Chapter 13
Riemannian Metrics

In this chapter, for the first time, we introduce *geometry* into smooth manifold theory. As is so much of this subject, our approach to geometry is modeled on the theory of finite-dimensional vector spaces. To define geometric concepts such as lengths and angles on a vector space, one uses an inner product. For manifolds, the appropriate structure is a *Riemannian metric*, which is essentially a choice of inner product on each tangent space, varying smoothly from point to point. A choice of Riemannian metric allows us to define geometric concepts such as lengths, angles, and distances on smooth manifolds.

Riemannian geometry is a deep subject in its own right. To develop all of the machinery needed for a complete treatment of it would require another whole book. (If you want to dig more deeply into it than we can here, you might start with [LeeRM], which gives a concise introduction to the subject, and for which you already have most of the necessary background.) But we can at least introduce the main definitions and some important examples.

After defining Riemannian metrics and the main constructions associated with them, we show how submanifolds of Riemannian manifolds inherit induced Riemannian metrics. Then we show how a Riemannian metric leads to a distance function, which allows us to consider connected Riemannian manifolds as metric spaces.

At the end of the chapter, we briefly describe a generalization of Riemannian metrics, called *pseudo-Riemannian metrics*, which play a central role in Einstein's general theory of relativity.

Riemannian Manifolds

The most important examples of symmetric tensors on a vector space are inner products. Any inner product allows us to define lengths of vectors and angles between them, and thus to do Euclidean geometry.

Transferring these ideas to manifolds, we obtain one of the most important applications of tensors to differential geometry. Let M be a smooth manifold with or without boundary. A ***Riemannian metric on M*** is a smooth symmetric covariant

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2-tensor field on M that is positive definite at each point. A ***Riemannian manifold*** is a pair (M, g), where M is a smooth manifold and g is a Riemannian metric on M. One sometimes simply says "M is a Riemannian manifold" if M is understood to be endowed with a specific Riemannian metric. A ***Riemannian manifold with boundary*** is defined similarly.

Note that a Riemannian metric is not the same thing as a metric in the sense of metric spaces, although the two concepts are related, as we will see below. Because of this ambiguity, we usually use the term "distance function" for a metric in the metric space sense, and reserve "metric" for a Riemannian metric. In any event, which type of metric is being considered should always be clear from the context.

If g is a Riemannian metric on M, then for each $p \in M$, the 2-tensor g_p is an inner product on T_pM. Because of this, we often use the notation $\langle v, w\rangle_g$ to denote the real number $g_p(v, w)$ for $v, w \in T_pM$.

In any smooth local coordinates $\left(x^i\right)$, a Riemannian metric can be written

$$g = g_{ij}\, dx^i \otimes dx^j,$$

where (g_{ij}) is a symmetric positive definite matrix of smooth functions. The symmetry of g allows us to write g also in terms of symmetric products as follows:

$$\begin{aligned} g &= g_{ij}\, dx^i \otimes dx^j \\ &= \tfrac{1}{2}\left(g_{ij}\, dx^i \otimes dx^j + g_{ji}\, dx^i \otimes dx^j\right) && (\text{since } g_{ij} = g_{ji}) \\ &= \tfrac{1}{2}\left(g_{ij}\, dx^i \otimes dx^j + g_{ij}\, dx^j \otimes dx^i\right) && (\text{switch } i \leftrightarrow j \text{ in the second term}) \\ &= g_{ij}\, dx^i\, dx^j && (\text{by Proposition 12.15(b)}). \end{aligned}$$

Example 13.1 (The Euclidean Metric). The simplest example of a Riemannian metric is the ***Euclidean metric*** $\overline{g}$ on $\mathbb{R}^n$, given in standard coordinates by

$$\overline{g} = \delta_{ij}\, dx^i\, dx^j,$$

where δ_{ij} is the Kronecker delta. It is common to abbreviate the symmetric product of a tensor α with itself by α^2, so the Euclidean metric can also be written

$$\overline{g} = \left(dx^1\right)^2 + \cdots + \left(dx^n\right)^2.$$

Applied to vectors $v, w \in T_p\mathbb{R}^n$, this yields

$$\overline{g}_p(v, w) = \delta_{ij} v^i w^j = \sum_{i=1}^{n} v^i w^i = v \cdot w.$$

In other words, $\overline{g}$ is the 2-tensor field whose value at each point is the Euclidean dot product. (As you may recall, we warned in Chapter 1 that expressions involving the Euclidean dot product are likely to violate our index conventions and therefore to require explicit summation signs. This can usually be avoided by writing the metric coefficients δ_{ij} explicitly, as in $\delta_{ij} v^i w^j$.) //

Example 13.2 (Product Metrics). If (M,g) and $\big(\widetilde{M},\widetilde{g}\big)$ are Riemannian manifolds, we can define a Riemannian metric $\widehat{g} = g \oplus \widetilde{g}$ on the product manifold $M \times \widetilde{M}$, called the ***product metric***, as follows:

$$\widehat{g}\Big(\big(v,\widetilde{v}\big),\big(w,\widetilde{w}\big)\Big) = g(v,w) + \widetilde{g}\big(\widetilde{v},\widetilde{w}\big) \tag{13.1}$$

for any $\big(v,\widetilde{v}\big),\big(w,\widetilde{w}\big) \in T_pM \oplus T_q\widetilde{M} \cong T_{(p,q)}\big(M \times \widetilde{M}\big)$. Given any local coordinates $\big(x^1,\dots,x^n\big)$ for M and $\big(y^1,\dots,y^m\big)$ for $\widetilde{M}$, we obtain local coordinates $\big(x^1,\dots,x^n,y^1,\dots,y^m\big)$ for $M \times \widetilde{M}$, and you can check that the product metric is represented locally by the block diagonal matrix

$$\big(\widehat{g}_{ij}\big) = \begin{pmatrix} g_{ij} & 0 \\ 0 & \widetilde{g}_{ij} \end{pmatrix}.$$

For example, it is easy to verify that the Euclidean metric on $\mathbb{R}^{n+m}$ is the same as the product metric determined by the Euclidean metrics on $\mathbb{R}^n$ and $\mathbb{R}^m$. //

One pleasant feature of Riemannian metrics is that they exist in great abundance. (For another approach, see Problem 13-18.)

Proposition 13.3 (Existence of Riemannian Metrics). *Every smooth manifold with or without boundary admits a Riemannian metric.*

Proof. Let M be a smooth manifold with or without boundary, and choose a covering of M by smooth coordinate charts $(U_\alpha,\varphi_\alpha)$. In each coordinate domain, there is a Riemannian metric $g_\alpha = \varphi_\alpha^*\overline{g}$, whose coordinate expression is $\delta_{ij}\,dx^i\,dx^j$. Let $\{\psi_\alpha\}$ be a smooth partition of unity subordinate to the cover $\{U_\alpha\}$, and define

$$g = \sum_\alpha \psi_\alpha g_\alpha,$$

with each term interpreted to be zero outside $\operatorname{supp}\psi_\alpha$. By local finiteness, there are only finitely many nonzero terms in a neighborhood of each point, so this expression defines a smooth tensor field. It is obviously symmetric, so only positivity needs to be checked. If $v \in T_pM$ is any nonzero vector, then

$$g_p(v,v) = \sum_\alpha \psi_\alpha(p) g_\alpha|_p(v,v).$$

This sum is nonnegative, because each term is nonnegative. At least one of the functions ψ_α is strictly positive at p (because they sum to 1). Because $g_\alpha|_p(v,v) > 0$, it follows that $g_p(v,v) > 0$. □

It is important to observe that there is an enormous amount of choice in the construction of a metric g for a given manifold, so there is nothing canonical about it. In particular, different metrics on the same manifold can have vastly different geometric properties. For example, Problem 13-20 describes four metrics on $\mathbb{R}^2$ that behave in strikingly different ways.

Below are just a few of the geometric constructions that can be defined on a Riemannian manifold (M,g) with or without boundary.

