

## Chapter 7

# Lie Groups

In this chapter we introduce *Lie groups*, which are smooth manifolds that are also groups in which multiplication and inversion are smooth maps. Besides providing many examples of interesting manifolds themselves, they are essential tools in the study of more general manifolds, primarily because of the role they play as groups of symmetries of other manifolds.

Our aim in this chapter is to introduce Lie groups and some of the tools for working with them, and to describe an abundant supply of examples. In subsequent chapters (especially Chapters 8, 20, and 21), we will develop many more properties and applications of Lie groups.

We begin with the definition of Lie groups and some of the basic structures associated with them, and then present a number of examples. Next we study *Lie group homomorphisms*, which are group homomorphisms that are also smooth maps. Then we introduce *Lie subgroups* (subgroups that are also smooth submanifolds), which lead to a number of new examples of Lie groups.

After explaining these basic ideas, we introduce actions of Lie groups on manifolds, which are the primary *raison d'être* of Lie groups. At the end of the chapter, we briefly touch on group representations.

The study of Lie groups was initiated in the late nineteenth century by the Norwegian mathematician Sophus Lie. Inspired by the way the French algebraist Évariste Galois had invented group theory and used it to analyze polynomial equations, Lie was interested in using symmetries, expressed in the form of group actions, to simplify problems in partial differential equations and geometry. However, Lie could not have conceived of the global objects that we now call Lie groups, for the simple reason that global topological notions such as manifolds (or even topological spaces!) had not yet been formulated. What Lie studied was essentially a local-coordinate version of Lie groups, now called *local Lie groups*. Despite the limitations imposed by the era in which he lived, he was able to lay much of the groundwork for our current understanding of Lie groups. We will describe his principal results in Chapter 20 (see Theorem 20.16).

## Basic Definitions

A **Lie group** is a smooth manifold  $G$  (without boundary) that is also a group in the algebraic sense, with the property that the multiplication map  $m: G \times G \rightarrow G$  and inversion map  $i: G \rightarrow G$ , given by

$$m(g, h) = gh, \quad i(g) = g^{-1},$$

are both smooth. A Lie group is, in particular, a **topological group** (a topological space with a group structure such that the multiplication and inversion maps are continuous).

The group operation in an arbitrary Lie group is denoted by juxtaposition, except in certain abelian groups such as  $\mathbb{R}^n$  in which the operation is usually written additively. It is traditional to denote the identity element of an arbitrary Lie group by the symbol  $e$  (for German *Einselement*, “unit element”), and we follow this convention, except in specific examples in which there are more common notations (such as  $I_n$  for the identity matrix in a matrix group, or  $0$  for the identity element in  $\mathbb{R}^n$ ).

The following alternative characterization of the smoothness condition is sometimes useful. (See also Problem 7-3 for a stronger result.)

**Proposition 7.1.** *If  $G$  is a smooth manifold with a group structure such that the map  $G \times G \rightarrow G$  given by  $(g, h) \mapsto gh^{-1}$  is smooth, then  $G$  is a Lie group.*

► **Exercise 7.2.** Prove Proposition 7.1.

If  $G$  is a Lie group, any element  $g \in G$  defines maps  $L_g, R_g: G \rightarrow G$ , called **left translation** and **right translation**, respectively, by

$$L_g(h) = gh, \quad R_g(h) = hg.$$

Because  $L_g$  can be expressed as the composition of smooth maps

$$G \xrightarrow{\iota_g} G \times G \xrightarrow{m} G,$$

where  $\iota_g(h) = (g, h)$  and  $m$  is multiplication, it follows that  $L_g$  is smooth. It is actually a diffeomorphism of  $G$ , because  $L_{g^{-1}}$  is a smooth inverse for it. Similarly,  $R_g: G \rightarrow G$  is a diffeomorphism. As we will see repeatedly below, many of the important properties of Lie groups follow from the fact that we can systematically map any point to any other by such a global diffeomorphism.

**Example 7.3 (Lie Groups).** Each of the following manifolds is a Lie group with the indicated group operation.

- (a) The **general linear group**  $\text{GL}(n, \mathbb{R})$  is the set of invertible  $n \times n$  matrices with real entries. It is a group under matrix multiplication, and it is an open submanifold of the vector space  $M(n, \mathbb{R})$ , as we observed in Example 1.27. Multiplication is smooth because the matrix entries of a product matrix  $AB$  are polynomials in the entries of  $A$  and  $B$ . Inversion is smooth by Cramer’s rule.

- (b) Let  $GL^+(n, \mathbb{R})$  denote the subset of  $GL(n, \mathbb{R})$  consisting of matrices with positive determinant. Because  $\det(AB) = (\det A)(\det B)$  and  $\det(A^{-1}) = 1/\det A$ , it is a subgroup of  $GL(n, \mathbb{R})$ ; and because it is the preimage of  $(0, \infty)$  under the continuous determinant function, it is an open subset of  $GL(n, \mathbb{R})$  and therefore an  $n^2$ -dimensional manifold. The group operations are the restrictions of those of  $GL(n, \mathbb{R})$ , so they are smooth. Thus  $GL^+(n, \mathbb{R})$  is a Lie group.
- (c) Suppose  $G$  is an arbitrary Lie group and  $H \subseteq G$  is an **open subgroup** (a subgroup that is also an open subset). By the same argument as in part (b),  $H$  is a Lie group with the inherited group structure and smooth manifold structure.
- (d) The **complex general linear group**  $GL(n, \mathbb{C})$  is the group of invertible complex  $n \times n$  matrices under matrix multiplication. It is an open submanifold of  $M(n, \mathbb{C})$  and thus a  $2n^2$ -dimensional smooth manifold, and it is a Lie group because matrix products and inverses are smooth functions of the real and imaginary parts of the matrix entries.
- (e) If  $V$  is any real or complex vector space,  $GL(V)$  denotes the set of invertible linear maps from  $V$  to itself. It is a group under composition. If  $V$  has finite dimension  $n$ , any basis for  $V$  determines an isomorphism of  $GL(V)$  with  $GL(n, \mathbb{R})$  or  $GL(n, \mathbb{C})$ , so  $GL(V)$  is a Lie group. The transition map between any two such isomorphisms is given by a map of the form  $A \mapsto BAB^{-1}$  (where  $B$  is the transition matrix between the two bases), which is smooth. Thus, the smooth manifold structure on  $GL(V)$  is independent of the choice of basis.
- (f) The real number field  $\mathbb{R}$  and Euclidean space  $\mathbb{R}^n$  are Lie groups under addition, because the coordinates of  $x - y$  are smooth (linear!) functions of  $(x, y)$ .
- (g) Similarly,  $\mathbb{C}$  and  $\mathbb{C}^n$  are Lie groups under addition.
- (h) The set  $\mathbb{R}^*$  of nonzero real numbers is a 1-dimensional Lie group under multiplication. (In fact, it is exactly  $GL(1, \mathbb{R})$  if we identify a  $1 \times 1$  matrix with the corresponding real number.) The subset  $\mathbb{R}^+$  of positive real numbers is an open subgroup, and is thus itself a 1-dimensional Lie group.
- (i) The set  $\mathbb{C}^*$  of nonzero complex numbers is a 2-dimensional Lie group under complex multiplication, which can be identified with  $GL(1, \mathbb{C})$ .
- (j) The circle  $\mathbb{S}^1 \subseteq \mathbb{C}^*$  is a smooth manifold and a group under complex multiplication. With appropriate angle functions as local coordinates on open subsets of  $\mathbb{S}^1$  (see Problem 1-8), multiplication and inversion have the smooth coordinate expressions  $(\theta_1, \theta_2) \mapsto \theta_1 + \theta_2$  and  $\theta \mapsto -\theta$ , and therefore  $\mathbb{S}^1$  is a Lie group, called the **circle group**.
- (k) Given Lie groups  $G_1, \dots, G_k$ , their **direct product** is the product manifold  $G_1 \times \dots \times G_k$  with the group structure given by componentwise multiplication:

$$(g_1, \dots, g_k)(g'_1, \dots, g'_k) = (g_1g'_1, \dots, g_kg'_k).$$

It is a Lie group, as you can easily check.

- (l) The  $n$ -torus  $\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$  is an  $n$ -dimensional abelian Lie group.
- (m) Any group with the discrete topology is a topological group, called a **discrete group**. If in addition the group is finite or countably infinite, then it is a zero-dimensional Lie group, called a **discrete Lie group**. //

## Lie Group Homomorphisms

If  $G$  and  $H$  are Lie groups, a **Lie group homomorphism from  $G$  to  $H$**  is a smooth map  $F: G \rightarrow H$  that is also a group homomorphism. It is called a **Lie group isomorphism** if it is also a diffeomorphism, which implies that it has an inverse that is also a Lie group homomorphism. In this case we say that  $G$  and  $H$  are **isomorphic Lie groups**.

### Example 7.4 (Lie Group Homomorphisms).

- (a) The inclusion map  $\mathbb{S}^1 \hookrightarrow \mathbb{C}^*$  is a Lie group homomorphism.
- (b) Considering  $\mathbb{R}$  as a Lie group under addition, and  $\mathbb{R}^*$  as a Lie group under multiplication, the map  $\exp: \mathbb{R} \rightarrow \mathbb{R}^*$  given by  $\exp(t) = e^t$  is smooth, and is a Lie group homomorphism because  $e^{(s+t)} = e^s e^t$ . The image of  $\exp$  is the open subgroup  $\mathbb{R}^+$  consisting of positive real numbers, and  $\exp: \mathbb{R} \rightarrow \mathbb{R}^+$  is a Lie group isomorphism with inverse  $\log: \mathbb{R}^+ \rightarrow \mathbb{R}$ .
- (c) Similarly,  $\exp: \mathbb{C} \rightarrow \mathbb{C}^*$  given by  $\exp(z) = e^z$  is a Lie group homomorphism. It is surjective but not injective, because its kernel consists of the complex numbers of the form  $2\pi i k$ , where  $k$  is an integer.
- (d) The map  $\varepsilon: \mathbb{R} \rightarrow \mathbb{S}^1$  defined by  $\varepsilon(t) = e^{2\pi i t}$  is a Lie group homomorphism whose kernel is the set  $\mathbb{Z}$  of integers. Similarly, the map  $\varepsilon^n: \mathbb{R}^n \rightarrow \mathbb{T}^n$  defined by  $\varepsilon^n(x^1, \dots, x^n) = (e^{2\pi i x^1}, \dots, e^{2\pi i x^n})$  is a Lie group homomorphism whose kernel is  $\mathbb{Z}^n$ .
- (e) The determinant function  $\det: \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}^*$  is smooth because  $\det A$  is a polynomial in the matrix entries of  $A$ . It is a Lie group homomorphism because  $\det(AB) = (\det A)(\det B)$ . Similarly,  $\det: \text{GL}(n, \mathbb{C}) \rightarrow \mathbb{C}^*$  is a Lie group homomorphism.
- (f) If  $G$  is a Lie group and  $g \in G$ , **conjugation by  $g$**  is the map  $C_g: G \rightarrow G$  given by  $C_g(h) = ghg^{-1}$ . Because group multiplication and inversion are smooth,  $C_g$  is smooth, and a simple computation shows that it is a group homomorphism. In fact, it is an isomorphism, because it has  $C_{g^{-1}}$  as an inverse. A subgroup  $H \subseteq G$  is said to be **normal** if  $C_g(H) = H$  for every  $g \in G$ . //

The next theorem is important for understanding many of the properties of Lie group homomorphisms.

