Chapter 15
Orientations

The purpose of this chapter is to introduce a subtle but important property of smooth manifolds called *orientation*. This word stems from the Latin *oriens* (“east”), and originally meant “turning toward the east” or more generally “positioning with respect to one’s surroundings.” Thus, an orientation of a line or curve is a simply a choice of direction along it. As we saw in Chapter 11, the sign of a line integral depends on a choice of preferred direction along the curve.

Mathematicians have extended the sense of the word “orientation” to higher-dimensional manifolds, as a choice between two inequivalent ways in which objects can be situated with respect to their surroundings. For 2-dimensional manifolds, an orientation is essentially a choice of which rotational direction should be considered “clockwise” and which “counterclockwise.” For 3-dimensional ones, it is a choice between “left-handedness” and “right-handedness.” The general definition of an orientation is an adaptation of these everyday concepts to arbitrary dimensions.

As we will see in this chapter, a vector space always has exactly two choices of orientation. In $\mathbb{R}^n$, there is a standard orientation that we can all agree on; but in other vector spaces, an arbitrary choice has to be made. For manifolds, the situation is much more complicated. On a sphere, it is possible to decide unambiguously which rotational direction is counterclockwise, by looking at the surface from the outside (Fig. 15.1). On the other hand, a Möbius band (Fig. 15.2) has the curious property that a figure moving around on the surface can come back to its starting point transformed into its mirror image, so it is impossible to decide consistently which of the two possible rotational directions on the surface to call “clockwise” and which “counterclockwise,” or which is the “front” side and which is the “back” side. The analogous phenomenon in a 3-manifold would be a right-handed person who takes a long trip and comes back left-handed. Manifolds like the sphere, in which it is possible to choose a consistent orientation, are said to be *orientable*; those like the Möbius band in which it is not possible are said to be *nonorientable*.

In this chapter we develop the theory of orientations of smooth manifolds. They have numerous applications, most notably in the theory of integration on manifolds, which we will study in Chapter 16.

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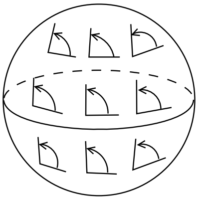

Fig. 15.1 A sphere is orientable

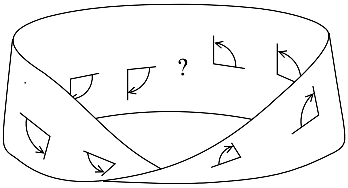

Fig. 15.2 A Möbius band is not orientable

We begin the chapter with an introduction to orientations of vector spaces, and then show how this theory can be carried over to manifolds. Next, we explore the ways in which orientations can be induced on hypersurfaces and on boundaries of manifolds with boundary. Then we treat the special case of orientations on Riemannian manifolds and Riemannian hypersurfaces. At the end of the chapter, we explore the close relationship between orientability and covering maps.

Orientations of Vector Spaces

We begin with orientations of vector spaces. We are all familiar with certain informal rules for singling out preferred ordered bases of $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ (see Fig. 15.3). We usually choose a basis for $\mathbb{R}^1$ that points to the right (i.e., in the positive direction). A natural family of preferred ordered bases for $\mathbb{R}^2$ consists of those for which the rotation from the first vector to the second is in the counterclockwise direction. And every student of vector calculus encounters "right-handed" bases in $\mathbb{R}^3$: these are the ordered bases (E_1, E_2, E_3) with the property that when the fingers of your right hand curl from E_1 to E_2, your thumb points in the direction of E_3.

Although "to the right," "counterclockwise," and "right-handed" are not mathematical terms, it is easy to translate the rules for selecting preferred bases of $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ into rigorous mathematical language: you can check that in all three cases, the preferred bases are the ones whose transition matrices from the standard basis have positive determinants.

In an abstract vector space for which there is no canonical basis, we no longer have any way to determine which bases are "correctly oriented." For example, if V is the vector space of polynomials in one real variable of degree at most 2, who is to say which of the ordered bases $\left(1, x, x^2\right)$ and $\left(x^2, x, 1\right)$ is "right-handed"? All we can say in general is what it means for two bases to have the "same orientation."

Thus we are led to introduce the following definition. Let V be a real vector space of dimension $n \geq 1$. We say that two ordered bases $(E_1, \ldots, E_n)$ and $\left(\widetilde{E}_1, \ldots, \widetilde{E}_n\right)$ for V are ***consistently oriented*** if the transition matrix $\left(B_i^j\right)$, defined by

$$E_i = B_i^j \widetilde{E}_j, \tag{15.1}$$

has positive determinant.

► **Exercise 15.1.** Show that being consistently oriented is an equivalence relation on the set of all ordered bases for V, and show that there are exactly two equivalence classes.

If $\dim V = n \geq 1$, we define an ***orientation for V*** as an equivalence class of ordered bases. If $(E_1, \dots, E_n)$ is any ordered basis for V, we denote the orientation that it determines by $[E_1, \dots, E_n]$, and the opposite orientation by $-[E_1, \dots, E_n]$. A vector space together with a choice of orientation is called an ***oriented vector space***. If V is oriented, then any ordered basis $(E_1, \dots, E_n)$ that is in the given orientation is said to be ***oriented*** or ***positively oriented***. Any basis that is not in the given orientation is said to be ***negatively oriented***.

For the special case of a zero-dimensional vector space V, we define an orientation of V to be simply a choice of one of the numbers ± 1.

Example 15.2. The orientation $[e_1, \dots, e_n]$ of $\mathbb{R}^n$ determined by the standard basis is called the ***standard orientation***. You should convince yourself that, in our usual way of representing the axes graphically, an oriented basis for $\mathbb{R}$ is one that points to the right; an oriented basis for $\mathbb{R}^2$ is one for which the rotation from the first basis vector to the second is counterclockwise; and an oriented basis for $\mathbb{R}^3$ is a right-handed one. (These can be taken as mathematical definitions for the words "right," "counterclockwise," and "right-handed.") The standard orientation for $\mathbb{R}^0$ is defined to be $+1$. //

There is an important connection between orientations and alternating tensors, expressed in the following proposition.

Proposition 15.3. *Let V be a vector space of dimension n. Each nonzero element $\omega \in \Lambda^n(V^*)$ determines an orientation $\mathcal{O}_\omega$ of V as follows: if $n \geq 1$, then $\mathcal{O}_\omega$ is the set of ordered bases $(E_1, \dots, E_n)$ such that $\omega(E_1, \dots, E_n) > 0$; while if $n = 0$, then $\mathcal{O}_\omega$ is $+1$ if $\omega > 0$, and -1 if $\omega < 0$. Two nonzero n-covectors determine the same orientation if and only if each is a positive multiple of the other.*

Proof. The 0-dimensional case is immediate, since a nonzero element of $\Lambda^0(V^*)$ is just a nonzero real number. Thus we may assume $n \geq 1$. Let ω be a nonzero element of $\Lambda^n(V^*)$, and let $\mathcal{O}_\omega$ denote the set of ordered bases on which ω gives positive values. We need to show that $\mathcal{O}_\omega$ is exactly one equivalence class.

Suppose (E_i) and $\left(\widetilde{E}_j\right)$ are any two ordered bases for V, and let $B\colon V \to V$ be the linear map sending E_j to $\widetilde{E}_j$. This means that $\widetilde{E}_j = BE_j = B_j^i E_i$, so B is the transition matrix between the two bases. By Proposition 14.9,

$$\omega\left(\widetilde{E}_1, \dots, \widetilde{E}_n\right) = \omega(BE_1, \dots, BE_n) = (\det B)\omega(E_1, \dots, E_n).$$

It follows that the basis $\left(\widetilde{E}_j\right)$ is consistently oriented with (E_i) if and only if $\omega(E_1, \dots, E_n)$ and $\omega\left(\widetilde{E}_1, \dots, \widetilde{E}_n\right)$ have the same sign, which is the same as saying that $\mathcal{O}_\omega$ is one equivalence class. The last statement then follows easily. □

If V is an oriented n-dimensional vector space and ω is an n-covector that determines the orientation of V as described in this proposition, we say that ω is a

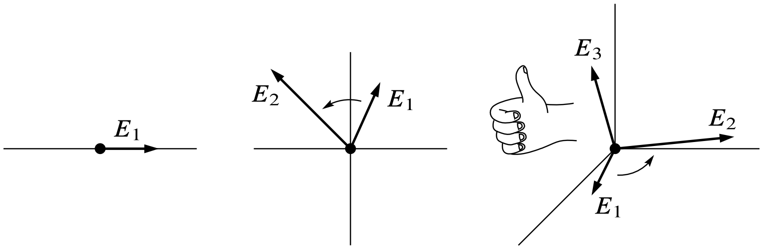

Fig. 15.3 Oriented bases for $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$

(positively) oriented n-covector. For example, the n-covector $e^1 \wedge \cdots \wedge e^n$ is positively oriented for the standard orientation on $\mathbb{R}^n$.