- The ***length*** or ***norm*** of a tangent vector $v \in T_pM$ is defined to be
$$|v|_g = \langle v, v\rangle_g^{1/2} = g_p(v,v)^{1/2}.$$
- The ***angle*** between two nonzero tangent vectors $v, w \in T_pM$ is the unique $\theta \in [0,\pi]$ satisfying
$$\cos\theta = \frac{\langle v, w\rangle_g}{|v|_g|w|_g}.$$
- Tangent vectors $v, w \in T_pM$ are said to be ***orthogonal*** if $\langle v, w\rangle_g = 0$. This means either one or both vectors are zero, or the angle between them is $\pi/2$.

One highly useful tool for the study of Riemannian manifolds is *orthonormal frames*. Let (M,g) be an n-dimensional Riemannian manifold with or without boundary. Just as we did for the case of $\mathbb{R}^n$ in Chapter 8, we say that a local frame $(E_1,\dots,E_n)$ for M on an open subset $U \subseteq M$ is an ***orthonormal frame*** if the vectors $(E_1|_p,\dots,E_n|_p)$ form an orthonormal basis for T_pM at each point $p \in U$, or equivalently if $\langle E_i, E_j\rangle_g = \delta_{ij}$.

Example 13.4. The coordinate frame $\left(\partial/\partial x^i\right)$ is a global orthonormal frame for $\mathbb{R}^n$ with the Euclidean metric. //

Example 13.5. The frame (E_1,E_2) on $\mathbb{R}^2 \smallsetminus \{0\}$ defined in Example 8.12 is a local orthonormal frame for $\mathbb{R}^2$. As we observed in Example 9.45, it is not a coordinate frame in any coordinates. //

The next proposition is proved in just the same way as Lemma 8.13, with the Euclidean dot product replaced by the inner product $\langle\cdot,\cdot\rangle_g$.

Proposition 13.6. *Suppose (M,g) is a Riemannian manifold with or without boundary, and (X_j) is a smooth local frame for M over an open subset $U \subseteq M$. Then there is a smooth orthonormal frame (E_j) over U such that* $\operatorname{span}\left(E_1|_p,\dots,E_j|_p\right) = \operatorname{span}\left(X_1|_p,\dots,X_j|_p\right)$ *for each $j = 1,\dots,n$ and each $p \in U$.*

► **Exercise 13.7.** Prove the preceding proposition.

Corollary 13.8 (Existence of Local Orthonormal Frames). *Let (M,g) be a Riemannian manifold with or without boundary. For each $p \in M$, there is a smooth orthonormal frame on a neighborhood of p.*

Proof. Start with a smooth coordinate frame and apply Proposition 13.6. □

Observe that Corollary 13.8 does *not* show that there are smooth coordinates on a neighborhood of p for which the *coordinate frame* is orthonormal. In fact, this is rarely the case, as we will see below.

Pullback Metrics

Suppose M, N are smooth manifolds with or without boundary, g is a Riemannian metric on N, and $F\colon M \to N$ is smooth. The pullback F^*g is a smooth 2-tensor

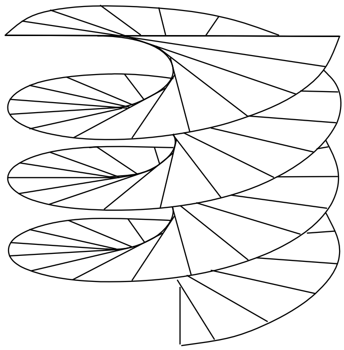

Fig. 13.1 A helicoid

field on M. If it is positive definite, it is a Riemannian metric on M, called the ***pullback metric*** determined by F. The next proposition shows when this is the case.

Proposition 13.9 (Pullback Metric Criterion). *Suppose $F: M \to N$ is a smooth map and g is a Riemannian metric on N. Then F^*g is a Riemannian metric on M if and only if F is a smooth immersion.*

► **Exercise 13.10.** Prove the preceding proposition.

If the coordinate representation for an immersion is known, then the pullback metric is easy to compute using the usual algorithm for computing pullbacks.

Example 13.11. Consider the smooth map $F: \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$F(u,v) = (u\cos v, u\sin v, v).$$

It is a proper injective smooth immersion, and thus it is an embedding by Proposition 4.22. Its image is a surface called a ***helicoid***; it looks like an infinitely wide spiral-shaped ramp (Fig. 13.1).The pullback metric $F^*\overline{g}$ can be computed by substituting the coordinate functions for F in place of x, y, z in the formula for $\overline{g}$:

$$\begin{aligned} F^*\overline{g} &= d(u\cos v)^2 + d(u\sin v)^2 + d(v)^2 \\ &= (\cos v\, du - u\sin v\, dv)^2 + (\sin v\, du + u\cos v\, dv)^2 + dv^2 \\ &= \cos^2 v\, du^2 - 2u\sin v\cos v\, du\, dv + u^2\sin^2 v\, dv^2 \\ &\quad + \sin^2 v\, du^2 + 2u\sin v\cos v\, du\, dv + u^2\cos^2 v\, dv^2 + dv^2 \\ &= du^2 + (u^2+1)dv^2. \end{aligned}$$

(By convention, when u is a real-valued function, the notation du^2 means the symmetric product $du\, du$, not $d\left(u^2\right)$). //

To transform a Riemannian metric under a change of coordinates, we use the same technique as we used for covector fields: think of the change of coordinates as the identity map expressed in terms of different coordinates for the domain and codomain, and use the formula of Corollary 12.28 to compute the pullback. As before, in practice this just amounts to substituting the formulas for one set of coordinates in terms of the other.

Example 13.12. To illustrate, we compute the coordinate expression for the Euclidean metric $\overline{g} = dx^2 + dy^2$ on $\mathbb{R}^2$ in polar coordinates. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ and expanding, we obtain

$$\begin{aligned}\overline{g} &= dx^2 + dy^2 = d(r\cos\theta)^2 + d(r\sin\theta)^2\\ &= (\cos\theta\,dr - r\sin\theta\,d\theta)^2 + (\sin\theta\,dr + r\cos\theta\,d\theta)^2\\ &= \left(\cos^2\theta + \sin^2\theta\right)dr^2 + \left(r^2\sin^2\theta + r^2\cos^2\theta\right)d\theta^2\\ &\quad + (-2r\cos\theta\sin\theta + 2r\sin\theta\cos\theta)dr\,d\theta\\ &= dr^2 + r^2\,d\theta^2. \end{aligned}$$

//

If (M,g) and $\left(\widetilde{M},\widetilde{g}\right)$ are both Riemannian manifolds, a smooth map $F\colon M \to \widetilde{M}$ is called a ***(Riemannian) isometry*** if it is a diffeomorphism that satisfies $F^*\widetilde{g} = g$. More generally, F is called a ***local isometry*** if every point $p \in M$ has a neighborhood U such that $F|_U$ is an isometry of U onto an open subset of $\widetilde{M}$; or equivalently, if F is a local diffeomorphism satisfying $F^*\widetilde{g} = g$.

If there exists a Riemannian isometry between (M,g) and $\left(\widetilde{M},\widetilde{g}\right)$, we say that they are ***isometric*** as Riemannian manifolds. If each point of M has a neighborhood that is isometric to an open subset of $\left(\widetilde{M},\widetilde{g}\right)$, then we say that (M,g) is ***locally isometric*** to $\left(\widetilde{M},\widetilde{g}\right)$. The study of properties of Riemannian manifolds that are invariant under (local or global) isometries is called ***Riemannian geometry***.

One such property is *flatness*. A Riemannian n-manifold (M,g) is said to be a ***flat Riemannian manifold***, and g is a ***flat metric***, if (M,g) is locally isometric to $(\mathbb{R}^n,\overline{g})$.

► **Exercise 13.13.** Suppose (M,g) and $\left(\widetilde{M},\widetilde{g}\right)$ are isometric Riemannian manifolds. Show that g is flat if and only if $\widetilde{g}$ is flat.

The next theorem is the key to deciding whether a Riemannian metric is flat.

