W_p).$$

Applying $d(\theta_{-t})_{\theta_t(p)}$ to both sides, we conclude

$$d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = W_p.$$

Differentiating with respect to t and applying the definition of the Lie derivative shows that $(\mathcal{L}_V W)_p = 0$. □

Commuting Frames

Suppose M is a smooth n-manifold. Recall that a *local frame for M* is an n-tuple (E_i) of vector fields defined on an open subset $U \subseteq M$ such that $\big(E_i|_p\big)$ forms a basis for T_pM at each $p \in U$. A smooth local frame (E_i) for M is called a ***commuting frame*** if $[E_i, E_j] = 0$ for all i and j. (Commuting frames are called ***holonomic frames*** by some authors.)

Example 9.45 (Commuting and Noncommuting Frames).

(a) The simplest examples of commuting frames are the coordinate frames. Given any smooth coordinate chart $\left(U,\left(x^i\right)\right)$ for a smooth manifold M, (8.10) shows that the coordinate frame $\left(\partial/\partial x^i\right)$ is a commuting frame.
(b) The frame (E_1, E_2) for $\mathbb{R}^2$ over $\mathbb{R}^2 \smallsetminus \{0\}$ defined by (8.3) is not a commuting frame, because a straightforward computation shows that

$$[E_1, E_2] = \frac{y}{r^2}\frac{\partial}{\partial x} - \frac{x}{r^2}\frac{\partial}{\partial y} \neq 0. \qquad //$$

Because every coordinate frame is a commuting frame, and because Lie brackets are invariantly defined, it follows that a necessary condition for a smooth frame to be expressible as a coordinate frame in some smooth chart is that it be a commuting frame. Thus, the computation above shows that (E_1, E_2) cannot be expressed as a coordinate frame for $\mathbb{R}^2$ with respect to any choice of smooth local coordinates.

The next theorem shows that commuting is also a sufficient condition for a smooth frame to be locally expressible as a coordinate frame.

Theorem 9.46 (Canonical Form for Commuting Vector Fields). *Let M be a smooth n-manifold, and let $(V_1,\dots,V_k)$ be a linearly independent k-tuple of smooth commuting vector fields on an open subset $W \subseteq M$. For each $p \in W$, there exists a smooth coordinate chart $\left(U,\left(s^i\right)\right)$ centered at p such that $V_i = \partial/\partial s^i$ for $i = 1,\dots,k$. If $S \subseteq W$ is an embedded codimension-k submanifold and p is a point of S such that T_pS is complementary to the span of $\left(V_1|_p,\dots,V_k|_p\right)$, then the coordinates can also be chosen such that $S \cap U$ is the slice defined by $s^1 = \dots = s^k = 0$.*

Proof. Let $p \in W$ be arbitrary. If no submanifold S is given, just let S be any smooth embedded codimension-k submanifold S whose tangent space at p is complementary to the span of $\left(V_1|_p,\dots,V_k|_p\right)$ (e.g., an appropriate coordinate slice). Let $\left(U,\left(x^i\right)\right)$ be a slice chart for S centered at p, with $U \subseteq W$, and with $S \cap U$ equal to the slice $\{x \in U : x^1 = \dots = x^k = 0\}$. Our assumptions ensure that the vectors $\left\{V_1|_p,\dots,V_k|_p, \partial/\partial x^{k+1}|_p,\dots,\partial/\partial x^n|_p\right\}$ span T_pM. Since the theorem is purely local, we may as well consider $V_1,\dots,V_k$ as vector fields on $U \subseteq \mathbb{R}^n$, and consider S to be the subset of U where the first k coordinates vanish. The basic idea of this proof is similar to that of the flowout theorem, except that we have to do a bit of extra work to make use of the hypothesis that the vector fields commute.

Let θ_i denote the flow of V_i for $i = 1,\dots,k$. There exist $\varepsilon > 0$ and a neighborhood Y of p in U such that the composition $(\theta_1)_{t_1} \circ (\theta_2)_{t_2} \circ \dots \circ (\theta_k)_{t_k}$ is defined on Y and maps Y into U whenever $|t_1|,\dots,|t_k|$ are all less than ε. (To see this, just choose $\varepsilon_k > 0$ and $U_k \subseteq U$ such that θ_k maps $(-\varepsilon_k,\varepsilon_k) \times U_k$ into U, and then inductively choose ε_i and U_i such that θ_i maps $(-\varepsilon_i,\varepsilon_i) \times U_i$ into U_{i+1}. Taking $\varepsilon = \min\{\varepsilon_i\}$ and $Y = U_1$ does the trick.)

Define $\Omega \subseteq \mathbb{R}^{n-k}$ by

$$\Omega = \left\{\left(s^{k+1},\dots,s^n\right) \in \mathbb{R}^{n-k} : \left(0,\dots,0,s^{k+1},\dots,s^n\right) \in Y\right\},$$

and define $\Phi\colon (-\varepsilon,\varepsilon)^k \times \Omega \to U$ by

$$\Phi\big(s^1,\dots,s^k,s^{k+1},\dots,s^n\big) = (\theta_1)_{s^1} \circ \cdots \circ (\theta_k)_{s^k}\big(0,\dots,0,s^{k+1},\dots,s^n\big).$$

By construction, $\Phi(\{0\} \times \Omega) = S \cap Y$.

We show next that $\partial/\partial s^i$ is Φ-related to V_i for $i = 1,\dots,k$. Because the flows θ_i commute, for any $i \in \{1,\dots,k\}$ and any $s_0 \in (-\varepsilon,\varepsilon)^k \times \Omega$ we have

$$\begin{aligned} d\Phi_{s_0}\left(\frac{\partial}{\partial s^i}\bigg|_{s_0}\right) f &= \frac{\partial}{\partial s^i}\bigg|_{s_0} f\left(\Phi\left(s^1,\dots,s^n\right)\right) \\ &= \frac{\partial}{\partial s^i}\bigg|_{s_0} f\Big((\theta_1)_{s^1} \circ \cdots \circ (\theta_k)_{s^k}\big(0,\dots,0,s^{k+1},\dots,s^n\big)\Big) \\ &= \frac{\partial}{\partial s^i}\bigg|_{s_0} f\Big((\theta_i)_{s^i} \circ (\theta_1)_{s^1} \circ \cdots \circ (\theta_{i-1})_{s^{i-1}} \circ (\theta_{i+1})_{s^{i+1}} \\ &\qquad \circ \cdots \circ (\theta_k)_{s^k}\big(0,\dots,0,s^{k+1},\dots,s^n\big)\Big). \end{aligned}$$

For any $q \in M$, $t \mapsto (\theta_i)_t(q)$ is an integral curve of V_i, so this last expression is equal to $V_i|_{\Phi(s_0)} f$, which proves the claim.