For any n-dimensional vector space V, the space $\Lambda^n(V^*)$ is 1-dimensional. Proposition 15.3 shows that choosing an orientation for V is equivalent to choosing one of the two components of $\Lambda^n(V^*) \smallsetminus \{0\}$. This formulation also works for 0-dimensional vector spaces, and explains why we have defined an orientation of a 0-dimensional space in the way we did.

Orientations of Manifolds

Let M be a smooth manifold with or without boundary. We define a ***pointwise orientation*** on M to be a choice of orientation of each tangent space. By itself, this is not a very useful concept, because the orientations of nearby points may have no relation to each other. For example, a pointwise orientation on $\mathbb{R}^n$ might switch randomly from point to point between the standard orientation and its opposite. In order for orientations to have some relationship with the smooth structure, we need an extra condition to ensure that the orientations of nearby tangent spaces are consistent with each other.

Let M be a smooth n-manifold with or without boundary, endowed with a pointwise orientation. If (E_i) is a local frame for TM, we say that (E_i) is ***(positively) oriented*** if $(E_1|_p, \dots, E_n|_p)$ is a positively oriented basis for T_pM at each point $p \in U$. A ***negatively oriented*** frame is defined analogously.

A pointwise orientation is said to be ***continuous*** if every point of M is in the domain of an oriented local frame. (Recall that by definition the vector fields that make up a local frame are continuous.) An ***orientation of M*** is a continuous pointwise orientation. We say that M is ***orientable*** if there exists an orientation for it, and ***nonorientable*** if not. An ***oriented manifold*** is an ordered pair $(M, \mathcal{O})$, where M is an orientable smooth manifold and $\mathcal{O}$ is a choice of orientation for M; an ***oriented manifold with boundary*** is defined similarly. For each $p \in M$, the orientation of T_pM determined by $\mathcal{O}$ is denoted by $\mathcal{O}_p$. When it is not important to name the orientation explicitly, we use the usual shorthand expression "M is an oriented smooth manifold" (or "manifold with boundary").

If M is zero-dimensional, this definition just means that an orientation of M is a choice of ± 1 attached to each of its points. The continuity condition is vacuous in this case, and the notion of oriented frames is not useful. Clearly, every 0-manifold is orientable.

► **Exercise 15.4.** Suppose M is an oriented smooth n-manifold with or without boundary, and $n \geq 1$. Show that every local frame with connected domain is either positively oriented or negatively oriented. Show that the connectedness assumption is necessary.

The next two propositions give ways of specifying orientations on manifolds that are more practical to use than the definition.

Proposition 15.5 (The Orientation Determined by an *n*-Form). *Let M be a smooth n-manifold with or without boundary. Any nonvanishing n-form ω on M determines a unique orientation of M for which ω is positively oriented at each point. Conversely, if M is given an orientation, then there is a smooth nonvanishing n-form on M that is positively oriented at each point.*

Remark. Because of this proposition, if M is a smooth n-manifold with or without boundary, any nonvanishing n-form on M is called an ***orientation form***. If M is oriented and ω is an orientation form determining the given orientation, we also say that ω is **(*positively*) *oriented***. It is easy to check that if ω and $\widetilde{\omega}$ are two positively oriented smooth forms on M, then $\widetilde{\omega} = f\omega$ for some strictly positive smooth real-valued function f. If M is a 0-manifold, a nonvanishing 0-form (i.e., real-valued function) assigns the orientation $+1$ to points where it is positive and -1 to points where it is negative.

Proof. Let ω be a nonvanishing n-form on M. Then ω defines a pointwise orientation by Proposition 15.3, so all we need to check is that it is continuous. This is trivially true when $n = 0$, so assume $n \geq 1$. Given $p \in M$, let (E_i) be any local frame on a connected neighborhood U of p, and let $\left(\varepsilon^i\right)$ be the dual coframe. On U, the expression for ω in this frame is $\omega = f\varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ for some continuous function f. The fact that ω is nonvanishing means that f is nonvanishing, and therefore

$$\omega(E_1, \ldots, E_n) = f \neq 0$$

at all points of U. Since U is connected, it follows that this expression is either always positive or always negative on U, and therefore the given frame is either positively oriented or negatively oriented. If negatively, we can replace E_1 by $-E_1$ to obtain a new frame that is positively oriented. Thus, the pointwise orientation determined by ω is continuous.

Conversely, suppose M is oriented, and let $\Lambda^n_+ T^*M \subseteq \Lambda^n T^*M$ be the open subset consisting of positively oriented n-covectors at all points of M. At any point $p \in M$, the intersection of $\Lambda^n_+ T^*M$ with the fiber $\Lambda^n(T^*_p M)$ is an open half-line, and therefore convex. By the usual partition-of-unity argument (see Problem 13-2), there exists a smooth global section of $\Lambda^n_+ T^*M$ (i.e., a positively oriented smooth global n-form). □

A smooth coordinate chart on an oriented smooth manifold with or without boundary is said to be **(*positively*) *oriented*** if the coordinate frame $\left(\partial/\partial x^i\right)$ is positively oriented, and ***negatively oriented*** if the coordinate frame is negatively oriented. A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ is said to be ***consistently oriented*** if for each α, β, the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian determinant everywhere on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

Proposition 15.6 (The Orientation Determined by a Coordinate Atlas). *Let M be a smooth positive-dimensional manifold with or without boundary. Given any consistently oriented smooth atlas for M, there is a unique orientation for M with the property that each chart in the given atlas is positively oriented. Conversely, if M is oriented and either $\partial M = \varnothing$ or $\dim M > 1$, then the collection of all oriented smooth charts is a consistently oriented atlas for M.*

Proof. First, suppose M has a consistently oriented smooth atlas. Each chart in the atlas determines a pointwise orientation at each point of its domain. Wherever two of the charts overlap, the transition matrix between their respective coordinate frames is the Jacobian matrix of the transition map, which has positive determinant by hypothesis, so they determine the same pointwise orientation at each point. The orientation thus determined is continuous because each point is in the domain of an oriented coordinate frame.

Conversely, assume M is oriented and either $\partial M = \varnothing$ or $\dim M > 1$. Each point is in the domain of a smooth chart, and if the chart is negatively oriented, we can replace x^1 by $-x^1$ to obtain a new chart that is positively oriented. The fact that these charts all are positively oriented guarantees that their transition maps have positive Jacobian determinants, so they form a consistently oriented atlas. (This does not work for boundary charts when $\dim M = 1$ because of our convention that the last coordinate is nonnegative in a boundary chart.) □

Proposition 15.7 (Product Orientations). *Suppose $M_1, \dots, M_k$ are orientable smooth manifolds. There is a unique orientation on $M_1 \times \dots \times M_k$, called the **product orientation**, with the following property: if for each $i = 1, \dots, k$, ω_i is an orientation form for the given orientation on M_i, then $\pi_1^*\omega_1 \wedge \dots \wedge \pi_k^*\omega_k$ is an orientation form for the product orientation.*

► **Exercise 15.8.** Prove the preceding proposition.

Proposition 15.9. *Let M be a connected, orientable, smooth manifold with or without boundary. Then M has exactly two orientations. If two orientations of M agree at one point, they are equal.*

► **Exercise 15.10.** Prove the preceding proposition.

Proposition 15.11 (Orientations of Codimension-0 Submanifolds). *Suppose M is an oriented smooth manifold with or without boundary, and $D \subseteq M$ is a smooth codimension-0 submanifold with or without boundary. Then the orientation of M restricts to an orientation of D. If ω is an orientation form for M, then $\iota_D^*\omega$ is an orientation form for D.*

► **Exercise 15.12.** Prove the preceding proposition.

Let M and N be oriented smooth manifolds with or without boundary, and suppose $F: M \to N$ is a local diffeomorphism. If M and N are positive-dimensional, we say that F is ***orientation-preserving*** if for each $p \in M$, the isomorphism dF_p takes oriented bases of T_pM to oriented bases of $T_{F(p)}N$, and ***orientation-reversing*** if it takes oriented bases of T_pM to negatively oriented bases of $T_{F(p)}N$. If M and N are 0-manifolds, then F is orientation-preserving if for every $p \in M$, the points p and $F(p)$ have the same orientation; and it is orientation-reversing if they have the opposite orientation.

