Chapter 16
Integration on Manifolds

In Chapter 11, we introduced line integrals of covector fields, which generalize ordinary integrals to the setting of curves in manifolds. It is also useful to generalize *multiple* integrals to manifolds. In this chapter, we carry out that generalization.

As we show in the beginning of this chapter, there is no way to define the integral of a *function* in a coordinate-independent way on a smooth manifold. On the other hand, differential forms turn out to have just the right properties for defining integrals intrinsically.

We begin the chapter with a heuristic discussion of the measurement of volume, to motivate the central role played by alternating tensors in integration theory. We will see that a k-covector on a vector space can be interpreted as "signed k-dimensional volume meter." This suggests that a k-form on a smooth manifold might be thought of as a way of assigning "signed volumes" to k-dimensional submanifolds. The purpose of this chapter is to make this rigorous.

First, we define the integral of a differential form over a domain in Euclidean space, and then we show how to use diffeomorphism invariance and partitions of unity to extend this definition to n-forms on oriented n-manifolds. The key feature of the definition is that it is invariant under orientation-preserving diffeomorphisms.

After developing the general theory of integration of differential forms, we prove one of the most important theorems in differential geometry: *Stokes's theorem*. It is a generalization of the fundamental theorem of calculus and of the fundamental theorem for line integrals, as well as of the three great classical theorems of vector analysis: Green's theorem for vector fields in the plane; the divergence theorem for vector fields in space; and (the classical version of) Stokes's theorem for surface integrals in $\mathbb{R}^3$. Then we extend the theorem to manifolds with corners, which will be useful in our treatment of de Rham cohomology in Chapters 17 and 18.

Next, we show how these ideas play out on a Riemannian manifold. We prove Riemannian versions of the divergence theorem and of Stokes's theorem for surface integrals, of which the classical theorems are special cases.

At the end of the chapter, we show how to extend the theory of integration to nonorientable manifolds by introducing *densities*, which are fields that can be integrated on any manifold, not just oriented ones.

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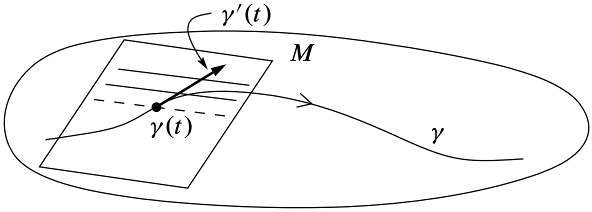

Fig. 16.1 A covector field as a "signed length meter"

The Geometry of Volume Measurement

How might we make coordinate-independent sense of multiple integrals? First, observe that there is no way to integrate real-valued *functions* in a coordinate-independent way on a manifold, at least without adding further structure such as a Riemannian metric. It is easy to see why, even in the simplest case: suppose $C \subseteq \mathbb{R}^n$ is a closed ball, and $f : C \to \mathbb{R}$ is the constant function $f(x) \equiv 1$. Then

$$\int_C f\, dV = \mathrm{Vol}(C),$$

which is clearly not invariant under coordinate transformations, even if we just restrict attention to linear ones.

On the other hand, in Chapter 11 we showed that covector fields could be integrated in a natural way along curves. Let us think a bit more geometrically about why this is so. A covector field on a manifold M assigns a number to each tangent vector, in such a way that multiplying the tangent vector by a constant has the effect of multiplying the resulting number by the same constant. Thus, a covector field can be thought of as assigning a "signed length meter" to each 1-dimensional subspace of each tangent space (Fig. 16.1), and it does so in a coordinate-independent way. Computing the line integral of a covector field, in effect, assigns a "length" to a curve by using this varying measuring scale along the points of the curve.

Now we wish to seek a kind of "field" that can be integrated in a coordinate-independent way over submanifolds of dimension $k > 1$. Its value at each point should be something that we can interpret as a "signed volume meter" on k-dimensional subspaces of the tangent space, a machine ω that accepts any k tangent vectors $(v_1, \dots, v_k)$ at a point and returns a number $\omega(v_1, \dots, v_k)$ that we might think of as the "signed volume" of the parallelepiped spanned by those vectors, measured according to a scale determined by ω.

The most obvious example of such a machine is the determinant in $\mathbb{R}^n$. For example, it is shown in most linear algebra texts that for any two vectors $v_1, v_2 \in \mathbb{R}^2$, $\det(v_1, v_2)$ is, up to a sign, the area of the parallelogram spanned by v_1, v_2. It is not hard to show (see Problem 16-1) that the analogous fact is true in all dimensions. The determinant, remember, is an example of an alternating tensor.

Let us consider what properties we might expect a general "signed k-dimensional volume meter" ω to have. To be consistent with our intuition about volume, multi-

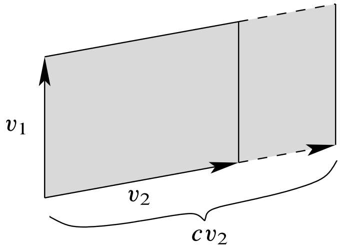

Fig. 16.2 Scaling by a constant

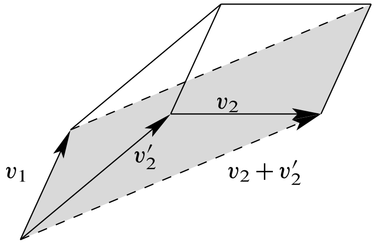

Fig. 16.3 Sum of two vectors

plying any one of the vectors by a constant should scale the volume by that same constant (Fig. 16.2),

and the volume of a k-dimensional parallelepiped formed by adding together two vectors in the ith place should be the sum of the volumes of the two parallelepipeds with the original vectors in the ith place (Fig. 16.3):

$$\omega(v_1, \dots, cv_i, \dots, v_k) = c\omega(v_1, \dots, v_i, \dots, v_k),$$
$$\omega\big(v_1, \dots, v_i + v_i', \dots, v_k\big) = \omega(v_1, \dots, v_i, \dots, v_k) + \omega\big(v_1, \dots, v_i', \dots, v_k\big).$$

(Note that the vectors in Fig. 16.3 are all assumed to lie in one plane.) This suggests that ω should be multilinear, and thus should be a covariant k-tensor.

There is one more property that we should expect: since a linearly dependent k-tuple of vectors spans a parallelepiped of zero k-dimensional volume, ω should give zero whenever it is applied to a such a k-tuple. By Lemma 14.1, this forces ω to be alternating. Thus, alternating tensor fields are promising objects for integrating in a coordinate-independent way. In this chapter, we show how this is done.

Integration of Differential Forms

Just as we began our treatment of line integrals by first defining integrals of 1-forms over intervals in $\mathbb{R}$, we begin here by defining integrals of n-forms over suitable subsets of $\mathbb{R}^n$. For the time being, let us restrict attention to the case $n \geq 1$. You should make sure that you are familiar with the basic properties of multiple integrals in $\mathbb{R}^n$, as summarized in Appendix C.

Recall that a *domain of integration* in $\mathbb{R}^n$ is a bounded subset whose boundary has measure zero. Let $D \subseteq \mathbb{R}^n$ be a domain of integration, and let ω be a (continuous) n-form on $\bar{D}$. Any such form can be written as $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$ for some continuous function $f\colon \bar{D} \to \mathbb{R}$. We define the ***integral of ω over D*** to be

$$\int_D \omega = \int_D f\, dV.$$

This can be written more suggestively as

$$\int_D f\, dx^1 \wedge \cdots \wedge dx^n = \int_D f\, dx^1 \cdots dx^n.$$

In simple terms, to compute the integral of a form such as $f\,dx^1 \wedge \cdots \wedge dx^n$, just "erase the wedges"!

Somewhat more generally, let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and suppose ω is a compactly supported n-form on U. We define

$$\int_U \omega = \int_D \omega,$$

where $D \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ is any domain of integration (such as a rectangle) containing $\operatorname{supp}\omega$, and ω is extended to be zero on the complement of its support. It is easy to check that this definition does not depend on what domain D is chosen.

Like the definition of the integral of a 1-form over an interval, our definition of the integral of an n-form might look like a trick of notation. The next proposition shows why it is natural.

Proposition 16.1. *Suppose D and E are open domains of integration in $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G\colon \bar{D} \to \bar{E}$ is a smooth map that restricts to an orientation-preserving or orientation-reversing diffeomorphism from D to E. If ω is an n-form on $\bar{E}$, then*

$$\int_D G^*\omega = \begin{cases} \displaystyle\int_E \omega & \text{if } G \text{ is orientation-preserving}, \\ \displaystyle -\int_E \omega & \text{if } G \text{ is orientation-reversing}. \end{cases}$$

Proof. Let us use $(y^1, \dots, y^n)$ to denote standard coordinates on E, and $(x^1, \dots, x^n)$ to denote those on D. Suppose first that G is orientation-preserving. With $\omega = f\,dy^1 \wedge \cdots \wedge dy^n$, the change of variables formula (Theorem C.26) together with formula (14.15) for pullbacks of n-forms yields

$$\begin{aligned} \int_E \omega &= \int_E f\,dV = \int_D (f \circ G)\,|\det DG|\,dV = \int_D (f \circ G)(\det DG)\,dV \\ &= \int_D (f \circ G)(\det DG)\,dx^1 \wedge \cdots \wedge dx^n = \int_D G^*\omega. \end{aligned}$$

If G is orientation-reversing, the same computation holds except that a negative sign is introduced when the absolute value signs are removed. □

We would like to extend this theorem to compactly supported n-forms defined on open subsets. However, since we cannot guarantee that arbitrary open subsets or arbitrary compact subsets are domains of integration, we need the following lemma.

Lemma 16.2. *Suppose U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and K is a compact subset of U. Then there is an open domain of integration D such that $K \subseteq D \subseteq \bar{D} \subseteq U$.*

Proof. For each $p \in K$, there is an open ball or half-ball containing p whose closure is contained in U. By compactness, finitely many such sets $B_1, \dots, B_m$ cover K (Fig. 16.4). Since the boundary of an open ball is a codimension-1 submanifold, and the boundary of an open half-ball is contained in a union of two such

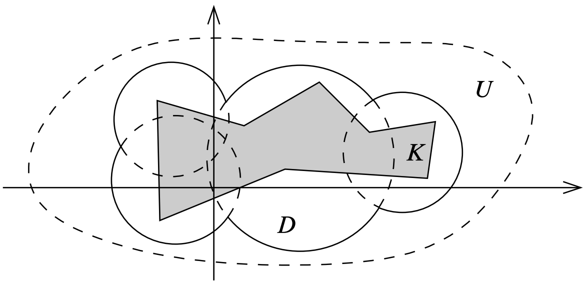

Fig. 16.4 A domain of integration containing a compact set

submanifolds, the boundary of each has measure zero by Corollary 6.12. The set $D = B_1 \cup \cdots \cup B_m$ is the required domain of integration. □

Proposition 16.3. *Suppose U, V are open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G\colon U \to V$ is an orientation-preserving or orientation-reversing diffeomorphism. If ω is a compactly supported n-form on V, then*

$$\int_V \omega = \pm \int_U G^*\omega,$$

with the positive sign if G is orientation-preserving, and the negative sign otherwise.

Proof. Let E be an open domain of integration such that $\operatorname{supp}\omega \subseteq E \subseteq \bar{E} \subseteq V$ (Fig. 16.5). Since diffeomorphisms take interiors to interiors, boundaries to boundaries, and sets of measure zero to sets of measure zero, $D = G^{-1}(E) \subseteq U$ is an open domain of integration containing $\operatorname{supp} G^*\omega$. The result follows from Proposition 16.1. □

Integration on Manifolds

Using the results of the previous section, we can now make sense of the integral of a differential form over an oriented manifold. Let M be an oriented smooth n-manifold with or without boundary, and let ω be an n-form on M. Suppose first that ω is compactly supported in the domain of a single smooth chart (U, φ) that is either positively or negatively oriented. We define the ***integral of ω over M*** to be

$$\int_M \omega = \pm \int_{\varphi(U)} \left(\varphi^{-1}\right)^* \omega, \tag{16.1}$$

with the positive sign for a positively oriented chart, and the negative sign otherwise. (See Fig. 16.6.) Since $(\varphi^{-1})^*\omega$ is a compactly supported n-form on the open subset $\varphi(U) \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, its integral is defined as discussed above.

Proposition 16.4. *With ω as above, $\int_M \omega$ does not depend on the choice of smooth chart whose domain contains* supp ω.

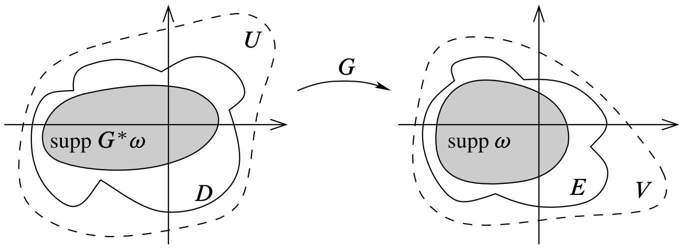

Fig. 16.5 Diffeomorphism invariance of the integral of a form on an open subset

Proof. Suppose (U,φ) and $\big(\widetilde{U},\widetilde{\varphi}\big)$ are two smooth charts such that $\operatorname{supp}\omega \subseteq U \cap \widetilde{U}$ (Fig. 16.7). If both charts are positively oriented or both are negatively oriented, then $\widetilde{\varphi}\circ\varphi^{-1}$ is an orientation-preserving diffeomorphism from $\varphi\big(U\cap\widetilde{U}\big)$ to $\widetilde{\varphi}\big(U\cap\widetilde{U}\big)$, so Proposition 16.3 implies that

$$\int_{\widetilde{\varphi}(\widetilde{U})}\big(\widetilde{\varphi}^{-1}\big)^*\omega = \int_{\widetilde{\varphi}(U\cap\widetilde{U})}\big(\widetilde{\varphi}^{-1}\big)^*\omega = \int_{\varphi(U\cap\widetilde{U})}\big(\widetilde{\varphi}\circ\varphi^{-1}\big)^*\big(\widetilde{\varphi}^{-1}\big)^*\omega$$
$$= \int_{\varphi(U\cap\widetilde{U})}\big(\varphi^{-1}\big)^*(\widetilde{\varphi})^*\big(\widetilde{\varphi}^{-1}\big)^*\omega = \int_{\varphi(U)}\big(\varphi^{-1}\big)^*\omega.$$

If the charts are oppositely oriented, then the two definitions given by (16.1) have opposite signs, but this is compensated by the fact that $\widetilde{\varphi}\circ\varphi^{-1}$ is orientation-reversing, so Proposition 16.3 introduces an extra negative sign into the computation above. In either case, the two definitions of $\int_M \omega$ agree. □

To integrate over an entire manifold, we combine this definition with a partition of unity. Suppose M is an oriented smooth n-manifold with or without boundary, and ω is a compactly supported n-form on M. Let $\{U_i\}$ be a finite open cover of $\operatorname{supp}\omega$ by domains of positively or negatively oriented smooth charts, and let $\{\psi_i\}$ be a subordinate smooth partition of unity. Define the ***integral of ω over M*** to be

$$\int_M \omega = \sum_i \int_M \psi_i\omega. \tag{16.2}$$

(The reason we allow for negatively oriented charts is that it may not be possible to find positively oriented boundary charts on a 1-manifold with boundary, as noted in the proof of Proposition 15.6.) Since for each i, the n-form $\psi_i\omega$ is compactly supported in U_i, each of the terms in this sum is well defined according to our discussion above. To show that the integral is well defined, we need only examine the dependence on the open cover and the partition of unity.