### Theorem 7.5. Every Lie group homomorphism has constant rank.

*Proof.* Let  $F: G \rightarrow H$  be a Lie group homomorphism, and let  $e$  and  $\tilde{e}$  denote the identity elements of  $G$  and  $H$ , respectively. Suppose  $g_0$  is an arbitrary element of  $G$ . We will show that  $dF_{g_0}$  has the same rank as  $dF_e$ . The fact that  $F$  is a homomorphism means that for all  $g \in G$ ,

$$F(L_{g_0}(g)) = F(g_0g) = F(g_0)F(g) = L_{F(g_0)}(F(g)),$$

or in other words,  $F \circ L_{g_0} = L_{F(g_0)} \circ F$ . Taking differentials of both sides at the identity and using Proposition 3.6(b), we find that

$$dF_{g_0} \circ d(L_{g_0})_e = d(L_{F(g_0)})_{\tilde{e}} \circ dF_e.$$

Left multiplication by any element of a Lie group is a diffeomorphism, so both  $d(L_{g_0})_e$  and  $d(L_{F(g_0)})_{\tilde{e}}$  are isomorphisms. Because composing with an isomorphism does not change the rank of a linear map, it follows that  $dF_{g_0}$  and  $dF_e$  have the same rank.  $\square$

**Corollary 7.6.** *A Lie group homomorphism is a Lie group isomorphism if and only if it is bijective.*

*Proof.* The global rank theorem shows that a bijective Lie group homomorphism is a diffeomorphism.  $\square$

### The Universal Covering Group

Covering space theory yields the following important result about Lie groups.

**Theorem 7.7 (Existence of a Universal Covering Group).** *Let  $G$  be a connected Lie group. There exists a simply connected Lie group  $\tilde{G}$ , called the **universal covering group of  $G$** , that admits a smooth covering map  $\pi: \tilde{G} \rightarrow G$  that is also a Lie group homomorphism.*

*Proof.* Let  $\tilde{G}$  be the universal covering manifold of  $G$  and  $\pi: \tilde{G} \rightarrow G$  be the corresponding smooth covering map. By Exercise 4.38,  $\pi \times \pi: \tilde{G} \times \tilde{G} \rightarrow G \times G$  is also a smooth covering map.

Let  $m: G \times G \rightarrow G$  and  $i: G \rightarrow G$  denote the multiplication and inversion maps of  $G$ , respectively, and let  $\tilde{e}$  be an arbitrary element of the fiber  $\pi^{-1}(e) \subseteq \tilde{G}$ . Since  $\tilde{G}$  is simply connected, the lifting criterion for covering maps (Proposition A.78) guarantees that the map  $m \circ (\pi \times \pi): \tilde{G} \times \tilde{G} \rightarrow G$  has a unique continuous lift  $\tilde{m}: \tilde{G} \times \tilde{G} \rightarrow \tilde{G}$  satisfying  $\tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$  and  $\pi \circ \tilde{m} = m \circ (\pi \times \pi)$ :

$$\begin{array}{ccc}
 \tilde{G} \times \tilde{G} & \xrightarrow{\tilde{m}} & \tilde{G} \\
 \pi \times \pi \downarrow & & \downarrow \pi \\
 G \times G & \xrightarrow{m} & G.
 \end{array} \tag{7.1}$$

Because  $\pi$  is a local diffeomorphism and  $\pi \circ \tilde{m} = m \circ (\pi \times \pi)$  is smooth, it follows from Exercise 4.10(a) that  $\tilde{m}$  is smooth. By the same reasoning,  $i \circ \pi: \tilde{G} \rightarrow G$  has a smooth lift  $\tilde{i}: \tilde{G} \rightarrow \tilde{G}$  satisfying  $\tilde{i}(\tilde{e}) = \tilde{e}$  and  $\pi \circ \tilde{i} = i \circ \pi$ :

$$\begin{array}{ccc}
 \tilde{G} & \xrightarrow{\tilde{i}} & \tilde{G} \\
 \pi \downarrow & & \downarrow \pi \\
 G & \xrightarrow{i} & G.
 \end{array} \tag{7.2}$$

We define multiplication and inversion in  $\tilde{G}$  by  $xy = \tilde{m}(x, y)$  and  $x^{-1} = \tilde{i}(x)$  for all  $x, y \in \tilde{G}$ . Then (7.1) and (7.2) can be rewritten as

$$\pi(xy) = \pi(x)\pi(y), \tag{7.3}$$

$$\pi(x^{-1}) = \pi(x)^{-1}. \quad (7.4)$$

It remains only to show that  $\tilde{G}$  is a group with these operations, for then it is a Lie group because  $\tilde{m}$  and  $\tilde{i}$  are smooth, and (7.3) shows that  $\pi$  is a homomorphism.

First we show that  $\tilde{e}$  is an identity for multiplication in  $\tilde{G}$ . Consider the map  $f: \tilde{G} \rightarrow \tilde{G}$  defined by  $f(x) = \tilde{e}x$ . Then (7.3) implies that  $\pi \circ f(x) = \pi(\tilde{e})\pi(x) = e\pi(x) = \pi(x)$ , so  $f$  is a lift of  $\pi: \tilde{G} \rightarrow G$ . The identity map  $\text{Id}_{\tilde{G}}$  is another lift of  $\pi$ , and it agrees with  $f$  at a point because  $f(\tilde{e}) = \tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$ , so the unique lifting property of covering maps (Proposition A.77(a)) implies that  $f = \text{Id}_{\tilde{G}}$ , or equivalently,  $\tilde{e}x = x$  for all  $x \in \tilde{G}$ . The same argument shows that  $x\tilde{e} = x$ .

Next, to show that multiplication in  $\tilde{G}$  is associative, consider the two maps  $\alpha_L, \alpha_R: \tilde{G} \times \tilde{G} \times \tilde{G} \rightarrow \tilde{G}$  defined by

$$\alpha_L(x, y, z) = (xy)z, \quad \alpha_R(x, y, z) = x(yz).$$

Then (7.3) applied repeatedly implies that

$$\pi \circ \alpha_L(x, y, z) = (\pi(x)\pi(y))\pi(z) = \pi(x)(\pi(y)\pi(z)) = \pi \circ \alpha_R(x, y, z),$$

so  $\alpha_L$  and  $\alpha_R$  are both lifts of the same map  $\alpha(x, y, z) = \pi(x)\pi(y)\pi(z)$ . Because  $\alpha_L$  and  $\alpha_R$  agree at  $(\tilde{e}, \tilde{e}, \tilde{e})$ , they are equal. A similar argument shows that  $x^{-1}x = xx^{-1} = \tilde{e}$ , so  $\tilde{G}$  is a group.  $\square$

► **Exercise 7.8.** Complete the proof of the preceding theorem by showing that  $x^{-1}x = xx^{-1} = \tilde{e}$ .

We also have the following uniqueness result.

**Theorem 7.9 (Uniqueness of the Universal Covering Group).** *For any connected Lie group  $G$ , the universal covering group is unique in the following sense: if  $\tilde{G}$  and  $\tilde{G}'$  are simply connected Lie groups that admit smooth covering maps  $\pi: \tilde{G} \rightarrow G$  and  $\pi': \tilde{G}' \rightarrow G$  that are also Lie group homomorphisms, then there exists a Lie group isomorphism  $\Phi: \tilde{G} \rightarrow \tilde{G}'$  such that  $\pi' \circ \Phi = \pi$ .*

*Proof.* See Problem 7-5.  $\square$

**Example 7.10 (Universal Covering Groups).**

(a) For each  $n$ , the map  $\varepsilon^n: \mathbb{R}^n \rightarrow \mathbb{T}^n$  given by

$$\varepsilon^n(x^1, \dots, x^n) = (e^{2\pi i x^1}, \dots, e^{2\pi i x^n})$$

is a Lie group homomorphism and a smooth covering map (see Example 7.4(d)). Since  $\mathbb{R}^n$  is simply connected, this shows that the universal covering group of  $\mathbb{T}^n$  is the additive Lie group  $\mathbb{R}^n$ .

(b) The Lie group homomorphism  $\exp: \mathbb{C} \rightarrow \mathbb{C}^*$  described in Example 7.4(c) is also a smooth covering map, so  $\mathbb{C}$  is the universal covering group of  $\mathbb{C}^*$ . //

## Lie Subgroups

Suppose  $G$  is a Lie group. A **Lie subgroup of  $G$**  is a subgroup of  $G$  endowed with a topology and smooth structure making it into a Lie group and an immersed submanifold of  $G$ . The following proposition shows that *embedded* subgroups are automatically Lie subgroups.

**Proposition 7.11.** *Let  $G$  be a Lie group, and suppose  $H \subseteq G$  is a subgroup that is also an embedded submanifold. Then  $H$  is a Lie subgroup.*

*Proof.* We need only check that multiplication  $H \times H \rightarrow H$  and inversion  $H \rightarrow H$  are smooth maps. Because multiplication is a smooth map from  $G \times G$  into  $G$ , its restriction is clearly smooth from  $H \times H$  into  $G$  (this is true even if  $H$  is merely immersed). Because  $H$  is a subgroup, multiplication takes  $H \times H$  into  $H$ , and since  $H$  is embedded, this is a smooth map into  $H$  by Corollary 5.30. A similar argument applies to inversion. This proves that  $H$  is a Lie subgroup.  $\square$

The simplest examples of embedded Lie subgroups are the open subgroups. The following lemma shows that the possibilities for open subgroups are limited.