► **Exercise 15.13.** Suppose M and N are oriented positive-dimensional smooth manifolds with or without boundary, and $F: M \to N$ is a local diffeomorphism. Show that the following are equivalent.

(a) F is orientation-preserving.
(b) With respect to any oriented smooth charts for M and N, the Jacobian matrix of F has positive determinant.
(c) For any positively oriented orientation form ω for N, the form $F^*\omega$ is positively oriented for M.

► **Exercise 15.14.** Show that a composition of orientation-preserving maps is orientation-preserving.

Here is another important method for constructing orientations.

Proposition 15.15 (The Pullback Orientation). *Suppose M and N are smooth manifolds with or without boundary. If $F: M \to N$ is a local diffeomorphism and N is oriented, then M has a unique orientation, called the **pullback orientation induced by F**, such that F is orientation-preserving.*

Proof. For each $p \in M$, there is a unique orientation on T_pM that makes the isomorphism $dF_p: T_pM \to T_{F(p)}N$ orientation-preserving. This defines a pointwise orientation on M, and provided it is continuous, it is the unique orientation on M with respect to which F is orientation-preserving. To see that it is continuous, just choose a smooth orientation form ω for N and note that $F^*\omega$ is a smooth orientation form for M. □

In the situation of the preceding proposition, if $\mathcal{O}$ denotes the given orientation on N, the pullback orientation on M is denoted by $F^*\mathcal{O}$.

► **Exercise 15.16.** Suppose $F: M \to N$ and $G: N \to P$ are local diffeomorphisms and $\mathcal{O}$ is an orientation on P. Show that $(G \circ F)^*\mathcal{O} = F^*(G^*\mathcal{O})$.

Recall that a smooth manifold is said to be *parallelizable* if it admits a smooth global frame.

Proposition 15.17. *Every parallelizable smooth manifold is orientable.*

Proof. Suppose M is parallelizable, and let $(E_1, \dots, E_n)$ be a global smooth frame for M. Define a pointwise orientation on M by declaring the basis $\left(E_1|_p, \dots, E_n|_p\right)$ to be positively oriented at each $p \in M$. This pointwise orientation is continuous, because every point of M is in the domain of the (global) oriented frame (E_i). □

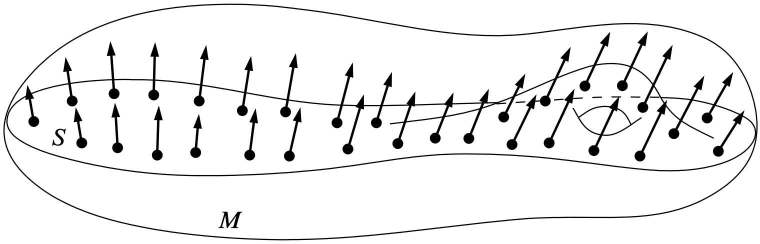

Fig. 15.4 A vector field along a submanifold

Example 15.18. The preceding proposition implies that Euclidean space $\mathbb{R}^n$, the n-torus $\mathbb{T}^n$, the spheres $\mathbb{S}^1$, $\mathbb{S}^3$, and $\mathbb{S}^7$, and products of them are all orientable, because they are all parallelizable. Therefore, any codimension-0 submanifold of one of these manifolds is also orientable. Likewise, every Lie group is orientable because it is parallelizable. //

In the case of Lie groups, we can say more. If G is a Lie group, an orientation of G is said to be ***left-invariant*** if L_g is orientation-preserving for every $g \in G$.

Proposition 15.19. *Every Lie group has precisely two left-invariant orientations, corresponding to the two orientations of its Lie algebra.*

► **Exercise 15.20.** Prove the preceding proposition.

Orientations of Hypersurfaces

If M is an oriented smooth manifold and S is a smooth submanifold of M (with or without boundary), S might not inherit an orientation from M, even if S is embedded. Clearly, it is not sufficient to restrict an orientation form from M to S, since the restriction of an n-form to a manifold of lower dimension must necessarily be zero. A useful example to consider is the Möbius band, which is not orientable (see Example 15.38 below), even though it can be embedded in $\mathbb{R}^3$.

In this section we focus our attention on immersed or embedded hypersurfaces (codimension-1 submanifolds). With one extra piece of information (a vector field that is nowhere tangent to the hypersurface), we can use an orientation on M to induce an orientation on a hypersurface in M.

Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is a smooth submanifold (immersed or embedded, with or without boundary). Recall (Example 10.10) that a *vector field along* S is a section of the ambient tangent bundle $TM|_S$, i.e., a continuous map $N: S \to TM$ with the property that $N_p \in T_pM$ for each $p \in S$ (Fig. 15.4). For example, any vector field on M restricts to a vector field along S, but in general, not every vector field along S is of this form (see Problem 10-9).

Proposition 15.21. *Suppose M is an oriented smooth n-manifold with or without boundary, S is an immersed hypersurface with or without boundary in M, and N is*

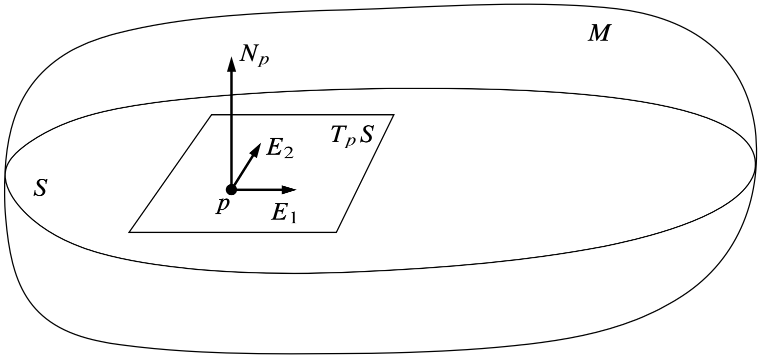

Fig. 15.5 The orientation induced by a nowhere tangent vector field

a vector field along S that is nowhere tangent to S. Then S has a unique orientation such that for each $p \in S$, $(E_1, \dots, E_{n-1})$ is an oriented basis for T_pS if and only if $(N_p, E_1, \dots, E_{n-1})$ is an oriented basis for T_pM. If ω is an orientation form for M, then $\iota_S^*(N \lrcorner \omega)$ is an orientation form for S with respect to this orientation, where $\iota_S \colon S \hookrightarrow M$ is inclusion.

Remark. See Fig. 15.5 for an illustration of the $n = 3$ case. When $n = 1$, since S is a 0-manifold, this proposition should be interpreted as follows: at each point $p \in S$, we assign the orientation $+1$ to p if N_p is an oriented basis for T_pM, and -1 if N_p is negatively oriented. With this understanding, the proof below goes through in the $n = 1$ case without modification.

Proof. Let ω be an orientation form for M. Then $\sigma = \iota_S^*(N \lrcorner \omega)$ is an $(n-1)$-form on S. (Recall that the pullback ι_S^* is really just restriction to vectors tangent to S.) It will follow that σ is an orientation form for S if we can show that it never vanishes. Given any basis $(E_1, \dots, E_{n-1})$ for T_pS, the fact that N is nowhere tangent to S implies that $(N_p, E_1, \dots, E_{n-1})$ is a basis for T_pM. The fact that ω is nonvanishing implies that

$$\sigma_p(E_1, \dots, E_{n-1}) = \omega_p(N_p, E_1, \dots, E_{n-1}) \neq 0.$$

Since $\sigma_p(E_1, \dots, E_{n-1}) > 0$ if and only if $\omega_p(N_p, E_1, \dots, E_{n-1}) > 0$, the orientation determined by σ is the one defined in the statement of the proposition. □

Example 15.22. The sphere $\mathbb{S}^n$ is a hypersurface in $\mathbb{R}^{n+1}$, to which the vector field $N = x^i \partial/\partial x^i$ is nowhere tangent, so this vector field induces an orientation on $\mathbb{S}^n$. This shows that all spheres are orientable. We define the ***standard orientation of*** $\mathbb{S}^n$ to be the orientation determined by N. Unless otherwise specified, we always use this orientation. (The standard orientation on $\mathbb{S}^0$ is the one that assigns the orientation $+1$ to the point $+1 \in \mathbb{S}^0$ and -1 to $-1 \in \mathbb{S}^0$.) //

Not every hypersurface admits a nowhere tangent vector field. (See Problem 15-6.) However, the following proposition gives a sufficient condition that holds in many cases.