Proposition 16.5. *The definition of $\int_M \omega$ given above does not depend on the choice of open cover or partition of unity.*

Proof. Suppose $\{\widetilde{U}_j\}$ is another finite open cover of $\operatorname{supp}\omega$ by domains of positively or negatively oriented smooth charts, and $\{\widetilde{\psi}_j\}$ is a subordinate smooth parti-

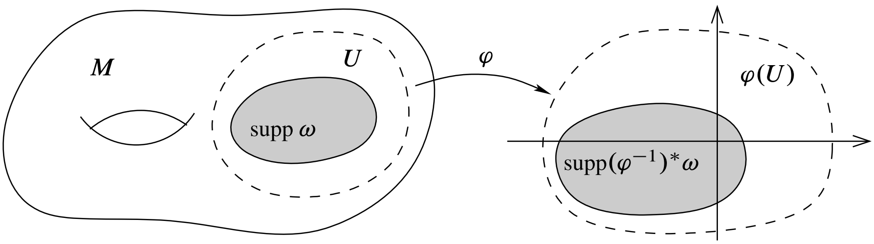

Fig. 16.6 The integral of a form over a manifold

tion of unity. For each i, we compute

$$\int_M \psi_i \omega = \int_M \Big(\sum_j \widetilde{\psi}_j\Big)\psi_i \omega = \sum_j \int_M \widetilde{\psi}_j \psi_i \omega.$$

Summing over i, we obtain

$$\sum_i \int_M \psi_i \omega = \sum_{i,j} \int_M \widetilde{\psi}_j \psi_i \omega.$$

Observe that each term in this last sum is the integral of a form that is compactly supported in a single smooth chart (e.g., in U_i), so by Proposition 16.4 each term is well defined, regardless of which coordinate map we use to compute it. The same argument, starting with $\int_M \widetilde{\psi}_j \omega$, shows that

$$\sum_j \int_M \widetilde{\psi}_j \omega = \sum_{i,j} \int_M \widetilde{\psi}_j \psi_i \omega.$$

Thus, both definitions yield the same value for $\int_M \omega$. □

As usual, we have a special definition in the zero-dimensional case. The integral of a compactly supported 0-form (i.e., a real-valued function) f over an oriented 0-manifold M is defined to be the sum

$$\int_M f = \sum_{p \in M} \pm f(p),$$

where we take the positive sign at points where the orientation is positive and the negative sign at points where it is negative. The assumption that f is compactly supported implies that there are only finitely many nonzero terms in this sum.

If $S \subseteq M$ is an oriented immersed k-dimensional submanifold (with or without boundary), and ω is a k-form on M whose restriction to S is compactly supported, we interpret $\int_S \omega$ to mean $\int_S \iota_S^* \omega$, where $\iota_S \colon S \hookrightarrow M$ is inclusion. In particular, if M is a compact, oriented, smooth n-manifold with boundary and ω is an $(n-1)$-form on M, we can interpret $\int_{\partial M} \omega$ unambiguously as the integral of $\iota_{\partial M}^* \omega$ over ∂M, where ∂M is always understood to have the induced orientation.

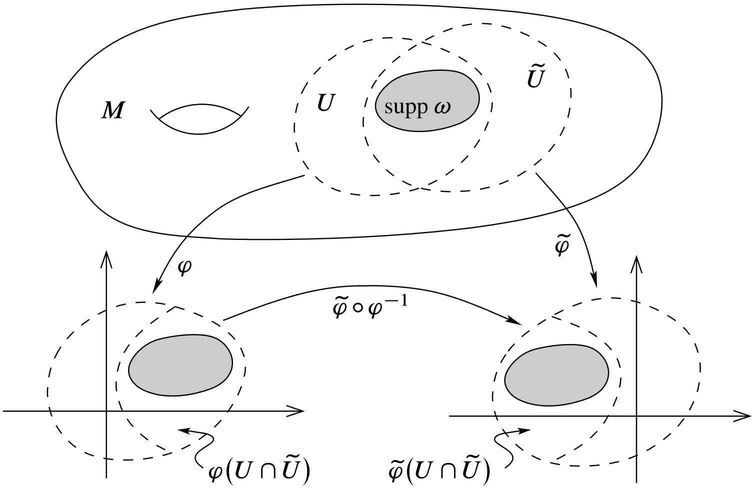

Fig. 16.7 Coordinate independence of the integral

It is worth remarking that it is possible to extend the definition of the integral to some noncompactly supported forms, and such integrals are important in many applications. However, in such cases the resulting multiple integrals are improper, so one must pay close attention to convergence issues. For the purposes we have in mind, the cases we have described here are quite sufficient.

Proposition 16.6 (Properties of Integrals of Forms). *Suppose M and N are nonempty oriented smooth n-manifolds with or without boundary, and ω, η are compactly supported n-forms on M.*

(a) LINEARITY: *If $a, b \in \mathbb{R}$, then*

$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta.$$

(b) ORIENTATION REVERSAL: *If $-M$ denotes M with the opposite orientation, then*

$$\int_{-M} \omega = -\int_M \omega.$$

(c) POSITIVITY: *If ω is a positively oriented orientation form, then $\int_M \omega > 0$.*

(d) DIFFEOMORPHISM INVARIANCE: *If $F\colon N \to M$ is an orientation-preserving or orientation-reversing diffeomorphism, then*

$$\int_M \omega = \begin{cases} \displaystyle\int_N F^*\omega & \text{if } F \text{ is orientation-preserving,} \\ \displaystyle-\int_N F^*\omega & \text{if } F \text{ is orientation-reversing.} \end{cases}$$

Proof. Parts (a) and (b) are left as an exercise. Suppose ω is a positively oriented orientation form for M. This means that if (U, φ) is a positively oriented smooth chart, then $\left(\varphi^{-1}\right)^* \omega$ is a positive function times $dx^1 \wedge \cdots \wedge dx^n$, and for a negatively oriented chart it is a negative function times the same form. Therefore, each term in the sum (16.2) defining $\int_M \omega$ is nonnegative, with at least one strictly positive term, thus proving (c).

To prove (d), it suffices to assume that ω is compactly supported in a single positively or negatively oriented smooth chart, because any compactly supported n-form on M can be written as a finite sum of such forms by means of a partition of unity. Thus, suppose (U, φ) is a positively oriented smooth chart on M whose domain contains the support of ω. When F is orientation-preserving, it is easy to check that $\left(F^{-1}(U), \varphi \circ F\right)$ is an oriented smooth chart on N whose domain contains the support of $F^*\omega$, and the result then follows immediately from Proposition 16.3. The cases in which the chart is negatively oriented or F is orientation-reversing then follow from this result together with (b). □

► **Exercise 16.7.** Prove parts (a) and (b) of the preceding proposition.

Although the definition of the integral of a form based on partitions of unity is very convenient for theoretical purposes, it is useless for doing actual computations. It is generally quite difficult to write down a smooth partition of unity explicitly, and even when one can be written down, one would have to be exceptionally lucky to be able to compute the resulting integrals (think of trying to integrate $e^{-1/x}$).

For computational purposes, it is much more convenient to "chop up" the manifold into a finite number of pieces whose boundaries are sets of measure zero, and compute the integral on each piece separately by means of local parametrizations. One way to do this is described below.

Proposition 16.8 (Integration Over Parametrizations). *Let M be an oriented smooth n-manifold with or without boundary, and let ω be a compactly supported n-form on M. Suppose $D_1, \ldots, D_k$ are open domains of integration in $\mathbb{R}^n$, and for $i = 1, \ldots, k$, we are given smooth maps $F_i \colon \bar{D}_i \to M$ satisfying*

(i) *F_i restricts to an orientation-preserving diffeomorphism from D_i onto an open subset $W_i \subseteq M$;*
(ii) *$W_i \cap W_j = \varnothing$ when $i \neq j$;*
(iii) $\operatorname{supp} \omega \subseteq \overline{W}_1 \cup \cdots \cup \overline{W}_k$.

Then

$$\int_M \omega = \sum_{i=1}^{k} \int_{D_i} F_i^* \omega. \tag{16.3}$$

Proof. As in the preceding proof, it suffices to assume that ω is supported in the domain of a single oriented smooth chart (U, φ). In fact, by restricting to sufficiently nice charts, we may assume that U is precompact, $Y = \varphi(U)$ is a domain of integration in $\mathbb{R}^n$ or $\mathbb{H}^n$, and φ extends to a diffeomorphism from $\overline{U}$ to $\overline{Y}$.

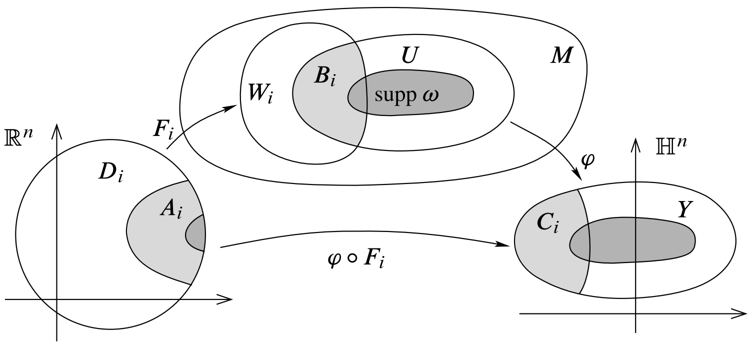

Fig. 16.8 Integrating over parametrizations

For each i, define open subsets $A_i \subseteq D_i$, $B_i \subseteq W_i$, and $C_i \subseteq Y$ (Fig. 16.8) by

$$A_i = F_i^{-1}\big(U \cap W_i\big), \quad B_i = U \cap W_i = F_i(A_i), \quad C_i = \varphi(B_i) = \varphi_i \circ F_i(A_i).$$

Because $\bar{D}_i$ is compact, it is straightforward to check that $\partial W_i \subseteq F_i(\partial D_i)$, and therefore ∂W_i has measure zero in M, and $\partial C_i = \varphi(\partial B_i)$ has measure zero in $\mathbb{R}^n$.

The support of $\left(\varphi^{-1}\right)^* \omega$ is contained in $\bar{C}_1 \cup \cdots \cup \bar{C}_k$, and any two of these sets intersect only on their boundaries, which have measure zero. Thus by Proposition C.23,

$$\int_M \omega = \int_Y \left(\varphi^{-1}\right)^* \omega = \sum_{i=1}^{k} \int_{C_i} \left(\varphi^{-1}\right)^* \omega.$$

The proof is completed by applying Proposition 16.1 to each term above, using the diffeomorphism $\varphi \circ F_i \colon A_i \to C_i$:

$$\int_{C_i} \left(\varphi^{-1}\right)^* \omega = \int_{A_i} (\varphi \circ F_i)^* \left(\varphi^{-1}\right)^* \omega = \int_{A_i} F_i^* \omega = \int_{D_i} F_i^* \omega.$$

Summing over i, we obtain (16.3). □

Example 16.9. Let us use this technique to compute the integral of a 2-form over $\mathbb{S}^2$, oriented as the boundary of $\bar{\mathbb{B}}^3$. Let ω be the following 2-form on $\mathbb{R}^3$:

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy.$$

Let D be the open rectangle $(0,\pi) \times (0,2\pi)$, and let $F \colon \bar{D} \to \mathbb{S}^2$ be the spherical coordinate parametrization $F(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$. Example 15.28 showed that $F|_D$ is orientation-preserving, so it satisfies the hypotheses of Proposition 16.8. Note that

$$\begin{aligned}
F^*\,dx &= \cos\varphi\cos\theta\,d\varphi - \sin\varphi\sin\theta\,d\theta,\\
F^*\,dy &= \cos\varphi\sin\theta\,d\varphi + \sin\varphi\cos\theta\,d\theta,\\
F^*\,dz &= -\sin\varphi\,d\varphi.
\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{S}^2}\omega &= \int_D \big(-\sin^3\varphi\cos^2\theta\,d\theta\wedge d\varphi + \sin^3\varphi\sin^2\theta\,d\varphi\wedge d\theta \\ &\qquad + \cos^2\varphi\sin\varphi\cos^2\theta\,d\varphi\wedge d\theta - \cos^2\varphi\sin\varphi\sin^2\theta\,d\theta\wedge d\varphi\big)\\ &= \int_D \sin\varphi\,d\varphi\wedge d\theta = \int_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\,d\theta = 4\pi. \qquad //\end{aligned}$$

It is worth remarking that the hypotheses of Proposition 16.8 can be relaxed somewhat. The requirement that each map F_i be smooth on $\bar{D}_i$ is included to ensure that the boundaries of the image sets W_i have measure zero and that the pullback forms $F_i^*\omega$ are continuous on $\bar{D}_i$. Provided the open subsets W_i together fill up all of M except for a set of measure zero, we can allow maps F_i that do not extend smoothly to the boundary, by interpreting the resulting integrals of unbounded forms either as improper Riemann integrals or as Lebesgue integrals. For example, if the closed upper hemisphere of $\mathbb{S}^2$ is parametrized by the map $F\colon \bar{\mathbb{B}}^2 \to \mathbb{S}^2$ given by $F(u,v) = \big(u, v, \sqrt{1-u^2-v^2}\big)$, then F is continuous but not smooth up to the boundary, but the conclusion of the proposition still holds. We leave it to the interested reader to work out reasonable conditions under which such a generalization of Proposition 16.8 holds.