Theorem 13.14. *For a Riemannian manifold (M,g), the following are equivalent:*

(a) *g is flat.*
(b) *Each point of M is contained in the domain of a smooth coordinate chart in which g has the coordinate representation $g = \delta_{ij}\,dx^i\,dx^j$.*
(c) *Each point of M is contained in the domain of a smooth coordinate chart in which the coordinate frame is orthonormal.*
(d) *Each point of M is contained in the domain of a commuting orthonormal frame.*

Proof. The implications (a) ⇒ (b) ⇒ (c) ⇒ (d) are easy consequences of the definitions, and are left to the reader. The remaining implication, (d) ⇒ (a), follows from the canonical form theorem for commuting frames: if (E_i) is a commuting orthonormal frame for g on an open subset $U \subseteq M$, then Theorem 9.46 implies that each $p \in V$ is contained in the domain of a smooth chart (U,φ) in which the coordinate frame is equal to (E_i). This means $\varphi_* E_i = \partial/\partial x^i$, so the diffeomorphism $\varphi\colon U \to \varphi(U)$ satisfies

$$\varphi^*\overline{g}(E_i,E_j) = \overline{g}(\varphi_* E_i, \varphi_* E_j) = \overline{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} = g(E_i,E_j).$$

Bilinearity then shows that $\varphi^*\overline{g} = g$, so φ is an isometry between $(U, g|_U)$ and $\varphi(U)$ with the Euclidean metric. This shows that g is flat. □

► **Exercise 13.15.** Complete the preceding proof by showing (a) ⇒ (b) ⇒ (c) ⇒ (d).

It is not at all obvious from the definitions that there exist Riemannian metrics that are not flat. In fact, in the 1-dimensional case, every metric is flat, as Problem 13-6 shows. Later in this chapter, we will use Theorem 13.14 to show that most surfaces of revolution in $\mathbb{R}^3$, including $\mathbb{S}^2$, are not flat.

Riemannian Submanifolds

Pullback metrics are especially important for submanifolds. If (M,g) is a Riemannian manifold with or without boundary, every submanifold $S \subseteq M$ (immersed or embedded, with or without boundary) automatically inherits a pullback metric $\iota^* g$, where $\iota\colon S \hookrightarrow M$ is inclusion. In this setting, the pullback metric is also called the ***induced metric*** on S. By definition, this means for $v,w \in T_pS$ that

$$\left(\iota^* g\right)(v,w) = g\big(d\iota_p(v), d\iota_p(w)\big) = g(v,w),$$

because $d\iota_p\colon T_pS \to T_pM$ is our usual identification of T_pS as a subspace of T_pM. Thus $\iota^* g$ is just the restriction of g to pairs of vectors tangent to S. With this metric, S is called a ***Riemannian submanifold (with or without boundary) of M***.

Example 13.16. The metric $\mathring{g} = \iota^*\overline{g}$ induced on $\mathbb{S}^n$ by the usual inclusion $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is called the ***round metric*** (or the ***standard metric***) on the sphere. //

If (M,g) is a Riemannian manifold and $\iota\colon S \hookrightarrow M$ is a Riemannian submanifold, it is usually easiest to compute the induced metric $\iota^* g$ in terms of a *local parametrization*; recall from Chapter 5 that this is an injective immersion X from an open subset $U \subseteq \mathbb{R}^k$ into M whose image is an open subset of S, and whose inverse is a smooth coordinate map for S. Since $\iota \circ X = X$, the coordinate representation of $\iota^* g$ is $X^*\left(\iota^* g\right) = X^* g$. The next two examples illustrate the procedure.

Example 13.17 (Induced Metrics in Graph Coordinates). Let $U \subseteq \mathbb{R}^n$ be an open subset, and let $S \subseteq \mathbb{R}^{n+1}$ be the graph of a smooth function $f\colon U \to \mathbb{R}$. The map $X\colon U \to \mathbb{R}^{n+1}$ given by $X\left(u^1,\dots,u^n\right) = \left(u^1,\dots,u^n, f(u)\right)$ is a smooth global parametrization of S and the induced metric on S is given in graph coordinates by

$$X^*\overline{g} = X^*\left(\left(dx^1\right)^2 + \cdots + \left(dx^{n+1}\right)^2\right) = \left(du^1\right)^2 + \cdots + (du^n)^2 + df^2.$$

For example, the upper hemisphere of $\mathbb{S}^2$ is parametrized by the map $X\colon \mathbb{B}^2 \to \mathbb{R}^3$ given by

$$X(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right).$$

In these coordinates, the round metric can be written

$$\begin{aligned} \mathring{g} = X^*\overline{g} &= du^2 + dv^2 + \left(\frac{u\,du + v\,dv}{\sqrt{1-u^2-v^2}}\right)^2 \\ &= \frac{\left(1-v^2\right)du^2 + \left(1-u^2\right)dv^2 + 2uv\,du\,dv}{1-u^2-v^2}. \end{aligned}$$ //

Example 13.18 (Induced Metrics on Surfaces of Revolution). Let C be an embedded 1-dimensional submanifold of the half-plane $\{(r,z) : r > 0\}$, and let S_C be the surface of revolution generated by C as described in Example 5.17. To compute the induced metric on S_C, choose any smooth local parametrization $\gamma(t) = \big(a(t), b(t)\big)$ for C, and note that the map $X(t,\theta) = \big(a(t)\cos\theta, a(t)\sin\theta, b(t)\big)$ yields a smooth local parametrization of S_C, provided that (t,θ) is restricted to a sufficiently small open subset of the plane. Thus we can compute

$$\begin{aligned} X^*\overline{g} &= d\big(a(t)\cos\theta\big)^2 + d\big(a(t)\sin\theta\big)^2 + d\big(b(t)\big)^2 \\ &= \big(a'(t)\cos\theta\,dt - a(t)\sin\theta\,d\theta\big)^2 \\ &\quad + \big(a'(t)\sin\theta\,dt + a(t)\cos\theta\,d\theta\big)^2 + \big(b'(t)\,dt\big)^2 \\ &= \big(a'(t)^2 + b'(t)^2\big)\,dt^2 + a(t)^2\,d\theta^2. \end{aligned}$$

In particular, if γ is a ***unit-speed curve***, meaning that $|\gamma'(t)|^2 = a'(t)^2 + b'(t)^2 = 1$, this reduces to the simple formula $dt^2 + a(t)^2\,d\theta^2$.

Here are some familiar examples of surfaces of revolution.

(a) The embedded torus described in Example 5.17 is the surface of revolution generated by the circle $(r-2)^2 + z^2 = 1$. Using the unit-speed parametrization $\gamma(t) = (2+\cos t, \sin t)$ for the circle, we obtain the formula $dt^2 + (2+\cos t)^2\,d\theta^2$ for the induced metric.
(b) The unit sphere (minus the north and south poles) is a surface of revolution whose generating curve is the semicircle parametrized by $\gamma(t) = (\sin t, \cos t)$ for $0 < t < \pi$. The induced metric is $dt^2 + \sin^2 t\,d\theta^2$.
(c) The unit cylinder $x^2 + y^2 = 1$ is a surface of revolution whose generating curve is the vertical line parametrized by $\gamma(t) = (1,t)$ for $t \in \mathbb{R}$. The induced metric is $dt^2 + d\theta^2$. //

Look again at the last example above. It shows that for each local parametrization of the cylinder given by $X(t, \theta) = (\cos\theta, \sin\theta, t)$, the induced metric $X^*\overline{g}$ is the Euclidean metric on the (t, θ)-plane. To put it another way, for any point p in the cylinder, a suitable restriction of X gives a Riemannian isometry between an open subset of $\left(\mathbb{R}^2, \overline{g}\right)$ and a neighborhood of p in the cylinder with its induced metric. Thus the induced metric on the cylinder is *flat*. A two-dimensional being living in the cylinder would not be able to distinguish its surroundings from the Euclidean plane by local geometric measurements. This illustrates that the question of whether a metric is flat or not can sometimes have an unexpected answer.

To develop adequate machinery to determine systematically which metrics are flat and which are not would require techniques that are beyond the scope of this book. Just as proving two topological spaces are not homeomorphic requires finding topological invariants that distinguish them, in order to prove two Riemannian manifolds are not locally isometric, one must introduce local invariants that are preserved by Riemannian isometries, and show that different metrics have different invariants. The fundamental invariant of a Riemannian metric is called its *curvature*; this is a quantitative measure of how far the metric deviates from flatness. See, for example, [LeeRM] for an account of the theory of Riemannian curvature.

For the present, we have to content ourselves with the next proposition, which answers the question for surfaces of revolution using a rather ad hoc method.