Proposition 15.23. *Let M be an oriented smooth manifold, and suppose $S \subseteq M$ is a regular level set of a smooth function $f\colon M \to \mathbb{R}$. Then S is orientable.*

Proof. Choose any Riemannian metric on M, and let $N = \operatorname{grad} f|_S$. The hypotheses imply that N is a nowhere tangent vector field along S, so the result follows from Proposition 15.21. □

Boundary Orientations

If M is a smooth manifold with boundary, then ∂M is an embedded hypersurface in M (see Theorem 5.11), and Problem 8-4 showed that there is always a smooth outward-pointing vector field along ∂M. Because an outward-pointing vector field is nowhere tangent to ∂M, it determines an orientation on ∂M.

Proposition 15.24 (The Induced Orientation on a Boundary). *Let M be an oriented smooth n-manifold with boundary, $n \geq 1$. Then ∂M is orientable, and all outward-pointing vector fields along ∂M determine the same orientation on ∂M.*

Remark. The orientation on ∂M determined by any outward-pointing vector field is called the ***induced orientation*** or the ***Stokes orientation*** on ∂M. (The second term is chosen because of the role this orientation plays in Stokes's theorem, to be discussed in Chapter 16.)

Proof. Let $n = \dim M$, let ω be an orientation form for M, and let N be a smooth outward-pointing vector field along ∂M. The $(n-1)$-form $\iota_{\partial M}^*(N \lrcorner\, \omega)$ is an orientation form for ∂M by Proposition 15.21, so ∂M is orientable.

To show that this orientation is independent of the choice of N, let $p \in \partial M$ be arbitrary, and let $\left(x^i\right)$ be smooth boundary coordinates for M on a neighborhood of p. If N and $\widetilde{N}$ are two different outward-pointing vector fields along ∂M, Proposition 5.41 shows that the last components $N^n(p)$ and $\widetilde{N}^n(p)$ are both negative. Both $\left(N_p, \partial/\partial x^1|_p, \dots, \partial/\partial x^{n-1}|_p\right)$ and $\left(\widetilde{N}_p, \partial/\partial x^1|_p, \dots, \partial/\partial x^{n-1}|_p\right)$ are bases for T_pM, and the transition matrix between them has determinant equal to $N^n(p)/\widetilde{N}^n(p) > 0$. Thus, both bases determine the same orientation for T_pM, so N and $\widetilde{N}$ determine the same orientation for $T_p\partial M$. (When $n = 1$, the bases in question are just (N_p) and $\left(\widetilde{N}_p\right)$, which determine the same orientation because they are both negative multiples of $\partial/\partial x^1|_p$.) □

Example 15.25. This proposition gives a simpler proof that $\mathbb{S}^n$ is orientable, because it is the boundary of the closed unit ball. The orientation thus induced on $\mathbb{S}^n$ is the standard one, as you can check. //

Example 15.26. Let us determine the induced orientation on $\partial\mathbb{H}^n$ when $\mathbb{H}^n$ itself has the standard orientation inherited from $\mathbb{R}^n$. We can identify $\partial\mathbb{H}^n$ with $\mathbb{R}^{n-1}$ under the correspondence $\left(x^1, \dots, x^{n-1}, 0\right) \leftrightarrow \left(x^1, \dots, x^{n-1}\right)$. Since the vector field $-\partial/\partial x^n$ is outward-pointing along $\partial\mathbb{H}^n$, the standard coordinate frame for $\mathbb{R}^{n-1}$ is positively oriented for $\partial\mathbb{H}^n$ if and only if $\left[-\partial/\partial x^n, \partial/\partial x^1, \dots, \partial/\partial x^{n-1}\right]$ is the

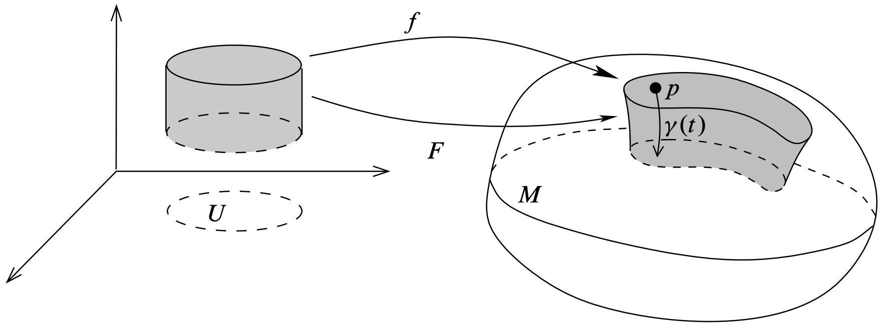

Fig. 15.6 Orientation criterion for a boundary parametrization

standard orientation for $\mathbb{R}^n$. This orientation satisfies

$$\begin{aligned}\left[-\partial/\partial x^n, \partial/\partial x^1, \dots, \partial/\partial x^{n-1}\right] &= -\left[\partial/\partial x^n, \partial/\partial x^1, \dots, \partial/\partial x^{n-1}\right] \\ &= (-1)^n \left[\partial/\partial x^1, \dots, \partial/\partial x^{n-1}, \partial/\partial x^n\right].\end{aligned}$$

Thus, the induced orientation on $\partial\mathbb{H}^n$ is equal to the standard orientation on $\mathbb{R}^{n-1}$ when n is even, but it is *opposite* to the standard orientation when n is odd. In particular, the standard coordinates on $\partial\mathbb{H}^n \approx \mathbb{R}^{n-1}$ are positively oriented if and only if n is even. (This fact will be important in the proof of Stokes's theorem in Chapter 16.) //

For many purposes, the most useful way of describing submanifolds is by means of local parametrizations. The next lemma gives a useful criterion for checking whether a local parametrization of a boundary is orientation-preserving.

Lemma 15.27. *Let M be an oriented smooth n-manifold with boundary. Suppose $U \subseteq \mathbb{R}^{n-1}$ is open, a, b are real numbers with $a < b$, and $F\colon (a,b] \times U \to M$ is a smooth embedding that restricts to an embedding of $\{b\} \times U$ into ∂M. Then the parametrization $f\colon U \to \partial M$ given by $f(x) = F(b,x)$ is orientation-preserving for ∂M if and only if F is orientation-preserving for M.*

Proof. Let x be an arbitrary point of U, and let $p = f(x) = F(b,x) \in \partial M$ (Fig. 15.6). The hypothesis that F is an embedding means that the linear map $dF_{(b,x)}\colon \left(T_b\mathbb{R} \oplus T_x\mathbb{R}^{n-1}\right) \to T_pM$ is bijective. Since the restriction of $dF_{(b,x)}$ to $T_x\mathbb{R}^{n-1}$ is equal to $df_x\colon T_x\mathbb{R}^{n-1} \to T_p\partial M$, which is already injective, it follows that $dF\left(\partial/\partial s|_{(b,x)}\right) \notin T_p\partial M$ (where s denotes the coordinate on $(a,b]$).

Define a smooth curve $\gamma\colon [0,\varepsilon) \to M$ by

$$\gamma(t) = F(b-t, x).$$

This curve satisfies $\gamma(0) = p$ and $\gamma'(0) = -dF\left(\partial/\partial s|_{(b,x)}\right) \notin T_p\partial M$. It follows that $-dF\left(\partial/\partial s|_{(b,x)}\right)$ is inward-pointing, and therefore $dF\left(\partial/\partial s|_{(b,x)}\right)$ is outward-pointing.

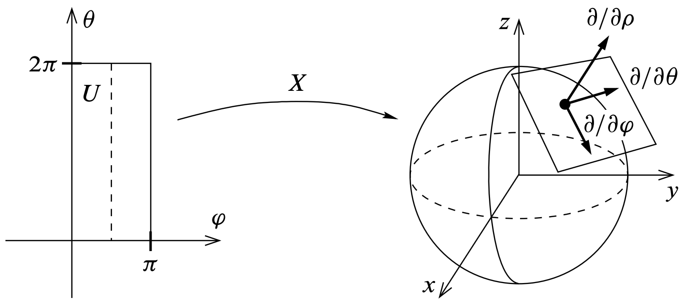

Fig. 15.7 Parametrizing the sphere via spherical coordinates

The definition of the induced orientation yields the following equivalences:

F is orientation-preserving for M

$\Leftrightarrow$ $\big(dF(\partial/\partial s), dF(\partial/\partial x^1), \dots, dF(\partial/\partial x^{n-1})\big)$ is oriented for TM

$\Leftrightarrow$ $\big(dF(\partial/\partial x^1), \dots, dF(\partial/\partial x^{n-1})\big)$ is oriented for $T\partial M$

$\Leftrightarrow$ $\big(df(\partial/\partial x^1), \dots, df(\partial/\partial x^{n-1})\big)$ is oriented for $T\partial M$

$\Leftrightarrow$ f is orientation-preserving for ∂M. □

Here is an illustration of how the lemma can be used.