Next we show that $d\Phi_0$ is invertible. The computation above shows that

$$d\Phi_0\left(\frac{\partial}{\partial s^i}\bigg|_0\right) = V_i|_p, \quad i = 1,\dots,k.$$

On the other hand, since $\Phi\big(0,\dots,0,s^{k+1},\dots,s^n\big) = \big(0,\dots,0,s^{k+1},\dots,s^n\big)$, it follows immediately that

$$d\Phi_0\left(\frac{\partial}{\partial s^i}\bigg|_0\right) = \frac{\partial}{\partial x^i}\bigg|_p, \quad i = k+1,\dots,n.$$

It follows that $d\Phi_0$ takes the basis $\big(\partial/\partial s^1|_0,\dots,\partial/\partial s^n|_0\big)$ for $T_0\mathbb{R}^n$ to the basis $\big(V_1|_p,\dots,V_k|_p,\partial/\partial x^{k+1}|_p,\dots,\partial/\partial x^n|_p\big)$ for T_pM. By the inverse function theorem, Φ is a diffeomorphism in a neighborhood of 0, and $\varphi = \Phi^{-1}$ is a smooth coordinate map that takes $\partial/\partial s^i$ to V_i for $i = 1,\dots,k$, and takes S to the slice $s^1 = \cdots = s^k = 0$. □

Just as in the case of a single vector field, the proof of Theorem 9.46 suggests a technique for finding explicit coordinates that put a set of commuting vector fields into canonical form, as long as their flows can be found explicitly. The method can be summarized as follows: Begin with an $(n-k)$-dimensional submanifold S whose tangent space at p is complementary to the span of $\big(V_1|_p,\dots,V_k|_p\big)$. Then define Φ by starting at an arbitrary point in S and following the k flows successively for k arbitrary times. Because the flows commute, it does not matter in which order they are applied. An example will help to clarify the procedure.

Example 9.47. Consider the following two vector fields on $\mathbb{R}^2$:

$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

A computation shows that $[V, W] = 0$. Example 9.8 showed that the flow of V is

$$\theta_t(x, y) = (x\cos t - y\sin t,\ x\sin t + y\cos t),$$

and an easy verification shows that the flow of W is

$$\eta_t(x, y) = \left(e^t x, e^t y\right).$$

At $p = (1,0)$, V_p and W_p are linearly independent. Because $k = n = 2$ in this case, we can take the subset S to be the single point $\{(1,0)\}$, and define $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi(s,t) = \eta_t \circ \theta_s(1,0) = \left(e^t\cos s, e^t\sin s\right).$$

In this case, we can solve for $(s,t) = \Phi^{-1}(x, y)$ explicitly in a neighborhood of $(1,0)$ to obtain the coordinate map

$$(s,t) = \left(\tan^{-1}(y/x), \log\sqrt{x^2+y^2}\right). \qquad //$$

Time-Dependent Vector Fields

All of the systems of differential equations we have encountered so far have been *autonomous* ones, meaning that when they are written in the form (9.1), the functions V^i on the right-hand sides do not depend explicitly on the independent variable t (see Appendix D). However, nonautonomous ODEs do arise in manifold theory, so it is worth exploring how the results of this chapter can be extended to cover this case. We will use this theory only in Chapter 22.

Let M be a smooth manifold. A ***time-dependent vector field on M*** is a continuous map $V\colon J\times M\to TM$, where $J\subseteq\mathbb{R}$ is an interval, such that $V(t,p)\in T_pM$ for each $(t,p)\in J\times M$. This means that for each $t\in J$, the map $V_t\colon M\to TM$ defined by $V_t(p) = V(t,p)$ is a vector field on M. If V is a time-dependent vector field on M, an ***integral curve of V*** is a differentiable curve $\gamma\colon J_0\to M$, where J_0 is an interval contained in J, such that

$$\gamma'(t) = V\big(t,\gamma(t)\big) \quad \text{for all } t\in J_0.$$

Every ordinary vector field $X\in\mathfrak{X}(M)$ determines a time-dependent vector field defined on $\mathbb{R}\times M$, just by setting $V(t,p) = X_p$. (It is occasionally useful to consider time-dependent vector fields defined on more general open subsets of $\mathbb{R}\times M$; but for simplicity we restrict attention to a product set $J\times M$, and leave it to the interested reader to figure out how the results need to be modified for the more general case.)

A time-dependent vector field might not generate a flow, because two integral curves starting at the same point but at different times might follow different paths, whereas all integral curves of a flow through a given point have the same image. As a substitute for the fundamental theorem on flows, we have the following theorem.

Theorem 9.48 (Fundamental Theorem on Time-Dependent Flows). *Let M be a smooth manifold, let $J \subseteq \mathbb{R}$ be an open interval, and let $V: J \times M \to TM$ be a smooth time-dependent vector field on M. There exist an open subset $\mathcal{E} \subseteq J \times J \times M$ and a smooth map $\psi: \mathcal{E} \to M$ called the* ***time-dependent flow of V***, *with the following properties*:

(a) *For each $t_0 \in J$ and $p \in M$, the set $\mathcal{E}^{(t_0,p)} = \{t \in J : (t, t_0, p) \in \mathcal{E}\}$ is an open interval containing t_0, and the smooth curve $\psi^{(t_0,p)}: \mathcal{E}^{(t_0,p)} \to M$ defined by $\psi^{(t_0,p)}(t) = \psi(t, t_0, p)$ is the unique maximal integral curve of V with initial condition $\psi^{(t_0,p)}(t_0) = p$.*
(b) *If $t_1 \in \mathcal{E}^{(t_0,p)}$ and $q = \psi^{(t_0,p)}(t_1)$, then $\mathcal{E}^{(t_1,q)} = \mathcal{E}^{(t_0,p)}$ and $\psi^{(t_1,q)} = \psi^{(t_0,p)}$.*
(c) *For each $(t_1, t_0) \in J \times J$, the set $M_{t_1,t_0} = \{p \in M : (t_1, t_0, p) \in \mathcal{E}\}$ is open in M, and the map $\psi_{t_1,t_0}: M_{t_1,t_0} \to M$ defined by $\psi_{t_1,t_0}(p) = \psi(t_1, t_0, p)$ is a diffeomorphism from M_{t_1,t_0} onto M_{t_0,t_1} with inverse ψ_{t_0,t_1}.*
(d) *If $p \in M_{t_1,t_0}$ and $\psi_{t_1,t_0}(p) \in M_{t_2,t_1}$, then $p \in M_{t_2,t_0}$ and*

$$\psi_{t_2,t_1} \circ \psi_{t_1,t_0}(p) = \psi_{t_2,t_0}(p). \tag{9.18}$$

Proof. This can be proved by following the outline of the proof of Theorem 9.12, using Theorem D.6 in place of Theorem D.1. However, it is much quicker to use the following trick to reduce it to the time-independent case.