**Lemma 7.12.** *Suppose  $G$  is a Lie group and  $H \subseteq G$  is an open subgroup. Then  $H$  is an embedded Lie subgroup. In addition,  $H$  is closed, so it is a union of connected components of  $G$ .*

*Proof.* If  $H$  is open in  $G$ , it is embedded by Proposition 5.1. In addition, every left coset  $gH = \{gh : h \in H\}$  is open in  $G$  because it is the image of the open subset  $H$  under the diffeomorphism  $L_g$ . Because  $G \setminus H$  is the union of the cosets of  $H$  other than  $H$  itself, it is open, and therefore  $H$  is closed in  $G$ . Because  $H$  is both open and closed, it is a union of components.  $\square$

If  $G$  is a group and  $S \subseteq G$ , the **subgroup generated by  $S$**  is the smallest subgroup containing  $S$  (i.e., the intersection of all subgroups containing  $S$ ).

► **Exercise 7.13.** Given a group  $G$  and a subset  $S \subseteq G$ , show that the subgroup generated by  $S$  is equal to the set of all elements of  $G$  that can be expressed as finite products of elements of  $S$  and their inverses.

**Proposition 7.14.** *Suppose  $G$  is a Lie group, and  $W \subseteq G$  is any neighborhood of the identity.*

- (a)  $W$  generates an open subgroup of  $G$ .
- (b) If  $W$  is connected, it generates a connected open subgroup of  $G$ .
- (c) If  $G$  is connected, then  $W$  generates  $G$ .

*Proof.* Let  $W \subseteq G$  be any neighborhood of the identity, and let  $H$  be the subgroup generated by  $W$ . As a matter of notation, if  $A$  and  $B$  are subsets of  $G$ , let us write

$$AB = \{ab : a \in A, b \in B\}, \quad A^{-1} = \{a^{-1} : a \in A\}. \quad (7.5)$$

For each positive integer  $k$ , let  $W_k$  denote the set of all elements of  $G$  that can be expressed as products of  $k$  or fewer elements of  $W \cup W^{-1}$ . By Exercise 7.13,  $H$  is

the union of all the sets  $W_k$  as  $k$  ranges over the positive integers. Now,  $W^{-1}$  is open because it is the image of  $W$  under the inversion map, which is a diffeomorphism. Thus,  $W_1 = W \cup W^{-1}$  is open, and for each  $k > 1$  we have

$$W_k = W_1 W_{k-1} = \bigcup_{g \in W_1} L_g(W_{k-1}).$$

Because each  $L_g$  is a diffeomorphism, it follows by induction that each  $W_k$  is open, and thus  $H$  is open.

Next suppose  $W$  is connected. Then  $W^{-1}$  is also connected because it is a diffeomorphic image of  $W$ , and  $W_1 = W \cup W^{-1}$  is connected because it is a union of connected sets with the identity in common. Therefore,  $W_2 = m(W_1 \times W_1)$  is connected because it is the image of a connected space under the continuous multiplication map  $m$ , and it follows by induction that  $W_k = m(W_1 \times W_{k-1})$  is connected for each  $k$ . Thus,  $H = \bigcup_k W_k$  is connected because it is a union of connected subsets with the identity in common.

Finally, assume  $G$  is connected. Since  $H$  is an open subgroup, it is also closed by Lemma 7.12, and it is not empty because it contains the identity. Thus  $H = G$ .  $\square$

If  $G$  is a Lie group, the connected component of  $G$  containing the identity is called the *identity component of  $G$* .

**Proposition 7.15.** *Let  $G$  be a Lie group and let  $G_0$  be its identity component. Then  $G_0$  is a normal subgroup of  $G$ , and is the only connected open subgroup. Every connected component of  $G$  is diffeomorphic to  $G_0$ .*

*Proof.* Problem 7-7.  $\square$

Now we move beyond the open subgroups to more general Lie subgroups. The following proposition shows how to produce many more examples of embedded Lie subgroups.

**Proposition 7.16.** *Let  $F : G \rightarrow H$  be a Lie group homomorphism. The kernel of  $F$  is a properly embedded Lie subgroup of  $G$ , whose codimension is equal to the rank of  $F$ .*

*Proof.* Because  $F$  has constant rank, its kernel  $F^{-1}(e)$  is a properly embedded submanifold of codimension equal to  $\text{rank } F$ . It is thus a Lie subgroup by Proposition 7.11.  $\square$

Complementary to the preceding result about kernels is the following result about images. (In Chapter 21, we will prove the analogous result for images of arbitrary Lie group homomorphisms, not just injective ones; see Theorem 21.27.)

**Proposition 7.17.** *If  $F : G \rightarrow H$  is an injective Lie group homomorphism, the image of  $F$  has a unique smooth manifold structure such that  $F(G)$  is a Lie subgroup of  $H$  and  $F : G \rightarrow F(G)$  is a Lie group isomorphism.*

*Proof.* Since a Lie group homomorphism has constant rank, it follows from the global rank theorem that  $F$  is a smooth immersion. Proposition 5.18 shows that

$F(G)$  has a unique smooth manifold structure such that it is an immersed submanifold of  $H$  and  $F$  is a diffeomorphism onto its image. It is a Lie group (because  $G$  is), and it is a subgroup for algebraic reasons, so it is a Lie subgroup. Because  $F: G \rightarrow F(G)$  is a group isomorphism and a diffeomorphism, it is a Lie group isomorphism.  $\square$

**Example 7.18 (Embedded Lie Subgroups).**

- (a) The subgroup  $GL^+(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$  described in Example 7.3(b) is an open subgroup and thus an embedded Lie subgroup.
- (b) The circle  $\mathbb{S}^1$  is an embedded Lie subgroup of  $\mathbb{C}^*$  because it is a subgroup and an embedded submanifold.
- (c) The set  $SL(n, \mathbb{R})$  of  $n \times n$  real matrices with determinant equal to 1 is called the *special linear group of degree  $n$* . Because  $SL(n, \mathbb{R})$  is the kernel of the Lie group homomorphism  $\det: GL(n, \mathbb{R}) \rightarrow \mathbb{R}^*$ , it is a properly embedded Lie subgroup. Because the determinant function is surjective, it is a smooth submersion by the global rank theorem, so  $SL(n, \mathbb{R})$  has dimension  $n^2 - 1$ .
- (d) Let  $n$  be a positive integer, and define a map  $\beta: GL(n, \mathbb{C}) \rightarrow GL(2n, \mathbb{R})$  by replacing each complex matrix entry  $a + ib$  with the  $2 \times 2$  block  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ :

$$\beta \begin{pmatrix} a_1^1 + ib_1^1 & \dots & a_1^n + ib_1^n \\ \vdots & & \vdots \\ a_n^1 + ib_n^1 & \dots & a_n^n + ib_n^n \end{pmatrix} = \begin{pmatrix} a_1^1 - b_1^1 & a_1^n - b_1^n \\ b_1^1 & a_1^1 & \dots & b_1^n & a_1^n \\ \vdots & & & \vdots & \\ a_n^1 - b_n^1 & a_n^n - b_n^n \\ b_n^1 & a_n^1 & \dots & b_n^n & a_n^n \end{pmatrix}.$$

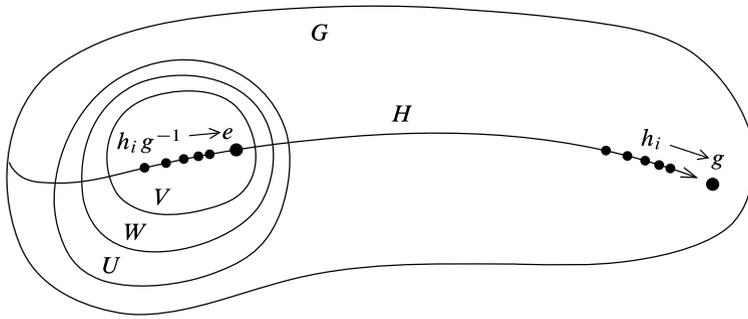
It is straightforward to verify that  $\beta$  is an injective Lie group homomorphism whose image is a properly embedded Lie subgroup of  $GL(2n, \mathbb{R})$ . Thus,  $GL(n, \mathbb{C})$  is isomorphic to this Lie subgroup of  $GL(2n, \mathbb{R})$ . (You can check that  $\beta$  arises naturally from the identification of  $(x^1 + iy^1, \dots, x^n + iy^n) \in \mathbb{C}^n$  with  $(x^1, y^1, \dots, x^n, y^n) \in \mathbb{R}^{2n}$ .)

- (e) The subgroup  $SL(n, \mathbb{C}) \subseteq GL(n, \mathbb{C})$  consisting of complex matrices of determinant 1 is called the *complex special linear group of degree  $n$* . It is the kernel of the Lie group homomorphism  $\det: GL(n, \mathbb{C}) \rightarrow \mathbb{C}^*$ . This homomorphism is surjective, so it is a smooth submersion by the global rank theorem. Therefore,  $SL(n, \mathbb{C}) = \text{Ker}(\det)$  is a properly embedded Lie subgroup whose codimension is equal to  $\dim \mathbb{C}^* = 2$  and whose dimension is therefore  $2n^2 - 2$ .  $\square$

Finally, here is an example of a Lie subgroup that is not embedded.

**Example 7.19 (A Dense Lie Subgroup of the Torus).** Let  $H \subseteq \mathbb{T}^2$  be the dense submanifold of the torus that is the image of the immersion  $\gamma: \mathbb{R} \rightarrow \mathbb{T}^2$  defined in Example 4.20. It is easy to check that  $\gamma$  is an injective Lie group homomorphism, and thus  $H$  is an immersed Lie subgroup of  $\mathbb{T}^2$  by Proposition 7.17.  $\square$

► **Exercise 7.20.** Let  $S \subseteq \mathbb{T}^3$  be the image of the subgroup  $H$  of the preceding example under the obvious embedding  $\mathbb{T}^2 \hookrightarrow \mathbb{T}^3$ . Show that  $S$  is a Lie subgroup of



**Fig. 7.1** An embedded Lie subgroup is closed

$\mathbb{T}^3$  that is not closed, embedded, or dense; but its closure is a properly embedded Lie subgroup of  $\mathbb{T}^3$ .

In Chapter 20, we will see that the subgroup  $S \subseteq \mathbb{T}^3$  described in the preceding exercise is typical of nonembedded Lie subgroups: they are all dense subgroups of properly embedded Lie subgroups (see Problem 20-10).