Example 15.28. Spherical coordinates (Example C.38) yield a smooth local parametrization of $\mathbb{S}^2$ as follows. Let U be the open rectangle $(0,\pi)\times(0,2\pi)\subseteq\mathbb{R}^2$, and let $X\colon U\to\mathbb{R}^3$ be the following map:

$$X(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

(Fig. 15.7).We can check whether X preserves or reverses orientation by using the fact that it is the restriction of the 3-dimensional spherical coordinate parametrization $F\colon (0,1]\times U\to\overline{\mathbb{B}}^3$ defined by

$$F(\rho,\varphi,\theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi).$$

Because $F(1,\varphi,\theta) = X(\varphi,\theta)$, the hypotheses of Lemma 15.27 are satisfied. By direct computation, the Jacobian determinant of F is $\rho^2\sin\varphi$, which is positive on $(0,1]\times U$. By virtue of Lemma 15.27, X is orientation-preserving. //

The Riemannian Volume Form

Let (M,g) be an oriented Riemannian manifold of positive dimension. We know from Proposition 13.6 that there is a smooth orthonormal frame $(E_1,\dots,E_n)$ in a neighborhood of each point of M. By replacing E_1 by $-E_1$ if necessary, we can find an *oriented* orthonormal frame in a neighborhood of each point.

Proposition 15.29. *Suppose (M, g) is an oriented Riemannian n-manifold with or without boundary, and $n \geq 1$. There is a unique smooth orientation form $\omega_g \in \Omega^n(M)$, called the **Riemannian volume form**, that satisfies*

$$\omega_g(E_1, \ldots, E_n) = 1 \tag{15.2}$$

for every local oriented orthonormal frame (E_i) for M.

Proof. Suppose first that such a form ω_g exists. If $(E_1, \ldots, E_n)$ is any local oriented orthonormal frame on an open subset $U \subseteq M$ and $\left(\varepsilon^1, \ldots, \varepsilon^n\right)$ is the dual coframe, we can write $\omega_g = f\, \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ on U. The condition (15.2) then reduces to $f = 1$, so

$$\omega_g = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n. \tag{15.3}$$

This proves that such a form is uniquely determined.

To prove existence, we would like to *define* ω_g in a neighborhood of each point by (15.3), so we need to check that this definition is independent of the choice of oriented orthonormal frame. If $\left(\widetilde{E}_1, \ldots, \widetilde{E}_n\right)$ is another oriented orthonormal frame, with dual coframe $\left(\widetilde{\varepsilon}^1, \ldots, \widetilde{\varepsilon}^n\right)$, let

$$\widetilde{\omega}_g = \widetilde{\varepsilon}^1 \wedge \cdots \wedge \widetilde{\varepsilon}^n.$$

We can write

$$\widetilde{E}_i = A_i^j E_j$$

for some matrix $\left(A_i^j\right)$ of smooth functions. The fact that both frames are orthonormal means that $\left(A_i^j(p)\right) \in \mathrm{O}(n)$ for each p, so $\det\left(A_i^j\right) = \pm 1$, and the fact that the two frames are consistently oriented forces the positive sign. We compute

$$\omega_g\left(\widetilde{E}_1, \ldots, \widetilde{E}_n\right) = \det\left(\varepsilon^j\left(\widetilde{E}_i\right)\right) = \det\left(A_i^j\right) = 1 = \widetilde{\omega}_g\left(\widetilde{E}_1, \ldots, \widetilde{E}_n\right).$$

Thus $\omega_g = \widetilde{\omega}_g$, so defining ω_g in a neighborhood of each point by (15.3) with respect to some smooth oriented orthonormal frame yields a global n-form. The resulting form is clearly smooth and satisfies (15.2) for every oriented orthonormal frame. □

► **Exercise 15.30.** Suppose (M, g) and $\left(\widetilde{M}, \widetilde{g}\right)$ are positive-dimensional Riemannian manifolds with or without boundary, and $F\colon M \to \widetilde{M}$ is a local isometry. Show that $F^*\omega_{\widetilde{g}} = \omega_g$.

Although the expression for the Riemannian volume form with respect to an oriented orthonormal frame is particularly simple, it is also useful to have an expression for it in coordinates.

Proposition 15.31. *Let (M, g) be an oriented Riemannian n-manifold with or without boundary, $n \geq 1$. In any oriented smooth coordinates $\left(x^i\right)$, the Riemannian volume form has the local coordinate expression*

$$\omega_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n,$$

where g_{ij} are the components of g in these coordinates.

Proof. Let $\left(U,\left(x^i\right)\right)$ be an oriented smooth chart, and let $p \in M$. In these coordinates, $\omega_g = f\, dx^1 \wedge \cdots \wedge dx^n$ for some positive coefficient function f. To compute f, let (E_i) be any smooth oriented orthonormal frame defined on a neighborhood of p, and let $\left(\varepsilon^i\right)$ be the dual coframe. If we write the coordinate frame in terms of the orthonormal frame as

$$\frac{\partial}{\partial x^i} = A_i^j E_j,$$

then we can compute

$$\begin{aligned} f &= \omega_g\left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\right) = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n\left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\right) \\ &= \det\left(\varepsilon^j\left(\frac{\partial}{\partial x^i}\right)\right) = \det\left(A_i^j\right). \end{aligned}$$

On the other hand, observe that

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle_g = \left\langle A_i^k E_k, A_j^l E_l\right\rangle_g = A_i^k A_j^l \langle E_k, E_l\rangle_g = \sum_k A_i^k A_j^k.$$

This last expression is the (i,j)-entry of the matrix product $A^T A$, where $A = \left(A_i^j\right)$. Thus,

$$\det(g_{ij}) = \det\left(A^T A\right) = \det A^T \det A = (\det A)^2,$$

from which it follows that $f = \det A = \pm\sqrt{\det(g_{ij})}$. Since both frames $\left(\partial/\partial x^i\right)$ and (E_j) are oriented, the sign must be positive. □

Hypersurfaces in Riemannian Manifolds

Let (M,g) be an oriented Riemannian manifold with or without boundary, and suppose $S \subseteq M$ is an immersed hypersurface with or without boundary. Any unit normal vector field along S is nowhere tangent to S, so it determines an orientation of S by Proposition 15.21. The next proposition gives a simple formula for the volume form of the induced metric on S with respect to this orientation.

Proposition 15.32. *Let (M,g) be an oriented Riemannian manifold with or without boundary, let $S \subseteq M$ be an immersed hypersurface with or without boundary, and let $\widetilde{g}$ denote the induced metric on S. Suppose N is a smooth unit normal vector field along S. With respect to the orientation of S determined by N, the volume form of $\left(S,\widetilde{g}\right)$ is given by*

$$\omega_{\widetilde{g}} = \iota_S^*(N \lrcorner\, \omega_g).$$

Proof. By Proposition 15.21, the $(n-1)$-form $\iota_S^*(N \lrcorner\, \omega_g)$ is an orientation form for S. To prove that it is the volume form for the induced Riemannian metric, we

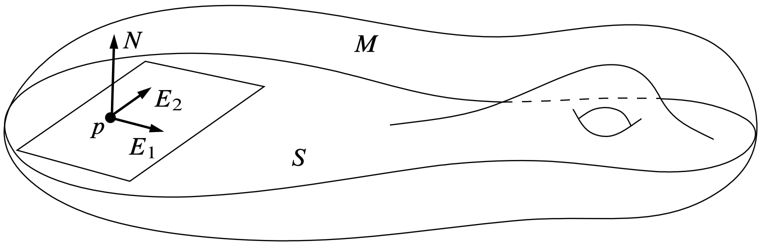

Fig. 15.8 A hypersurface in a Riemannian manifold

need only show that it gives the value 1 whenever it is applied to an oriented orthonormal frame for S. Thus, let $(E_1, \dots, E_{n-1})$ be such a frame. At each point $p \in S$, the basis $(N_p, E_1|_p, \dots, E_{n-1}|_p)$ is orthonormal (Fig. 15.8),and is oriented for T_pM (this is the definition of the orientation determined by N). Thus

$$\iota_S^*(N \lrcorner \omega_g)(E_1, \dots, E_{n-1}) = \omega_g(N, E_1, \dots, E_{n-1}) = 1,$$

which proves the result. □

The result of Proposition 15.32 takes on particular importance in the case of a Riemannian manifold with boundary, because of the following proposition.

