Chapter 6
Sard's Theorem

This chapter introduces a powerful tool in smooth manifold theory, *Sard's theorem*, which says that the set of critical values of a smooth function has measure zero. This theorem is fundamental in differential topology (the study of properties of smooth manifolds that are preserved by diffeomorphisms or by smooth deformations).

Before we begin, we need to extend the notion of sets of measure zero to manifolds. These are sets that are "small" in a sense that is closely related to having zero volume (even though we do not yet have a way to measure volume quantitatively on manifolds); they include such things as countable unions of submanifolds of positive codimension. With this tool in hand, we then prove Sard's theorem itself.

After proving Sard's theorem, we use it to prove three important results about smooth manifolds. The first result is the *Whitney embedding theorem*, which says that every smooth manifold can be smoothly embedded in some Euclidean space. (This justifies our habit of visualizing manifolds as subsets of $\mathbb{R}^n$.) The second result is the *Whitney approximation theorem*, which comes in two versions: every continuous real-valued or vector-valued function can be uniformly approximated by smooth ones, and every continuous map between smooth manifolds is homotopic to a smooth map. The third result is the *transversality homotopy theorem*, which says, among other things, that embedded submanifolds can always be deformed slightly so that they intersect "nicely" in a certain sense that we will make precise.

We will use some basic properties of sets of measure zero in the theory of integration in Chapter 16, and we will use the Whitney approximation theorems in our treatment of line integrals and de Rham cohomology in Chapters 16–18.

Sets of Measure Zero

An important notion in integration theory is that certain subsets of $\mathbb{R}^n$, called *sets of measure zero*, are so "thin" that they are negligible in integrals. In this section, we show how to define sets of measure zero in manifolds, and show that smooth maps between manifolds of the same dimension take sets of measure zero to sets of measure zero.

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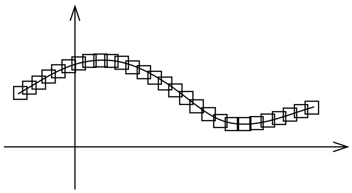

Fig. 6.1 A set of measure zero

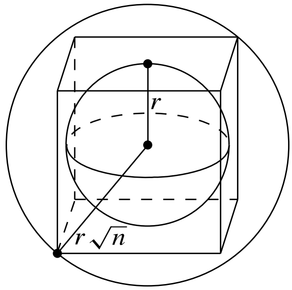

Fig. 6.2 *Balls* and *cubes*

Recall what it means for a set $A \subseteq \mathbb{R}^n$ to have measure zero (see Appendix C): for any $\delta > 0$, A can be covered by a countable collection of open rectangles, the sum of whose volumes is less than δ (Fig. 6.1).

► **Exercise 6.1.** Show that open rectangles can be replaced by open balls or open cubes in the definition of subsets of measure zero. [Hint: for cubes, first show that every open rectangle $R \subseteq \mathbb{R}^n$ can be covered by finitely many open cubes, the sum of whose volumes is no more than 2^n times the volume of R. For balls, Fig. 6.2 suggests the main idea.]

We need the following technical lemma about sets of measure zero. If you are familiar with the theory of Lebesgue measure, you will notice that this result follows easily from Fubini's theorem for integrals of measurable functions over product sets; but this is an elementary proof that does not depend on measure theory.

Lemma 6.2. *Suppose $A \subseteq \mathbb{R}^n$ is a compact subset whose intersection with $\{c\} \times \mathbb{R}^{n-1}$ has $(n-1)$-dimensional measure zero for every $c \in \mathbb{R}$. Then A has n-dimensional measure zero.*

Proof. Choose an interval $[a,b] \subseteq \mathbb{R}$ such that $A \subseteq [a,b] \times \mathbb{R}^{n-1}$. For each $c \in [a,b]$, let $A_c \subseteq \mathbb{R}^{n-1}$ denote the compact subset $\{x \in \mathbb{R}^{n-1} : (c,x) \in A\}$.