Consider the smooth vector field $\widetilde{V}$ on $J \times M$ defined by

$$\widetilde{V}_{(s,p)} = \left(\left.\frac{\partial}{\partial s}\right|_s, V(s,p) \right),$$

where s is the standard coordinate on $J \subseteq \mathbb{R}$, and we identify $T_{(s,p)}(J \times M)$ with $T_s J \oplus T_p M$ as usual (see Proposition 3.14). Let $\widetilde{\theta}: \widetilde{\mathcal{D}} \to J \times M$ denote the flow of $\widetilde{V}$. If we write the component functions of $\widetilde{\theta}$ as

$$\widetilde{\theta}\big(t,(s,p)\big) = \Big(\alpha\big(t,(s,p)\big), \beta\big(t,(s,p)\big)\Big),$$

then $\alpha: \widetilde{\mathcal{D}} \to J$ and $\beta: \widetilde{\mathcal{D}} \to M$ satisfy

$$\begin{aligned} &\frac{\partial \alpha}{\partial t}\big(t,(s,p)\big) = 1, && \alpha\big(0,(s,p)\big) = s,\\ &\frac{\partial \beta}{\partial t}\big(t,(s,p)\big) = V\Big(\alpha\big(t,(s,p)\big), \beta\big(t,(s,p)\big)\Big), && \beta\big(0,(s,p)\big) = p. \end{aligned}$$

It follows immediately that $\alpha(t,(s,p)) = t + s$, and therefore β satisfies

$$\frac{\partial \beta}{\partial t}\big(t,(s,p)\big) = V\Big(t+s, \beta\big(t,(s,p)\big)\Big). \tag{9.19}$$

Let $\mathcal{E}$ be the subset of $\mathbb{R} \times J \times M$ defined by

$$\mathcal{E} = \Big\{(t, t_0, p) : \big(t - t_0, (t_0, p)\big) \in \widetilde{\mathcal{D}}\Big\}.$$

Clearly, $\mathcal{E}$ is open in $\mathbb{R} \times J \times M$ because $\widetilde{\mathcal{D}}$ is. Moreover, since α maps $\widetilde{\mathcal{D}}$ into J, if $(t, t_0, p) \in \mathcal{E}$, then $t = \alpha\big(t - t_0, (t_0, p)\big) \in J$, which implies that $\mathcal{E} \subseteq J \times J \times M$.

The fact that each set $M_{t_1,t_0} = \{p \in M : (t_1, t_0, p) \in \mathcal{E}\}$ is open in M follows immediately from the fact that $\mathcal{E}$ is open.

Now define $\psi\colon \mathcal{E} \to M$ by

$$\psi(t, t_0, p) = \beta\big(t - t_0, (t_0, p)\big).$$

Then ψ is smooth because β is, and it follows from (9.19) that $\psi^{(t_0,p)}(t) = \psi(t, t_0, p)$ is an integral curve of V with initial condition $\psi^{(t_0,p)}(t_0) = p$.

To prove uniqueness, suppose $t_0 \in J$ and $\gamma\colon J_0 \to M$ is any integral curve of V defined on some open interval $J_0 \subseteq J$ containing t_0 and satisfying $\gamma(t_0) = p$. Define a smooth curve $\widetilde{\gamma}\colon J_0 \to J \times M$ by $\widetilde{\gamma}(t) = \big(t, \gamma(t)\big)$. Then $\widetilde{\gamma}$ is easily seen to be an integral curve of $\widetilde{V}$ with initial condition $\widetilde{\gamma}(t_0) = (t_0, p)$. By uniqueness and maximality of integral curves of $\widetilde{V}$, we must have $\widetilde{\gamma}(t) = \widetilde{\theta}\big(t - t_0, (t_0, p)\big)$ on its whole domain, which implies that the domain of γ is contained in that of $\psi^{(t_0,p)}$, and $\gamma = \psi^{(t_0,p)}$ on that domain. It follows that $\psi^{(t_0,p)}$ is the unique maximal integral curve of V passing through p at $t = t_0$. This completes the proof of (a).

To prove (b), suppose $t_1 \in \mathcal{E}^{(t_0,p)}$ and $q = \psi^{(t_0,p)}(t_1)$. Then both $\psi^{(t_1,q)}$ and $\psi^{(t_0,p)}$ are integral curves of V that pass through q when $t = t_1$, so by uniqueness and maximality they must have the same domain and be equal on that domain.

Next, we prove (d). Suppose $p \in M_{t_1,t_0}$ and $\psi_{t_1,t_0}(p) \in M_{t_2,t_1}$, and set $q = \psi_{t_1,t_0}(p) = \psi^{(t_0,p)}(t_1)$. Then (b) implies that $\psi^{(t_1,q)}(t_2) = \psi^{(t_0,p)}(t_2)$. Unwinding the definitions yields (9.18).

Finally, we prove (c). Suppose $(t_1, t_0) \in J \times J$. We have already noted that M_{t_1,t_0} is open in M. To show that $\psi_{t_1,t_0}\big(M_{t_1,t_0}\big) \subseteq M_{t_0,t_1}$, let p be a point of M_{t_1,t_0}, and set $q = \psi_{t_1,t_0}(p)$. Part (b) implies that $\mathcal{E}^{(t_0,p)} = \mathcal{E}^{(t_1,q)}$, and thus $t_0 \in \mathcal{E}^{(t_0,p)} = \mathcal{E}^{(t_1,q)}$. This is equivalent to $(t_0, t_1, q) \in \mathcal{E}$, which in turn means $q \in M_{t_0,t_1}$ as claimed. To see that $\psi_{t_1,t_0}\colon M_{t_1,t_0} \to M_{t_0,t_1}$ is a diffeomorphism, just note that the same argument as above implies that $\psi_{t_0,t_1}\big(M_{t_0,t_1}\big) \subseteq M_{t_1,t_0}$, and then (d) implies that $\psi_{t_1,t_0} \circ \psi_{t_0,t_1}(p) = \psi_{t_1,t_1}(p) = p$ for all $p \in M_{t_0,t_1}$, and similarly that $\psi_{t_0,t_1} \circ \psi_{t_1,t_0}(q) = q$ for $q \in M_{t_1,t_0}$. □

► **Exercise 9.49.** Let M be a smooth manifold. Suppose X is a (time-independent) smooth vector field on M, and $\theta\colon \mathcal{D} \to M$ is its flow. Let V be the time-dependent vector field defined by $V(t, p) = X_p$. Show that the time-dependent flow of V is given by $\psi(t, t_0, p) = \theta(t - t_0, p)$, with domain $\mathcal{E} = \{(t, t_0, p) : (t - t_0, p) \in \mathcal{D}\}$.