Proposition 13.19 (Flatness Criterion for Surfaces of Revolution). *Let $C \subseteq H$ be a connected embedded 1-dimensional submanifold of the half-plane $H = \{(r, z) : r > 0\}$, and let S_C be the surface of revolution generated by C. The induced metric on S_C is flat if and only if C is part of a straight line.*

Proof. First assume C is part of a straight line. Then it has a parametrization of the form $\gamma(t) = (Pt + K, Qt + L)$ for some constants P, Q, K, L with P and Q not both zero. By rescaling the t variable, we may assume that γ is unit-speed. If $Q = 0$, then S_C is an open subset of the plane $z = L$ and is therefore flat. If $P = 0$, then S_C is part of the cylinder $x^2 + y^2 = K^2$, which can be shown to be flat in the same way as we did for the unit cylinder in Example 13.18(c). On the other hand, if neither P nor Q is zero, then S_C is part of a cone, and Example 13.18 shows that the induced metric is $dt^2 + (Pt + K)^2\,d\theta^2$. In a neighborhood of any point, the change of coordinates $(u, v) = \big((t + K/P)\cos P\theta, (t + K/P)\sin P\theta\big)$ pulls the Euclidean metric $du^2 + dv^2$ back to $dt^2 + (Pt + K)^2\,d\theta^2$, so this metric is flat. (Think of slitting a paper cone along one side and flattening it out.)

Conversely, assuming that S_C is flat, we will show that C is part of a straight line. Let $\gamma(t) = \big(a(t), b(t)\big)$ be a local parametrization of C. Using the result of Problem 13-5, we may assume that γ is unit-speed, so that $a'(t)^2 + b'(t)^2 = 1$. As in Example 13.18, the induced metric is $dt^2 + a^2 d\theta^2$. Thus the local frame

(E_1, E_2) given by

$$E_1 = \frac{\partial}{\partial t}, \qquad E_2 = \frac{1}{a}\frac{\partial}{\partial \theta},$$

is orthonormal. Any other orthonormal frame $(\widetilde{E}_1, \widetilde{E}_2)$ can be written in the form

$$\widetilde{E}_1 = uE_1 + vE_2 = u\frac{\partial}{\partial t} + \frac{v}{a}\frac{\partial}{\partial \theta},$$
$$\pm\widetilde{E}_2 = vE_1 - uE_2 = v\frac{\partial}{\partial t} - \frac{u}{a}\frac{\partial}{\partial \theta},$$

for some functions u and v depending smoothly on (t,θ) and satisfying $u^2 + v^2 = 1$. Because the metric is flat, it is possible to choose u and v such that $(\widetilde{E}_1, \widetilde{E}_2)$ is a commuting orthonormal frame (Theorem 13.14). Using formula (8.8) for Lie brackets in coordinates, this implies

$$0 = \pm[\widetilde{E}_1, \widetilde{E}_2] = \left(u\frac{\partial v}{\partial t} + \frac{v}{a}\frac{\partial v}{\partial \theta}\right)\frac{\partial}{\partial t} - \left(u\frac{\partial}{\partial t}\left(\frac{u}{a}\right) + \frac{v}{a}\frac{\partial}{\partial \theta}\left(\frac{u}{a}\right)\right)\frac{\partial}{\partial \theta}$$
$$- \left(v\frac{\partial u}{\partial t} - \frac{u}{a}\frac{\partial u}{\partial \theta}\right)\frac{\partial}{\partial t} - \left(v\frac{\partial}{\partial t}\left(\frac{v}{a}\right) - \frac{u}{a}\frac{\partial}{\partial \theta}\left(\frac{v}{a}\right)\right)\frac{\partial}{\partial \theta}.$$

To simplify this expression, we use the shorthand notations $f_\theta = \partial f/\partial\theta$ and $f_t = \partial f/\partial t$ for any function f. Note that $u^2 + v^2 = 1$ implies $uu_\theta + vv_\theta = uu_t + vv_t = 0$, and the fact that a depends only on t implies $a_\theta = 0$ and $a_t = a'$. Inserting these relations into the formula above and simplifying, we obtain

$$0 = (uv_t - vu_t)\frac{\partial}{\partial t} + \left(\frac{a' - vu_\theta + uv_\theta}{a^2}\right)\frac{\partial}{\partial \theta},$$

which implies

$$uv_t - vu_t = 0, \tag{13.2}$$
$$vu_\theta - uv_\theta = a'. \tag{13.3}$$

Because $u^2 + v^2 \equiv 1$, each point has a neighborhood on which either u or v is nonzero. On any open subset where $v \neq 0$, (13.2) implies that the t-derivative of u/v is zero. Thus we can write $u = fv$, where f is some function of θ alone. Then $u^2 + v^2 = 1$ implies that $v^2(f^2 + 1) = 1$, so $v = \pm 1/\sqrt{f^2 + 1}$ is also a function of θ alone, and so is $u = \pm\sqrt{1 - v^2}$. A similar argument applies where $u \neq 0$. But then (13.3) implies that a' is independent of t, so it is constant, and consequently so is $b' = \pm\sqrt{1 - (a')^2}$. It follows that a and b are affine functions of t, so each point of C has a neighborhood contained in a straight line. Since we are assuming C is connected, it follows that all of C is contained in a single straight line. □

Corollary 13.20. *The round metric on $\mathbb{S}^2$ is not flat.* □

The Normal Bundle

Suppose (M,g) is an n-dimensional Riemannian manifold with or without boundary, and $S \subseteq M$ is a k-dimensional Riemannian submanifold (also with or without boundary). Just as we did for submanifolds of $\mathbb{R}^n$, for any $p \in S$ we say that a vector $v \in T_pM$ is ***normal to S*** if v is orthogonal to every vector in T_pS with respect to the inner product $\langle \cdot,\cdot \rangle_g$. The ***normal space to S at p*** is the subspace $N_pS \subseteq T_pM$ consisting of all vectors that are normal to S at p, and the ***normal bundle of S*** is the subset $NS \subseteq TM$ consisting of the union of all the normal spaces at points of S. The projection $\pi_{NS}\colon NS \to S$ is defined as the restriction to NS of $\pi\colon TM \to M$. The following proposition is proved in the same way as Corollary 10.36.

Proposition 13.21 (The Normal Bundle to a Riemannian Submanifold). *Let (M,g) be a Riemannian n-manifold with or without boundary. For any immersed k-dimensional submanifold $S \subseteq M$ with or without boundary, the normal bundle NS is a smooth rank-$(n-k)$ subbundle of $TM|_S$. For each $p \in S$, there is a smooth frame for NS on a neighborhood of p that is orthonormal with respect to g.*

► **Exercise 13.22.** Prove the preceding proposition.

The Riemannian Distance Function

One of the most important tools that a Riemannian metric gives us is the ability to define lengths of curves. Suppose (M,g) is a Riemannian manifold with or without boundary. If $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment, the ***length of γ*** is

$$L_g(\gamma) = \int_a^b \left|\gamma'(t)\right|_g dt.$$

Because $|\gamma'(t)|_g$ is continuous at all but finitely many values of t, and has well-defined limits from the left and right at those points, the integral is well defined.

► **Exercise 13.23.** Suppose $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment and $a < c < b$. Show that

$$L_g(\gamma) = L_g\left(\gamma|_{[a,c]}\right) + L_g\left(\gamma|_{[c,b]}\right).$$

► **Exercise 13.24.** Show that lengths of curves are local isometry invariants of Riemannian manifolds. More precisely, suppose (M,g) and $\left(\widetilde{M},\widetilde{g}\right)$ are Riemannian manifolds with or without boundary, and $F\colon M \to \widetilde{M}$ is a local isometry. Show that $L_{\widetilde{g}}(F\circ\gamma) = L_g(\gamma)$ for every piecewise smooth curve segment γ in M.

It is an extremely important fact that length is independent of parametrization in the following sense. In Chapter 11 we defined a *reparametrization* of a piecewise smooth curve segment $\gamma\colon [a,b] \to M$ to be a curve segment of the form $\widetilde{\gamma} = \gamma\circ\varphi$, where $\varphi\colon [c,d] \to [a,b]$ is a diffeomorphism.

Proposition 13.25 (Parameter Independence of Length). *Let* (M,g) *be a Riemannian manifold with or without boundary, and let* $\gamma\colon [a,b]\to M$ *be a piecewise smooth curve segment. If* $\widetilde{\gamma}$ *is a reparametrization of* γ*, then* $\mathrm{L}_g(\widetilde{\gamma}) = \mathrm{L}_g(\gamma)$.