Let $\delta > 0$ be given. Our hypothesis implies that for each $c \in [a,b]$, the set A_c is covered by finitely many $(n-1)$-dimensional open cubes $C_1, \dots, C_k$ with total volume less than δ. Let U_c be the open subset $C_1 \cup \dots \cup C_k \subseteq \mathbb{R}^{n-1}$. Because A is compact, there must be an open interval J_c containing c such that the intersection of A with $J_c \times \mathbb{R}^{n-1}$ is contained in $J_c \times U_c$, for otherwise there would be a sequence of points $(c_i, x_i) \in A$ such that $c_i \to c$ and $x_i \notin U_c$; but then passing to a convergent subsequence we obtain $x_i \to x \in A_c \smallsetminus U_c$, which contradicts the fact that $A_c \subseteq U_c$.

The intervals $\{J_c : c \in [a,b]\}$ form an open cover of $[a,b]$, so there are finitely many numbers $c_1 < \dots < c_m$ such that the intervals $J_{c_1}, \dots, J_{c_m}$ cover $[a,b]$. By shrinking the intervals J_{c_i} where they overlap if necessary, we can arrange that the combined lengths of $J_{c_1}, \dots, J_{c_m}$ add up to no more than $2|b-a|$. It follows that A is contained in $\left(J_{c_1} \times U_{c_1}\right) \cup \dots \cup (J_{c_m} \times U_{c_m})$, which is a union of finitely many open rectangles with combined volume less than $2\delta|b-a|$. Since this can be made as small as desired, it follows that A has n-dimensional measure zero. □

The most important sets of measure zero are graphs of continuous functions.

Proposition 6.3. *Suppose A is an open or closed subset of $\mathbb{R}^{n-1}$ or $\mathbb{H}^{n-1}$, and $f: A \to \mathbb{R}$ is a continuous function. Then the graph of f has measure zero in $\mathbb{R}^n$.*

Proof. First assume A is compact. We prove the theorem in this case by induction on n. When $n = 1$, it is trivial because the graph of f is at most a single point. To prove the inductive step, we appeal to Lemma 6.2. For each $c \in \mathbb{R}$, the intersection of the graph of f with $\{c\} \times \mathbb{R}^{n-1}$ is just the graph of the restriction of f to $\{x \in A : x^1 = c\}$, which is in turn the graph of a continuous function of $n-2$ variables. It follows by induction that each such graph has $(n-1)$-dimensional measure zero, and thus by Lemma 6.2, the graph of f itself has n-dimensional measure zero.

If A is noncompact, it is a countable union of compact subsets by Proposition A.60, so the graph of f is a countable union of sets of measure zero. □

Corollary 6.4. *Every proper affine subspace of $\mathbb{R}^n$ has measure zero in $\mathbb{R}^n$.*

Proof. Let $S \subseteq \mathbb{R}^n$ be a proper affine subspace. Suppose first that $\dim S = n-1$. Then there is at least one coordinate axis, say the x^i-axis, that is not parallel to S, and in that case S is the graph of an affine function of the form $x^i = F\left(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n\right)$, so it has measure zero by Proposition 6.3. If $\dim S < n-1$, then S is contained in some affine subspace of dimension $n-1$, so it follows from Proposition C.18(b) that S has measure zero. □

Our goal is to extend the notion of measure zero in a diffeomorphism-invariant fashion to subsets of manifolds. Because a manifold does not come with a metric, volumes of cubes or balls do not make sense, so we cannot simply use the same definition. However, the key is provided by the next proposition, which implies that the condition of having measure zero is diffeomorphism-invariant for subsets of $\mathbb{R}^n$.

Proposition 6.5. *Suppose $A \subseteq \mathbb{R}^n$ has measure zero and $F: A \to \mathbb{R}^n$ is a smooth map. Then $F(A)$ has measure zero.*

Proof. By definition, for each $p \in A$, F has an extension to a smooth map, which we still denote by F, on a neighborhood of p in $\mathbb{R}^n$. Shrinking this neighborhood if necessary, we may assume that there is an open ball U containing p such that F extends smoothly to $\bar{U}$. By Proposition A.16, A is covered by countably many such precompact open subsets, so $F(A)$ is the union of countably many sets of the form $F\left(A \cap \bar{U}\right)$. Thus, it suffices to show that each such set has measure zero.