Integration on Lie Groups

Let G be a Lie group. A covariant tensor field A on G is said to be ***left-invariant*** if $L_g^*A = A$ for all $g\in G$.

Proposition 16.10. *Let G be a compact Lie group endowed with a left-invariant orientation. Then G has a unique positively oriented left-invariant n-form ω_G with the property that $\int_G \omega_G = 1$.*

Proof. If $\dim G = 0$, we just let ω_G be the constant function $1/k$, where k is the cardinality of G. Otherwise, let $E_1,\dots,E_n$ be a left-invariant global frame on G (i.e., a basis for the Lie algebra of G). By replacing E_1 with $-E_1$ if necessary, we may assume that this frame is positively oriented. Let $\varepsilon^1,\dots,\varepsilon^n$ be the dual coframe. Left invariance of E_j implies that

$$\big(L_g^*\varepsilon^i\big)(E_j) = \varepsilon^i(L_{g*}E_j) = \varepsilon^i(E_j) = \delta^i_j,$$

which shows that $L_g^*\varepsilon^i = \varepsilon^i$, so ε^i is left-invariant.

Let $\omega_G = \varepsilon^1\wedge\cdots\wedge\varepsilon^n$. Then

$$L_g^*(\omega_G) = L_g^*\varepsilon^1\wedge\cdots\wedge L_g^*\varepsilon^n = \varepsilon^1\wedge\cdots\wedge\varepsilon^n = \omega_G,$$

so ω_G is left-invariant as well. Because $\omega_G(E_1,\dots,E_n) = 1 > 0$, ω_G is an orientation form for the given orientation. Clearly, any positive constant multiple of ω_G is also a left-invariant orientation form. Conversely, if $\widetilde{\omega}_G$ is any other left-invariant

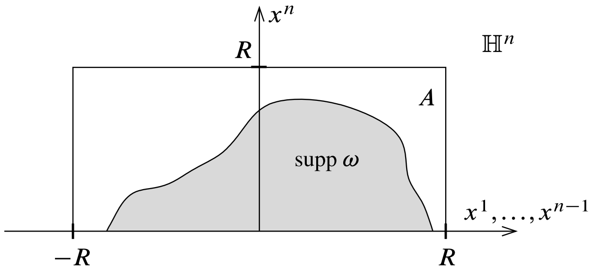

Fig. 16.9 Proof of Stokes's theorem

orientation form, we can write $\widetilde{\omega}_G|_e = c\omega_G|_e$ for some positive number c. Using left-invariance, we find that

$$\widetilde{\omega}_G|_g = L^*_{g^{-1}}\widetilde{\omega}_G|_e = cL^*_{g^{-1}}\omega_G|_e = c\omega_G|_g,$$

which proves that $\widetilde{\omega}_G$ is a positive constant multiple of ω_G.

Since G is compact and oriented, $\int_G \omega_G$ is a positive real number, so we can define $\widetilde{\omega}_G = \left(\int_G \omega_G\right)^{-1}\omega_G$. Clearly, $\widetilde{\omega}_G$ is the unique positively oriented left-invariant orientation form with integral 1. □

Remark. The orientation form whose existence is asserted in this proposition is called the ***Haar volume form on G***. Similarly, the map $f \mapsto \int_G f\,\omega_G$ is called the ***Haar integral***. Observe that the proof above did not use the fact that G was compact until the last paragraph; thus every Lie group has a left-invariant orientation form that is uniquely defined up to a constant multiple. It is only in the compact case, however, that we can use the volume normalization to single out a unique one.

Stokes's Theorem

In this section we state and prove the central result in the theory of integration on manifolds, Stokes's theorem. It is a far-reaching generalization of the fundamental theorem of calculus and of the classical theorems of vector calculus.

Theorem 16.11 (Stokes's Theorem). *Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{16.4}$$

Remark. The statement of this theorem is concise and elegant, but it requires a bit of interpretation. First, as usual, ∂M is understood to have the induced (Stokes) orientation, and the ω on the right-hand side is to be interpreted as $\iota^*_{\partial M}\omega$. If $\partial M = \varnothing$, then the right-hand side is to be interpreted as zero. When M is 1-dimensional, the right-hand integral is really just a finite sum.

With these understandings, we proceed with the proof of the theorem. You should check that it works correctly when $n = 1$ and when $\partial M = \varnothing$.

Proof. We begin with a very special case: suppose M is the upper half-space $\mathbb{H}^n$ itself. Then because ω has compact support, there is a number $R > 0$ such that supp ω is contained in the rectangle $A = [-R, R] \times \cdots \times [-R, R] \times [0, R]$ (Fig. 16.9). We can write ω in standard coordinates as

$$\omega = \sum_{i=1}^{n} \omega_i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

where the hat means that dx^i is omitted. Therefore,

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} d\omega_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} \, dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \, dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Thus we compute

$$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= \sum_{i=1}^{n} (-1)^{i-1} \int_A \frac{\partial \omega_i}{\partial x^i} \, dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) \, dx^1 \cdots dx^n. \end{aligned}$$

We can change the order of integration in each term so as to do the x^i integration first. By the fundamental theorem of calculus, the terms for which $i \neq n$ reduce to

$$\begin{aligned} &\sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) \, dx^1 \cdots dx^n \\ &\quad = \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x) \, dx^i \, dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &\quad = \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \Big[\omega_i(x) \Big]_{x^i=-R}^{x^i=R} dx^1 \cdots \widehat{dx^i} \cdots dx^n = 0, \end{aligned}$$

because we have chosen R large enough that $\omega = 0$ when $x^i = \pm R$. The only term that might not be zero is the one for which $i = n$. For that term we have

$$\begin{aligned}\int_{\mathbb{H}^n} d\omega &= (-1)^{n-1}\int_{-R}^{R}\cdots\int_{-R}^{R}\int_0^R \frac{\partial\omega_n}{\partial x^n}(x)dx^n\,dx^1\cdots dx^{n-1}\\ &= (-1)^{n-1}\int_{-R}^{R}\cdots\int_{-R}^{R}\Big[\omega_n(x)\Big]_{x^n=0}^{x^n=R} dx^1\cdots dx^{n-1}\\ &= (-1)^n\int_{-R}^{R}\cdots\int_{-R}^{R}\omega_n\left(x^1,\dots,x^{n-1},0\right)dx^1\cdots dx^{n-1}, \qquad (16.5)\end{aligned}$$

because $\omega_n = 0$ when $x^n = R$.

To compare this to the other side of (16.4), we compute as follows:

$$\int_{\partial\mathbb{H}^n}\omega = \sum_i\int_{A\cap\partial\mathbb{H}^n}\omega_i\left(x^1,\dots,x^{n-1},0\right)dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n.$$

Because x^n vanishes on $\partial\mathbb{H}^n$, the pullback of dx^n to the boundary is identically zero (see Exercise 11.30). Thus, the only term above that is nonzero is the one for which $i = n$, which becomes

$$\int_{\partial\mathbb{H}^n}\omega = \int_{A\cap\partial\mathbb{H}^n}\omega_n\left(x^1,\dots,x^{n-1},0\right)dx^1\wedge\cdots\wedge dx^{n-1}.$$

Taking into account the fact that the coordinates $\left(x^1,\dots,x^{n-1}\right)$ are positively oriented for $\partial\mathbb{H}^n$ when n is even and negatively oriented when n is odd (Example 15.26), we find that this is equal to (16.5).

Next we consider another special case: $M = \mathbb{R}^n$. In this case, the support of ω is contained in a cube of the form $A = [-R,R]^n$. Exactly the same computation goes through, except that in this case the $i = n$ term vanishes like all the others, so the left-hand side of (16.4) is zero. Since M has empty boundary in this case, the right-hand side is zero as well.

Now let M be an arbitrary smooth manifold with boundary, but consider an $(n-1)$-form ω that is compactly supported in the domain of a single positively or negatively oriented smooth chart (U,φ). Assuming that φ is a positively oriented boundary chart, the definition yields

$$\int_M d\omega = \int_{\mathbb{H}^n}\left(\varphi^{-1}\right)^* d\omega = \int_{\mathbb{H}^n} d\left(\left(\varphi^{-1}\right)^*\omega\right).$$

By the computation above, this is equal to

$$\int_{\partial\mathbb{H}^n}\left(\varphi^{-1}\right)^*\omega, \qquad (16.6)$$

where $\partial\mathbb{H}^n$ is given the induced orientation. Since $d\varphi$ takes outward-pointing vectors on ∂M to outward-pointing vectors on $\mathbb{H}^n$ (by Proposition 5.41), it follows that $\varphi|_{U\cap\partial M}$ is an orientation-preserving diffeomorphism onto $\varphi(U)\cap\partial\mathbb{H}^n$, and thus (16.6) is equal to $\int_{\partial M} \omega$. For a negatively oriented smooth boundary chart, the same argument applies with an additional negative sign on each side of the equation. For an interior chart, we get the same computations with $\mathbb{H}^n$ replaced by $\mathbb{R}^n$. This proves the theorem in this case.

Finally, let ω be an arbitrary compactly supported smooth $(n-1)$-form. Choosing a cover of supp ω by finitely many domains of positively or negatively oriented smooth charts $\{U_i\}$, and choosing a subordinate smooth partition of unity $\{\psi_i\}$, we can apply the preceding argument to $\psi_i \omega$ for each i and obtain

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \psi_i \omega = \sum_i \int_M d(\psi_i \omega) = \sum_i \int_M d\psi_i \wedge \omega + \psi_i \, d\omega$$
$$= \int_M d\Big(\sum_i \psi_i\Big) \wedge \omega + \int_M \Big(\sum_i \psi_i\Big) d\omega = 0 + \int_M d\omega,$$

because $\sum_i \psi_i \equiv 1$. □

Example 16.12. Let M be a smooth manifold and suppose $\gamma \colon [a,b] \to M$ is a smooth embedding, so that $S = \gamma([a,b])$ is an embedded 1-submanifold with boundary in M. If we give S the orientation such that γ is orientation-preserving, then for any smooth function $f \in C^\infty(M)$, Stokes's theorem says that

$$\int_\gamma df = \int_{[a,b]} \gamma^* df = \int_S df = \int_{\partial S} f = f\big(\gamma(b)\big) - f\big(\gamma(a)\big).$$

Thus Stokes's theorem reduces to the fundamental theorem for line integrals (Theorem 11.39) in this case. In particular, when $\gamma \colon [a,b] \to \mathbb{R}$ is the inclusion map, then Stokes's theorem is just the ordinary fundamental theorem of calculus. //

Two special cases of Stokes's theorem arise so frequently that they are worthy of special note. The proofs are immediate.

Corollary 16.13 (Integrals of Exact Forms). *If M is a compact oriented smooth manifold without boundary, then the integral of every exact form over M is zero*:

$$\int_M d\omega = 0 \quad \text{if } \partial M = \varnothing. \qquad \square$$

Corollary 16.14 (Integrals of Closed Forms over Boundaries). *Suppose M is a compact oriented smooth manifold with boundary. If ω is a closed form on M, then the integral of ω over ∂M is zero*:

$$\int_{\partial M} \omega = 0 \quad \text{if } d\omega = 0 \text{ on } M. \qquad \square$$

These results have the following extremely useful applications to submanifolds.

Corollary 16.15. *Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an oriented compact smooth k-dimensional submanifold* (*without boundary*), *and ω is a closed k-form on M. If $\int_S \omega \neq 0$, then both of the following are true*:

(a) *ω is not exact on M.*
(b) *S is not the boundary of an oriented compact smooth submanifold with boundary in M.* □

Example 16.16. It follows from the computation of Example 11.36 that the closed 1-form $\omega = (x\,dy - y\,dx)/\left(x^2 + y^2\right)$ has nonzero integral over $\mathbb{S}^1$. We already observed that ω is not exact on $\mathbb{R}^2 \smallsetminus \{0\}$. The preceding corollary tells us in addition that $\mathbb{S}^1$ is not the boundary of a compact regular domain in $\mathbb{R}^2 \smallsetminus \{0\}$. //

The following classical result is an easy application of Stokes's theorem.

Theorem 16.17 (Green's Theorem). *Suppose D is a compact regular domain in $\mathbb{R}^2$, and P, Q are smooth real-valued functions on D. Then*

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\partial D} P\,dx + Q\,dy.$$

Proof. This is just Stokes's theorem applied to the 1-form $P\,dx + Q\,dy$. □

Manifolds with Corners

In many applications of Stokes's theorem it is necessary to deal with geometric objects such as triangles, squares, or cubes that are topological manifolds with boundary, but are not smooth manifolds with boundary because they have "corners." It is easy to generalize Stokes's theorem to this setting, and we do so in this section.

Let $\overline{\mathbb{R}}^n_+$ denote the subset of $\mathbb{R}^n$ where all of the coordinates are nonnegative:

$$\overline{\mathbb{R}}^n_+ = \left\{\left(x^1, \dots, x^n\right) \in \mathbb{R}^n : x^1 \ge 0, \dots, x^n \ge 0\right\}.$$

This space is the model for the type of corners we are concerned with.