Proof. First suppose that γ is smooth, and $\varphi\colon [c,d]\to[a,b]$ is a diffeomorphism such that $\widetilde{\gamma}=\gamma\circ\varphi$. The fact that φ is a diffeomorphism implies that either $\varphi'>0$ or $\varphi'<0$ everywhere. Let us assume first that $\varphi'>0$. We have

$$\begin{aligned}\mathrm{L}_g(\widetilde{\gamma}) &= \int_c^d \left|\widetilde{\gamma}'(t)\right|_g dt = \int_c^d \left|\frac{d}{dt}(\gamma\circ\varphi)(t)\right|_g dt\\ &= \int_c^d \left|\varphi'(t)\gamma'\big(\varphi(t)\big)\right|_g dt = \int_c^d \left|\gamma'\big(\varphi(t)\big)\right|_g \varphi'(t)\,dt\\ &= \int_a^b \left|\gamma'(s)\right|_g ds = \mathrm{L}_g(\gamma),\end{aligned}$$

where the next-to-last equality uses the change of variables formula for integrals.

In the case $\varphi'<0$, we just need to introduce two sign changes into the above calculation. The sign changes once when $\varphi'(t)$ is moved outside the absolute value signs, because $|\varphi'(t)| = -\varphi'(t)$. Then it changes again when we change variables, because φ reverses the direction of the integral. Since the two sign changes cancel each other, the result is the same.

If γ is only piecewise smooth, we just apply the same argument on each subinterval on which it is smooth. □

Using curve segments as "measuring tapes," we can define distances between points on a Riemannian manifold. Suppose (M,g) is a connected Riemannian manifold. (The theory is most straightforward when $\partial M=\varnothing$, so we assume that for the rest of this section.) For any $p,q\in M$, the ***(Riemannian) distance from p to q***, denoted by $d_g(p,q)$, is defined to be the infimum of $\mathrm{L}_g(\gamma)$ over all piecewise smooth curve segments γ from p to q. Because any pair of points in M can be joined by a piecewise smooth curve segment (Proposition 11.33), this is well defined.

Example 13.26. In $(\mathbb{R}^n,\overline{g})$, Problem 13-10 shows that any straight line segment is the shortest piecewise smooth curve segment between its endpoints. Therefore, the distance function $d_{\overline{g}}$ is equal to the usual Euclidean distance:

$$d_{\overline{g}}(x,y) = |x-y|. \qquad //$$

► **Exercise 13.27.** Suppose (M,g) and $\big(\widetilde{M},\widetilde{g}\big)$ are connected Riemannian manifolds and $F\colon M\to\widetilde{M}$ is a Riemannian isometry. Show that $d_{\widetilde{g}}(F(p),F(q)) = d_g(p,q)$ for all $p,q\in M$.

We will see below that the Riemannian distance function turns M into a metric space whose topology is the same as the given manifold topology. The key is the following technical lemma, which shows that every Riemannian metric is locally comparable to the Euclidean metric in coordinates.

Lemma 13.28. *Let g be a Riemannian metric on an open subset $U \subseteq \mathbb{R}^n$. Given a compact subset $K \subseteq U$, there exist positive constants c, C such that for all $x \in K$ and all $v \in T_x\mathbb{R}^n$,*

$$c|v|_{\overline{g}} \le |v|_g \le C|v|_{\overline{g}}. \tag{13.4}$$

Proof. For any compact subset $K \subseteq U$, let $L \subseteq T\mathbb{R}^n$ be the set

$$L = \left\{(x,v) \in T\mathbb{R}^n : x \in K,\ |v|_{\overline{g}} = 1\right\}.$$

Under the canonical identification of $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$, L is just the product set $K \times \mathbb{S}^{n-1}$ and therefore is compact. Because the norm $|v|_g$ is continuous and strictly positive on L, there are positive constants c, C such that $c \le |v|_g \le C$ whenever $(x,v) \in L$. If $x \in K$ and v is a nonzero vector in $T_x\mathbb{R}^n$, let $\lambda = |v|_{\overline{g}}$. Then $\left(x, \lambda^{-1}v\right) \in L$, so by homogeneity of the norm,

$$|v|_g = \lambda\left|\lambda^{-1}v\right|_g \le \lambda C = C|v|_{\overline{g}}.$$

A similar computation shows that $|v|_g \ge c|v|_{\overline{g}}$. The same inequalities are trivially true when $v = 0$. □

Theorem 13.29 (Riemannian Manifolds as Metric Spaces). *Let (M,g) be a connected Riemannian manifold. With the Riemannian distance function, M is a metric space whose metric topology is the same as the original manifold topology.*

Proof. It is immediate from the definition that $d_g(p,q) \ge 0$. Because every constant curve segment has length zero, it follows that $d_g(p,p) = 0$; and $d_g(p,q) = d_g(q,p)$ follows from the fact that any curve segment from p to q can be reparametrized to go from q to p. Suppose γ_1 and γ_2 are piecewise smooth curve segments from p to q and q to r, respectively (Fig. 13.2), and let γ be a piecewise smooth curve segment that first follows γ_1 and then follows γ_2 (reparametrized if necessary). Then

$$d_g(p,r) \le \mathrm{L}_g(\gamma) = \mathrm{L}_g(\gamma_1) + \mathrm{L}_g(\gamma_2).$$

Taking the infimum over all such γ_1 and γ_2, we find that $d_g(p,r) \le d_g(p,q) + d_g(q,r)$. (This is one reason why it is important to define the distance function using piecewise smooth curves instead of just smooth ones.)

To complete the proof that (M, d_g) is a metric space, we need only show that $d_g(p,q) > 0$ if $p \ne q$. For this purpose, let $p, q \in M$ be distinct points, and let U be a smooth coordinate domain containing p but not q. Use the coordinate map as usual to identify U with an open subset in $\mathbb{R}^n$, and let $\overline{g}$ denote the Euclidean metric in these coordinates. If V is a regular coordinate ball of radius ε centered at p such that $\overline{V} \subseteq U$, Lemma 13.28 shows that there are positive constants c, C such that (13.4) is satisfied whenever $x \in \overline{V}$ and $v \in T_xM$. Then for any piecewise smooth curve segment γ lying entirely in $\overline{V}$, it follows that

$$c\mathrm{L}_{\overline{g}}(\gamma) \le \mathrm{L}_g(\gamma) \le C\mathrm{L}_{\overline{g}}(\gamma).$$

Suppose $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment from p to q. Let t_0 be the infimum of all $t \in [a,b]$ such that $\gamma(t) \notin \overline{V}$ (Fig. 13.3). It follows that

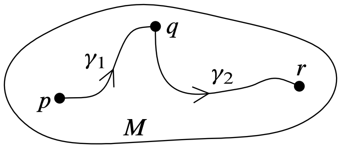

Fig. 13.2 The triangle inequality

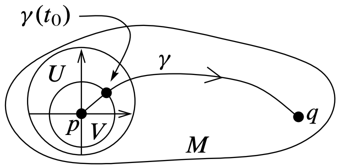

Fig. 13.3 Positivity of d_g

$\gamma(t_0) \in \partial V$ by continuity, and $\gamma(t) \in \overline{V}$ for $a \le t \le t_0$. Thus,

$$\mathrm{L}_g(\gamma) \ge \mathrm{L}_g\left(\gamma|_{[a,t_0]}\right) \ge c\mathrm{L}_{\overline{g}}\left(\gamma|_{[a,t_0]}\right) \ge c d_{\overline{g}}\big(p, \gamma(t_0)\big) = c\varepsilon.$$

Taking the infimum over all such γ, we conclude that $d_g(p,q) \ge c\varepsilon > 0$.

Finally, to show that the metric topology generated by d_g is the same as the given manifold topology on M, we need to show that the open subsets in the manifold topology are open in the metric topology, and vice versa. Suppose, first, that $U \subseteq M$ is open in the manifold topology. Let $p \in U$, and let V be a regular coordinate ball of radius ε around p such that $\overline{V} \subseteq U$ as above. The argument in the previous paragraph shows that $d_g(p,q) \ge c\varepsilon$ whenever $q \notin \overline{V}$. The contrapositive of this statement is that $d_g(p,q) < c\varepsilon$ implies $q \in \overline{V} \subseteq U$, or in other words, the metric ball of radius $c\varepsilon$ around p is contained in U. This shows that U is open in the metric topology.