Example 9.50. Define a time-dependent vector field V on $\mathbb{R}^n$ by

$$V(t, x) = \frac{1}{t} x^i \left.\frac{\partial}{\partial x^i}\right|_x, \qquad (t, x) \in (0, \infty) \times \mathbb{R}^n.$$

Suppose $t_0 \in (0, \infty)$ and $x_0 \in \mathbb{R}^n$ are arbitrary, and let $\gamma(t) = \big(x^1(t), \dots, x^n(t)\big)$ denote the integral curve of V with initial condition $\gamma(t_0) = x_0$. Then the components of γ satisfy the following nonautonomous system of differential equations:

$$\dot{x}^i(t) = \frac{1}{t} x^i(t),$$
$$x^i(t_0) = x_0^i.$$

The maximal solution to this system, as you can easily check, is $x^i(t) = tx_0^i/t_0$, defined for all $t > 0$. Therefore, the time-dependent flow of V is given by $\psi(t,t_0,x) = tx/t_0$ for $(t,t_0,x) \in (0,\infty) \times (0,\infty) \times \mathbb{R}^n$. //

First-Order Partial Differential Equations

One of the most powerful applications of the theory of flows is to partial differential equations. In its most general form, a ***partial differential equation*** (***PDE***) is any equation that relates an unknown function of two or more variables with its partial derivatives up to some order and with the independent variables. The ***order*** of the PDE is the highest-order derivative of the unknown function that appears.

The number of specialized techniques that have been developed to solve partial differential equations is staggering. (For an introduction to the general theory, you can consult one of the many excellent introductory books on the subject, such as [Eva98, Fol95, Joh91].) However, it is a remarkable fact that real-valued *first-order* PDEs can be reduced to *ordinary* differential equations by means of the theory of flows, and thus can be solved using only ODEs and a little differential-geometric insight but no specialized PDE theory. In this section, we describe how this is done for two special classes of first-order equations: first, linear equations; and then, somewhat more generally, quasilinear equations (which we define below). A PDE that is not quasilinear is said to be ***fully nonlinear***; we will show how to treat fully nonlinear first-order equations in Chapter 22.

In coordinates, any first-order PDE for a single unknown function can be written

$$F\left(x^1,\dots,x^n,u(x),\frac{\partial u}{\partial x^1}(x),\dots,\frac{\partial u}{\partial x^n}(x)\right) = 0, \tag{9.20}$$

where u is an unknown function of n variables and F is a given smooth function of $2n+1$ variables. (Smoothness is not strictly necessary, but we assume it throughout for simplicity.) The theory we will describe applies only when F and u are real-valued, so we assume that as well. (There is also a fascinating theory of complex-valued first-order PDEs, but it requires entirely different methods.)

Without further restrictions, most PDEs have a multitude of solutions—for example, the PDE $\partial u/\partial x = 0$ in the plane is solved by any smooth function u that depends on y alone—so in order to get a unique solution one generally stipulates that the solution should satisfy some extra conditions. For first-order equations, the appropriate condition is to specify "initial values" on a hypersurface: given a smooth hypersurface $S \subseteq \mathbb{R}^n$ and a smooth function $\varphi\colon S \to \mathbb{R}$, we seek a smooth function u that solves the PDE and also satisfies the initial condition

$$u|_S = \varphi. \tag{9.21}$$

The problem of finding a solution to (9.20) in a neighborhood of S subject to the initial condition (9.21) is called a ***Cauchy problem***.

Not every Cauchy problem has a solution: for example, in $\mathbb{R}^2$, the equation $\partial u/\partial x = 1$ has no solution with $u = 0$ on the x-axis, because the equation and the initial condition contradict each other. To avoid such difficulties, one usually assumes that the Cauchy problem (9.20)–(9.21) is *noncharacteristic*, meaning that there is a certain geometric relationship between the equation and the initial data, which is sufficient to guarantee the existence of a solution near S. As we study the Cauchy problem in increasing generality, we will describe the noncharacteristic condition separately for each type of equation we treat. The most general form of the condition is given at the end of Chapter 22.

Linear Equations

The first type of equation we will treat is a first-order ***linear PDE***, which is one that depends linearly or affinely on the unknown function and its derivatives. In coordinate form, the most general such equation can be written

$$a^1(x)\frac{\partial u}{\partial x^1}(x) + \cdots + a^n(x)\frac{\partial u}{\partial x^n}(x) + b(x)u(x) = f(x), \tag{9.22}$$

where $a^1, \dots, a^n$, b, and f are smooth, real-valued functions defined on some open subset $\Omega \subseteq \mathbb{R}^n$, and u is an unknown smooth function on Ω.

It should come as no surprise that flows of vector fields play a role in the solution of (9.22), because the first n terms on the left-hand side represent the action on u of a smooth vector field $A \in \mathfrak{X}(\Omega)$:

$$A_x = a^1(x)\left.\frac{\partial}{\partial x^1}\right|_x + \cdots + a^n(x)\left.\frac{\partial}{\partial x^n}\right|_x. \tag{9.23}$$

In terms of A, we can rewrite (9.22) in the simple form $Au + bu = f$. In this form, it makes sense on any smooth manifold, and is no more difficult to solve in that generality, so we state our first theorem in that context. The Cauchy problem for $Au + bu = f$ with initial hypersurface S is said to be ***noncharacteristic*** if A is nowhere tangent to S.