In general, smooth submanifolds can be closed without being embedded (as is, for example, the figure-eight curve of Example 5.19) or embedded without being closed (as is the open unit ball in  $\mathbb{R}^n$ ). However, as the next theorem shows, Lie subgroups have the remarkable property that closedness and embeddedness are not independent. This means that every embedded Lie subgroup is properly embedded.

**Theorem 7.21.** *Suppose  $G$  is a Lie group and  $H \subseteq G$  is a Lie subgroup. Then  $H$  is closed in  $G$  if and only if it is embedded.*

*Proof.* Assume first that  $H$  is embedded in  $G$ . To prove that  $H$  is closed, let  $g$  be an arbitrary point of  $\bar{H}$ . Then there is a sequence  $(h_i)$  of points in  $H$  converging to  $g$  (Fig. 7.1). Let  $U$  be the domain of a slice chart for  $H$  containing the identity, and let  $W$  be a smaller neighborhood of  $e$  such that  $\bar{W} \subseteq U$ . By Problem 7-6, there is a neighborhood  $V$  of  $e$  with the property that  $g_1 g_2^{-1} \in W$  whenever  $g_1, g_2 \in V$ .

Because  $h_i g^{-1} \rightarrow e$ , by discarding finitely many terms of the sequence we may assume that  $h_i g^{-1} \in V$  for all  $i$ . This implies that

$$h_i h_j^{-1} = (h_i g^{-1}) (h_j g^{-1})^{-1} \in W$$

for all  $i$  and  $j$ . Fixing  $j$  and letting  $i \rightarrow \infty$ , we find that  $h_i h_j^{-1} \rightarrow g h_j^{-1} \in \bar{W} \subseteq U$ . Since  $H \cap U$  is a slice, it is closed in  $U$ , and therefore  $g h_j^{-1} \in H$ , which implies  $g \in H$ . Thus  $H$  is closed.

Conversely, assume  $H$  is a closed Lie subgroup, and let  $m = \dim H$  and  $n = \dim G$ . We need to show that  $H$  is an embedded submanifold of  $G$ . If  $m = n$ , then  $H$  is embedded by Proposition 5.21(a), so we may assume henceforth that  $m < n$ .

It suffices to show that for some  $h_1 \in H$ , there is a neighborhood  $U_1$  of  $h_1$  in  $G$  such that  $H \cap U_1$  is an embedded submanifold of  $U_1$ ; for then if  $h$  is any other point

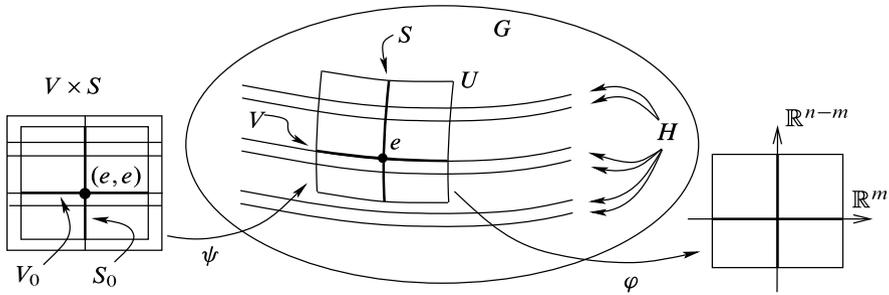


Fig. 7.2 Finding a slice chart

of  $H$ , right translation  $R_{hh^{-1}}: G \rightarrow G$  is a diffeomorphism of  $G$  that takes  $H$  to  $H$ , and takes  $U_1$  to a neighborhood  $U'_1$  of  $h$  such that  $U'_1 \cap H$  is embedded in  $U'_1$ , so it follows from the local slice criterion that  $H$  is an embedded submanifold of  $G$ .

Because every immersed submanifold is locally embedded (Proposition 5.22), there exist a neighborhood  $V$  of  $e$  in  $H$  and a slice chart  $(U, \varphi)$  for  $V$  in  $G$  centered at  $e$ . By shrinking  $U$  if necessary, we may assume that it is a coordinate cube, and  $U \cap V$  is the set of points whose coordinates are of the form  $(x^1, \dots, x^m, 0, \dots, 0)$ . Let  $S \subseteq U$  be the set of points with coordinates of the form  $(0, \dots, 0, x^{m+1}, \dots, x^n)$ ; it is the slice “perpendicular” to  $U \cap V$  in these coordinates. Then  $S$  is an embedded submanifold of  $U$  and hence of  $G$ . Note that in these coordinates,  $T_e V$  is spanned by the first  $m$  coordinate vectors and  $T_e S$  by the last  $n - m$ , so  $T_e G = T_e V \oplus T_e S$  (Fig. 7.2).

Now consider the map  $\psi: V \times S \rightarrow G$  obtained by restricting group multiplication:  $\psi(v, s) = vs$ . Since  $\psi(v, e) = v$  for  $v \in V$  and  $\psi(e, s) = s$  for  $s \in S$ , it follows easily that the differential of  $\psi$  at  $(e, e)$  satisfies  $d\psi(X, 0) = X$  and  $d\psi(0, Y) = Y$  for  $X \in T_e V$ ,  $Y \in T_e S$ , and therefore  $d\psi_{(e,e)}$  is bijective. By the inverse function theorem, there are connected neighborhoods  $W_0$  of  $(e, e)$  in  $V \times S$  and  $U_0$  of  $e$  in  $G$  such that  $\psi: W_0 \rightarrow U_0$  is a diffeomorphism. Shrinking the neighborhoods if necessary, we may assume that  $W_0 = V_0 \times S_0$ , where  $V_0$  and  $S_0$  are neighborhoods of  $e$  in  $V$  and  $S$ , respectively.

Let  $K = S_0 \cap H$ . There are two things we need to show about the set  $K$ :

- (a)  $\psi(V_0 \times K) = H \cap U_0$ .
- (b)  $K$  is a discrete set in the topology of  $H$ .

To prove (a), let  $(v, s) \in V_0 \times S_0$  be arbitrary. Since  $H$  is a subgroup and  $V_0 \subseteq H$ , it follows that  $vs \in H$  if and only if  $s \in H$ , which is to say that  $\psi(v, s) \in H \cap U_0$  if and only if  $(v, s) \in V_0 \times K$ . To prove (b), suppose  $h \in K$ . Right translation  $R_h: H \rightarrow H$  is a diffeomorphism of  $H$  taking  $e$  to  $h$  and taking  $V_0$  to a neighborhood  $V_h$  of  $h$  in  $H$ . Note that  $V_h = R_h(V_0) = \psi(V_0 \times \{h\})$ , while  $K = \psi(\{e\} \times K)$ . Since  $\psi$  is injective on  $V_0 \times S_0$ , it follows that

$$V_h \cap K = \psi(\{e\} \times \{h\}) = \{h\}.$$

Thus each point  $h \in K$  is isolated in  $H$ , which implies that  $K$  is discrete.

Since  $K$  is a discrete subset of the manifold  $H$ , it is countable, and since  $H$  is closed in  $G$ , it follows that  $K = S_0 \cap H$  is closed in  $S_0$ . Thus, by Corollary A.59, there is a point  $h_1 \in K$  that is isolated in  $S_0$ . (This step fails if  $H$  is not closed—for example, if  $H$  were a dense subgroup of the torus, then  $K$  would be dense in  $S_0$ .) This means there is a neighborhood  $S_1$  of  $h_1$  in  $S_0$  such that  $S_1 \cap H = \{h_1\}$ . Then  $U_1 = \psi(V_0 \times S_1)$  is a neighborhood of  $h_1$  in  $G$  with the property that  $U_1 \cap H$  is the slice  $\psi(V_0 \times \{h_1\})$  in  $U_1$ . As explained at the beginning of the proof, the existence of such a neighborhood for one point of  $H$  implies that  $H$  is embedded.  $\square$

In Chapter 20, we will be able to prove a significantly strengthened form of this theorem, called the *closed subgroup theorem*, which asserts that every subgroup of a Lie group that is topologically a closed subset (but not assumed to be a submanifold) is automatically a properly embedded Lie subgroup.

## Group Actions and Equivariant Maps

The most important applications of Lie groups to smooth manifold theory involve actions by Lie groups on other manifolds. If  $G$  is a group and  $M$  is a set, a **left action of  $G$  on  $M$**  is a map  $G \times M \rightarrow M$ , often written as  $(g, p) \mapsto g \cdot p$ , that satisfies

$$\begin{aligned} g_1 \cdot (g_2 \cdot p) &= (g_1 g_2) \cdot p && \text{for all } g_1, g_2 \in G \text{ and } p \in M; \\ e \cdot p &= p && \text{for all } p \in M. \end{aligned} \tag{7.6}$$

A **right action** is defined analogously as a map  $M \times G \rightarrow M$  with the appropriate composition law:

$$\begin{aligned} (p \cdot g_1) \cdot g_2 &= p \cdot (g_1 g_2) && \text{for all } g_1, g_2 \in G \text{ and } p \in M; \\ p \cdot e &= p && \text{for all } p \in M. \end{aligned}$$

If  $M$  is a topological space and  $G$  is a topological group, an action of  $G$  on  $M$  is said to be a **continuous action** if the defining map  $G \times M \rightarrow M$  or  $M \times G \rightarrow M$  is continuous. In this case,  $M$  is said to be a **(left or right)  $G$ -space**. If in addition  $M$  is a smooth manifold with or without boundary,  $G$  is a Lie group, and the defining map is smooth, then the action is said to be a **smooth action**. We are primarily interested in smooth actions of Lie groups on smooth manifolds.

Sometimes it is useful to give a name to an action, such as  $\theta: G \times M \rightarrow M$ , with the action of a group element  $g$  on a point  $p$  usually written as  $\theta_g(p)$ . In terms of this notation, the conditions (7.6) for a left action read

$$\begin{aligned} \theta_{g_1} \circ \theta_{g_2} &= \theta_{g_1 g_2}, \\ \theta_e &= \text{Id}_M, \end{aligned} \tag{7.7}$$

while for a right action the first equation is replaced by

$$\theta_{g_2} \circ \theta_{g_1} = \theta_{g_1 g_2}.$$

For a smooth action, each map  $\theta_g: M \rightarrow M$  is a diffeomorphism, because  $\theta_{g^{-1}}$  is a smooth inverse for it.