Proposition 15.33. *Suppose M is any Riemannian manifold with boundary. There is a unique smooth outward-pointing unit normal vector field N along ∂M.*

Proof. First, we prove uniqueness. At any point $p \in \partial M$, the subspace $(T_p\partial M)^\perp \subseteq T_pM$ is 1-dimensional, so there are exactly two unit vectors at p that are normal to ∂M. Since any unit normal vector N is nowhere tangent to ∂M, it must have nonzero x^n-component in any smooth boundary chart. Thus, exactly one of the two choices of unit normal has negative x^n-component, which is equivalent to being outward-pointing.

To prove existence, let $f: M \to \mathbb{R}$ be a boundary defining function (Proposition 5.43), and let N be the restriction to ∂M of the unit vector field $-\operatorname{grad} f / |\operatorname{grad} f|_g$. Because $df \neq 0$ at points of ∂M, N is well-defined and smooth on ∂M. Then N is normal to ∂M by Problem 13-21, and outward pointing by Exercise 5.44, because $Nf = -\langle \operatorname{grad} f, \operatorname{grad} f \rangle_g / |\operatorname{grad} f|_g = -|\operatorname{grad} f|_g < 0$. □

The next corollary is immediate.

Corollary 15.34. *If (M, g) is an oriented Riemannian manifold with boundary and $\tilde{g}$ is the induced Riemannian metric on ∂M, then the volume form of $\tilde{g}$ is*

$$\omega_{\tilde{g}} = \iota_{\partial M}^*(N \lrcorner \omega_g),$$

where N is the outward unit normal vector field along ∂M. □

Orientations and Covering Maps

Although it is often easy to prove that a given smooth manifold is orientable by constructing an orientation for it, proving that a manifold is *not* orientable can be much trickier. The theory of covering spaces provides one of the most useful techniques for doing so. In this section, we explore the close relationship between orientability and covering maps.

Our first result is a simple application of pullback orientations.

Proposition 15.35. *If $\pi\colon E \to M$ is a smooth covering map and M is orientable, then E is also orientable.*

Proof. Because a covering map is a local diffeomorphism, this follows immediately from Proposition 15.15. □

The next theorem is more interesting. If G is a Lie group acting smoothly on a smooth manifold E (on the left, say), we say the action is an ***orientation-preserving action*** if for each $g \in G$, the diffeomorphism $x \mapsto g \cdot x$ is orientation-preserving.

Theorem 15.36. *Suppose E is a connected, oriented, smooth manifold with or without boundary, and $\pi\colon E \to M$ is a smooth normal covering map. Then M is orientable if and only if the action of $\operatorname{Aut}_\pi(E)$ on E is orientation-preserving.*

Proof. Let $\mathcal{O}_E$ denote the given orientation on E. First suppose M is orientable, and let q be an arbitrary point in E. Because M is connected, it has exactly two orientations, and one of them has the property that $d\pi_q\colon T_qE \to T_{\pi(q)}M$ is orientation-preserving. Call that orientation $\mathcal{O}_M$. The pullback orientation $\pi^*\mathcal{O}_M$ agrees with the given orientation at q, so it must be equal to $\mathcal{O}_E$ by Proposition 15.9. Suppose $\varphi \in \operatorname{Aut}_\pi(E)$. The fact that $\pi \circ \varphi = \pi$ implies that

$$\varphi^*\mathcal{O}_E = \varphi^*(\pi^*\mathcal{O}_M) = (\pi \circ \varphi)^*\mathcal{O}_M = \pi^*\mathcal{O}_M = \mathcal{O}_E.$$

Thus, φ is orientation-preserving.

Conversely, suppose the action of $\operatorname{Aut}_\pi(E)$ is orientation-preserving, and let $p \in M$. If $U \subseteq M$ is any evenly covered neighborhood of p, there is a smooth section $\sigma\colon U \to E$, which induces an orientation $\sigma^*\mathcal{O}_E$ on U. Suppose $\sigma_1\colon U \to E$ is any other smooth local section over U. Because π is a normal covering, $\operatorname{Aut}_\pi(E)$ acts transitively on each fiber of π, so there is a covering automorphism φ such that $\sigma_1(p) = \varphi\big(\sigma(p)\big)$. Then $\varphi \circ \sigma$ is a local section of π that agrees with σ_1 at p, and thus $\sigma_1 = \varphi \circ \sigma$ on all of U. Because φ is orientation-preserving, $\sigma_1^*\mathcal{O}_E = \sigma^*\varphi^*\mathcal{O}_E = \sigma^*\mathcal{O}_E$, so the orientations induced by σ and σ_1 are equal. Thus, we can define a global orientation $\mathcal{O}_M$ on M by defining it on each evenly covered open subset to be the pullback orientation induced by any local section; the argument above shows that the orientations so defined agree where they overlap. □

Here are two applications of the preceding theorem.

Example 15.37 (Orientability of Projective Spaces). For $n \geq 1$, consider the smooth covering map $q\colon \mathbb{S}^n \to \mathbb{RP}^n$ of Example 4.35. The only nontrivial covering automorphism of q is the antipodal map $\alpha(x) = -x$. Problem 15-3 shows that α is orientation-preserving if and only if n is odd, so it follows that $\mathbb{RP}^n$ is orientable if and only if n is odd. //

Example 15.38 (The Möbius Bundle and the Möbius Band). Let E be the total space of the Möbius bundle (Example 10.3). The quotient map $q\colon \mathbb{R}^2 \to E$ used to define E is a smooth normal covering map, and the covering automorphism group is isomorphic to $\mathbb{Z}$, acting on $\mathbb{R}^2$ by $n \cdot (x,y) = \big(x+n, (-1)^n y\big)$. (You can check this directly from the definitions, or you can accept this for now and wait until we have developed more machinery in Chapter 21, where a simpler proof is available; see Problem 21-9.) For n odd, the diffeomorphism $(x,y) \mapsto n \cdot (x,y)$ of $\mathbb{R}^2$ pulls back the orientation form $dx \wedge dy$ to $-dx \wedge dy$, so the action of $\operatorname{Aut}_\pi(E)$ is not orientation-preserving. Thus, Theorem 15.36 shows that E is not orientable.

For each $r > 0$, the image under q of the rectangle $[0,1] \times [-r,r]$ is a Möbius band M_r. Because q restricts to a smooth covering map from $\mathbb{R} \times [-r,r]$ to M_r, the same argument shows that a Möbius band is not orientable either. //

The Orientation Covering

Next we show that every nonorientable smooth manifold M has an orientable two-sheeted covering manifold. The fiber over a point $p \in M$ will correspond to the two orientations of T_pM.

In order to handle the orientable and nonorientable cases in a uniform way, it is useful to expand our definition of covering maps slightly, by allowing "covering spaces" that are not connected. If N and M are topological spaces, let us say that a map $\pi\colon N \to M$ is a ***generalized covering map*** if it satisfies all of the requirements for a covering map except that N might not be connected: this means that N is locally path-connected, π is surjective and continuous, and each point $p \in M$ has a neighborhood that is evenly covered by π. If in addition N and M are smooth manifolds with or without boundary and π is a local diffeomorphism, we say it is a ***generalized smooth covering map***.

Lemma 15.39. *Suppose N and M are topological spaces and $\pi\colon N \to M$ is a generalized covering map. If M is connected, then the restriction of π to each component of N is a covering map.*

Proof. Suppose W is a component of N. If U is any open subset of M that is evenly covered by π, then each component of $\pi^{-1}(U)$ is connected and therefore contained in a single component of N. It follows that $\big(\pi|_W\big)^{-1}(U) = \pi^{-1}(U) \cap W$ is either the empty set or a nonempty disjoint union of components of $\pi^{-1}(U)$, each of which is mapped homeomorphically onto U by $\pi|_W$. In particular, this means that each point in $\pi(W)$ has a neighborhood that is evenly covered by $\pi|_W$.

To complete the proof, we just need to show that $\pi|_W$ is surjective. Because π is a local homeomorphism, $\pi(W)$ is an open subset of M. On the other hand, if

$p \in M \smallsetminus \pi(W)$, and U is a neighborhood of p that is evenly covered by π, then the discussion in the preceding paragraph shows that $\left(\pi|_W\right)^{-1}(U) = \varnothing$, which implies that $U \subseteq M \smallsetminus \pi(W)$. Therefore, $\pi(W)$ is closed in M. Because W is not empty, $\pi(W)$ is all of M. □

Let M be a connected, smooth, positive-dimensional manifold with or without boundary, and let $\widehat{M}$ denote the set of orientations of all tangent spaces to M:

$$\widehat{M} = \left\{(p, \mathcal{O}_p) : p \in M \text{ and } \mathcal{O}_p \text{ is an orientation of } T_pM\right\}.$$

Define the projection $\widehat{\pi}\colon \widehat{M} \to M$ by sending an orientation of T_pM to the point p itself: $\widehat{\pi}(p, \mathcal{O}_p) = p$. Since each tangent space has exactly two orientations, each fiber of this map has cardinality 2. The map $\widehat{\pi}\colon \widehat{M} \to M$ is called the ***orientation covering of*** M.