Theorem 9.51 (The Linear First-Order Cauchy Problem). *Let M be a smooth manifold. Suppose we are given an embedded hypersurface $S \subseteq M$, a smooth vector field $A \in \mathfrak{X}(M)$ that is nowhere tangent to S, and functions $b, f \in C^\infty(M)$ and $\varphi \in C^\infty(S)$. Then for some neighborhood U of S in M, there exists a unique solution $u \in C^\infty(U)$ to the noncharacteristic Cauchy problem*

$$Au + bu = f, \tag{9.24}$$

$$u|_S = \varphi. \tag{9.25}$$

Proof. The flowout theorem gives us a neighborhood $\mathcal{O}_\delta$ of $\{0\} \times S$ in $\mathbb{R} \times S$, a neighborhood U of S in M, and a diffeomorphism $\Phi\colon \mathcal{O}_\delta \to U$ that satisfies $\Phi(0, p) = p$ for $p \in S$ and $\Phi_*(\partial/\partial t) = A$. Let us write $\widehat{u} = u \circ \Phi$, $\widehat{f} = f \circ \Phi$, and $\widehat{b} = b \circ \Phi$. Proposition 8.16 shows that $\partial\widehat{u}/\partial t = (Au) \circ \Phi$. Thus, $u \in C^\infty(U)$ satisfies (9.24)–(9.25) if and only if $\widehat{u}$ satisfies

$$\begin{aligned} \frac{\partial \widehat{u}}{\partial t}(t,p) &= \widehat{f}(t,p) - \widehat{b}(t,p)\widehat{u}(t,p), \quad (t,p) \in \mathcal{O}_\delta, \\ \widehat{u}(0,p) &= \varphi(p), \quad p \in S. \end{aligned} \tag{9.26}$$

For each fixed $p \in S$, this is a linear first-order ODE initial value problem for $\widehat{u}$ on the interval $-\delta(p) < t < \delta(p)$. As is shown in ODE texts, such a problem always has a unique solution on the whole interval, which can be written explicitly as

$$\widehat{u}(t,p) = e^{-B(t,p)}\left(\varphi(p) + \int_0^t \widehat{f}(\tau,p)e^{B(\tau,p)}\,d\tau\right), \quad \text{where}$$
$$B(\tau,p) = \int_0^\tau \widehat{b}(\sigma,p)\,d\sigma.$$

This is a smooth function of (t,p) (as can be seen by choosing local coordinates for S and differentiating under the integral signs). Therefore, $u = \widehat{u} \circ \Phi^{-1}$ is the unique solution on U to (9.24)–(9.25). □

This proof shows how to write down an explicit solution to the Cauchy problem, provided the flow of the vector field A can be found explicitly. The computations are usually easiest if we first choose a (local or global) parametrization $X\colon \Omega \to S$, and substitute $X(s)$ for p in (9.26). This amounts to using the canonical coordinates of Theorem 9.22 to transform the Cauchy problem to an ODE.

Example 9.52 (A Linear Cauchy Problem). Suppose we wish to solve the following Cauchy problem for a smooth function $u(x,y)$ in the plane:

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = x, \tag{9.27}$$

$$u(x,0) = x \quad \text{when } x > 0. \tag{9.28}$$

The vector field acting on u on the left-hand side of (9.27) is the vector field W of Example 9.23. The initial hypersurface S is the positive x-axis, and this problem is noncharacteristic because W is nowhere tangent to S. (Notice that this would not be the case if we took S to be the entire x-axis.) Using the computations of Example 9.23, we find that the transformation $(x,y) = \Psi(t,s) = (s\cos t, s\sin t)$ pushes $\partial/\partial t$ forward to W, and thus transforms (9.27)–(9.28) to the ODE initial value problem

$$\begin{aligned} \frac{\partial \widehat{u}}{\partial t}(t,s) &= s\cos t, \\ \widehat{u}(0,s) &= s. \end{aligned}$$

This is solved by $\widehat{u}(t,s) = s\sin t + s$. Substituting for (t,s) in terms of (x,y) using (9.13), we obtain the solution $u(x,y) = y + \sqrt{x^2+y^2}$ to the original problem. //

Quasilinear Equations

The preceding results extend easily to certain nonlinear partial differential equations. A PDE is called ***quasilinear*** if it can be written as an affine equation in the highest-order derivatives of the unknown function, with coefficients that may depend on the function itself and its derivatives of lower order. Thus, in coordinates, a ***quasilinear first-order PDE*** is a differential equation of the form

$$a^1\big(x,u(x)\big)\frac{\partial u}{\partial x^1}+\cdots+a^n\big(x,u(x)\big)\frac{\partial u}{\partial x^n}=f\big(x,u(x)\big) \tag{9.29}$$

for an unknown real-valued function $u(x^1,\dots,x^n)$, where $a^1,\dots,a^n$ and f are smooth real-valued functions defined on some open subset $W\subseteq\mathbb{R}^{n+1}$. (For simplicity, this time we concentrate only on the local problem, and restrict our attention to open subsets of Euclidean space.)

We wish to solve a Cauchy problem for this equation with initial condition

$$u|_S=\varphi, \tag{9.30}$$

where $S\subseteq\mathbb{R}^n$ is a smooth, embedded hypersurface, and $\varphi\colon S\to\mathbb{R}$ is a smooth function whose graph is contained in W. A quasilinear Cauchy problem is said to be ***noncharacteristic*** if the vector field A^φ along S defined by

$$A^\varphi\big|_x=a^1\big(x,\varphi(x)\big)\left.\frac{\partial}{\partial x^1}\right|_x+\cdots+a^n\big(x,\varphi(x)\big)\left.\frac{\partial}{\partial x^n}\right|_x \tag{9.31}$$

is nowhere tangent to S. (Notice that in this case the noncharacteristic condition depends on the initial value φ, not just on the initial hypersurface.) We will show that a noncharacteristic Cauchy problem always has local solutions. (As we will see below, finding global solutions can be problematic because of the lack of uniqueness.)

Theorem 9.53 (The Quasilinear Cauchy Problem). *If the Cauchy problem* (9.29)–(9.30) *is noncharacteristic, then for each $p\in S$ there exists a neighborhood U of p in M on which there exists a unique solution u to* (9.29)–(9.30).

Proof. The key is to convert the dependent variable u to an additional independent variable. (This is a trick that is useful in many different contexts.) Define the ***characteristic vector field*** for (9.29) to be the vector field ξ on $W\subseteq\mathbb{R}^{n+1}$ given by

$$\xi_{(x,z)}=a^1(x,z)\left.\frac{\partial}{\partial x^1}\right|_{(x,z)}+\cdots+a^n(x,z)\left.\frac{\partial}{\partial x^n}\right|_{(x,z)}+f(x,z)\left.\frac{\partial}{\partial z}\right|_{(x,z)}, \tag{9.32}$$

where we write $(x,z)=\big(x^1,\dots,x^n,z\big)$. Suppose u is a smooth function defined on an open subset $V\subseteq\mathbb{R}^n$ whose graph $\Gamma(u)=\big\{\big(x,u(x)\big):x\in V\big\}$ is contained in W. Then (9.29) holds if and only if $\xi\big(z-u(x)\big)=0$ at all points of $\Gamma(u)$. Since $z-u(x)$ is a defining function for $\Gamma(u)$, it follows from Corollary 5.39 that u solves (9.29) if and only if ξ is tangent to $\Gamma(u)$. The idea is to construct the graph of u as the flowout by ξ from a suitable initial submanifold.