For left actions, we generally use the notations  $g \cdot p$  and  $\theta_g(p)$  interchangeably. The latter notation contains a bit more information, and is useful when it is important to specify the particular action under consideration, while the former is often more convenient when the action is understood. For right actions, the notation  $p \cdot g$  is generally preferred because of the way composition works.

A right action can always be converted to a left action by the trick of defining  $g \cdot p$  to be  $p \cdot g^{-1}$ , and a left action can similarly be converted to a right action. Thus, any results about left actions can be translated into results about right actions, and vice versa. We usually focus our attention on left actions, because their group law (7.7) has the property that multiplication of group elements corresponds to composition of maps. However, there are some circumstances in which right actions arise naturally; we will see several such actions later in the book.

Lie group actions typically arise in situations involving some kind of symmetry. For example, if  $M$  is a vector space or smooth manifold endowed with a metric or other geometric structure, the set of diffeomorphisms of  $M$  that preserve the structure (called the *symmetry group* of the structure) frequently turns out to be a Lie group acting smoothly on  $M$ .

Throughout the book, we use the following standard terminology regarding group actions. Suppose  $\theta: G \times M \rightarrow M$  is a left action of a group  $G$  on a set  $M$ . (The definitions for right actions are similar. For these definitions, no continuity or smoothness assumption is necessary.)

- For each  $p \in M$ , the *orbit of  $p$* , denoted by  $G \cdot p$ , is the set of all images of  $p$  under the action by elements of  $G$ :

$$G \cdot p = \{g \cdot p : g \in G\}.$$

- For each  $p \in M$ , the *isotropy group* or *stabilizer of  $p$* , denoted by  $G_p$ , is the set of elements of  $G$  that fix  $p$ :

$$G_p = \{g \in G : g \cdot p = p\}.$$

The definition of a group action guarantees that  $G_p$  is a subgroup of  $G$ .

- The action is said to be *transitive* if for every pair of points  $p, q \in M$ , there exists  $g \in G$  such that  $g \cdot p = q$ , or equivalently if the only orbit is all of  $M$ .
- The action is said to be *free* if the only element of  $G$  that fixes any element of  $M$  is the identity:  $g \cdot p = p$  for some  $p \in M$  implies  $g = e$ , or equivalently if every isotropy group is trivial.

Here are some examples of Lie group actions on manifolds. We will see more in Chapter 21.

**Example 7.22 (Lie Group Actions).**

- (a) If  $G$  is any Lie group and  $M$  is any smooth manifold, the **trivial action of  $G$  on  $M$**  is defined by  $g \cdot p = p$  for all  $g \in G$  and  $p \in M$ . It is a smooth action, for which each orbit is a single point and each isotropy group is all of  $G$ .
- (b) The **natural action of  $GL(n, \mathbb{R})$  on  $\mathbb{R}^n$**  is the left action given by matrix multiplication:  $(A, x) \mapsto Ax$ , considering  $x \in \mathbb{R}^n$  as a column matrix. This is an action because  $I_n x = x$  and matrix multiplication is associative:  $(AB)x = A(Bx)$ . It is smooth because the components of  $Ax$  depend polynomially on the matrix entries of  $A$  and the components of  $x$ . Because any nonzero vector can be taken to any other by some invertible linear transformation, there are exactly two orbits:  $\{0\}$  and  $\mathbb{R}^n \setminus \{0\}$ .
- (c) Every Lie group  $G$  acts smoothly on itself by left translation. Given any two points  $g_1, g_2 \in G$ , there is a unique left translation of  $G$  taking  $g_1$  to  $g_2$ , namely left translation by  $g_2 g_1^{-1}$ ; thus the action is both free and transitive. More generally, if  $H$  is a Lie subgroup of  $G$ , then the restriction of the multiplication map to  $H \times G \rightarrow G$  defines a smooth and free (but generally not transitive) left action of  $H$  on  $G$ . Similar observations apply to right translations.
- (d) Every Lie group acts smoothly on itself by conjugation:  $g \cdot h = ghg^{-1}$ .
- (e) An action of a discrete group  $\Gamma$  on a manifold  $M$  is smooth if and only if for each  $g \in \Gamma$ , the map  $p \mapsto g \cdot p$  is a smooth map from  $M$  to itself. Thus, for example,  $\mathbb{Z}^n$  acts smoothly and freely on  $\mathbb{R}^n$  by left translation:

$$(m^1, \dots, m^n) \cdot (x^1, \dots, x^n) = (m^1 + x^1, \dots, m^n + x^n). \quad //$$

Another important class of Lie group actions arises from covering maps. Suppose  $E$  and  $M$  are topological spaces, and  $\pi : E \rightarrow M$  is a (topological) covering map. An **automorphism of  $\pi$**  (also called a **deck transformation** or **covering transformation**) is a homeomorphism  $\varphi : E \rightarrow E$  such that  $\pi \circ \varphi = \pi$ :

$$\begin{array}{ccc}
 E & \xrightarrow{\varphi} & E \\
 \searrow \pi & & \swarrow \pi \\
 & M &
 \end{array} \tag{7.8}$$

The set  $\text{Aut}_\pi(E)$  of all automorphisms of  $\pi$ , called the **automorphism group of  $\pi$** , is a group under composition, acting on  $E$  on the left. It can be shown that  $\text{Aut}_\pi(E)$  acts transitively on each fiber of  $\pi$  if and only if  $\pi$  is a **normal covering map**, which means that  $\pi_*(\pi_1(E, q))$  is a normal subgroup of  $\pi_1(M, \pi(q))$  for every  $q \in E$  (see, for example, [LeeTM, Cor. 12.5]).

**Proposition 7.23.** *Suppose  $E$  and  $M$  are smooth manifolds with or without boundary, and  $\pi : E \rightarrow M$  is a smooth covering map. With the discrete topology, the automorphism group  $\text{Aut}_\pi(E)$  is a zero-dimensional Lie group acting smoothly and freely on  $E$ .*

*Proof.* Suppose  $\varphi \in \text{Aut}_\pi(E)$  is an automorphism that fixes a point  $p \in E$ . Simply by rotating diagram (7.8), we can consider  $\varphi$  as a lift of  $\pi$ :

$$\begin{array}{ccc} & & E \\ & \nearrow \varphi & \downarrow \pi \\ E & \xrightarrow{\pi} & M. \end{array}$$

Since the identity map of  $E$  is another such lift that agrees with  $\varphi$  at  $p$ , the unique lifting property of covering maps (Proposition A.77(a)) guarantees that  $\varphi = \text{Id}_E$ . Thus, the action of  $\text{Aut}_\pi(E)$  is free.

To show that  $\text{Aut}_\pi(E)$  is a Lie group, we need only verify that it is countable. Let  $q \in E$  be arbitrary, let  $p = \pi(q) \in M$ , and let  $U \subseteq M$  be an evenly covered neighborhood of  $p$ . Because  $E$  is second-countable,  $\pi^{-1}(U)$  has countably many components, and because each component contains exactly one point of  $\pi^{-1}(p)$ , it follows that  $\pi^{-1}(p)$  is countable. Let  $\theta^{(q)}: \text{Aut}_\pi(E) \rightarrow E$  be the map  $\theta^{(q)}(\varphi) = \varphi(q)$ . Then  $\theta^{(q)}$  maps  $\text{Aut}_\pi(E)$  into  $\pi^{-1}(p)$ , and the fact that the action is free implies that it is injective; thus  $\text{Aut}_\pi(E)$  is countable.

Smoothness of the action follows from Theorem 4.29.  $\square$

### Equivariant Maps

For some manifolds with group actions, there is an easily verified sufficient condition for a smooth map to have constant rank. Suppose  $G$  is a Lie group, and  $M$  and  $N$  are both smooth manifolds endowed with smooth left or right  $G$ -actions. A map  $F: M \rightarrow N$  is said to be **equivariant** with respect to the given  $G$ -actions if for each  $g \in G$ ,

$$F(g \cdot p) = g \cdot F(p) \quad (\text{for left actions}),$$

$$F(p \cdot g) = F(p) \cdot g \quad (\text{for right actions}).$$

Equivalently, if  $\theta$  and  $\varphi$  are the given actions on  $M$  and  $N$ , respectively,  $F$  is equivariant if the following diagram commutes for each  $g \in G$ :

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \theta_g \downarrow & & \downarrow \varphi_g \\ M & \xrightarrow{F} & N. \end{array}$$

This condition is also expressed by saying that  $F$  **intertwines  $\theta$  and  $\varphi$** .

**Example 7.24.** Let  $v = (v^1, \dots, v^n) \in \mathbb{R}^n$  be any fixed nonzero vector. Define smooth left actions of  $\mathbb{R}$  on  $\mathbb{R}^n$  and  $\mathbb{T}^n$  by

$$\begin{aligned} t \cdot (x^1, \dots, x^n) &= (x^1 + tv^1, \dots, x^n + tv^n), \\ t \cdot (z^1, \dots, z^n) &= (e^{2\pi itv^1} z^1, \dots, e^{2\pi itv^n} z^n), \end{aligned}$$

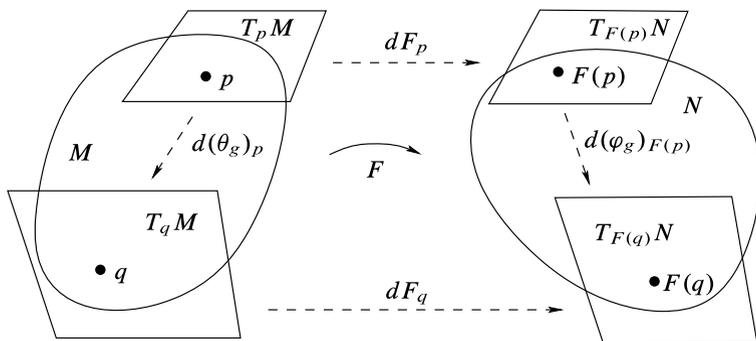


Fig. 7.3 The equivariant rank theorem

for  $t \in \mathbb{R}$ ,  $(x^1, \dots, x^n) \in \mathbb{R}^n$ , and  $(z^1, \dots, z^n) \in \mathbb{T}^n$ . The smooth map  $\varepsilon^n: \mathbb{R}^n \rightarrow \mathbb{T}^n$  given by  $\varepsilon^n(x^1, \dots, x^n) = (e^{2\pi i x^1}, \dots, e^{2\pi i x^n})$  is equivariant with respect to these actions. //

The following generalization of Theorem 7.5 is an extremely useful tool for proving that certain maps have constant rank.