Conversely, suppose that W is open in the metric topology, and let $p \in W$. Let V be a regular coordinate ball of radius r around p, let $\overline{g}$ be the Euclidean metric on $\overline{V}$ determined by the given coordinates, and let c, C be positive constants such that (13.4) is satisfied for $v \in T_qM$, $q \in \overline{V}$. Let $\varepsilon < r$ be a positive number small enough that the metric ball around p of radius $C\varepsilon$ is contained in W, and let V_ε be the set of points in $\overline{V}$ whose Euclidean distance from p is less than ε. If $q \in V_\varepsilon$, let γ be the straight-line segment in coordinates from p to q. Using Lemma 13.28 as above, we conclude that

$$d_g(p,q) \le \mathrm{L}_g(\gamma) \le C\mathrm{L}_{\overline{g}}(\gamma) < C\varepsilon.$$

This shows that V_ε is contained in the metric ball of radius $C\varepsilon$ around p, and therefore in W. Since V_ε is a neighborhood of p in the manifold topology, this shows that W is open in the manifold topology as well. □

As a consequence of this theorem, all of the terminology of metric spaces can be carried over to connected Riemannian manifolds. Thus, a connected Riemannian manifold (M,g) is said to be ***complete***, and g is said to be a ***complete Riemannian metric***, if (M, d_g) is a complete metric space (i.e., if every Cauchy sequence in M converges to a point in M); and a subset $B \subseteq M$ is said to be ***bounded*** if there exists a constant K such that $d_g(x,y) \le K$ for all $x, y \in B$. Problems 13-17 and 13-18 outline two different proofs that every connected smooth manifold admits a complete Riemannian metric.

Recall that a topological space is said to be ***metrizable*** if it admits a distance function whose metric topology is the same as the given topology.

Corollary 13.30. *Every smooth manifold with or without boundary is metrizable.*

Proof. First suppose M is a smooth manifold without boundary, and choose any Riemannian metric g on M. If M is connected, Theorem 13.29 shows that M is metrizable. More generally, let $\{M_i\}$ be the connected components of M, and choose a point $p_i \in M_i$ for each i. For $x \in M_i$ and $y \in M_j$, define $d_g(x, y)$ as in Theorem 13.29 when $i = j$, and otherwise

$$d_g(x, y) = d_g(x, p_i) + 1 + d_g(p_j, y).$$

(Think of building a "bridge" of length 1 between each pair of chosen points p_i, p_j in different components, so to get from x to y, you have to go to p_i, cross the bridge to p_j, and then go from p_j to y.) It is straightforward to check that this is a distance function that induces the given topology on M. Finally, if M has nonempty boundary, just embed M into its double (Example 9.32), and note that a subspace of a metrizable topological space is always metrizable. □

The Tangent–Cotangent Isomorphism

Another convenient feature of every Riemannian metric is that it provides a natural isomorphism between the tangent and cotangent bundles. Given a Riemannian metric g on a smooth manifold M with or without boundary, we define a bundle homomorphism $\widehat{g}\colon TM \to T^*M$ as follows. For each $p \in M$ and each $v \in T_pM$, we let $\widehat{g}(v) \in T_p^*M$ be the covector defined by

$$\widehat{g}(v)(w) = g_p(v, w) \quad \text{for all } w \in T_pM.$$

To see that this is a smooth bundle homomorphism, it is easiest to consider its action on smooth vector fields:

$$\widehat{g}(X)(Y) = g(X, Y) \quad \text{for } X, Y \in \mathfrak{X}(M).$$

Because $\widehat{g}(X)(Y)$ is linear over $C^\infty(M)$ as a function of Y, it follows from the tensor characterization lemma (Lemma 12.24) that $\widehat{g}(X)$ is a smooth covector field; and because $\widehat{g}(X)$ is linear over $C^\infty(M)$ as a function of X, this defines $\widehat{g}$ as a smooth bundle homomorphism by the bundle homomorphism characterization lemma (Lemma 10.29). As usual, we use the same symbol for both the pointwise bundle homomorphism $\widehat{g}\colon TM \to T^*M$ and the linear map on sections $\widehat{g}\colon \mathfrak{X}(M) \to \mathfrak{X}^*(M)$.

Note that $\widehat{g}$ is injective at each point, because $\widehat{g}(v) = 0$ for some $v \in T_pM$ implies

$$0 = \widehat{g}(v)(v) = \langle v, v \rangle_g,$$

which in turn implies $v = 0$. For dimensional reasons, therefore, $\widehat{g}$ is bijective, so it is a bundle isomorphism (see Proposition 10.26).

In any smooth coordinates $\left(x^i\right)$, we can write $g = g_{ij}\, dx^i\, dx^j$. Thus, if X and Y are smooth vector fields, we have

$$\widehat{g}(X)(Y) = g_{ij} X^i Y^j,$$

which implies that the covector field $\widehat{g}(X)$ has the coordinate expression

$$\widehat{g}(X) = g_{ij} X^i\, dx^j.$$

In other words, $\widehat{g}$ is the bundle homomorphism whose matrix with respect to coordinate frames for TM and T^*M is the same as the matrix of g itself. (Actually, it is the transpose of the matrix of g, but because (g_{ij}) is symmetric, these are the same.)

It is customary to denote the components of the covector field $\widehat{g}(X)$ by

$$\widehat{g}(X) = X_j\, dx^j, \quad \text{where } X_j = g_{ij} X^i.$$

Because of this, one says that $\widehat{g}(X)$ is obtained from X by ***lowering an index***. The notation $X^\flat$ is frequently used for $\widehat{g}(X)$, because the symbol $\flat$ ("flat") is used in musical notation to indicate that a tone is to be lowered.

The matrix of the inverse map $\widehat{g}^{-1}\colon T_p^*M \to T_pM$ is thus the inverse of (g_{ij}). (Because (g_{ij}) is the matrix of the isomorphism $\widehat{g}$, it is invertible at each point.) We let $\left(g^{ij}\right)$ denote the matrix-valued function whose value at $p \in M$ is the inverse of the matrix $\left(g_{ij}(p)\right)$, so that

$$g^{ij} g_{jk} = g_{kj} g^{ji} = \delta^i_k.$$

Because g_{ij} is a symmetric matrix, so is g^{ij}, as you can easily check. Thus for a covector field $\omega \in \mathfrak{X}^*(M)$, the vector field $\widehat{g}^{-1}(\omega)$ has the coordinate representation

$$\widehat{g}^{-1}(\omega) = \omega^i \frac{\partial}{\partial x^i}, \quad \text{where } \omega^i = g^{ij}\omega_j.$$

We use the notation $\omega^\sharp$ ("ω-sharp") for $\widehat{g}^{-1}(\omega)$, and say that $\omega^\sharp$ is obtained from ω by ***raising an index***. Because the symbols $\flat$ and $\sharp$ are borrowed from musical notation, these two inverse isomorphisms are frequently called the ***musical isomorphisms***. A handy mnemonic device for keeping the flat and sharp operations straight is to remember that the value of $\omega^\sharp$ at each point is a vector, which we visualize as a (sharp) arrow; while the value of $X^\flat$ is a covector, which we visualize by means of its (flat) level sets.

The most important use of the sharp operation is to reinstate the gradient as a vector field on Riemannian manifolds. For any smooth real-valued function f on a Riemannian manifold (M, g) with or without boundary, we define a vector field called the ***gradient of f*** by

$$\operatorname{grad} f = (df)^\sharp = \widehat{g}^{-1}(df).$$

Unraveling the definitions, we see that for any $X \in \mathfrak{X}(M)$, the gradient satisfies

$$\langle \operatorname{grad} f, X\rangle_g = \widehat{g}(\operatorname{grad} f)(X) = df(X) = Xf.$$

Thus grad f is the unique vector field that satisfies

$$\langle \operatorname{grad} f, X \rangle_g = Xf \quad \text{for every vector field } X,$$

or equivalently,

$$\langle \operatorname{grad} f, \cdot \rangle_g = df.$$

In smooth coordinates, grad f has the expression

$$\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

In particular, this shows that grad f is smooth. On $\mathbb{R}^n$ with the Euclidean metric, this reduces to

$$\operatorname{grad} f = \delta^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

Thus our new definition of the gradient in this case coincides with the gradient from elementary calculus. In other coordinates, however, the gradient does not generally have the same form.