Let $\Gamma(\varphi) = \{(x, \varphi(x)) : x \in S\}$ denote the graph of φ; it is an $(n-1)$-dimensional embedded submanifold of W. The projection $\pi : W \to \mathbb{R}^n$ onto the first n variables maps $\Gamma(\varphi)$ diffeomorphically onto S, so if ξ were tangent to $\Gamma(\varphi)$ at some point $(x, \varphi(x))$, then $d\pi(\xi_{(x,\varphi(x))})$ would be tangent to S at x. However, a direct computation using (9.32) and (9.31) shows that

$$d\pi\left(\xi_{(x,\varphi(x))}\right) = A^{\varphi}\big|_x,$$

so the noncharacteristic assumption guarantees that ξ is nowhere tangent to $\Gamma(\varphi)$.

We can apply the flowout theorem to the vector field ξ starting on $\Gamma(\varphi) \subseteq W$ to obtain an immersed n-dimensional submanifold $\mathcal{S} \subseteq W$ containing $\Gamma(\varphi)$, such that ξ is everywhere tangent to $\mathcal{S}$. If we can show that $\mathcal{S}$ is the graph of a smooth function u, at least locally near $\Gamma(\varphi)$, then u will be a solution to our problem.

Let $p \in S$ be arbitrary. At $(p, \varphi(p)) \in \Gamma(\varphi) \subseteq \mathcal{S}$, the tangent space to $\mathcal{S}$ is spanned by the vector $\xi_{(p,\varphi(p))}$ together with $T_{(p,\varphi(p))}\Gamma(\varphi)$. The restriction of π to $\Gamma(\varphi)$ is a diffeomorphism onto S, so $d\pi$ maps $T_{(p,\varphi(p))}\Gamma(\varphi)$ isomorphically onto T_pS. On the other hand, as we noted above, $d\pi$ takes $\xi_{(p,\varphi(p))}$ to $A^{\varphi}|_p$. By the noncharacteristic assumption, $A^{\varphi}|_p \notin T_pS$, so $d\pi$ is injective on $T_{(p,\varphi(p))}\mathcal{S}$, and thus for dimensional reasons $T_{(p,\varphi(p))}\mathbb{R}^{n+1} = T_{(p,\varphi(p))}\mathcal{S} \oplus \operatorname{Ker} d\pi_{(p,\varphi(p))}$. Because $\operatorname{Ker} d\pi_{(p,\varphi(p))}$ is the tangent space to $\{p\} \times \mathbb{R}$, it follows that $\mathcal{S}$ intersects $\{p\} \times \mathbb{R}$ transversely at $(p, \varphi(p))$. By Corollary 6.33, there exist a neighborhood V of $(p, \varphi(p))$ in $\mathcal{S}$ and a neighborhood U of p in $\mathbb{R}^n$ such that V is the graph of a smooth function $u : U \to \mathbb{R}$. This function solves the Cauchy problem in U.

To prove uniqueness, we might need to shrink U. Because $\mathcal{S}$ is a flowout, it is the image of some open subset $\mathcal{O}_{\mathcal{S}} \subseteq \mathbb{R} \times \Gamma(\varphi)$ under the flow of ξ. Choose V small enough that it is the image under the flow of a set of the form $(-\varepsilon, \varepsilon) \times Y \subseteq \mathcal{O}_{\mathcal{S}}$, for some $\varepsilon > 0$ and some neighborhood Y of $(p, \varphi(p))$ in $\Gamma(\varphi)$. With this assumption, $\Gamma(u)$ is exactly the union of the images of the integral curves of ξ starting at points of Y and flowing for time $|t| < \varepsilon$. Suppose $\widetilde{u}$ is any other solution to the same Cauchy problem on the same open subset U. As we noted above, this means that ξ is tangent to the graph of $\widetilde{u}$, and the initial condition ensures that $Y = \Gamma(\varphi) \cap U \subseteq \Gamma(\widetilde{u})$. Since the graph of $\widetilde{u}$ is a properly embedded submanifold of $U \times \mathbb{R}$, Problem 9-2 shows that each integral curve of ξ in $U \times \mathbb{R}$ starting at a point of Y must lie entirely in the graph of $\widetilde{u}$. Thus $\Gamma(\widetilde{u}) \supseteq \Gamma(u)$. But then $\Gamma(\widetilde{u})$ cannot contain any points that are not in $\Gamma(u)$ and still be the graph of a function, so $\widetilde{u} = u$ on U. □

To find an explicit solution to a quasilinear Cauchy problem, we begin by choosing a smooth local parametrization of S, written as $s \mapsto X(s)$ for $s = (s^2, \dots, s^n) \in \Omega \subseteq \mathbb{R}^{n-1}$. Then the map $\widetilde{X} : \Omega \to \mathbb{R}^{n+1}$ given by $\widetilde{X}(s) = (X(s), \varphi(X(s)))$ is a local parametrization of $\Gamma(\varphi)$, and a local parametrization of $\mathcal{S}$ is given by

$$\Psi(t, s) = \theta_t\left(\widetilde{X}(s)\right),$$

where θ is the flow of ξ. To rewrite $\mathcal{S}$ as a graph, just invert the map $\pi \circ \Psi : \Omega \to \mathbb{R}^n$ locally by solving for $(x^1, \dots, x^n)$ in terms of $(t, s^2, \dots, s^n)$; then the z-component of Ψ, written as a function of $(x^1, \dots, x^n)$, is a solution to the Cauchy problem.

Example 9.54 (A Quasilinear Cauchy Problem). Suppose we wish to solve the following quasilinear Cauchy problem in the plane:

$$(u+1)\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0,$$
$$u(x,0)=x.$$

The initial hypersurface S is the x-axis, and the initial value is $\varphi(x,0)=x$. The vector field A^{φ} is $(x+1)\partial/\partial x+\partial/\partial y$, which is nowhere tangent to the x-axis, so this problem is noncharacteristic.