Proposition 15.40 (Properties of the Orientation Covering). *Suppose M is a connected, smooth, positive-dimensional manifold with or without boundary, and let $\widehat{\pi}\colon \widehat{M} \to M$ be its orientation covering. Then $\widehat{M}$ can be given the structure of a smooth, oriented manifold with or without boundary, with the following properties:*

(a) *$\widehat{\pi}\colon \widehat{M} \to M$ is a generalized smooth covering map.*
(b) *A connected open subset $U \subseteq M$ is evenly covered by $\widehat{\pi}$ if and only if U is orientable.*
(c) *If $U \subseteq M$ is an evenly covered open subset, then every orientation of U is the pullback orientation induced by a local section of $\widehat{\pi}$ over U.*

Proof. We first topologize $\widehat{M}$ by defining a basis for it. For each pair $(U, \mathcal{O})$, where U is an open subset of M and $\mathcal{O}$ is an orientation on U, define a subset $\widehat{U}_{\mathcal{O}} \subseteq \widehat{M}$ as follows:

$$\widehat{U}_{\mathcal{O}} = \left\{(p, \mathcal{O}_p) \in \widehat{M} : p \in U \text{ and } \mathcal{O}_p \text{ is the orientation of } T_pM \text{ determined by } \mathcal{O}\right\}.$$

We will show that the collection of all subsets of the form $\widehat{U}_{\mathcal{O}}$ is a basis for a topology on $\widehat{M}$. Given an arbitrary point $(p, \mathcal{O}_p) \in \widehat{M}$, let U be an orientable neighborhood of p in M, and let $\mathcal{O}$ be an orientation on it. After replacing $\mathcal{O}$ by $-\mathcal{O}$ if necessary, we may assume that the given orientation $\mathcal{O}_p$ is same as the orientation of T_pM determined by $\mathcal{O}$. It follows that $(p, \mathcal{O}_p) \in \widehat{U}_{\mathcal{O}}$, so the collection of all sets of the form $\widehat{U}_{\mathcal{O}}$ covers $\widehat{M}$. If $\widehat{U}_{\mathcal{O}}$ and $\widehat{U}'_{\mathcal{O}'}$ are two such sets and $(p, \mathcal{O}_p)$ is a point in their intersection, then $\mathcal{O}_p$ is the orientation of T_pM determined by both $\mathcal{O}$ and $\mathcal{O}'$. If V is the component of $U \cap U'$ containing p, then the restricted orientations $\mathcal{O}|_V$ and $\mathcal{O}'|_V$ agree at p and therefore are identical by Proposition 15.9, so it follows that $\widehat{U}_{\mathcal{O}} \cap \widehat{U}'_{\mathcal{O}'}$ contains the basis set $\widehat{V}_{\mathcal{O}|_V}$. Thus, we have defined a topology on $\widehat{M}$. Note that for each orientable open subset $U \subseteq M$ and each orientation $\mathcal{O}$ of U, $\widehat{\pi}$ maps the basis set $\widehat{U}_{\mathcal{O}}$ bijectively onto U. Because the orientable open subsets form a basis for the topology of M, this implies that $\widehat{\pi}$ restricts to a homeomorphism from $\widehat{U}_{\mathcal{O}}$ to U. In particular, $\widehat{\pi}$ is a local homeomorphism.

Next we show that with this topology, $\widehat{\pi}$ is a generalized covering map. Suppose $U \subseteq M$ is an orientable connected open subset and $\mathcal{O}$ is an orientation for U.

Then $\widehat{\pi}^{-1}(U)$ is the disjoint union of open subsets $\widehat{U}_{\mathcal{O}}$ and $\widehat{U}_{-\mathcal{O}}$, and $\widehat{\pi}$ restricts to a homeomorphism from each of these sets to U. Thus, each such set U is evenly covered, and it follows that $\widehat{\pi}$ is a generalized covering map. By Lemma 15.39, $\widehat{\pi}$ restricts to an ordinary covering map on each component of $\widehat{M}$, and so Proposition 4.40 shows that each such component is a topological n-manifold with or without boundary and has a unique smooth structure making $\widehat{\pi}$ into a smooth covering map. These smooth structures combine to give a smooth structure on all of $\widehat{M}$. This completes the proof of (a).

Next we give $\widehat{M}$ an orientation. Let $\widehat{p} = (p, \mathcal{O}_p)$ be a point in $\widehat{M}$. By definition, $\mathcal{O}_p$ is an orientation of T_pM, so we can give $T_{\widehat{p}}\widehat{M}$ the unique orientation $\widehat{\mathcal{O}}_{\widehat{p}}$ such that $d\widehat{\pi}_{\widehat{p}}\colon T_{\widehat{p}}\widehat{M} \to T_pM$ is orientation-preserving. This defines a pointwise orientation $\widehat{\mathcal{O}}$ on $\widehat{M}$. On each basis open subset $\widehat{U}_{\mathcal{O}}$, the orientation $\widehat{\mathcal{O}}$ agrees with the pullback orientation induced from $(U, \mathcal{O})$ by (the restriction of) $\widehat{\pi}$, so it is continuous.

Next we prove (b). We showed earlier that every orientable connected open subset of M is evenly covered by $\widehat{\pi}$. Conversely, if $U \subseteq M$ is any evenly covered open subset, then there is a smooth local section $\sigma\colon U \to \widehat{M}$ of $\widehat{\pi}$ by Proposition 4.36, which pulls $\widehat{\mathcal{O}}$ back to an orientation on U by Proposition 15.15.

Finally, to prove (c), assume $U \subseteq M$ is evenly covered and therefore orientable. Given any orientation $\mathcal{O}$ of U, define a section $\sigma\colon U \to \widehat{M}$ by setting $\sigma(p) = (p, \mathcal{O}_p)$. To see that σ is continuous, suppose $\widehat{U}'_{\mathcal{O}'}$ is any basis open subset of $\widehat{M}$. Then for each component V of $U \cap U'$, the restricted orientations $\mathcal{O}|_V$ and $\mathcal{O}'|_V$ must either agree or disagree on all of V, so $\sigma^{-1}\big(\widehat{U}'_{\mathcal{O}'}\big)$ is a union of such components and therefore open. □

Theorem 15.41 (Orientation Covering Theorem). *Suppose M is a connected smooth manifold with or without boundary, and let $\widehat{\pi}\colon \widehat{M} \to M$ be its orientation covering.*

(a) *If M is orientable, then $\widehat{M}$ has exactly two components, and the restriction of $\widehat{\pi}$ to each component is a diffeomorphism onto M.*
(b) *If M is nonorientable, then $\widehat{M}$ is connected, and $\widehat{\pi}$ is a two-sheeted smooth covering map.*

Proof. If M is orientable, then Proposition 15.40(b) shows that M is evenly covered by $\widehat{\pi}$, which means that $\widehat{M}$ has two components, each mapped diffeomorphically onto M.

Now assume M is nonorientable. We show first that $\widehat{M}$ is connected. Let W be a component of $\widehat{M}$. Lemma 15.39 shows that $\widehat{\pi}|_W$ is a covering map, so its fibers all have the same cardinality. Because the fibers of $\widehat{\pi}$ have cardinality 2 and W is not empty, the fibers of $\widehat{\pi}|_W$ must have cardinality 1 or 2. If the cardinality were 1, then $\widehat{\pi}|_W$ would be an injective smooth covering map and thus a diffeomorphism, and its inverse would be a smooth section of $\widehat{\pi}$, which would induce an orientation on M. Thus, the cardinality must be 2, which implies that $W = \widehat{M}$. Because $\widehat{M}$ is connected, $\widehat{\pi}$ is a covering map by Lemma 15.39, and because it is a local diffeomorphism it is a smooth covering map. □

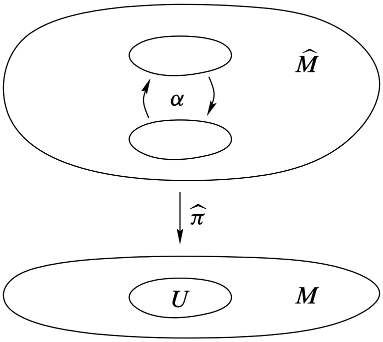

Fig. 15.9 The nontrivial covering automorphism of $\widehat{M}$

The orientation covering is sometimes called the ***oriented double covering of M***. There are other ways of constructing it besides the one we have given here, but as the next theorem shows, the specific details of the construction do not matter, because they all yield isomorphic covering manifolds.