► **Exercise 16.18.** Prove that $\overline{\mathbb{R}}^n_+$ is homeomorphic to the upper half-space $\mathbb{H}^n$.

Suppose M is a topological n-manifold with boundary. A ***chart with corners*** for M is a pair (U, φ), where $U \subseteq M$ is open and φ is a homeomorphism from U to a (relatively) open subset $\widehat{U} \subseteq \overline{\mathbb{R}}^n_+$ (Fig. 16.10). Two charts with corners (U, φ), (V, ψ) are smoothly compatible if the composite map $\varphi \circ \psi^{-1} \colon \psi(U \cap V) \to \varphi(U \cap V)$ is smooth. (As usual, this means that it admits a smooth extension in an open neighborhood of each point.)

A ***smooth structure with corners*** on a topological manifold with boundary is a maximal collection of smoothly compatible interior charts and charts with corners whose domains cover M. A topological manifold with boundary together with a smooth structure with corners is called a ***smooth manifold with corners***. Any chart with corners in the given smooth structure with corners is called a ***smooth chart with corners*** for M.

Example 16.19. Any closed rectangle in $\mathbb{R}^n$ is a smooth n-manifold with corners. //

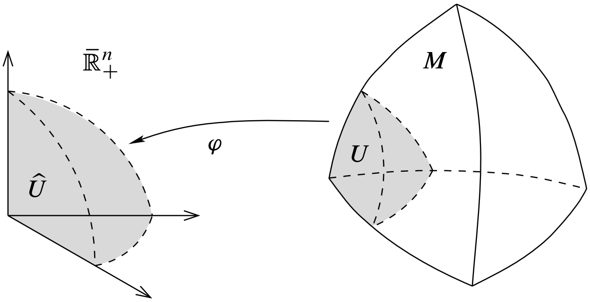

Fig. 16.10 A chart with corners

Because of the result of Exercise 16.18, charts with corners are topologically indistinguishable from boundary charts. Thus, from the topological point of view there is no difference between manifolds with boundary and manifolds with corners. The difference is in the smooth structure, because in dimensions greater than 1, the compatibility condition for charts with corners is different from that for boundary charts. In the case $n = 1$, though, $\bar{\mathbb{R}}^1_+$ is actually equal to $\mathbb{H}^1$, so smooth 1-manifolds with corners are no different from smooth manifolds with boundary.

The boundary of $\bar{\mathbb{R}}^n_+$ in $\mathbb{R}^n$ is the set of points at which at least one coordinate vanishes. The points in $\bar{\mathbb{R}}^n_+$ at which more than one coordinate vanishes are called its ***corner points***. For example, the corner points of $\bar{\mathbb{R}}^3_+$ are the origin together with all the points on the positive x-, y-, and z-axes.

Proposition 16.20 (Invariance of Corner Points). *Let M be a smooth n-manifold with corners, $n \geq 2$, and let $p \in M$. If $\varphi(p)$ is a corner point for some smooth chart with corners (U, φ), then the same is true for every such chart whose domain contains p.*

Proof. Suppose (U, φ) and (V, ψ) are two smooth charts with corners such that $\varphi(p)$ is a corner point but $\psi(p)$ is not (Fig. 16.11). To simplify notation, let us assume without loss of generality that $\varphi(p)$ has coordinates $\left(x^1, \dots, x^k, 0, \dots, 0\right)$ with $k \leq n-2$. Then $\psi(V)$ contains an open subset of some $(n-1)$-dimensional linear subspace $S \subseteq \mathbb{R}^n$, with $\psi(p) \in S$. (If $\psi(p) \in \partial\bar{\mathbb{R}}^n_+$, take S to be the unique subspace defined by an equation of the form $x^i = 0$ that contains $\psi(p)$. If $\psi(p)$ is an interior point, any $(n-1)$-dimensional subspace containing $\psi(p)$ will do.)

Let $S' = S \cap \psi(U \cap V)$, and let $\alpha\colon S' \to \mathbb{R}^n$ be the restriction of $\varphi \circ \psi^{-1}$ to S'. Because $\varphi \circ \psi^{-1}$ is a diffeomorphism from $\psi(U \cap V)$ to $\varphi(U \cap V)$, it follows that $\psi \circ \varphi^{-1} \circ \alpha$ is the identity of S', and therefore $d\alpha_{\psi(p)}$ is an injective linear map. Let $T = d\alpha_{\psi(p)}(T_{\psi(p)}S) \subseteq \mathbb{R}^n$. Because T is $(n-1)$-dimensional, it must contain a vector v such that one of the last two components, v^{n-1} or v^n, is nonzero (otherwise, T would be contained in a codimension-2 subspace). Renumbering the coordinates and replacing v by $-v$ if necessary, we may assume that $v^n < 0$.

Now let $\gamma\colon (-\varepsilon, \varepsilon) \to S$ be a smooth curve such that $\gamma(0) = p$ and $d\alpha\left(\gamma'(0)\right) = v$. Then $\alpha \circ \gamma(t)$ has negative x^n coordinate for small $t > 0$, which contradicts the fact that α takes its values in $\bar{\mathbb{R}}^n_+$. □

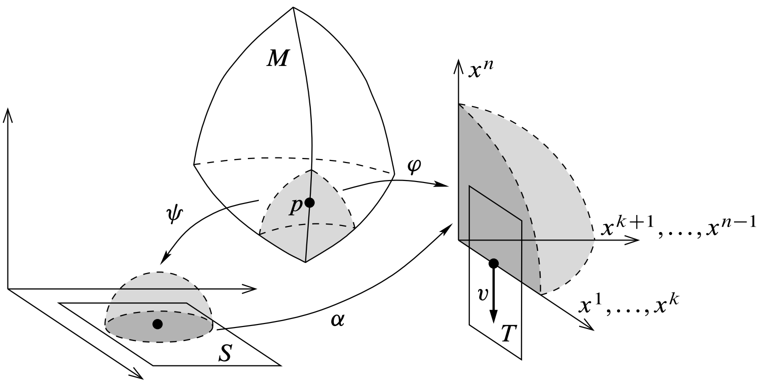

Fig. 16.11 Invariance of corner points

If M is a smooth manifold with corners, a point $p \in M$ is called a ***corner point*** if $\varphi(p)$ is a corner point in $\overline{\mathbb{R}}^n_+$ with respect to some (and hence every) smooth chart with corners (U, φ). Similarly, p is called a ***boundary point*** if $\varphi(p) \in \partial\overline{\mathbb{R}}^n_+$ with respect to some (hence every) such chart. For example, the set of corner points of the unit cube $[0,1]^3 \subseteq \mathbb{R}^3$ is the union of its eight vertices and twelve edges.

Every smooth manifold with or without boundary is also a smooth manifold with corners (but with no corner points). Conversely, a smooth manifold with corners is a smooth manifold with boundary if and only if it has no corner points. The boundary of a smooth manifold with corners, however, is in general not a smooth manifold with corners (e.g., think of the boundary of a cube). In fact, even the boundary of $\overline{\mathbb{R}}^n_+$ itself is not a smooth manifold with corners. It is, however, a union of finitely many such: $\partial\overline{\mathbb{R}}^n_+ = H_1 \cup \cdots \cup H_n$, where

$$H_i = \left\{ \left(x^1, \dots, x^n\right) \in \overline{\mathbb{R}}^n_+ : x^i = 0 \right\} \tag{16.7}$$

is an $(n-1)$-dimensional smooth manifold with corners contained in the subspace defined by $x^i = 0$.

The usual flora and fauna of smooth manifolds—smooth maps, partitions of unity, tangent vectors, covectors, tensors, differential forms, orientations, and integrals of differential forms—can be defined on smooth manifolds with corners in exactly the same way as we have done for smooth manifolds and smooth manifolds with boundary, using smooth charts with corners in place of smooth boundary charts. The details are left to the reader.

In addition, for Stokes's theorem we need to integrate a differential form over the boundary of a smooth manifold with corners. Since the boundary is not itself a smooth manifold with corners, this requires a separate (albeit routine) definition. Let M be an oriented smooth n-manifold with corners, and suppose ω is an $(n-1)$-form on ∂M that is compactly supported in the domain of a single oriented smooth

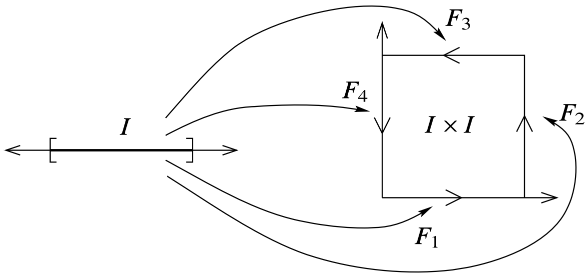

Fig. 16.12 Parametrizing the boundary of the square

chart with corners (U, φ). We define the integral of ω over ∂M by

$$\int_{\partial M} \omega = \sum_{i=1}^{n} \int_{H_i} \left(\varphi^{-1}\right)^* \omega,$$

where H_i, defined by (16.7), is given the induced orientation as part of the boundary of the set where $x^i \geq 0$. In other words, we simply integrate ω in coordinates over the codimension-1 portion of the boundary. Finally, if ω is an arbitrary compactly supported $(n-1)$-form on M, we define the integral of ω over ∂M by piecing together with a partition of unity just as in the case of a manifold with boundary.

In practice, of course, one does not evaluate such integrals by using partitions of unity. Instead, one "chops up" the boundary into pieces that can be parametrized by domains of integration, just as for ordinary manifolds with or without boundary. The following proposition is an analogue of Proposition 16.8.

Proposition 16.21. *The statement of Proposition 16.8 is true if M is replaced by the boundary of a compact, oriented, smooth n-manifold with corners.*

► **Exercise 16.22.** Show how the proof of Proposition 16.8 needs to be adapted to prove Proposition 16.21.

Example 16.23. Let $I \times I = [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$, and suppose ω is a 1-form on $\partial(I \times I)$. Then it is not hard to check that the maps $F_i \colon I \to I \times I$ given by

$$\begin{aligned} F_1(t) &= (t, 0), \qquad & F_2(t) &= (1, t), \\ F_3(t) &= (1-t, 1), \qquad & F_4(t) &= (0, 1-t), \end{aligned} \tag{16.8}$$

satisfy the hypotheses of Proposition 16.21. (These four curve segments in sequence traverse the boundary of $I \times I$ in the counterclockwise direction; see Fig. 16.12.) Therefore,

$$\int_{\partial(I \times I)} \omega = \int_{F_1} \omega + \int_{F_2} \omega + \int_{F_3} \omega + \int_{F_4} \omega. \qquad //$$

► **Exercise 16.24.** Verify the claims of the preceding example.

The next theorem is the main result of this section.

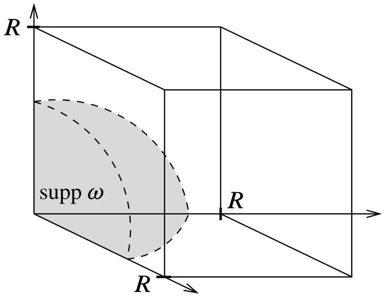

Fig. 16.13 Stokes's theorem for manifolds with corners

Theorem 16.25 (Stokes's Theorem on Manifolds with Corners). *Let M be an oriented smooth n-manifold with corners, and let ω be a compactly supported smooth $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. The proof is nearly identical to the proof of Stokes's theorem proper, so we just indicate where changes need to be made. By means of smooth charts and a partition of unity, we may reduce the theorem to the case in which either $M = \mathbb{R}^n$ or $M = \overline{\mathbb{R}}^n_+$. The $\mathbb{R}^n$ case yields zero on both sides of the equation, just as before. In the case of a chart with corners, ω is supported in some cube $[0, R]^n$ (Fig. 16.13), and we calculate exactly as in the proof of Theorem 16.11:

$$\begin{aligned}
\int_{\overline{\mathbb{R}}^n_+} d\omega &= \sum_{i=1}^n (-1)^{i-1} \int_0^R \cdots \int_0^R \frac{\partial \omega_i}{\partial x^i}(x)\, dx^1 \cdots dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \int_0^R \cdots \int_0^R \frac{\partial \omega_i}{\partial x^i}(x)\, dx^i\, dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \int_0^R \cdots \int_0^R \left[\omega_i(x)\right]_{x^i=0}^{x^i=R} dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= \sum_{i=1}^n (-1)^{i} \int_0^R \cdots \int_0^R \omega_i\left(x^1, \dots, 0, \dots, x^n\right) dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= \sum_{i=1}^n \int_{H_i} \omega = \int_{\partial \overline{\mathbb{R}}^n_+} \omega.
\end{aligned}$$

(The factor $(-1)^i$ disappeared because the induced orientation on H_i is $(-1)^i$ times that of the standard coordinates $\left(x^1, \dots, \widehat{x^i}, \dots, x^n\right)$.) This completes the proof. □

The preceding theorem has the following important application.