**Theorem 7.25 (Equivariant Rank Theorem).** *Let  $M$  and  $N$  be smooth manifolds and let  $G$  be a Lie group. Suppose  $F: M \rightarrow N$  is a smooth map that is equivariant with respect to a transitive smooth  $G$ -action on  $M$  and any smooth  $G$ -action on  $N$ . Then  $F$  has constant rank. Thus, if  $F$  is surjective, it is a smooth submersion; if it is injective, it is a smooth immersion; and if it is bijective, it is a diffeomorphism.*

*Proof.* Let  $\theta$  and  $\varphi$  denote the  $G$ -actions on  $M$  and  $N$ , respectively, and let  $p$  and  $q$  be arbitrary points in  $M$ . Choose  $g \in G$  such that  $\theta_g(p) = q$ . (Such a  $g$  exists because we are assuming that  $G$  acts transitively on  $M$ .) Because  $\varphi_g \circ F = F \circ \theta_g$ , the following diagram commutes (see Fig. 7.3):

$$\begin{array}{ccc}
 T_p M & \xrightarrow{dF_p} & T_{F(p)} N \\
 d(\theta_g)_p \downarrow & & \downarrow d(\varphi_g)_{F(p)} \\
 T_q M & \xrightarrow{dF_q} & T_{F(q)} N.
 \end{array}$$

Because the vertical linear maps in this diagram are isomorphisms, the horizontal ones have the same rank. In other words, the rank of  $F$  is the same at any two arbitrary points  $p, q \in M$ , so  $F$  has constant rank. The final statement follows from the global rank theorem. □

Here is an important application of the equivariant rank theorem. Suppose  $G$  is a Lie group,  $M$  is a smooth manifold, and  $\theta: G \times M \rightarrow M$  is a smooth left action. (The definitions for right actions are analogous.) For each  $p \in M$ , define a map

$\theta^{(p)}: G \rightarrow M$  by

$$\theta^{(p)}(g) = g \cdot p. \quad (7.9)$$

This is often called the **orbit map**, because its image is the orbit  $G \cdot p$ . In addition, the preimage  $(\theta^{(p)})^{-1}(p)$  is the isotropy group  $G_p$ .

**Proposition 7.26 (Properties of the Orbit Map).** *Suppose  $\theta$  is a smooth left action of a Lie group  $G$  on a smooth manifold  $M$ . For each  $p \in M$ , the orbit map  $\theta^{(p)}: G \rightarrow M$  is smooth and has constant rank, so the isotropy group  $G_p = (\theta^{(p)})^{-1}(p)$  is a properly embedded Lie subgroup of  $G$ . If  $G_p = \{e\}$ , then  $\theta^{(p)}$  is an injective smooth immersion, so the orbit  $G \cdot p$  is an immersed submanifold of  $M$ .*

*Remark.* It is a fact that every orbit is an immersed submanifold of  $M$ , not just the ones corresponding to trivial isotropy groups; but the proof of that fact will have to wait until Chapter 21 (see Problem 21-17).

*Proof.* The orbit map is smooth because it is equal to the composition

$$G \approx G \times \{p\} \hookrightarrow G \times M \xrightarrow{\theta} M.$$

It follows from the definition of a group action that  $\theta^{(p)}$  is equivariant with respect to the action of  $G$  on itself by left translation and the given action on  $M$ :

$$\theta^{(p)}(g'g) = (g'g) \cdot p = g' \cdot (g \cdot p) = g' \cdot \theta^{(p)}(g).$$

Since  $G$  acts transitively on itself, the equivariant rank theorem shows that  $\theta^{(p)}$  has constant rank. Thus,  $G_p$  is a properly embedded submanifold by Theorem 5.12, and a Lie subgroup by Proposition 7.11.

Now suppose  $G_p = \{e\}$ . If  $\theta^{(p)}(g') = \theta^{(p)}(g)$ , then

$$g' \cdot p = g \cdot p \Rightarrow (g^{-1}g') \cdot p = p \Rightarrow g^{-1}g' = e \Rightarrow g = g',$$

showing that  $\theta^{(p)}$  is injective. By the equivariant rank theorem, it is a smooth immersion, and thus the orbit (endowed with a suitable topology and smooth structure) is an immersed submanifold by Proposition 5.18.  $\square$

Next we use the equivariant rank theorem to identify some important Lie subgroups of the general linear groups.

**Example 7.27 (The Orthogonal Group).** A real  $n \times n$  matrix  $A$  is said to be **orthogonal** if as a linear map  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  it preserves the Euclidean dot product:

$$(Ax) \cdot (Ay) = x \cdot y \quad \text{for all } x, y \in \mathbb{R}^n.$$

The set  $O(n)$  of all orthogonal  $n \times n$  matrices is a subgroup of  $GL(n, \mathbb{R})$ , called the **orthogonal group of degree  $n$** . It is easy to check that a matrix  $A$  is orthogonal if and only if it takes the standard basis of  $\mathbb{R}^n$  to an orthonormal basis, which is equivalent to the columns of  $A$  being orthonormal. Since the  $(i, j)$ -entry of the matrix  $A^T A$

(where  $A^T$  represents the transpose of  $A$ ) is the dot product of the  $i$ th and  $j$ th columns of  $A$ , this condition is also equivalent to the requirement that  $A^T A = I_n$ .

Define a smooth map  $\Phi: \text{GL}(n, \mathbb{R}) \rightarrow \text{M}(n, \mathbb{R})$  by  $\Phi(A) = A^T A$ . Then  $\text{O}(n)$  is equal to the level set  $\Phi^{-1}(I_n)$ . To show that  $\Phi$  has constant rank and therefore that  $\text{O}(n)$  is an embedded Lie subgroup, we show that  $\Phi$  is equivariant with respect to suitable right actions of  $\text{GL}(n, \mathbb{R})$ . Let  $\text{GL}(n, \mathbb{R})$  act on itself by right multiplication, and define a right action of  $\text{GL}(n, \mathbb{R})$  on  $\text{M}(n, \mathbb{R})$  by

$$X \cdot B = B^T X B \quad \text{for } X \in \text{M}(n, \mathbb{R}), B \in \text{GL}(n, \mathbb{R}).$$

It is easy to check that this is a smooth action, and  $\Phi$  is equivariant because

$$\Phi(AB) = (AB)^T (AB) = B^T A^T AB = B^T \Phi(A) B = \Phi(A) \cdot B.$$

Thus,  $\text{O}(n)$  is a properly embedded Lie subgroup of  $\text{GL}(n, \mathbb{R})$ . It is compact because it is closed and bounded in  $\text{M}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ : closed because it is a level set of  $\Phi$ , and bounded because every  $A \in \text{O}(n)$  has columns of norm 1, and therefore satisfies  $|A| = \sqrt{n}$ .

To determine the dimension of  $\text{O}(n)$ , we need to compute the rank of  $\Phi$ . Because the rank is constant, it suffices to compute it at the identity  $I_n \in \text{GL}(n, \mathbb{R})$ . Thus for any  $B \in T_{I_n} \text{GL}(n, \mathbb{R}) = \text{M}(n, \mathbb{R})$ , let  $\gamma: (-\varepsilon, \varepsilon) \rightarrow \text{GL}(n, \mathbb{R})$  be the curve  $\gamma(t) = I_n + tB$ , and compute

$$d\Phi_{I_n}(B) = \left. \frac{d}{dt} \right|_{t=0} \Phi \circ \gamma(t) = \left. \frac{d}{dt} \right|_{t=0} (I_n + tB)^T (I_n + tB) = B^T + B.$$

From this formula, it is evident that the image of  $d\Phi_{I_n}$  is contained in the vector space of *symmetric* matrices. Conversely, if  $B \in \text{M}(n, \mathbb{R})$  is an arbitrary symmetric  $n \times n$  matrix, then  $d\Phi_{I_n}(\frac{1}{2}B) = B$ . It follows that the image of  $d\Phi_{I_n}$  is exactly the space of symmetric matrices. This is a linear subspace of  $\text{M}(n, \mathbb{R})$  of dimension  $n(n+1)/2$ , because each symmetric matrix is uniquely determined by its values on and above the main diagonal. It follows that  $\text{O}(n)$  is an embedded Lie subgroup of dimension  $n^2 - n(n+1)/2 = n(n-1)/2$ . //

**Example 7.28 (The Special Orthogonal Group).** The *special orthogonal group of degree  $n$*  is defined as  $\text{SO}(n) = \text{O}(n) \cap \text{SL}(n, \mathbb{R}) \subseteq \text{GL}(n, \mathbb{R})$ . Because every matrix  $A \in \text{O}(n)$  satisfies

$$1 = \det I_n = \det(A^T A) = (\det A)(\det A^T) = (\det A)^2,$$

it follows that  $\det A = \pm 1$  for all  $A \in \text{O}(n)$ . Therefore,  $\text{SO}(n)$  is the open subgroup of  $\text{O}(n)$  consisting of matrices of positive determinant, and is therefore also an embedded Lie subgroup of dimension  $n(n-1)/2$  in  $\text{GL}(n, \mathbb{R})$ . It is a compact group because it is a closed subset of  $\text{O}(n)$ . //

**Example 7.29 (The Unitary Group).** For any complex matrix  $A$ , the *adjoint of  $A$*  is the matrix  $A^*$  formed by conjugating the entries of  $A$  and taking the transpose:  $A^* = \overline{A^T}$ . Observe that  $(AB)^* = (\overline{A\overline{B}})^T = \overline{B^T} \overline{A^T} = B^* A^*$ . For any positive integer  $n$ , the *unitary group of degree  $n$*  is the subgroup  $\text{U}(n) \subseteq \text{GL}(n, \mathbb{C})$  consisting

of complex  $n \times n$  matrices whose columns form an orthonormal basis for  $\mathbb{C}^n$  with respect to the Hermitian dot product  $z \cdot w = \sum_i z^i \overline{w^i}$ . It is straightforward to check that  $U(n)$  consists of those matrices  $A$  such that  $A^*A = I_n$ . Problem 7-13 shows that it is a properly embedded Lie subgroup of  $GL(n, \mathbb{C})$  of dimension  $n^2$ . //

**Example 7.30 (The Special Unitary Group).** The group  $SU(n) = U(n) \cap SL(n, \mathbb{C})$  is called the *complex special unitary group of degree  $n$* . Problem 7-14 shows that it is a properly embedded  $(n^2 - 1)$ -dimensional Lie subgroup of  $U(n)$ . Since the composition of smooth embeddings  $SU(n) \hookrightarrow U(n) \hookrightarrow GL(n, \mathbb{C})$  is again a smooth embedding, this implies that  $SU(n)$  is also embedded in  $GL(n, \mathbb{C})$ . //

### Semidirect Products

Group actions give us a powerful new way to construct Lie groups. Suppose  $H$  and  $N$  are Lie groups, and  $\theta: H \times N \rightarrow N$  is a smooth left action of  $H$  on  $N$ . It is said to be an *action by automorphisms* if for each  $h \in H$ , the map  $\theta_h: N \rightarrow N$  is a group automorphism of  $N$  (i.e., an isomorphism from  $N$  to itself). Given such an action, we define a new Lie group  $N \rtimes_{\theta} H$ , called a *semidirect product of  $H$  and  $N$* , as follows. As a smooth manifold,  $N \rtimes_{\theta} H$  is just the Cartesian product  $N \times H$ ; but the group multiplication is defined by

$$(n, h)(n', h') = (n\theta_h(n'), hh'). \quad (7.10)$$

Sometimes, if the action of  $H$  on  $N$  is understood or irrelevant, the semidirect product is denoted simply by  $N \rtimes H$ .