The characteristic vector field is the following vector field on $\mathbb{R}^3$:

$$\xi=(z+1)\frac{\partial}{\partial x}+\frac{\partial}{\partial y}.$$

We wish to find the flowout of ξ starting from $\Gamma(\varphi)$. We can parametrize S by $X(s)=(s,0)$ for $s\in\mathbb{R}$, and then $\Gamma(\varphi)$ is parametrized by $\widetilde{X}(s)=(s,0,s)$. Solving the system of ODEs associated to ξ with initial conditions $(x,y,z)=(s,0,s)$, we find that the flowout of ξ starting from $\widetilde{X}(s)$ is parametrized by

$$\Psi(t,s)=(s+t+st,t,s).$$

The image of this map is the graph of our solution. To reparametrize the graph in terms of x and y, we invert the map $\pi\circ\Psi$; that is, we solve $(x,y)=(s+t+st,t)$ (locally) for t and s, yielding

$$(t,s)=\left(y,\frac{x-y}{1+y}\right),$$

and therefore on the flowout manifold we have $z=s=(x-y)/(1+y)$. The solution to our Cauchy problem is $u(x,y)=(x-y)/(1+y)$. Note that it is defined only in a neighborhood of S (the set where $y>-1$), not on the whole plane. //

The integral curves of ξ in $\mathbb{R}^{n+1}$ are called the ***characteristic curves*** (or ***characteristics***) of the PDE (9.29). This solution technique, which boils down to constructing the graph of u as a union of characteristic curves, is called the ***method of characteristics***. (For linear equations, the term *characteristic curves* is also sometimes applied to the integral curves of the vector field A defined by (9.23). Of course, the method described above for quasilinear equations can be applied to linear ones as well, but the technique of Example 9.52 is usually easier in the linear case.)

Theorems 9.51 and 9.53 only assert existence and uniqueness of a solution in a neighborhood of the initial submanifold. Cauchy problems do not always admit global solutions, and when global solutions do exist, they might not be unique (see Problem 9-23 for some examples). The basic problem is that the characteristic curves passing through the initial hypersurface might not reach all points of the manifold. Also, quasilinear problems have the added complication that the characteristic curves in $\mathbb{R}^{n+1}$ might cease to be transverse to the fibers of the projection

$\mathbb{R}^{n+1} \to \mathbb{R}^n$, or even if they are transverse, their projections into $\mathbb{R}^n$ starting at different points of $\Gamma(\varphi)$ might cross each other, even though the characteristic curves themselves do not. At any point in the image of two or more characteristic curves, the procedure above would produce two different values for u. Nevertheless, in specific cases, it is often possible to identify a neighborhood of S on which a unique solution exists by analyzing the behavior of the characteristics. For instance, consider the solution u that we produced on the set $U = \{(x,y) : y > -1\}$ in Example 9.54. Its graph contains the entire maximal integral curve starting at each point of $\Gamma(\varphi)$. Because of this, the argument we used to prove local uniqueness in Theorem 9.53 actually proves that the solution is globally unique in this case.

Problems

9-1. Suppose M is a smooth manifold, $X \in \mathfrak{X}(M)$, and γ is a maximal integral curve of X.

(a) We say γ is ***periodic*** if there is a number $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. Show that exactly one of the following holds:

- γ is constant.
- γ is injective.
- γ is periodic and nonconstant.

(b) Show that if γ is periodic and nonconstant, then there exists a unique positive number T (called the ***period of γ***) such that $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$ for some $k \in \mathbb{Z}$.

(c) Show that the image of γ is an immersed submanifold of M, diffeomorphic to $\mathbb{R}$, $\mathbb{S}^1$, or $\mathbb{R}^0$.

(*Used on pp. 398, 560.*)

9-2. Suppose M is a smooth manifold, $S \subseteq M$ is an immersed submanifold, and V is a smooth vector field on M that is tangent to S.

(a) Show that for any integral curve γ of V such that $\gamma(t_0) \in S$, there exists $\varepsilon > 0$ such that $\gamma\big((t_0 - \varepsilon, t_0 + \varepsilon)\big) \subseteq S$.

(b) Now assume S is properly embedded. Show that every integral curve that intersects S is contained in S.

(c) Give a counterexample to (b) if S is not closed.

(*Used on pp. 243, 491.*)

9-3. Compute the flow of each of the following vector fields on $\mathbb{R}^2$:

(a) $V = y\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$.

(b) $W = x\dfrac{\partial}{\partial x} + 2y\dfrac{\partial}{\partial y}$.

(c) $X = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.

(d) $Y = x\dfrac{\partial}{\partial y} + y\dfrac{\partial}{\partial x}$.

9-4. For any integer $n \geq 1$, define a flow on the odd-dimensional sphere $\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$ by $\theta(t, z) = e^{it}z$. Show that the infinitesimal generator of θ is a smooth nonvanishing vector field on $\mathbb{S}^{2n-1}$. [Remark: in the case $n = 2$, the integral curves of X are the curves γ_z of Problem 3-6, so this provides a simpler proof that each γ_z is smooth.] (*Used on p. 435.*)

9-5. Suppose M is a smooth, compact manifold that admits a nowhere vanishing smooth vector field. Show that there exists a smooth map $F: M \to M$ that is homotopic to the identity and has no fixed points.

9-6. Prove Lemma 9.19 (the escape lemma).

9-7. Let M be a connected smooth manifold. Show that the group of diffeomorphisms of M acts transitively on M: that is, for any $p, q \in M$, there is a diffeomorphism $F: M \to M$ such that $F(p) = q$. [Hint: first prove that if $p, q \in \mathbb{B}^n$ (the open unit ball in $\mathbb{R}^n$), there is a compactly supported smooth vector field on $\mathbb{B}^n$ whose flow θ satisfies $\theta_1(p) = q$.]

9-8. Let M be a smooth manifold and let $S \subseteq M$ be a compact embedded submanifold. Suppose $V \in \mathfrak{X}(M)$ is a smooth vector field that is nowhere tangent to S. Show that there exists $\varepsilon > 0$ such that the flow of V restricts to a smooth embedding $\Phi: (-\varepsilon, \varepsilon) \times S \to M$.

9-9. Suppose M is a smooth manifold and $S \subseteq M$ is an embedded hypersurface (not necessarily compact). Suppose further that there is a smooth vector field V defined on a neighborhood of S and nowhere tangent to S. Show that S has a neighborhood in M diffeomorphic to $(-1, 1) \times S$, under a diffeomorphism that restricts to the obvious identification $\{0\} \times S \approx S$. [Hint: using the notation of the flowout theorem, show that $\mathcal{O}_\delta \approx \mathcal{O}_1$.]