Theorem 15.42 (Uniqueness of the Orientation Covering). *Let M be a nonorientable connected smooth manifold with or without boundary, and let $\widehat{\pi}\colon \widehat{M} \to M$ be its orientation covering. If $\widetilde{M}$ is an oriented smooth manifold with or without boundary that admits a two-sheeted smooth covering map $\widetilde{\pi}\colon \widetilde{M} \to M$, then there exists a unique orientation-preserving diffeomorphism $\varphi\colon \widetilde{M} \to \widehat{M}$ such that $\widehat{\pi} \circ \varphi = \widetilde{\pi}$.*

Proof. See Problem 15-11. □

By invoking a little more covering space theory, we obtain the following sufficient topological condition for orientability. If G is a group and $H \subseteq G$ is a subgroup, the ***index of H in G*** is the cardinality of the set of left cosets of H in G. (If H is a normal subgroup, it is just the cardinality of the quotient group G/H.)

Theorem 15.43. *Let M be a connected smooth manifold with or without boundary, and suppose the fundamental group of M has no subgroup of index 2. Then M is orientable. In particular, if M is simply connected then it is orientable.*

Proof. Suppose M is not orientable, and let $\widehat{\pi}\colon \widehat{M} \to M$ be its orientation covering, which is an honest covering map in this case. Choose any point $q \in \widehat{M}$, and let $p = \widehat{\pi}(q) \in M$. Let $\alpha\colon \widehat{M} \to \widehat{M}$ be the map that interchanges the two points in each fiber of $\widehat{\pi}$ (Fig. 15.9).To prove that α is smooth, suppose $U \subseteq M$ is any evenly covered open subset and $U_0, U_1 \subseteq \widehat{M}$ are the two components of $\widehat{\pi}^{-1}(U)$. Since $\widehat{\pi}$ restricts to a diffeomorphism from each component onto U, we can write $\alpha|_{U_0} = \left(\widehat{\pi}|_{U_1}\right)^{-1} \circ \left(\widehat{\pi}|_{U_0}\right)$, which is smooth. Similarly, $\alpha|_{U_1}$ is also smooth. Since the collection of all such sets U_0, U_1 is an open covering of $\widehat{M}$, it follows that α is smooth, and it is a covering automorphism because it satisfies $\widehat{\pi} \circ \alpha = \widehat{\pi}$. In fact, since a covering automorphism is determined by what it does to one point, α is the unique nontrivial element of the automorphism group $\operatorname{Aut}_{\widehat{\pi}}(\widehat{M})$, which is therefore equal to the two-element group $\{\operatorname{Id}_{\widehat{M}}, \alpha\}$. Because the automorphism group acts transitively on fibers, $\widehat{\pi}$ is a normal covering map. Let H denote the subgroup $\widehat{\pi}_*(\pi_1(\widehat{M}, q))$ of $\pi_1(M, p)$. A fundamental result in the theory of covering spaces (see, e.g., [LeeTM, Chap. 12]) is that the quotient group $\pi_1(M, p)/H$ is isomorphic to $\operatorname{Aut}_{\widehat{\pi}}(\widehat{M})$. Therefore, H has index 2 in $\pi_1(M, p)$. □

Problems

15-1. Suppose M is a smooth manifold that is the union of two orientable open submanifolds with connected intersection. Show that M is orientable. Use this to give another proof that $\mathbb{S}^n$ is orientable.

15-2. Suppose M and N are oriented smooth manifolds with or without boundary, and $F: M \to N$ is a local diffeomorphism. Show that if M is connected, then F is either orientation-preserving or orientation-reversing.

15-3. Suppose $n \geq 1$, and let $\alpha: \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map: $\alpha(x) = -x$. Show that α is orientation-preserving if and only if n is odd. [Hint: consider the map $F: \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ given by $F(x) = -x$, and use Corollary 15.34.] (*Used on pp. 393, 435.*)

15-4. Let θ be a smooth flow on an oriented smooth manifold with or without boundary. Show that for each $t \in \mathbb{R}$, θ_t is orientation-preserving wherever it is defined. (*Used on p. 425.*)

15-5. Let M be a smooth manifold with or without boundary. Show that the total spaces of TM and T^*M are orientable.

15-6. Let $U \subseteq \mathbb{R}^3$ be the open subset $\left\{(x, y, z) : \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 < 1\right\}$ (the solid torus bounded by the torus of revolution of Example 5.17). Define a map $F: \mathbb{R}^2 \to U$ by

$$F(u, v) = \big(\cos 2\pi u(2 + \tanh v \cos \pi u),\\ \sin 2\pi u(2 + \tanh v \cos \pi u), \tanh v \sin \pi u\big).$$

(a) Show that F descends to a smooth embedding of E into U, where E is the total space of the Möbius bundle of Example 10.3.
(b) Let S be the image of F. Show that S is a properly embedded smooth submanifold of U.
(c) Show that there is no unit normal vector field along S.
(d) Show that S has no global defining function in U.

15-7. Suppose M is an oriented Riemannian manifold with or without boundary, and $S \subseteq M$ is an oriented smooth hypersurface with or without boundary. Show that there is a unique smooth unit normal vector field along S that determines the given orientation of S.

15-8. Suppose M is an orientable Riemannian manifold, and $S \subseteq M$ is an immersed or embedded submanifold with or without boundary. Prove the following statements.

(a) If S has trivial normal bundle, then S is orientable.

(b) If S is an orientable hypersurface, then S has trivial normal bundle.

15-9. Let S be an oriented, embedded, 2-dimensional submanifold with boundary in $\mathbb{R}^3$, and let $C = \partial S$ with the induced orientation. By Problem 15-7, there is a unique smooth unit normal vector field N on S that determines the orientation. Let T be the oriented unit tangent vector field on C, and let V be the unique unit vector field tangent to S along C that is orthogonal to T and inward-pointing. Show that (T_p, V_p, N_p) is an oriented orthonormal basis for $\mathbb{R}^3$ at each $p \in C$.

15-10. Characteristic Property of the Orientation Covering: Let M be a connected nonorientable smooth manifold with or without boundary, and let $\widehat{\pi}\colon \widehat{M} \to M$ be its orientation covering. Prove that if X is any oriented smooth manifold with or without boundary, and $F\colon X \to M$ is any local diffeomorphism, then there exists a unique orientation-preserving local diffeomorphism $\widehat{F}\colon X \to \widehat{M}$ such that $\widehat{\pi} \circ \widehat{F} = F$:

$$\begin{array}{ccc} & & \widehat{M} \\ & \overset{\widehat{F}}{\nearrow} & \downarrow \widehat{\pi} \\ X & \xrightarrow[F]{} & M. \end{array}$$

15-11. Prove Theorem 15.42 (uniqueness of the orientation covering). [Hint: use Problem 15-10.]

15-12. Show that every orientation-reversing diffeomorphism of $\mathbb{R}$ has a fixed point.

15-13. Classification of Smooth 1-Manifolds: Let M be a connected smooth 1-manifold. Show that M is diffeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$, as follows:

(a) First, do the case in which M is orientable by showing that M admits a nonvanishing smooth vector field and using Problem 9-1.

(b) Now let M be arbitrary, and prove that M is orientable by showing that its universal covering manifold is diffeomorphic to $\mathbb{R}$ and using the result of Problem 15-12.

Conclude that the smooth structures on both $\mathbb{R}$ and $\mathbb{S}^1$ are unique up to diffeomorphism.

15-14. Classification of Smooth 1-Manifolds with Boundary: Show that every connected smooth 1-manifold with nonempty boundary is diffeomorphic to either $[0, 1]$ or $[0, \infty)$. [Hint: use the double.]

15-15. Let M be a nonorientable embedded hypersurface in $\mathbb{R}^n$, and let NM be its normal bundle with projection $\pi_{NM}: NM \to M$. Show that the set

$$W = \{(x, v) \in NM : |v| = 1\}$$

is an embedded submanifold of NM, and the restriction of π_{NM} to W is a smooth covering map isomorphic to the orientation covering of M. [Hint: consider the orientation determined by $v \lrcorner (dx^1 \wedge \cdots \wedge dx^n)$.]

15-16. Let E be the total space of the Möbius bundle as in Example 15.38. Show that the orientation covering of E is diffeomorphic to the cylinder $\mathbb{S}^1 \times \mathbb{R}$.