► **Exercise 7.31.** Verify that (7.10) does indeed define a Lie group structure on the manifold  $N \times H$ , with  $(e, e)$  as identity and  $(n, h)^{-1} = (\theta_{h^{-1}}(n^{-1}), h^{-1})$ .

**Example 7.32 (The Euclidean Group).** If we consider  $\mathbb{R}^n$  as a Lie group under addition, then the natural action of  $O(n)$  on  $\mathbb{R}^n$  is an action by automorphisms. The resulting semidirect product  $E(n) = \mathbb{R}^n \rtimes O(n)$  is called the *Euclidean group*; its multiplication is given by  $(b, A)(b', A') = (b + Ab', AA')$ . It acts on  $\mathbb{R}^n$  via

$$(b, A) \cdot x = b + Ax.$$

This action preserves lines, distances, and angle measures, and thus all of the relationships of Euclidean geometry. //

The next proposition details some basic properties of the semidirect product. Recall the notation  $AB$  defined in (7.5) for subsets  $A, B$  of a group  $G$ .

**Proposition 7.33 (Properties of Semidirect Products).** *Suppose  $N$  and  $H$  are Lie groups, and  $\theta$  is a smooth action of  $H$  on  $N$  by automorphisms. Let  $G = N \rtimes_{\theta} H$ .*

- (a) *The subsets  $\tilde{N} = N \times \{e\}$  and  $\tilde{H} = \{e\} \times H$  are closed Lie subgroups of  $G$  isomorphic to  $N$  and  $H$ , respectively.*

- (b)  $\tilde{N}$  is a normal subgroup of  $\tilde{G}$ .  
 (c)  $\tilde{N} \cap \tilde{H} = \{e, e\}$  and  $\tilde{N}\tilde{H} = G$ .

► **Exercise 7.34.** Prove the preceding proposition.

Thanks to the next theorem, many Lie groups can be realized as semidirect products of suitable subgroups.

**Theorem 7.35 (Characterization of Semidirect Products).** *Suppose  $G$  is a Lie group, and  $N, H \subseteq G$  are closed Lie subgroups such that  $N$  is normal,  $N \cap H = \{e\}$ , and  $NH = G$ . Then the map  $(n, h) \mapsto nh$  is a Lie group isomorphism between  $N \rtimes_{\theta} H$  and  $G$ , where  $\theta: H \times N \rightarrow N$  is the action by conjugation:  $\theta_h(n) = hnh^{-1}$ .*

*Proof.* Problem 7-18. □

Under the hypotheses of Theorem (7.35), we say that  $G$  is the **internal semidirect product of  $N$  and  $H$** . Some examples are described in Problem 7-20.

## Representations

Most of the Lie groups we have seen so far can be realized as Lie subgroups of  $\mathrm{GL}(n, \mathbb{R})$  or  $\mathrm{GL}(n, \mathbb{C})$ . It is natural to ask whether all Lie groups are of this form. The key to studying this question is the theory of *group representations*.

Recall that if  $V$  is a finite-dimensional real or complex vector space,  $\mathrm{GL}(V)$  denotes the group of invertible linear transformations of  $V$ , which is a Lie group isomorphic to  $\mathrm{GL}(n, \mathbb{R})$  or  $\mathrm{GL}(n, \mathbb{C})$  for  $n = \dim V$ . If  $G$  is a Lie group, a **(finite-dimensional) representation of  $G$**  is a Lie group homomorphism from  $G$  to  $\mathrm{GL}(V)$  for some  $V$ . (Although it is useful for many applications to consider also the case in which  $V$  is infinite-dimensional, in this book we consider only finite-dimensional representations.)

If a representation  $\rho: G \rightarrow \mathrm{GL}(V)$  is injective, it is said to be **faithful**. In that case, it follows from Proposition 7.17 that the image of  $\rho$  is a Lie subgroup of  $\mathrm{GL}(V)$ , and  $\rho$  gives a Lie group isomorphism between  $G$  and  $\rho(G) \subseteq \mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{R})$  or  $\mathrm{GL}(n, \mathbb{C})$ . Thus, a Lie group admits a faithful representation if and only if it is isomorphic to a Lie subgroup of  $\mathrm{GL}(n, \mathbb{R})$  or  $\mathrm{GL}(n, \mathbb{C})$  for some  $n$ . Not every Lie group admits such a representation. We do not yet have the technology to construct a counterexample, but Problem 21-26 asks you to prove that the universal covering group of  $\mathrm{SL}(2, \mathbb{R})$  has no faithful representation and therefore is not isomorphic to any matrix group.

Representation theory is a vast subject, with applications to fields as diverse as differential geometry, differential equations, harmonic analysis, number theory, quantum physics, and engineering; we can do no more than touch on it here.

### Example 7.36 (Lie Group Representations).

- (a) If  $G$  is any Lie subgroup of  $\mathrm{GL}(n, \mathbb{R})$ , the inclusion map  $G \hookrightarrow \mathrm{GL}(n, \mathbb{R}) = \mathrm{GL}(\mathbb{R}^n)$  is a faithful representation, called the **defining representation of  $G$** . The defining representation of a Lie subgroup of  $\mathrm{GL}(n, \mathbb{C})$  is defined similarly.

- (b) The inclusion map  $\mathbb{S}^1 \hookrightarrow \mathbb{C}^* \cong \mathrm{GL}(1, \mathbb{C})$  is a faithful representation of the circle group. More generally, the map  $\rho: \mathbb{T}^n \rightarrow \mathrm{GL}(n, \mathbb{C})$  given by

$$\rho(z^1, \dots, z^n) = \begin{pmatrix} z^1 & 0 & \dots & 0 \\ 0 & z^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & z^n \end{pmatrix}$$

is a faithful representation of  $\mathbb{T}^n$ .

- (c) Let  $\sigma: \mathbb{R}^n \rightarrow \mathrm{GL}(n+1, \mathbb{R})$  be the map that sends  $x \in \mathbb{R}^n$  to the matrix  $\sigma(x)$  defined in block form by

$$\sigma(x) = \begin{pmatrix} I_n & x \\ 0 & 1 \end{pmatrix},$$

where  $I_n$  is the  $n \times n$  identity matrix and  $x$  is regarded as an  $n \times 1$  column matrix. A straightforward computation shows that  $\sigma$  is a faithful representation of the additive Lie group  $\mathbb{R}^n$ .

- (d) Another faithful representation of  $\mathbb{R}^n$  is the map  $\mathbb{R}^n \rightarrow \mathrm{GL}(n, \mathbb{R})$  that sends  $(x^1, \dots, x^n)$  to the diagonal matrix whose diagonal entries are  $(e^{x^1}, \dots, e^{x^n})$ .  
 (e) Yet another representation of  $\mathbb{R}^n$  is the map  $\mathbb{R}^n \rightarrow \mathrm{GL}(n, \mathbb{C})$  sending  $x$  to the diagonal matrix with diagonal entries  $(e^{2\pi i x^1}, \dots, e^{2\pi i x^n})$ . This one is not faithful, because its kernel is the subgroup  $\mathbb{Z}^n \subseteq \mathbb{R}^n$ .  
 (f) Let  $E(n)$  be the Euclidean group (Example 7.32). A faithful representation of  $E(n)$  is given by the map  $\rho: E(n) \rightarrow \mathrm{GL}(n+1, \mathbb{R})$  defined in block form by

$$\rho(b, A) = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix},$$

where  $b$  is considered as a column matrix.

- (g) For positive integers  $n$  and  $d$ , let  $\mathcal{P}_d^n$  denote the vector space of real-valued polynomial functions  $p: \mathbb{R}^n \rightarrow \mathbb{R}$  of degree at most  $d$ . For any matrix  $A \in \mathrm{GL}(n, \mathbb{R})$ , define a linear map  $\tau_d^n(A): \mathcal{P}_d^n \rightarrow \mathcal{P}_d^n$  by

$$\tau_d^n(A)p = p \circ A^{-1}.$$

Problem 7-24 shows that the map  $\tau_d^n: \mathrm{GL}(n, \mathbb{R}) \rightarrow \mathrm{GL}(\mathcal{P}_d^n)$  is a faithful representation. //

There is a close connection between representations and group actions. Let  $G$  be a Lie group and  $V$  be a finite-dimensional vector space. An action of  $G$  on  $V$  is said to be a **linear action** if for each  $g \in G$ , the map from  $V$  to itself given by  $x \mapsto g \cdot x$  is linear. For example, if  $\rho: G \rightarrow \mathrm{GL}(V)$  is a representation of  $G$ , there is an associated smooth linear action of  $G$  on  $V$  given by  $g \cdot x = \rho(g)x$ . The next proposition shows that every linear action is of this type.