9-10. For each vector field in Problem 9-3, find smooth coordinates in a neighborhood of $(1, 0)$ for which the given vector field is a coordinate vector field.

9-11. Prove Theorem 9.24 (the boundary flowout theorem). [Hint: define Φ first in boundary coordinates and use uniqueness to glue together the local definitions. To obtain an embedding, make sure $\delta(p)$ is no more than half of the first time the integral curve starting at p hits the boundary (if it ever does).]

9-12. Suppose M_1 and M_2 are connected smooth n-manifolds and $M_1 \# M_2$ is their smooth connected sum (see Example 9.31). Show that the smooth structure on $M_1 \# M_2$ can be chosen in such a way that there are open subsets $\widetilde{M}_1, \widetilde{M}_2 \subseteq M_1 \# M_2$ that are diffeomorphic to $M_1 \smallsetminus \{p_1\}$ and $M_2 \smallsetminus \{p_2\}$, respectively, such that $\widetilde{M}_1 \cup \widetilde{M}_2 = M_1 \# M_2$ and $\widetilde{M}_1 \cap \widetilde{M}_2$ is diffeomorphic to $(-1, 1) \times \mathbb{S}^{n-1}$. (*Used on p. 465.*)

9-13. Prove that the conclusions of Theorems 5.29 and 5.53(b) (restricting the codomain of a smooth map to a submanifold or submanifold with boundary) remain true if M is allowed to be a smooth manifold with boundary.

9-14. Use the double to prove Theorem 6.18 (the Whitney immersion theorem) in the case that M has nonempty boundary.

9-15. Prove Theorem 9.35 (canonical form near a regular point on the boundary). [Hint: consider M as a regular domain in its double, and start with coordinates in which x^n is a local defining function for ∂M in $D(M)$.]

9-16. Give an example of smooth vector fields V, $\widetilde{V}$, and W on $\mathbb{R}^2$ such that $V = \widetilde{V} = \partial/\partial x$ along the x-axis but $\mathcal{L}_V W \neq \mathcal{L}_{\widetilde{V}} W$ at the origin. [Remark: this shows that it is really necessary to know the vector field V to compute $(\mathcal{L}_V W)_p$; it is not sufficient just to know the vector V_p, or even to know the values of V along an integral curve of V.]

9-17. For each k-tuple of vector fields on $\mathbb{R}^3$ shown below, either find smooth coordinates $\left(s^1, s^2, s^3\right)$ in a neighborhood of $(1,0,0)$ such that $V_i = \partial/\partial s^i$ for $i = 1, \dots, k$, or explain why there are none.

(a) $k = 2;\ V_1 = \dfrac{\partial}{\partial x},\ V_2 = \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}.$

(b) $k = 2;\ V_1 = (x+1)\dfrac{\partial}{\partial x} - (y+1)\dfrac{\partial}{\partial y},\ V_2 = (x+1)\dfrac{\partial}{\partial x} + (y+1)\dfrac{\partial}{\partial y}.$

(c) $k = 3;\ V_1 = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x},\ V_2 = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y},\ V_3 = z\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial z}.$

9-18. Define vector fields X and Y on the plane by

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}.$$

Compute the flows θ, ψ of X and Y, and verify that the flows do not commute by finding explicit open intervals J and K containing 0 such that $\theta_s \circ \psi_t$ and $\psi_t \circ \theta_s$ are both defined for all $(s,t) \in J \times K$, but they are unequal for some such (s,t).

9-19. Let M be $\mathbb{R}^3$ with the z-axis removed. Define $V, W \in \mathfrak{X}(M)$ by

$$V = \frac{\partial}{\partial x} - \frac{y}{x^2+y^2}\frac{\partial}{\partial z}, \qquad W = \frac{\partial}{\partial y} + \frac{x}{x^2+y^2}\frac{\partial}{\partial z},$$

and let θ and ψ be the flows of V and W, respectively. Prove that V and W commute, but there exist $p \in M$ and $s, t \in \mathbb{R}$ such that $\theta_t \circ \psi_s(p)$ and $\psi_s \circ \theta_t(p)$ are both defined but are not equal.

9-20. Suppose M is a compact smooth manifold and $V\colon J \times M \to TM$ is a smooth time-dependent vector field on M. Show that the domain of the time-dependent flow of V is all of $J \times J \times M$.

9-21. Let M be a smooth manifold. A ***smooth isotopy of M*** is a smooth map $H\colon M \times J \to M$, where $J \subseteq \mathbb{R}$ is an interval, such that for each $t \in J$, the map $H_t\colon M \to M$ defined by $H_t(p) = H(p,t)$ is a diffeomorphism. (In particular, if J is the unit interval, then H is a homotopy from H_0 to H_1 through diffeomorphisms.) This problem shows that smooth isotopies are closely related to time-dependent flows.

(a) Suppose $J \subseteq \mathbb{R}$ is an open interval and $H: M \times J \to M$ is a smooth isotopy. Show that the map $V: J \times M \to TM$ defined by

$$V(t, p) = \frac{\partial}{\partial t} H(p, t)$$

is a smooth time-dependent vector field on M, whose time-dependent flow is given by $\psi(t, t_0, p) = H_t \circ H_{t_0}^{-1}(p)$ with domain $J \times J \times M$.

(b) Conversely, suppose J is an open interval and $V: J \times M \to M$ is a smooth time-dependent vector field on M whose time-dependent flow is defined on $J \times J \times M$. For any $t_0 \in J$, show that the map $H: M \times J \to M$ defined by $H(t, p) = \psi(t, t_0, p)$ is a smooth isotopy of M.

9-22. Here are three Cauchy problems in $\mathbb{R}^2$. For each one, find an explicit solution $u(x, y)$ in a neighborhood of the initial submanifold:

(a) $y\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = x, \qquad u(x,0) = \sin x.$

(b) $\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0, \qquad u(x,1) = e^{-x}.$

(c) $\dfrac{\partial u}{\partial x} + u\dfrac{\partial u}{\partial y} = y, \qquad u(0,y) = 0.$

9-23. Consider again the Cauchy problems in Problem 9-22. Show that (a) has a unique global solution; (b) has a global solution, but it is not unique; and (c) has no global solutions. [Hint: consider which characteristic curves intersect the initial submanifold and which do not.]

9-24. Prove the converse to Euler's homogeneous function theorem (Problem 8-2): if $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfies $Vf = cf$, where V is the Euler vector field and $c \in \mathbb{R}$, then f is positively homogeneous of degree c.