Theorem 16.26. *Suppose M is a smooth manifold and $\gamma_0, \gamma_1 \colon [a,b] \to M$ are path-homotopic piecewise smooth curve segments. For every closed 1-form ω on M,*

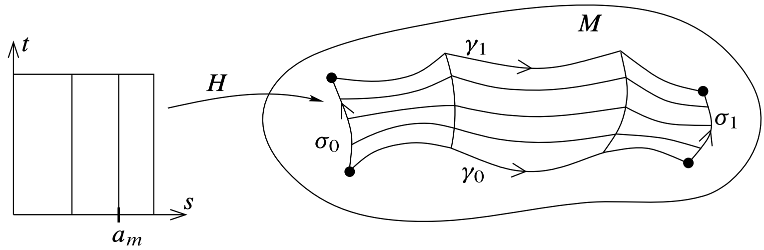

Fig. 16.14 Homotopic piecewise smooth curve segments

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof. By means of an affine reparametrization, we may as well assume for simplicity that $[a,b] = [0,1]$. Assume first that γ_0 and γ_1 are smooth. By Theorem 6.29, γ_0 and γ_1 are smoothly homotopic relative to $\{0,1\}$. Let $H\colon I \times I \to M$ be such a smooth homotopy. Since ω is closed, we have

$$\int_{I\times I} d(H^*\omega) = \int_{I\times I} H^*\, d\omega = 0.$$

On the other hand, $I \times I$ is a smooth manifold with corners, so Stokes's theorem implies

$$0 = \int_{I\times I} d(H^*\omega) = \int_{\partial(I\times I)} H^*\omega.$$

Using the parametrization of $\partial(I \times I)$ given in Example 16.23 together with Proposition 11.34(d), we obtain

$$\begin{aligned} 0 = \int_{\partial(I\times I)} H^*\omega &= \int_{F_1} H^*\omega + \int_{F_2} H^*\omega + \int_{F_3} H^*\omega + \int_{F_4} H^*\omega \\ &= \int_{H\circ F_1} \omega + \int_{H\circ F_2} \omega + \int_{H\circ F_3} \omega + \int_{H\circ F_4} \omega, \end{aligned}$$

where F_1, F_2, F_3, F_4 are defined by (16.8). The fact that H is a homotopy relative to $\{0,1\}$ means that $H \circ F_2$ and $H \circ F_4$ are constant maps, and therefore the second and fourth terms above are zero. The theorem then follows from the facts that $H \circ F_1 = \gamma_0$ and $H \circ F_3$ is a backward reparametrization of γ_1.

Next we consider the general case of piecewise smooth curves. We cannot simply apply the preceding result on each subinterval where γ_0 and γ_1 are smooth, because the restricted curves may not start and end at the same points. Instead, we prove the following more general claim: *Let $\gamma_0, \gamma_1\colon I \to M$ be piecewise smooth curve segments* (*not necessarily with the same endpoints*)*, and suppose $H\colon I \times I \to M$ is any homotopy between them* (*Fig.* 16.14)*. Define curve segments $\sigma_0, \sigma_1\colon I \to M$ by*

$$\sigma_0(t) = H(0,t), \qquad \sigma_1(t) = H(1,t),$$

and let $\widetilde{\sigma}_0, \widetilde{\sigma}_1$ be any smooth curve segments that are path-homotopic to σ_0, σ_1 respectively. Then

$$\int_{\gamma_1} \omega - \int_{\gamma_0} \omega = \int_{\widetilde{\sigma}_1} \omega - \int_{\widetilde{\sigma}_0} \omega. \tag{16.9}$$

When specialized to the case in which γ_0 and γ_1 are path-homotopic, this implies the theorem, because σ_0 and σ_1 are constant maps in that case.

Since γ_0 and γ_1 are piecewise smooth, there are only finitely many points $\{a_1, \dots, a_m\}$ in $(0,1)$ at which either γ_0 or γ_1 is not smooth. We prove the claim by induction on the number m of such points. When $m = 0$, both curves are smooth, and by Theorem 6.29 we may replace the given homotopy H by a smooth homotopy $\widetilde{H}$. Recall from the proof of Theorem 6.29 that the smooth homotopy $\widetilde{H}$ can actually be taken to be homotopic to H relative to $I \times \{0\} \cup I \times \{1\}$. Thus, for $i = 0, 1$, the curve $\widetilde{\sigma}_i(t) = \widetilde{H}(i,t)$ is a smooth curve segment that is path-homotopic to σ_i. In this setting, (16.9) just reduces to the integration formula of Example 16.23. Note that the integrals over $\widetilde{\sigma}_0$ and $\widetilde{\sigma}_1$ do not depend on which smooth curves path-homotopic to σ_0 and σ_1 are chosen, by the smooth case proved above.

Now let γ_0, γ_1 be homotopic piecewise smooth curves with m nonsmooth points $\{a_1, \dots, a_m\}$, and suppose the claim is true for curves with fewer than m such points. For $i = 0, 1$, let γ_i' be the restriction of γ_i to $[0, a_m]$, and let γ_i'' be its restriction to $[a_m, 1]$. Let $\sigma\colon I \to M$ be the curve segment $\sigma(t) = H(a_m, t)$, and let $\widetilde{\sigma}$ by any smooth curve segment that is path-homotopic to σ. Then, since γ_i' and γ_i'' have fewer than m nonsmooth points, the inductive hypothesis implies

$$\begin{aligned}
\int_{\gamma_1} \omega - \int_{\gamma_0} \omega &= \left(\int_{\gamma_1'} \omega - \int_{\gamma_0'} \omega\right) + \left(\int_{\gamma_1''} \omega - \int_{\gamma_0''} \omega\right) \\
&= \left(\int_{\widetilde{\sigma}} \omega - \int_{\widetilde{\sigma}_0} \omega\right) + \left(\int_{\widetilde{\sigma}_1} \omega - \int_{\widetilde{\sigma}} \omega\right) \\
&= \int_{\widetilde{\sigma}_1} \omega - \int_{\widetilde{\sigma}_0} \omega.
\end{aligned}$$

□

Corollary 16.27. *On a simply connected smooth manifold, every closed* 1*-form is exact.*

Proof. Suppose M is simply connected and ω is a closed 1-form on M. Since every piecewise smooth closed curve segment in M is path-homotopic to a constant curve, the preceding theorem shows that the integral of ω over every such curve is equal to 0. Thus, ω is conservative and therefore exact. □

Integration on Riemannian Manifolds

In this section we explore what happens when the theory of integration and Stokes's theorem are specialized to Riemannian manifolds.

Integration of Functions on Riemannian Manifolds

We noted at the beginning of the chapter that real-valued functions cannot be integrated in a coordinate-independent way on an arbitrary manifold. However, with the additional structures of a Riemannian metric and an orientation, we can recover the notion of the integral of a real-valued function.

Suppose (M, g) is an oriented Riemannian manifold with or without boundary, and let ω_g denote its Riemannian volume form. If f is a compactly supported continuous real-valued function on M, then $f\omega_g$ is a compactly supported n-form, so we can define the ***integral of f over M*** to be $\int_M f\omega_g$. If M itself is compact, we define the ***volume of M*** by $\mathrm{Vol}(M) = \int_M \omega_g$.

Because of these definitions, the Riemannian volume form is often denoted by dV_g (or dA_g or ds_g in the 2-dimensional or 1-dimensional case, respectively). Then the integral of f over M is written $\int_M f\, dV_g$, and the volume of M as $\int_M dV_g$. Be warned, however, that this notation is *not* meant to imply that the volume form is the exterior derivative of an $(n-1)$-form; in fact, as we will see when we study de Rham cohomology, this is never the case on a compact manifold. You should just interpret dV_g as a notational convenience.

Proposition 16.28. *Let (M, g) be a nonempty oriented Riemannian manifold with or without boundary, and suppose f is a compactly supported continuous real-valued function on M satisfying $f \geq 0$. Then $\int_M f\, dV_g \geq 0$, with equality if and only if $f \equiv 0$.*

Proof. If f is supported in the domain of a single oriented smooth chart (U, φ), then Proposition 15.31 shows that

$$\int_M f\, dV_g = \int_{\varphi(U)} f(x)\sqrt{\det(g_{ij})}\, dx^1 \cdots dx^n \geq 0.$$

The same inequality holds in a negatively oriented chart because the negative sign from the chart cancels the negative sign in the expression for dV_g. The general case follows from this one, because $\int_M f\, dV_g$ is equal to a sum of terms like $\int_M \psi_i f\, dV_g$, where each integrand $\psi_i f$ is nonnegative and supported in a single smooth chart. If in addition f is positive somewhere, then it is positive on a nonempty open subset by continuity, so at least one of the integrals in this sum is positive. On the other hand, if f is identically zero, then clearly $\int_M f\, dV_g = 0$. □

► **Exercise 16.29.** Suppose (M, g) is an oriented Riemannian manifold and $f\colon M \to \mathbb{R}$ is continuous and compactly supported. Prove that $\left|\int_M f\, dV_g\right| \leq \int_M |f|\, dV_g$.

The Divergence Theorem

Let (M, g) be an oriented Riemannian n-manifold (with or without boundary). We can generalize the classical divergence operator to this setting as follows. Multiplication by the Riemannian volume form defines a smooth bundle isomorphism

$*\colon C^\infty(M) \to \Omega^n(M)$:

$$* f = f\, dV_g. \tag{16.10}$$

In addition, as we did in Chapter 14 in the case of $\mathbb{R}^3$, we define a smooth bundle isomorphism $\beta\colon \mathfrak{X}(M) \to \Omega^{n-1}(M)$ as follows:

$$\beta(X) = X \lrcorner\, dV_g. \tag{16.11}$$

We need the following technical lemma.

Lemma 16.30. *Let (M,g) be an oriented Riemannian manifold with or without boundary. Suppose $S \subseteq M$ is an immersed hypersurface with the orientation determined by a unit normal vector field N, and $\tilde{g}$ is the induced metric on S. If X is any vector field along S, then*

$$\iota_S^*\big(\beta(X)\big) = \langle X, N\rangle_g\, dV_{\tilde{g}}. \tag{16.12}$$

Proof. Define two vector fields $X^\top$ and $X^\perp$ along S by

$$X^\perp = \langle X, N\rangle_g N,$$
$$X^\top = X - X^\perp.$$

Then $X = X^\perp + X^\top$, where $X^\perp$ is normal to S and $X^\top$ is tangent to it. Using this decomposition,

$$\beta(X) = X^\perp \lrcorner\, dV_g + X^\top \lrcorner\, dV_g.$$

Now pull back to S. Proposition 15.32 shows that the first term simplifies to

$$\iota_S^*\big(X^\perp \lrcorner\, dV_g\big) = \langle X, N\rangle_g \iota_S^*(N \lrcorner\, dV_g) = \langle X, N\rangle_g dV_{\tilde{g}}.$$

Thus (16.12) will be proved if we can show that $\iota_S^*\big(X^\top \lrcorner\, dV_g\big) = 0$. If $X_1, \dots, X_{n-1}$ are any vectors tangent to S, then

$$\big(X^\top \lrcorner\, dV_g\big)(X_1, \dots, X_{n-1}) = dV_g\big(X^\top, X_1, \dots, X_{n-1}\big) = 0,$$

because any n-tuple of vectors in an $(n-1)$-dimensional vector space is linearly dependent. □

Define the ***divergence operator*** $\operatorname{div}\colon \mathfrak{X}(M) \to C^\infty(M)$ by

$$\operatorname{div} X = *^{-1} d\big(\beta(X)\big),$$

or equivalently,

$$d(X \lrcorner\, dV_g) = (\operatorname{div} X) dV_g.$$

► **Exercise 16.31.** Show that divergence operator on an oriented Riemannian manifold does not depend on the choice of orientation, and conclude that it is invariantly defined on all Riemannian manifolds.

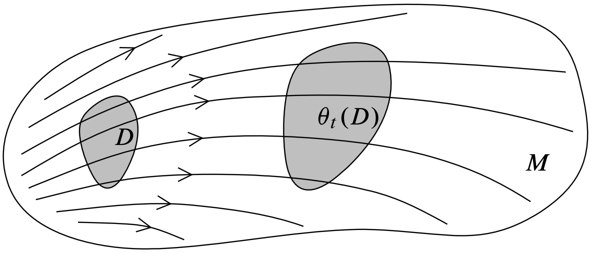

Fig. 16.15 Geometric interpretation of the divergence

The next theorem is a fundamental result about vector fields on Riemannian manifolds. In the special case of a compact regular domain in $\mathbb{R}^3$, it is often referred to as ***Gauss's theorem***. (Later in the chapter, we will show that this theorem holds on nonorientable manifolds as well; see Theorem 16.48.)

Theorem 16.32 (The Divergence Theorem). *Let (M, g) be an oriented Riemannian manifold with boundary. For any compactly supported smooth vector field X on M,*

$$\int_M (\operatorname{div} X)\, dV_g = \int_{\partial M} \langle X, N\rangle_g \, dV_{\widetilde{g}},$$

where N is the outward-pointing unit normal vector field along ∂M and $\widetilde{g}$ is the induced Riemannian metric on ∂M.

Proof. By Stokes's theorem,

$$\int_M (\operatorname{div} X)\, dV_g = \int_M d\big(\beta(X)\big) = \int_{\partial M} \iota_S^* \beta(X).$$

The divergence theorem then follows from Lemma 16.30. □

The term "divergence" is used because of the following geometric interpretation. A smooth flow θ on M is said to be ***volume-preserving*** if for every compact regular domain D, we have $\operatorname{Vol}\big(\theta_t(D)\big) = \operatorname{Vol}(D)$ whenever the domain of θ_t contains D. It is called ***volume-increasing***, ***volume-decreasing***, ***volume-nonincreasing***, or ***volume-nondecreasing*** if for every such D, $\operatorname{Vol}\big(\theta_t(D)\big)$ is strictly increasing, strictly decreasing, nonincreasing, or nondecreasing, respectively, as a function of t. Note that the properties of flow domains ensure that if D is contained in the domain of θ_t for some t, then the same is true for all times between 0 and t.

The next proposition shows that the divergence of a vector field can be interpreted as a measure of the tendency of its flow to "spread out," or diverge (see Fig. 16.15).

Proposition 16.33 (Geometric Interpretation of the Divergence). *Let M be an oriented Riemannian manifold, let $X \in \mathfrak{X}(M)$, and let θ be the flow of X. Then θ is*

(a) *volume-preserving if and only if* $\operatorname{div} X = 0$ *everywhere on M.*
(b) *volume-nondecreasing if and only if* $\operatorname{div} X \geq 0$ *everywhere on M.*

(c) *volume-nonincreasing if and only if* $\operatorname{div} X \leq 0$ *everywhere on* M.
(d) *volume-increasing if and only if* $\operatorname{div} X > 0$ *on a dense subset of* M.
(e) *volume-decreasing if and only if* $\operatorname{div} X < 0$ *on a dense subset of* M.