Example 13.31. Let us compute the gradient of a function $f \in C^\infty\left(\mathbb{R}^2\right)$ with respect to the Euclidean metric in polar coordinates. From Example 13.12 we see that the matrix of $\overline{g}$ in polar coordinates is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & r^2 \end{smallmatrix}\right)$, so its inverse matrix is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/r^2 \end{smallmatrix}\right)$. Inserting this into the formula for the gradient, we obtain

$$\operatorname{grad} f = \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}. \qquad //$$

Problem 13-21 shows that the gradient of a function f on a Riemannian manifold has the same geometric interpretation as it has in Euclidean space: its direction is the direction in which f is increasing fastest, and is orthogonal to the level sets of f; and its length is the maximum directional derivative of f in any direction.

Pseudo-Riemannian Metrics

An important generalization of Riemannian metrics is obtained by relaxing the positivity requirement. A symmetric 2-tensor g on a vector space V is said to be ***nondegenerate*** if the linear map $\widehat{g}\colon V \to V^*$ defined by $\widehat{g}(v)(w) = g(v, w)$ is an isomorphism, or equivalently if for every nonzero $v \in V$ there exists $w \in V$ such that $g(v, w) \neq 0$. Just as any inner product can be transformed to the Euclidean one by switching to an orthonormal basis, every nondegenerate symmetric 2-tensor can be transformed by a change of basis to one whose matrix is diagonal with all entries equal to ± 1 (the proof is an adaptation of the Gram–Schmidt algorithm). The numbers r and s of positive and negative diagonal entries, respectively, are independent of the choice of basis (a fact known as *Sylvester's law of inertia*; see [FIS03] for a proof). Thus the ordered pair (r, s), called the ***signature of g***, is an invariant of g.

A ***pseudo-Riemannian metric*** on a smooth manifold M is a smooth symmetric 2-tensor field whose value is nondegenerate at each point, with the same signature everywhere on M. Pseudo-Riemannian metrics with signature $(n-1,1)$ (or $(1,n-1)$, depending on the convention used) are called ***Lorentz metrics***; they play a central role in physics, where they are used to model gravitation in Einstein's general theory of relativity.

We do not pursue the subject of pseudo-Riemannian metrics any further, except to note that the proof of the existence of Riemannian metrics does not carry over to the pseudo-Riemannian case, since it is not generally true that a linear combination of nondegenerate 2-tensors with positive coefficients is necessarily nondegenerate. Indeed, not every manifold admits a Lorentz metric (cf. [HE73, p. 39]).

Problems

13-1. If (M,g) is a Riemannian n-manifold with or without boundary, let $UM \subseteq TM$ be the subset $UM = \{(x,v) \in TM : |v|_g = 1\}$, called the ***unit tangent bundle of*** $\boldsymbol{M}$. Show that UM is a smooth fiber bundle over M with model fiber $\mathbb{S}^{n-1}$.

13-2. In the proof of Proposition 13.3 we used a partition of unity to patch together locally defined Riemannian metrics to obtain a global one. A crucial part of the proof was verifying that the global tensor field so obtained was positive definite. The key to the success of this argument is the fact that the set of inner products on a given tangent space is a convex subset of the vector space of all symmetric 2-tensors. This problem outlines a generalization of this construction to arbitrary vector bundles. Suppose that E is a smooth vector bundle over a smooth manifold M with or without boundary, and $V \subseteq E$ is an open subset with the property that for each $p \in M$, the intersection of V with the fiber E_p is convex and nonempty. By a "section of V," we mean a (local or global) section of E whose image lies in V.

(a) Show that there exists a smooth global section of V.

(b) Suppose $\sigma\colon A \to V$ is a smooth section of V defined on a closed subset $A \subseteq M$. (This means that σ extends to a smooth section of V in a neighborhood of each point of A.) Show that there exists a smooth global section $\widetilde{\sigma}$ of V whose restriction to A is equal to σ. Show that if V contains the image of the zero section of E, then $\widetilde{\sigma}$ can be chosen to be supported in any predetermined neighborhood of A.

(*Used on pp. 381, 430.*)

13-3. Let M be a smooth manifold. Prove the following statements.

(a) If there exists a global nonvanishing vector field on M, then there exists a global *smooth* nonvanishing vector field. [Hint: imitate the proof of Theorem 6.21, with the constants $F(x_i)$ replaced by constant-coefficient vector fields in coordinates, and with absolute values replaced by norms in some Riemannian metric.]

(b) If there exists a linearly independent k-tuple of vector fields on M, then there exists such a k-tuple of *smooth* vector fields.

13-4. Let $\mathring{g}$ denote the round metric on $\mathbb{S}^n$. Compute the coordinate representation of $\mathring{g}$ in stereographic coordinates (see Problem 1-7).

13-5. Suppose (M, g) is a Riemannian manifold. A smooth curve $\gamma: J \to M$ is said to be a ***unit-speed curve*** if $|\gamma'(t)|_g \equiv 1$. Prove that every smooth curve with nowhere-vanishing velocity has a unit-speed reparametrization. (*Used on p. 335.*)

13-6. Prove that every Riemannian 1-manifold is flat. [Hint: use Problem 13-5. Note that this implies the round metric on $\mathbb{S}^1$ is flat!]

13-7. Show that a product of flat metrics is flat.

13-8. Let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \subseteq \mathbb{C}^n$, and let g be the metric on $\mathbb{T}^n$ induced from the Euclidean metric on $\mathbb{C}^n$ (identified with $\mathbb{R}^{2n}$). Show that g is flat.

13-9. Let $H \subseteq \mathbb{R}^3$ be the helicoid (the image of the embedding $F: \mathbb{R}^2 \to \mathbb{R}^3$ of Example 13.11), and let $C \subseteq \mathbb{R}^3$ be the ***catenoid***, which is the surface of revolution generated by the curve $\gamma(t) = (\cosh t, t)$. Show that H is locally isometric to C but not globally isometric.

13-10. Show that the shortest path between two points in Euclidean space is a straight line segment. More precisely, for $x, y \in \mathbb{R}^n$, let $\gamma: [0,1] \to \mathbb{R}^n$ be the curve segment $\gamma(t) = x + t(y - x)$, and show that any other piecewise smooth curve segment $\widetilde{\gamma}$ from x to y satisfies $L_{\bar{g}}(\widetilde{\gamma}) > L_{\bar{g}}(\gamma)$ unless $\widetilde{\gamma}$ is a reparametrization of γ. [Hint: first, consider the case in which both x and y lie on the x^1-axis.] (*Used on p. 338.*)

13-11. Let $M = \mathbb{R}^2 \smallsetminus \{0\}$, and let g be the restriction to M of the Euclidean metric $\bar{g}$. Show that there are points $p, q \in M$ for which there is no piecewise smooth curve segment γ from p to q in M with $L_g(\gamma) = d_g(p, q)$.

13-12. Consider $\mathbb{R}^n$ as a Riemannian manifold with the Euclidean metric $\bar{g}$.

(a) Suppose $U, V \subseteq \mathbb{R}^n$ are connected open sets, $\varphi, \psi: U \to V$ are Riemannian isometries, and for some $p \in U$ they satisfy $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Show that $\varphi = \psi$. [Hint: first, use Problem 13-10 to show that φ and ψ take lines to lines.]

(b) Show that the set of maps from $\mathbb{R}^n$ to itself given by the action of $E(n)$ on $\mathbb{R}^n$ described in Example 7.32 is the full group of Riemannian isometries of $(\mathbb{R}^n, \bar{g})$.

13-13. Let (M, g) be a Riemannian manifold. A smooth vector field V on M is called a ***Killing vector field for g*** (named after the late nineteenth/early twentieth-century German mathematician Wilhelm Killing) if the flow of V acts by isometries of g.

(a) Show that the set of all Killing vector fields on M constitutes a Lie subalgebra of $\mathfrak{X}(M)$. [Hint: see Corollary 9.39(c).]