**Proposition 7.37.** *Let  $G$  be a Lie group and  $V$  be a finite-dimensional vector space. A smooth left action of  $G$  on  $V$  is linear if and only if it is of the form  $g \cdot x = \rho(g)x$  for some representation  $\rho$  of  $G$ .*

*Proof.* Every action induced by a representation is evidently linear. To prove the converse, assume that we are given a linear action of  $G$  on  $V$ . The hypothesis implies that for each  $g \in G$  there is a linear map  $\rho(g) \in \text{GL}(V)$  such that  $g \cdot x = \rho(g)x$  for all  $x \in V$ . The fact that the action satisfies (7.6) translates to  $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$ , so  $\rho: G \rightarrow \text{GL}(V)$  is a group homomorphism. Thus, to show that it is a Lie group representation, we need only show that it is smooth. Choose a basis  $(E_i)$  for  $V$ , and for each  $i$  let  $\pi^i: V \rightarrow \mathbb{R}$  be the projection onto the  $i$ th coordinate with respect to this basis:  $\pi^i(x^j E_j) = x^i$ . If we let  $\rho_j^i(g)$  denote the matrix entries of  $\rho(g)$  with respect to this basis, it follows that  $\rho_j^i(g) = \pi^i(g \cdot E_j)$ , so each function  $\rho_j^i$  is a composition of smooth functions. Because the matrix entries form global smooth coordinates for  $\text{GL}(V)$ , this implies that  $\rho$  is smooth.  $\square$

### Problems

- 7-1. Show that for any Lie group  $G$ , the multiplication map  $m: G \times G \rightarrow G$  is a smooth submersion. [Hint: use local sections.]
- 7-2. Let  $G$  be a Lie group.
  - (a) Let  $m: G \times G \rightarrow G$  denote the multiplication map. Using Proposition 3.14 to identify  $T_{(e,e)}(G \times G)$  with  $T_eG \oplus T_eG$ , show that the differential  $dm_{(e,e)}: T_eG \oplus T_eG \rightarrow T_eG$  is given by

$$dm_{(e,e)}(X, Y) = X + Y.$$

[Hint: compute  $dm_{(e,e)}(X, 0)$  and  $dm_{(e,e)}(0, Y)$  separately.]

- (b) Let  $i: G \rightarrow G$  denote the inversion map. Show that  $di_e: T_eG \rightarrow T_eG$  is given by  $di_e(X) = -X$ .

(Used on pp. 203, 522.)

- 7-3. Our definition of Lie groups includes the requirement that both the multiplication map and the inversion map are smooth. Show that smoothness of the inversion map is redundant: if  $G$  is a smooth manifold with a group structure such that the multiplication map  $m: G \times G \rightarrow G$  is smooth, then  $G$  is a Lie group. [Hint: show that the map  $F: G \times G \rightarrow G \times G$  defined by  $F(g, h) = (g, gh)$  is a bijective local diffeomorphism.]
- 7-4. Let  $\det: \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}$  denote the determinant function. Use Corollary 3.25 to compute the differential of  $\det$ , as follows.
  - (a) For any  $A \in \text{M}(n, \mathbb{R})$ , show that

$$\left. \frac{d}{dt} \right|_{t=0} \det(I_n + tA) = \text{tr } A,$$

where  $\text{tr}(A_j^i) = \sum_i A_j^i$  is the trace of  $A$ . [Hint: the defining equation (B.3) expresses  $\det(I_n + tA)$  as a polynomial in  $t$ . What is the linear term?]

(b) For  $X \in \text{GL}(n, \mathbb{R})$  and  $B \in T_X \text{GL}(n, \mathbb{R}) \cong \text{M}(n, \mathbb{R})$ , show that

$$d(\det)_X(B) = (\det X) \text{tr}(X^{-1}B). \quad (7.11)$$

[Hint:  $\det(X + tB) = \det(X) \det(I_n + tX^{-1}B)$ .]

(Used on p. 203.)

- 7-5. Prove Theorem 7.9 (uniqueness of the universal covering group).  
 7-6. Suppose  $G$  is a Lie group and  $U$  is any neighborhood of the identity. Show that there exists a neighborhood  $V$  of the identity such that  $V \subseteq U$  and  $gh^{-1} \in U$  whenever  $g, h \in V$ . (Used on pp. 159, 556.)  
 7-7. Prove Proposition 7.15 (properties of the identity component of a Lie group).  
 7-8. Suppose a connected topological group  $G$  acts continuously on a discrete space  $K$ . Show that the action is trivial. (Used on p. 562.)  
 7-9. Show that the formula

$$A \cdot [x] = [Ax]$$

defines a smooth, transitive left action of  $\text{GL}(n+1, \mathbb{R})$  on  $\mathbb{R}\mathbb{P}^n$ .

- 7-10. Repeat Problem 7-9 for  $\text{GL}(n+1, \mathbb{C})$  and  $\mathbb{C}\mathbb{P}^n$  (see Problems 1-9 and 4-5).  
 7-11. Considering  $\mathbb{S}^{2n+1}$  as the unit sphere in  $\mathbb{C}^{n+1}$ , define an action of  $\mathbb{S}^1$  on  $\mathbb{S}^{2n+1}$ , called the **Hopf action**, by

$$z \cdot (w^1, \dots, w^{n+1}) = (zw^1, \dots, zw^{n+1}).$$

Show that this action is smooth and its orbits are disjoint unit circles in  $\mathbb{C}^{n+1}$  whose union is  $\mathbb{S}^{2n+1}$ . (Used on p. 560.)

- 7-12. Use the equivariant rank theorem to give another proof of Theorem 7.5 by showing that every Lie group homomorphism  $F: G \rightarrow H$  is equivariant with respect to suitable smooth  $G$ -actions on  $G$  and  $H$ .  
 7-13. For each  $n \geq 1$ , prove that  $\text{U}(n)$  is a properly embedded  $n^2$ -dimensional Lie subgroup of  $\text{GL}(n, \mathbb{C})$ . (See Example 7.29.)  
 7-14. For each  $n \geq 1$ , prove that  $\text{SU}(n)$  is a properly embedded  $(n^2 - 1)$ -dimensional Lie subgroup of  $\text{U}(n)$ . (See Example 7.30.)  
 7-15. Show that  $\text{SO}(2)$ ,  $\text{U}(1)$ , and  $\mathbb{S}^1$  are all isomorphic as Lie groups.  
 7-16. Prove that  $\text{SU}(2)$  is diffeomorphic to  $\mathbb{S}^3$ . (Used on pp. 179, 563.)  
 7-17. Determine which of the following Lie groups are compact:

$$\text{GL}(n, \mathbb{R}), \text{SL}(n, \mathbb{R}), \text{GL}(n, \mathbb{C}), \text{SL}(n, \mathbb{C}), \text{U}(n), \text{SU}(n).$$

- 7-18. Prove Theorem 7.35 (characterization of semidirect products).  
 7-19. Suppose  $G$ ,  $N$ , and  $H$  are Lie groups. Prove that  $G$  is isomorphic to a semidirect product  $N \rtimes H$  if and only if there are Lie group homomorphisms  $\varphi: G \rightarrow H$  and  $\psi: H \rightarrow G$  such that  $\varphi \circ \psi = \text{Id}_H$  and  $\text{Ker } \varphi \cong N$ .  
 7-20. Prove that the following Lie groups are isomorphic to semidirect products as shown. [Hint: Use Problem 7-19.]

- (a)  $O(n) \cong SO(n) \times O(1)$ .
- (b)  $U(n) \cong SU(n) \times U(1)$ .
- (c)  $GL(n, \mathbb{R}) \cong SL(n, \mathbb{R}) \times \mathbb{R}^*$ .
- (d)  $GL(n, \mathbb{C}) \cong SL(n, \mathbb{C}) \times \mathbb{C}^*$ .

- 7-21. Prove that when  $n > 1$ , none of the groups in Problem 7-20 are isomorphic to direct products of the indicated groups. [Hint: the **center** of a group  $G$  is the set of all elements that commute with every element of  $G$ . Show that isomorphic groups have isomorphic centers.]
- 7-22. Let  $\mathbb{H} = \mathbb{C} \times \mathbb{C}$  (considered as a real vector space), and define a bilinear product  $\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$  by

$$(a, b)(c, d) = (ac - d\bar{b}, \bar{a}d + cb), \quad \text{for } a, b, c, d \in \mathbb{C}.$$

With this product,  $\mathbb{H}$  is a 4-dimensional algebra over  $\mathbb{R}$ , called the algebra of **quaternions**. For each  $p = (a, b) \in \mathbb{H}$ , define  $p^* = (\bar{a}, -b)$ . It is useful to work with the basis  $(\mathbb{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$  for  $\mathbb{H}$  defined by

$$\mathbb{1} = (1, 0), \quad \mathbf{i} = (i, 0), \quad \mathbf{j} = (0, 1), \quad \mathbf{k} = (0, -i).$$

It is straightforward to verify that this basis satisfies

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbb{1}, \quad \mathbb{1}q = q\mathbb{1} = q \quad \text{for all } q \in \mathbb{H}, \\ \mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i} = \mathbf{k}, \quad \mathbf{j}\mathbf{k} = -\mathbf{k}\mathbf{j} = \mathbf{i}, \quad \mathbf{k}\mathbf{i} = -\mathbf{i}\mathbf{k} = \mathbf{j}, \\ \mathbb{1}^* = \mathbb{1}, \quad \mathbf{i}^* = -\mathbf{i}, \quad \mathbf{j}^* = -\mathbf{j}, \quad \mathbf{k}^* = -\mathbf{k}. \end{aligned}$$

A quaternion  $p$  is said to be **real** if  $p^* = p$ , and **imaginary** if  $p^* = -p$ . Real quaternions can be identified with real numbers via the correspondence  $x \leftrightarrow x\mathbb{1}$ .

- (a) Show that quaternionic multiplication is associative but not commutative.
  - (b) Show that  $(pq)^* = q^*p^*$  for all  $p, q \in \mathbb{H}$ .
  - (c) Show that  $\langle p, q \rangle = \frac{1}{2}(p^*q + q^*p)$  is an inner product on  $\mathbb{H}$ , whose associated norm satisfies  $|pq| = |p| |q|$ .
  - (d) Show that every nonzero quaternion has a two-sided multiplicative inverse given by  $p^{-1} = |p|^{-2}p^*$ .
  - (e) Show that the set  $\mathbb{H}^*$  of nonzero quaternions is a Lie group under quaternionic multiplication.  
(Used on pp. 200, 200, 562.)
- 7-23. Let  $\mathbb{H}^*$  be the Lie group of nonzero quaternions (Problem 7-22), and let  $\mathcal{S} \subseteq \mathbb{H}^*$  be the set of unit quaternions. Show that  $\mathcal{S}$  is a properly embedded Lie subgroup of  $\mathbb{H}^*$ , isomorphic to  $SU(2)$ . (Used on pp. 200, 562.)
- 7-24. Prove that each of the maps  $\tau_d^n: GL(n, \mathbb{R}) \rightarrow GL(\mathcal{P}_d^n)$  described in Example 7.36(g) is a faithful representation of  $GL(n, \mathbb{R})$ .