Proof. First we establish some preliminary results. For each $t \in \mathbb{R}$, let M_t be the domain of θ_t. If D is a compact regular domain contained in M_t, then θ_t is an orientation-preserving diffeomorphism from D to $\theta_t(D)$ by the result of Problem 15-4, so

$$\operatorname{Vol}\big(\theta_t(D)\big) = \int_{\theta_t(D)} dV_g = \int_D \theta_t^* \, dV_g.$$

Because the integrand on the right depends smoothly on (t, p) in the domain of θ, we can differentiate this expression with respect to t by differentiating under the integral sign. (Strictly speaking, we should use a partition of unity to express the integral as a sum of integrals over domains in $\mathbb{R}^n$, and then differentiate under the integral signs there; but the result is the same. The details are left to you.)

Using Cartan's magic formula for the Lie derivative of the Riemannian volume form, we obtain

$$\mathcal{L}_X dV_g = X \lrcorner\, d(dV_g) + d(X \lrcorner\, dV_g) = (\operatorname{div} X)\, dV_g,$$

because $d(dV_g)$ is an $(n+1)$-form on an n-manifold. Then Proposition 12.36 implies

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=t_0} \operatorname{Vol}\big(\theta_t(D)\big) &= \int_D \left.\frac{\partial}{\partial t}\right|_{t=t_0} (\theta_t^* \, dV_g) = \int_D \theta_{t_0}^*(\mathcal{L}_X dV_g) \\ &= \int_D \theta_{t_0}^*\big((\operatorname{div} X)\, dV_g\big) = \int_{\theta_{t_0}(D)} (\operatorname{div} X)\, dV_g. \end{aligned} \tag{16.13}$$

Now we can prove the "if" parts of all five equivalences. If $\operatorname{div} X \equiv 0$, then it follows from (16.13) that $\operatorname{Vol}\big(\theta_t(D)\big)$ is a constant function of t for every D, and thus θ is volume-preserving. Similarly, an inequality of the form $\operatorname{div} X \geq 0$ or $\operatorname{div} X \leq 0$ implies that $\operatorname{Vol}\big(\theta_t(D)\big)$ is nondecreasing or nonincreasing, respectively. For part (d), suppose that $\operatorname{div} X > 0$ on a dense subset of M, and let D be a compact regular domain in M. Then $\operatorname{div} X \geq 0$ everywhere by continuity, so $\operatorname{Vol}\big(\theta_t(D)\big)$ is nondecreasing by the argument above. Because $\operatorname{Int} D$ is an open subset of M (by Proposition 5.1), $\operatorname{Int} \theta_t(D)$ is open for each t such that $D \subseteq M_t$, and therefore by density there is a point in $\operatorname{Int} \theta_t(D)$ where $\operatorname{div} X > 0$. Proposition 16.28 then shows that $\int_{\theta_t(D)} (\operatorname{div} X) dV_g > 0$, and thus $\operatorname{Vol}\big(\theta_t(D)\big)$ is strictly increasing by (16.13). A similar argument proves (e).

To prove the converses, we prove their contrapositives. We begin with (b). If there is a point where $\operatorname{div} X < 0$, then by continuity there is an open subset $U \subseteq M$ on which $\operatorname{div} X < 0$. The argument in the first part of the proof shows that X generates a volume-decreasing flow on U. In particular, for any regular coordinate ball B such that $\overline{B} \subseteq U$ and any $t > 0$ small enough to ensure that $\theta_t\big(\overline{B}\big) \subseteq U$, we have $\operatorname{Vol}\big(\theta_t\big(\overline{B}\big)\big) < \operatorname{Vol}\big(\overline{B}\big)$, which implies that θ is not volume-nondecreasing. The same argument with inequalities reversed proves (c). If $\operatorname{div} X$ is not identically zero, then there is an open subset on which it is either strictly positive or strictly negative, and then the argument above shows that it is not volume-preserving on that set, thus proving (a).

Next, consider (d). If the subset of M where $\operatorname{div} X > 0$ is not dense, there is an open subset $U \subseteq M$ on which $\operatorname{div} X \le 0$. Then (c) shows that θ is volume-nonincreasing on U, so it cannot be volume-increasing on M. The argument for (e) is similar. □

Surface Integrals

The original theorem that bears the name of Stokes concerned "surface integrals" of vector fields over surfaces in $\mathbb{R}^3$. Using the version of Stokes's theorem that we have proved, we cam generalize this to surfaces in Riemannian 3-manifolds.

Let (M, g) be an oriented Riemannian 3-manifold. Define the ***curl operator***, denoted by $\operatorname{curl}\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$, by

$$\operatorname{curl} X = \beta^{-1} d\left(X^\flat\right),$$

where $\beta\colon \mathfrak{X}(M) \to \Omega^2(M)$ is defined in (16.11). Unwinding the definitions, we see that this is equivalent to

$$(\operatorname{curl} X) \mathbin{\lrcorner} dV_g = d\left(X^\flat\right). \tag{16.14}$$

The operators div, grad, and curl on an oriented Riemannian 3-manifold M are related by the following commutative diagram analogous to (14.27):

$$\begin{array}{ccccccc} C^\infty(M) & \xrightarrow{\operatorname{grad}} & \mathfrak{X}(M) & \xrightarrow{\operatorname{curl}} & \mathfrak{X}(M) & \xrightarrow{\operatorname{div}} & C^\infty(M) \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \flat} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle *} \\ \Omega^0(M) & \xrightarrow[d]{} & \Omega^1(M) & \xrightarrow[d]{} & \Omega^2(M) & \xrightarrow[d]{} & \Omega^3(M). \end{array} \tag{16.15}$$

The identities $\operatorname{curl} \circ \operatorname{grad} \equiv 0$ and $\operatorname{div} \circ \operatorname{curl} \equiv 0$ follow from $d \circ d \equiv 0$ just as they do in the Euclidean case. The curl operator is defined only in dimension 3 because it is only in that case that $\Lambda^2 T^*M$ is isomorphic to TM (via the map $\beta\colon X \mapsto X \mathbin{\lrcorner} dV_g$).

Now suppose $S \subseteq M$ is a compact 2-dimensional submanifold with or without boundary, and N is a smooth unit normal vector field along S. Let dA denote the Riemannian volume form on S with respect to the induced metric $\iota_S^* g$ and the orientation determined by N, so that $dA = \iota_S^*(N \mathbin{\lrcorner} dV_g)$ by Proposition 15.32. (See Fig. 16.16.) For any smooth vector field X defined on M, the ***surface integral of X over S*** (with respect to the given choice of unit normal field) is defined as

$$\int_S \langle X, N \rangle_g \, dA.$$

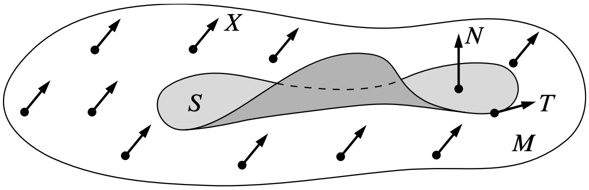

Fig. 16.16 The setup for a surface integral

The next result, in the special case in which $M = \mathbb{R}^3$, is the theorem usually referred to as *Stokes's theorem* in multivariable calculus texts.

Theorem 16.34 (Stokes's Theorem for Surface Integrals). *Suppose M is an oriented Riemannian 3-manifold with or without boundary, and S is a compact oriented 2-dimensional smooth submanifold with boundary in M. For any smooth vector field X on M,*

$$\int_S \langle \operatorname{curl} X, N \rangle_g \, dA = \int_{\partial S} \langle X, T \rangle_g \, ds,$$

where N is the smooth unit normal vector field along S that determines its orientation, ds is the Riemannian volume form for ∂S (with respect to the metric and orientation induced from S), and T is the unique positively oriented unit tangent vector field on ∂S.

Proof. The general version of Stokes's theorem applied to the 1-form $X^\flat$ yields

$$\int_S d\left(X^\flat\right) = \int_{\partial S} X^\flat.$$

Thus the theorem follows from the following two identities:

$$\iota_S^* d\left(X^\flat\right) = \langle \operatorname{curl} X, N \rangle_g \, dA, \tag{16.16}$$

$$\iota_{\partial S}^* X^\flat = \langle X, T \rangle_g \, ds. \tag{16.17}$$

Equation (16.16) is just the defining equation (16.14) for the curl combined with the result of Lemma 16.30. To prove (16.17), we note that $\iota_{\partial S}^* X^\flat$ is a smooth 1-form on a 1-manifold, and thus must be equal to $f\,ds$ for some smooth function f on ∂S. To evaluate f, we note that $ds(T) = 1$, and so the definition of $X^\flat$ yields

$$f = f\,ds(T) = X^\flat(T) = \langle X, T \rangle_g.$$

This proves (16.17) and thus the theorem. □

Densities

Although differential forms are natural objects to integrate on manifolds, and are essential for use in Stokes's theorem, they have the disadvantage of requiring oriented manifolds in order for their integrals to be defined. There is a way to define integration on nonorientable manifolds as well, which we describe in this section.

In the theory of integration of differential forms, the crucial place where orientations entered the picture was in our proof of the diffeomorphism-invariance of the integral (Proposition 16.1), because the transformation law for an n-form on an n-manifold under a change of coordinates involves the Jacobian determinant of the transition map, while the transformation law for integrals involves the absolute value of the determinant. We had to restrict attention to orientation-preserving diffeomorphisms so that we could freely remove the absolute value signs. In this section we define objects whose transformation law involves the absolute value of the determinant, so that we no longer have this sign problem.

We begin, as always, in the linear-algebraic setting. Let V be an n-dimensional vector space. A ***density on V*** is a function

$$\mu\colon \underbrace{V\times\cdots\times V}_{n\text{ copies}}\to\mathbb{R}$$

satisfying the following condition: if $T\colon V\to V$ is any linear map, then

$$\mu(Tv_1,\dots,Tv_n)=|\det T|\mu(v_1,\dots,v_n). \tag{16.18}$$

(Compare this with the corresponding formula (14.2) for n-forms.) Observe that a density is *not* a tensor, because it is not linear over $\mathbb{R}$ in any of its arguments. Let $\mathcal{D}(V)$ denote the set of all densities on V.

Proposition 16.35 (Properties of Densities). *Let V be a vector space of dimension $n\geq 1$.*

(a) *$\mathcal{D}(V)$ is a vector space under the obvious vector operations*:

$$(c_1\mu_1+c_2\mu_2)(v_1,\dots,v_n)=c_1\mu_1(v_1,\dots,v_n)+c_2\mu_2(v_1,\dots,v_n).$$

(b) *If $\mu_1,\mu_2\in\mathcal{D}(V)$ and $\mu_1(E_1,\dots,E_n)=\mu_2(E_1,\dots,E_n)$ for some basis (E_i) of V, then $\mu_1=\mu_2$.*

(c) *If $\omega\in\Lambda^n(V^*)$, the map $|\omega|\colon V\times\cdots\times V\to\mathbb{R}$ defined by*

$$|\omega|\,(v_1,\dots,v_n)=|\omega(v_1,\dots,v_n)|$$

is a density.

(d) *$\mathcal{D}(V)$ is 1-dimensional, spanned by $|\omega|$ for any nonzero $\omega\in\Lambda^n(V^*)$.*

Proof. Part (a) is immediate from the definition. For part (b), suppose μ_1 and μ_2 give the same value when applied to $(E_1,\dots,E_n)$. If $v_1,\dots,v_n$ are arbitrary vectors in V, let $T\colon V\to V$ be the unique linear map that takes E_i to v_i for $i=1,\dots,n$. It follows that

$$\begin{aligned}\mu_1(v_1,\dots,v_n)&=\mu_1(TE_1,\dots,TE_n)\\&=|\det T|\,\mu_1(E_1,\dots,E_n)\\&=|\det T|\,\mu_2(E_1,\dots,E_n)\end{aligned}$$

$$
\begin{aligned}
&= \mu_2(TE_1, \ldots, TE_n) \\
&= \mu_2(v_1, \ldots, v_n).
\end{aligned}
$$

Part (c) follows from Proposition 14.9:

$$
\begin{aligned}
|\omega|(Tv_1, \ldots, Tv_n) &= |\omega(Tv_1, \ldots, Tv_n)| \\
&= |(\det T)\omega(v_1, \ldots, v_n)| \\
&= |\det T|\, |\omega|\, (v_1, \ldots, v_n).
\end{aligned}
$$

Finally, to prove (d), suppose ω is any nonzero element of $\Lambda^n(V^*)$. If μ is an arbitrary element of $\mathcal{D}(V)$, it suffices to show that $\mu = c\,|\omega|$ for some $c \in \mathbb{R}$. Let (E_i) be a basis for V, and define $a, b \in \mathbb{R}$ by

$$
\begin{aligned}
a &= |\omega|\,(E_1, \ldots, E_n) = |\omega(E_1, \ldots, E_n)|, \\
b &= \mu(E_1, \ldots, E_n).
\end{aligned}
$$

Because $\omega \neq 0$, it follows that $a \neq 0$. Thus, μ and $(b/a)|\omega|$ give the same result when applied to $(E_1, \ldots, E_n)$, so they are equal by part (b). □

A ***positive density on*** V is a density μ satisfying $\mu(v_1, \ldots, v_n) > 0$ whenever $(v_1, \ldots, v_n)$ is a linearly independent n-tuple. A ***negative density*** is defined similarly. If ω is a nonzero element of $\Lambda^n(V^*)$, then it is clear that $|\omega|$ is a positive density; more generally, a density $c\,|\omega|$ is positive, negative, or zero if and only if c has the same property. Thus, each density on V is either positive, negative, or zero, and the set of positive densities is a convex subset of $\mathcal{D}(V)$ (namely, a half-line).

Now let M be a smooth manifold with or without boundary. The set

$$
\mathcal{D}M = \coprod_{p \in M} \mathcal{D}(T_pM)
$$

is called the ***density bundle of*** M. Let $\pi\colon \mathcal{D}M \to M$ be the natural projection map taking each element of $\mathcal{D}(T_pM)$ to p.