(b) Show that a smooth vector field V on M is a Killing vector field if and only if it satisfies the following equation in each smooth local coordinate chart:

$$V^k \frac{\partial g_{ij}}{\partial x^k} + g_{jk}\frac{\partial V^k}{\partial x^i} + g_{ik}\frac{\partial V^k}{\partial x^j} = 0.$$

13-14. Let $K \subseteq \mathfrak{X}(\mathbb{R}^n)$ denote the Lie algebra of Killing vector fields with respect to the Euclidean metric (see Problem 13-13), and let $K_0 \subseteq K$ denote the subspace consisting of fields that vanish at the origin.

(a) Show that the map

$$V \mapsto \left(\frac{\partial V^i}{\partial x^j}(0)\right)$$

is an injective linear map from K_0 to $\mathfrak{o}(n)$. [Hint: If V is in the kernel of this map and θ is its flow, show that the linear map $d(\theta_t)_0 \colon T_0\mathbb{R}^n \to T_0\mathbb{R}^n$ is independent of t, and use the result of Problem 13-12(a).]

(b) Show that the following vector fields form a basis for K:

$$\frac{\partial}{\partial x^i}, \quad 1 \le i \le n; \qquad x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}, \quad 1 \le i < j \le n.$$

13-15. Let (M,g) be a Riemannian manifold, and let $\widehat{g}$ be the product metric on $M \times \mathbb{R}$ determined by g and the Euclidean metric on $\mathbb{R}$. Let $X = 0 \oplus d/dt$ be the product vector field on $M \times \mathbb{R}$ determined by the zero vector field on M and the standard coordinate vector field d/dt on $\mathbb{R}$ (see Problem 8-17). Show that X is a Killing vector field for $\left(M \times \mathbb{R}, \widehat{g}\right)$.

13-16. Suppose $g = f(t)dt^2$ is a Riemannian metric on $\mathbb{R}$. Show that g is complete if and only if both of the following improper integrals diverge:

$$\int_0^\infty \sqrt{f(t)}\,dt, \qquad \int_{-\infty}^0 \sqrt{f(t)}\,dt.$$

13-17. Let M be a connected noncompact smooth manifold and let g be a Riemannian metric on M. Prove that there exists a positive function $h \in C^\infty(M)$ such that the Riemannian metric $\widetilde{g} = hg$ is complete. Use this to prove that every connected smooth manifold admits a complete Riemannian metric. [Hint: let $f \colon M \to \mathbb{R}$ be an exhaustion function, and show that h can be chosen so that f is bounded on $\widetilde{g}$-bounded sets.]

13-18. Suppose (M,g) is a connected Riemannian manifold, $S \subseteq M$ is a connected embedded submanifold, and $\widetilde{g}$ is the induced Riemannian metric on S.

(a) Prove that $d_{\widetilde{g}}(p,q) \ge d_g(p,q)$ for $p,q \in S$.

(b) Prove that if (M,g) is complete and S is properly embedded, then $\left(S,\widetilde{g}\right)$ is complete.

(c) Use (b) together with the Whitney embedding theorem to prove (without quoting Proposition 13.3 or Problem 13-17) that every connected smooth manifold admits a complete Riemannian metric.

13-19. The following example shows that the converse of Problem 13-18(b) does not hold. Define $F\colon \mathbb{R} \to \mathbb{R}^2$ by $F(t) = \big((e^t+1)\cos t, (e^t+1)\sin t\big)$. Show that F is an embedding that is not proper, yet $\mathbb{R}$ is complete in the metric induced from the Euclidean metric on $\mathbb{R}^2$.

13-20. Consider the embeddings $F_1, F_2, F_3, F_4\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined as follows:

$$F_1(u,v) = (u,v,0);$$
$$F_2(u,v) = (u, e^v, 0);$$
$$F_3(u,v) = (u, v, u^2+v^2);$$
$$F_4(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

For each i, let $g_i = F_i^*\bar{g}$. For each of the Riemannian manifolds $(\mathbb{R}^2, g_1)$ through $(\mathbb{R}^2, g_4)$, answer the following questions: Is it bounded? Is it complete? Is it flat? Prove your answers correct. [Hint: if you have trouble analyzing g_4, look at Problem 1-7.]

13-21. Let (M,g) be a Riemannian manifold, let $f \in C^\infty(M)$, and let $p \in M$ be a regular point of f.

(a) Show that among all unit vectors $v \in T_pM$, the directional derivative vf is greatest when v points in the same direction as $\operatorname{grad} f|_p$, and the length of $\operatorname{grad} f|_p$ is equal to the value of the directional derivative in that direction.

(b) Show that $\operatorname{grad} f|_p$ is normal to the level set of f through p.

(*Used on p. 391.*)

13-22. For any smooth manifold M with or without boundary, show that the vector bundles TM and T^*M are smoothly isomorphic over M. [Remark: Problem 11-18 shows that this isomorphism cannot be *natural*, in the sense that there does not exist a rule that assigns to every smooth manifold M a bundle isomorphism $\lambda_M\colon TM \to T^*M$ in such a way that for every diffeomorphism $F\colon M \to N$, the two bundle isomorphisms λ_M and λ_N are related by $\lambda_M = dF^* \circ \lambda_N \circ dF$.]

13-23. Is there a smooth covector field on $\mathbb{S}^2$ that vanishes at exactly one point?

13-24. Let M be a compact smooth n-manifold, and suppose f is a smooth real-valued function on M that has only finitely many critical points $\{p_1, \dots, p_k\}$, with corresponding critical values $\{c_1, \dots, c_k\}$ labeled so that $c_1 \le \dots \le c_k$. For any $a < b \in \mathbb{R}$, define $M_a = f^{-1}(a)$, $M_{[a,b]} = f^{-1}\big([a,b]\big)$, and $M_{(a,b)} = f^{-1}\big((a,b)\big)$. If a and b are regular values, note that M_a and M_b are embedded hypersurfaces in M, $M_{(a,b)}$ is an open submanifold, and $M_{[a,b]}$ is a regular domain by Proposition 5.47 (see Fig. 13.4).

(a) Choose a Riemannian metric g on M, let X be the vector field $X = \operatorname{grad} f / |\operatorname{grad} f|_g^2$ on $M \smallsetminus \{p_1, \dots, p_k\}$, and let θ denote the flow of X. Show that $f\big(\theta_t(p)\big) = f(p) + t$ whenever $\theta_t(p)$ is defined.

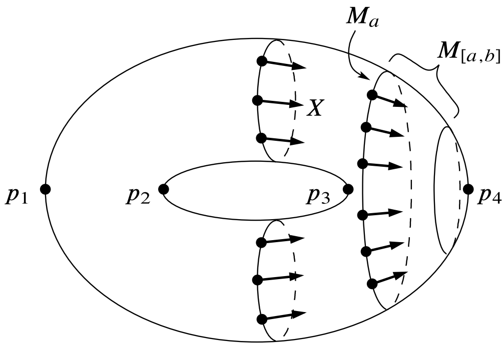

Fig. 13.4 The setup for Problem 13-24

(b) Let $[a, b] \subseteq \mathbb{R}$ be a compact interval containing no critical values of f. Show that θ restricts to a diffeomorphism from $[0, b-a] \times M_a$ to $M_{[a,b]}$.

[Remark: this result shows that M can be decomposed as a union of simpler "building blocks"—the product sets $M_{[c_i+\varepsilon, c_{i+1}-\varepsilon]} \approx I \times M_{c_i+\varepsilon}$, and the neighborhoods $M_{(c_i-\varepsilon, c_i+\varepsilon)}$ of the critical points. This is the starting point of ***Morse theory***, which is one of the deepest applications of differential geometry to topology. The next step would be to analyze the behavior of f near each critical point, and use this analysis to determine exactly how the level sets change topologically when crossing a critical level. See [Mil63] for an excellent introduction.]

13-25. Suppose M is a smooth manifold that admits a proper smooth function $f: M \to \mathbb{R}$ with no critical points. Show that M is diffeomorphic to $N \times \mathbb{R}$ for some compact smooth manifold N. [Hint: let $X = \operatorname{grad} f / |\operatorname{grad} f|_g^2$, defined with respect to some Riemannian metric on M. Show that X is complete, and use its flowout to define the diffeomorphism.]