Proposition 16.36. *If M is a smooth manifold with or without boundary, its density bundle is a smooth line bundle over M.*

Proof. We will construct local trivializations and use the vector bundle chart lemma (Lemma 10.6). Let $\big(U, (x^i)\big)$ be any smooth coordinate chart on M, and let $\omega = dx^1 \wedge \cdots \wedge dx^n$. Proposition 16.35 shows that $|\omega_p|$ is a basis for $\mathcal{D}(T_pM)$ at each point $p \in U$. Therefore, the map $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}$ given by

$$
\Phi\big(c|\omega_p|\big) = (p, c)
$$

is a bijection.

Now suppose $\big(\widetilde{U}, (\widetilde{x}^j)\big)$ is another smooth chart with $U \cap \widetilde{U} \neq \varnothing$. Let $\widetilde{\omega} = d\widetilde{x}^1 \wedge \cdots \wedge d\widetilde{x}^n$, and define $\widetilde{\Phi}\colon \pi^{-1}\big(\widetilde{U}\big) \to \widetilde{U} \times \mathbb{R}$ correspondingly:

$$
\widetilde{\Phi}\,\big(c|\widetilde{\omega}_p|\big) = (p, c).
$$

It follows from the transformation law (14.16) for n-forms under changes of coordinates that

$$\Phi \circ \widetilde{\Phi}^{-1}(p,c) = \Phi\big(c|\widetilde{\omega}_p|\big) = \Phi\left(c\left|\det\left(\frac{\partial \widetilde{x}^j}{\partial x^i}\right)\right||\omega_p|\right)$$
$$= \left(p, c\left|\det\left(\frac{\partial \widetilde{x}^j}{\partial x^i}\right)\right|\right).$$

Thus, the hypotheses of Lemma 10.6 are satisfied, with the transition functions equal to $\left|\det\left(\partial \widetilde{x}^j/\partial x^i\right)\right|$. □

If M is a smooth n-manifold with or without boundary, a section of $\mathcal{D}M$ is called a ***density on M***. (One might choose to call such a section a "density field" to distinguish it from a density on a vector space, but we do not do so.) If μ is a density and f is a continuous real-valued function, then $f\mu$ is again a density, which is smooth if both f and μ are. A density on M is said to be positive or negative if its value at each point has that property. Any nonvanishing n-form ω determines a positive density $|\omega|$, defined by $|\omega|_p = |\omega_p|$ for each $p \in M$. If ω is a nonvanishing n-form on an open subset $U \subseteq M$, then any density μ on U can be written $\mu = f|\omega|$ for some real-valued function f.

One important fact about densities is that every smooth manifold admits a global smooth positive density, without any orientability assumptions.

Proposition 16.37. *If M is a smooth manifold with or without boundary, there exists a smooth positive density on M.*

Proof. Because the set of positive elements of $\mathcal{D}M$ is an open subset whose intersection with each fiber is convex, the usual partition of unity argument (Problem 13-2) allows us to piece together local positive densities to obtain a global smooth positive density. □

It is important to understand that this proposition works because positivity of a density is a well-defined property, independent of any choices of coordinates or orientations. There is no corresponding existence result for orientation forms because without a choice of orientation, there is no way to decide which n-forms are positive.

Under smooth maps, densities pull back in the same way as differential forms. If $F: M \to N$ is a smooth map between n-manifolds (with or without boundary) and μ is a density on N, we define a density $F^*\mu$ on M by

$$(F^*\mu)_p(v_1,\dots,v_n) = \mu_{F(p)}\big(dF_p(v_1),\dots,dF_p(v_n)\big).$$

Proposition 16.38. *Let $G: P \to M$ and $F: M \to N$ be smooth maps between n-manifolds with or without boundary, and let μ be a density on N.*

(a) *For any $f \in C^\infty(N)$, $F^*(f\mu) = (f \circ F)F^*\mu$.*
(b) *If ω is an n-form on N, then $F^*|\omega| = |F^*\omega|$.*
(c) *If μ is smooth, then $F^*\mu$ is a smooth density on M.*
(d) *$(F \circ G)^*\mu = G^*(F^*\mu)$.*

► **Exercise 16.39.** Prove the preceding proposition.

The next result shows how to compute the pullback of a density in coordinates. It is an analogue for densities of Proposition 14.20.

Proposition 16.40. *Suppose $F\colon M \to N$ is a smooth map between n-manifolds with or without boundary. If (x^i) and (y^j) are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:*

$$F^*\left(u\left|dy^1\wedge\cdots\wedge dy^n\right|\right) = (u\circ F)\left|\det DF\right|\left|dx^1\wedge\cdots\wedge dx^n\right|, \tag{16.19}$$

where DF represents the matrix of partial derivatives of F in these coordinates.

Proof. Using Propositions 14.20 and 16.38, we obtain

$$\begin{aligned} F^*\left(u\left|dy^1\wedge\cdots\wedge dy^n\right|\right) &= (u\circ F)F^*\left|dy^1\wedge\cdots\wedge dy^n\right| \\ &= (u\circ F)\left|F^*\left(dy^1\wedge\cdots\wedge dy^n\right)\right| \\ &= (u\circ F)\left|(\det DF)\,dx^1\wedge\cdots\wedge dx^n\right| \\ &= (u\circ F)\left|\det DF\right|\left|dx^1\wedge\cdots\wedge dx^n\right|. \end{aligned}$$ □

Now we turn to integration. As we did with forms, we begin by defining integrals of densities on subsets of $\mathbb{R}^n$. If $D \subseteq \mathbb{R}^n$ is a domain of integration and μ is a density on $\bar{D}$, we can write $\mu = f\left|dx^1\wedge\cdots\wedge dx^n\right|$ for some uniquely determined continuous function $f\colon \bar{D}\to\mathbb{R}$. We define the ***integral of μ over D*** by

$$\int_D \mu = \int_D f\,dV,$$

or more suggestively,

$$\int_D f\left|dx^1\wedge\cdots\wedge dx^n\right| = \int_D f\,dx^1\cdots dx^n.$$

Similarly, if U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and μ is compactly supported in U, we define

$$\int_U \mu = \int_D \mu,$$

where D is any domain of integration containing the support of μ. The key fact is that this is diffeomorphism-invariant.

Proposition 16.41. *Suppose U and V are open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G\colon U\to V$ is a diffeomorphism. If μ is a compactly supported density on V, then*

$$\int_V \mu = \int_U G^*\mu.$$

Proof. The proof is essentially identical to that of Proposition 16.3, using (16.19) instead of (14.15). □

Now let M be a smooth n-manifold (with or without boundary). If μ is a density on M whose support is contained in the domain of a single smooth chart (U,φ), the ***integral of μ over M*** is defined as

$$\int_M \mu = \int_{\varphi(U)} \left(\varphi^{-1}\right)^* \mu.$$

This is extended to arbitrary densities μ by setting

$$\int_M \mu = \sum_i \int_M \psi_i \mu,$$

where $\{\psi_i\}$ is a smooth partition of unity subordinate to an open cover of M by smooth charts. The fact that this is independent of the choices of coordinates or partition of unity follows just as in the case of forms.

The following proposition is proved in the same way as Proposition 16.6.

Proposition 16.42 (Properties of Integrals of Densities). *Suppose M and N are smooth n-manifolds with or without boundary, and μ,η are compactly supported densities on M.*

(a) LINEARITY: *If $a,b \in \mathbb{R}$, then*

$$\int_M a\mu + b\eta = a\int_M \mu + b\int_M \eta.$$

(b) POSITIVITY: *If μ is a positive density, then $\int_M \mu > 0$.*
(c) DIFFEOMORPHISM INVARIANCE: *If $F\colon N \to M$ is a diffeomorphism, then $\int_M \mu = \int_N F^*\mu$.*

► **Exercise 16.43.** Prove Proposition 16.42.

Just as for forms, integrals of densities are usually computed by cutting the manifold into pieces and parametrizing each piece, just as in Proposition 16.8. The details are left to the reader.

► **Exercise 16.44.** Formulate and prove an analogue of Proposition 16.8 for densities.

The Riemannian Density

Densities are particularly useful on Riemannian manifolds.

Proposition 16.45 (The Riemannian Density). *Let (M,g) be a Riemannian manifold with or without boundary. There is a unique smooth positive density μ_g on M, called the* ***Riemannian density****, with the property that*

$$\mu_g(E_1,\dots,E_n) = 1 \tag{16.20}$$

for any local orthonormal frame (E_i).

Proof. Uniqueness is immediate, because any two densities that agree on a basis must be equal. Given any point $p \in M$, let U be a connected smooth coordinate neighborhood of p. Since U is diffeomorphic to an open subset of Euclidean space, it is orientable. Any choice of orientation of U uniquely determines a Riemannian volume form ω_g on U, with the property that $\omega_g(E_1, \dots, E_n) = 1$ for any oriented orthonormal frame. If we put $\mu_g = |\omega_g|$, it follows easily that μ_g is a smooth positive density on U satisfying (16.20). If U and V are two overlapping smooth coordinate neighborhoods, the two definitions of μ_g agree where they overlap by uniqueness, so this defines μ_g globally. □

► **Exercise 16.46.** Let (M, g) be an oriented Riemannian manifold with or without boundary and let ω_g be its Riemannian volume form.

(a) Show that the Riemannian density of M is given by $\mu_g = |\omega_g|$.
(b) For any compactly supported continuous function $f: M \to \mathbb{R}$, show that

$$\int_M f\mu_g = \int_M f\omega_g.$$

► **Exercise 16.47.** Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are Riemannian manifolds with or without boundary, and $F: M \to \widetilde{M}$ is a local isometry. Show that $F^*\mu_{\widetilde{g}} = \mu_g$.

Because of Exercise 16.46(b), it is customary to denote the Riemannian density simply by dV_g, and to specify when necessary whether the notation refers to a density or a form. If $f: M \to \mathbb{R}$ is a compactly supported continuous function, the ***integral of f over M*** is defined to be $\int_M f\, dV_g$. Exercise 16.46 shows that when M is oriented, it does not matter whether we interpret dV_g as the Riemannian volume form or the Riemannian density. (If the orientation of M is changed, then both the integral and dV_g change signs, so the result is the same.) When M is not orientable, however, we have no choice but to interpret it as a density.

One of the most useful applications of densities is that they enable us to generalize the divergence theorem to nonorientable manifolds. If X is a smooth vector field on M, Exercise 16.31 shows that the divergence of X can be defined even when M is not orientable. The next theorem shows that the divergence theorem holds in that case as well.

Theorem 16.48 (The Divergence Theorem in the Nonorientable Case). *Suppose (M, g) is a nonorientable Riemannian manifold with boundary. For any compactly supported smooth vector field X on M,*

$$\int_M (\operatorname{div} X)\mu_g = \int_{\partial M} \langle X, N\rangle_g\, \mu_{\widetilde{g}}, \tag{16.21}$$

where N is the outward-pointing unit normal vector field along ∂M, $\widetilde{g}$ is the induced Riemannian metric on ∂M, and μ_g, $\mu_{\widetilde{g}}$ are the Riemannian densities of g and $\widetilde{g}$, respectively.

Proof. Let $\widehat{\pi}: \widehat{M} \to M$ be the orientation covering of M. Problem 5-12 shows that $\widehat{\pi}$ restricts to a smooth covering map from each component of $\partial\widehat{M}$ to a component of ∂M, so in the terminology of Chapter 15, $\widehat{\pi}\colon \partial\widehat{M} \to \partial M$ is a generalized covering map.

Define metrics $\widehat{g} = \widehat{\pi}^* g$ on $\widehat{M}$ and $\overline{g} = \widehat{\pi}^*\widetilde{g}$ on $\partial\widehat{M}$. Denote the Riemannian volume forms of $\widehat{g}$ and $\overline{g}$ by $\omega_{\widehat{g}}$ and $\omega_{\overline{g}}$, respectively, and their Riemannian densities by $\mu_{\widehat{g}}$ and $\mu_{\overline{g}}$. Because $\widehat{\pi}$ is a local isometry, it is easy to check that the outward unit normal $\widehat{N}$ along $\partial\widehat{M}$ is $\widehat{\pi}$-related to N. Moreover, it follows from Problem 8-18(a) that there is a unique smooth vector field $\widehat{X}$ on $\widehat{M}$ that is $\widehat{\pi}$-related to X.

Since $\widehat{M}$ is an oriented smooth Riemannian manifold with boundary, we can apply the usual divergence theorem to it to obtain

$$\begin{aligned}
2\int_M (\operatorname{div} X)\mu_g &= \int_{\widehat{M}} \widehat{\pi}^*\big((\operatorname{div} X)\mu_g\big) && \text{(by Problem 16-3)}\\
&= \int_{\widehat{M}} \big(\operatorname{div} \widehat{X}\big)\mu_{\widehat{g}} && (\widehat{\pi} \text{ is a local isometry})\\
&= \int_{\widehat{M}} \big(\operatorname{div} \widehat{X}\big)\,\omega_{\widehat{g}} && \text{(by Exercise 16.46(b))}\\
&= \int_{\partial\widehat{M}} \langle \widehat{X}, \widehat{N}\rangle_{\widehat{g}}\,\omega_{\overline{g}} && \text{(divergence theorem on } \widehat{M})\\
&= \int_{\partial\widehat{M}} \langle \widehat{X}, \widehat{N}\rangle_{\widehat{g}}\,\mu_{\overline{g}} && \text{(by Exercise 16.46(b))}\\
&= \int_{\partial\widehat{M}} \big(\widehat{\pi}|_{\partial\widehat{M}}\big)^*\big(\langle X, N\rangle_g \mu_{\widetilde{g}}\big) && (\widehat{\pi}|_{\partial\widehat{M}} \text{ is a local isometry})\\
&= 2\int_{\partial M} \langle X, N\rangle_g\,\mu_{\widetilde{g}} && \text{(by Problem 16-3).}
\end{aligned}$$

Dividing both sides by 2 yields (16.21). □

Problems

16-1. Let $v_1, \dots, v_n$ be any n linearly independent vectors in $\mathbb{R}^n$, and let P be the n-dimensional parallelepiped they span:

$$P = \{t_1 v_1 + \cdots + t_n v_n : 0 \le t_i \le 1\}.$$

Show that $\operatorname{Vol}(P) = |\det(v_1, \dots, v_n)|$. (*Used on p. 401.*)

16-2. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{R}^4$ denote the 2-torus, defined as the set of points (w, x, y, z) such that $w^2 + x^2 = y^2 + z^2 = 1$, with the product orientation determined by the standard orientation on $\mathbb{S}^1$. Compute $\int_{\mathbb{T}^2}\omega$, where ω is the following 2-form on $\mathbb{R}^4$:

$$\omega = xyz\,dw \wedge dy.$$

16-3. Suppose E and M are smooth n-manifolds with or without boundary, and $\pi\colon E \to M$ is a smooth k-sheeted covering map or generalized covering map.

(a) Show that if E and M are oriented and π is orientation-preserving, then $\int_E \pi^*\omega = k \int_M \omega$ for any compactly supported n-form ω on M.
(b) Show that $\int_E \pi^*\mu = k \int_M \mu$ whenever μ is a compactly supported density on M.

16-4. Suppose M is an oriented compact smooth manifold with boundary. Show that there does not exist a retraction of M onto its boundary. [Hint: if the retraction is smooth, consider an orientation form on ∂M.]

16-5. Suppose M and N are oriented, compact, connected, smooth manifolds, and $F, G\colon M \to N$ are homotopic diffeomorphisms. Show that F and G are either both orientation-preserving or both orientation-reversing. [Hint: use Theorem 6.29 and Stokes's theorem on $M \times I$.]

16-6. THE HAIRY BALL THEOREM: *There exists a nowhere-vanishing vector field on* $\mathbb{S}^n$ *if and only if* n *is odd.* ("You cannot comb the hair on a ball.") Prove this by showing that the following are equivalent:
(a) There exists a nowhere-vanishing vector field on $\mathbb{S}^n$.
(b) There exists a continuous map $V\colon \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in \mathbb{S}^n$.
(c) The antipodal map $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to $\mathrm{Id}_{\mathbb{S}^n}$.
(d) The antipodal map $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving.
(e) n is odd.
[Hint: use Problems 9-4, 15-3, and 16-5.]

16-7. Show that any finite product $M_1 \times \cdots \times M_k$ of smooth manifolds with corners is again a smooth manifold with corners. Give a counterexample to show that a finite product of smooth manifolds with boundary need not be a smooth manifold with boundary.

16-8. Suppose M is a smooth manifold with corners, and let $\mathcal{C}$ denote the set of corner points of M. Show that $M \smallsetminus \mathcal{C}$ is a smooth manifold with boundary.

16-9. Let ω be the $(n-1)$-form on $\mathbb{R}^n \smallsetminus \{0\}$ defined by

$$\omega = |x|^{-n} \sum_{i=1}^{n} (-1)^{i-1} x^i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \qquad (16.22)$$

(a) Show that $\iota^*_{\mathbb{S}^{n-1}}\omega$ is the Riemannian volume form of $\mathbb{S}^{n-1}$ with respect to the round metric and the standard orientation.
(b) Show that ω is closed but not exact on $\mathbb{R}^n \smallsetminus \{0\}$.

16-10. Let D denote the torus of revolution in $\mathbb{R}^3$ obtained by revolving the circle $(r-2)^2 + z^2 = 1$ around the z-axis (Example 5.17), with its induced Riemannian metric and with the orientation determined by the outward unit normal.
(a) Compute the surface area of D.
(b) Compute the integral over D of the function $f(x,y,z) = z^2 + 1$.
(c) Compute the integral over D of the 2-form $\omega = z\, dx \wedge dy$.

16-11. Let (M, g) be a Riemannian n-manifold with or without boundary. In any smooth local coordinates $\left(x^i\right)$, show that

$$\operatorname{div}\left(X^i \frac{\partial}{\partial x^i}\right) = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(X^i \sqrt{\det g}\right),$$

where $\det g = \det(g_{kl})$ is the determinant of the component matrix of g in these coordinates.

16-12. Let (M, g) be a compact Riemannian manifold with boundary, let $\widetilde{g}$ denote the induced Riemannian metric on ∂M, and let N be the outward unit normal vector field along ∂M.

(a) Show that the divergence operator satisfies the following product rule for $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$:

$$\operatorname{div}(fX) = f \operatorname{div} X + \langle \operatorname{grad} f, X \rangle_g.$$

(b) Prove the following "integration by parts" formula:

$$\int_M \langle \operatorname{grad} f, X \rangle_g \, dV_g = \int_{\partial M} f \langle X, N \rangle_g \, dV_{\widetilde{g}} - \int_M (f \operatorname{div} X) \, dV_g.$$

(c) Explain what this has to do with integration by parts.

16-13. Let (M, g) be a Riemannian n-manifold with or without boundary. The linear operator $\Delta\colon C^\infty(M) \to C^\infty(M)$ defined by $\Delta u = -\operatorname{div}(\operatorname{grad} u)$ is called the ***(geometric) Laplacian***. Show that the Laplacian is given in any smooth local coordinates by

$$\Delta u = -\frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(g^{ij} \sqrt{\det g} \frac{\partial u}{\partial x^j}\right).$$

Conclude that on $\mathbb{R}^n$ with the Euclidean metric and standard coordinates,

$$\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{(\partial x^i)^2}.$$

[Remark: there is no general agreement about the sign convention for the Laplacian on a Riemannian manifold, and many authors define Δ to be the negative of the operator we have defined. Although the geometric Laplacian defined here is the opposite of the traditional Laplacian on $\mathbb{R}^n$, it has two distinct advantages: our Laplacian has nonnegative eigenvalues (see Problem 16-15), and it agrees with the Laplace–Beltrami operator defined on differential forms (see Problems 17-2 and 17-3). When reading any book or article that mentions the Laplacian, you have to be careful to determine which sign convention the author is using.] (*Used on p. 465.*)

16-14. Let (M, g) be a Riemannian manifold with or without boundary. A function $u \in C^\infty(M)$ is said to be ***harmonic*** if $\Delta u = 0$ (see Problem 16-13).

(a) Suppose M is compact, and prove ***Green's identities***:

$$\int_M u\Delta v\, dV_g = \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g\, dV_g - \int_{\partial M} uNv\, dV_{\widetilde{g}},$$

$$\int_M (u\Delta v - v\Delta u)\, dV_g = \int_{\partial M} (vNu - uNv) dV_{\widetilde{g}},$$

where N and $\widetilde{g}$ are as in Problem 16-12.

(b) Show that if M is compact and connected and $\partial M = \varnothing$, the only harmonic functions on M are the constants.

(c) Show that if M is compact and connected, $\partial M \neq \varnothing$, and u, v are harmonic functions on M whose restrictions to ∂M agree, then $u \equiv v$.

16-15. Let (M, g) be a compact connected Riemannian manifold without boundary, and let Δ be its geometric Laplacian. A real number λ is called an ***eigenvalue of*** $\boldsymbol{\Delta}$ if there exists a smooth real-valued function u on M, not identically zero, such that $\Delta u = \lambda u$. In this case, u is called an ***eigenfunction*** corresponding to λ.

(a) Prove that 0 is an eigenvalue of Δ, and that all other eigenvalues are strictly positive.

(b) Prove that if u and v are eigenfunctions corresponding to distinct eigenvalues, then $\int_M uv\, dV_g = 0$.

16-16. Let M be a compact connected Riemannian n-manifold with nonempty boundary. A number $\lambda \in \mathbb{R}$ is called a ***Dirichlet eigenvalue for M*** if there exists a smooth real-valued function u on M, not identically zero, such that $\Delta u = \lambda u$ and $u|_{\partial M} = 0$. Similarly, λ is called a ***Neumann eigenvalue*** if there exists such a u satisfying $\Delta u = \lambda u$ and $Nu|_{\partial M} = 0$, where N is the outward unit normal.

(a) Show that every Dirichlet eigenvalue is strictly positive.

(b) Show that 0 is a Neumann eigenvalue, and all other Neumann eigenvalues are strictly positive.

16-17. DIRICHLET'S PRINCIPLE: Suppose M is a compact connected Riemannian n-manifold with nonempty boundary. Prove that a function $u \in C^\infty(M)$ is harmonic if and only if it minimizes $\int_M |\operatorname{grad} u|_g^2\, dV_g$ among all smooth functions with the same boundary values. [Hint: for any function $f \in C^\infty(M)$ that vanishes on ∂M, expand $\int_M |\operatorname{grad}(u + \varepsilon f)|_g^2\, dV_g$ and use Problem 16-12.]

16-18. Let (M, g) be an oriented Riemannian n-manifold. This problem outlines an important generalization of the operator $*\colon C^\infty(M) \to \Omega^n(M)$ defined in this chapter.

(a) For each $k = 1, \dots, n$, show that g determines a unique inner product on $\Lambda^k(T_p^*M)$ (denoted by $\langle\cdot,\cdot\rangle_g$, just like the inner product on T_pM) satisfying

$$\left\langle \omega^1 \wedge \dots \wedge \omega^k, \eta^1 \wedge \dots \wedge \eta^k \right\rangle_g = \det\left(\left\langle (\omega^i)^\sharp, (\eta^j)^\sharp \right\rangle_g\right)$$

whenever $\omega^1, \ldots, \omega^k, \eta^1, \ldots, \eta^k$ are covectors at p. [Hint: define the inner product locally by declaring $\{\varepsilon^I|_p : I \text{ is increasing}\}$ to be an orthonormal basis for $\Lambda^k(T_p^*M)$ whenever (ε^i) is the coframe dual to a local orthonormal frame, and then prove that the resulting inner product is independent of the choice of frame.]

(b) Show that the Riemannian volume form dV_g is the unique positively oriented n-form that has unit norm with respect to this inner product.

(c) For each $k = 0, \ldots, n$, show that there is a unique smooth bundle homomorphism $*\colon \Lambda^k T^*M \to \Lambda^{n-k} T^*M$ satisfying

$$\omega \wedge * \eta = \langle \omega, \eta \rangle_g \, dV_g$$

for all smooth k-forms ω, η. (For $k = 0$, interpret the inner product as ordinary multiplication.) This map is called the ***Hodge star operator***. [Hint: first prove uniqueness, and then define $*$ locally by setting

$$*\left(\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}\right) = \pm \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_{n-k}}$$

in terms of an orthonormal coframe (ε^i), where the indices $j_1, \ldots, j_{n-k}$ are chosen so that $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is some permutation of $(1, \ldots, n)$.]

(d) Show that $*\colon \Lambda^0 T^*M \to \Lambda^n T^*M$ is given by $* f = f \, dV_g$.

(e) Show that $* * \omega = (-1)^{k(n-k)} \omega$ if ω is a k-form.

16-19. Consider $\mathbb{R}^n$ as a Riemannian manifold with the Euclidean metric and the standard orientation.

(a) Calculate $* dx^i$ for $i = 1, \ldots, n$.

(b) Calculate $*\left(dx^i \wedge dx^j\right)$ in the case $n = 4$.

16-20. Let M be an oriented Riemannian 4-manifold. A 2-form ω on M is said to be ***self-dual*** if $* \omega = \omega$, and ***anti-self-dual*** if $* \omega = -\omega$.

(a) Show that every 2-form ω on M can be written uniquely as a sum of a self-dual form and an anti-self-dual form.

(b) On $M = \mathbb{R}^4$ with the Euclidean metric, determine the self-dual and anti-self-dual forms in standard coordinates.

16-21. Let (M, g) be an oriented Riemannian manifold and $X \in \mathfrak{X}(M)$. Show that

$$X \lrcorner \, dV_g = * X^\flat,$$

$$\operatorname{div} X = * d * X^\flat,$$

and, when $\dim M = 3$,

$$\operatorname{curl} X = \left(* dX^\flat\right)^\sharp.$$

16-22. Let (M, g) be a compact, oriented Riemannian n-manifold. For $1 \le k \le n$, define a map $d^*\colon \Omega^k(M) \to \Omega^{k-1}(M)$ by $d^*\omega = (-1)^{n(k+1)+1} * d * \omega$, where $*$ is the Hodge star operator defined in Problem 16-18. Extend this definition to 0-forms by defining $d^*\omega = 0$ for $\omega \in \Omega^0(M)$.

(a) Show that $d^* \circ d^* = 0$.

(b) Show that the formula

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle_g \, dV_g$$

defines an inner product on $\Omega^k(M)$ for each k, where $\langle \cdot, \cdot \rangle_g$ is the pointwise inner product on forms defined in Problem 16-18.

(c) Show that $(d^*\omega, \eta) = (\omega, d\eta)$ for all $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{k-1}(M)$.

(*Used on p. 464.*)

16-23. This problem illustrates another approach to proving that certain Riemannian metrics are not flat. Let $\mathbb{B}^2$ be the unit disk in $\mathbb{R}^2$, and let g be the Riemannian metric on $\mathbb{B}^2$ given by

$$g = \frac{dx^2 + dy^2}{1 - x^2 - y^2}.$$

(a) Show that if g is flat, then for sufficiently small $r > 0$, the volume of the g-metric ball $\overline{B}^g_r(0)$ satisfies $\operatorname{Vol}_g\big(\overline{B}^g_r(0)\big) = \pi r^2$.

(b) For any $v \in \mathbb{B}^2$, by computing the g-length of the straight line from 0 to v, show that $d_g(0, v) \le \tanh^{-1}|v|$ (where $\tanh^{-1}$ denotes the inverse hyperbolic tangent function). Conclude that for any $r > 0$, the g-metric ball $\overline{B}^g_r(0)$ contains the Euclidean ball $\overline{B}_{\tanh r}(0) = \{v : |v| \le \tanh r\}$.

(c) Show that $\operatorname{Vol}_g\big(\overline{B}^g_r(0)\big) \ge \pi \sinh^2 r > \pi r^2$, and therefore g is not flat.