Since $\bar{U}$ is compact, there is a constant C such that $|DF(x)| \le C$ for all $x \in \bar{U}$. Using the Lipschitz estimate for smooth functions (Proposition C.29), we have

$$\left|F(x) - F(x')\right| \le C\,|x - x'| \tag{6.1}$$

for all $x, x' \in \bar{U}$.

Given $\delta > 0$, choose a countable cover $\{B_j\}$ of $A \cap \bar{U}$ by open balls satisfying $\sum_j \operatorname{Vol}(B_j) < \delta$. Then by (6.1), $F\left(\bar{U} \cap B_j\right)$ is contained in a ball $\widetilde{B}_j$ whose radius is no more than C times that of B_j (Fig. 6.3). We conclude that $F\left(A \cap \bar{U}\right)$ is contained

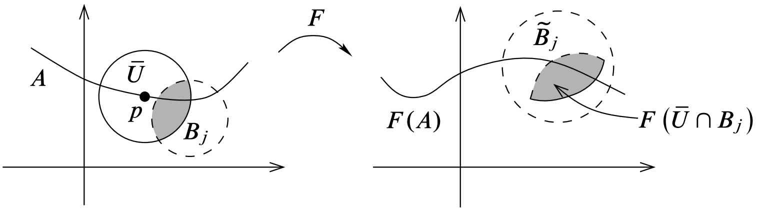

Fig. 6.3 The image of a set of measure zero

in the collection of balls $\{\widetilde{B}_j\}$, the sum of whose volumes is at most $C^n\delta$. Since this can be made as small as desired, it follows that $F(A \cap \bar{U})$ has measure zero. □

If M is a smooth n-manifold with or without boundary, we say that a subset $A \subseteq M$ has ***measure zero in M*** if for every smooth chart (U, φ) for M, the subset $\varphi(A \cap U) \subseteq \mathbb{R}^n$ has n-dimensional measure zero. The following lemma shows that we need only check this condition for a single collection of smooth charts whose domains cover A.

Lemma 6.6. *Let M be a smooth n-manifold with or without boundary and $A \subseteq M$. Suppose that for some collection $\{(U_\alpha, \varphi_\alpha)\}$ of smooth charts whose domains cover A, $\varphi_\alpha(A \cap U_\alpha)$ has measure zero in $\mathbb{R}^n$ for each α. Then A has measure zero in M.*

Proof. Let (V, ψ) be an arbitrary smooth chart. We need to show that $\psi(A \cap V)$ has measure zero. Some countable collection of the U_α's covers $A \cap V$. For each such U_α, we have

$$\psi(A \cap V \cap U_\alpha) = \left(\psi \circ \varphi_\alpha^{-1}\right) \circ \varphi_\alpha(A \cap V \cap U_\alpha).$$

(See Fig. 6.4.)Now, $\varphi_\alpha(A \cap V \cap U_\alpha)$ is a subset of $\varphi_\alpha(A \cap U_\alpha)$, which has measure zero in $\mathbb{R}^n$ by hypothesis. By Proposition 6.5 applied to $\psi \circ \varphi_\alpha^{-1}$, therefore, $\psi(A \cap V \cap U_\alpha)$ has measure zero. Since $\psi(A \cap V)$ is the union of countably many such sets, it too has measure zero. □

► **Exercise 6.7.** Let M be a smooth manifold with or without boundary. Show that a countable union of sets of measure zero in M has measure zero.

Proposition 6.8. *Suppose M is a smooth manifold with or without boundary and $A \subseteq M$ has measure zero in M. Then $M \smallsetminus A$ is dense in M.*

Proof. If $M \smallsetminus A$ is not dense, then A contains a nonempty open subset of M, which implies that there is a smooth chart (V, ψ) such that $\psi(A \cap V)$ contains a nonempty open subset of $\mathbb{R}^n$ (where $n = \dim M$). Because $\psi(A \cap V)$ has measure zero in $\mathbb{R}^n$, this contradicts Corollary C.25. □

Theorem 6.9. *Suppose M and N are smooth n-manifolds with or without boundary, $F\colon M \to N$ is a smooth map, and $A \subseteq M$ is a subset of measure zero. Then $F(A)$ has measure zero in N.*

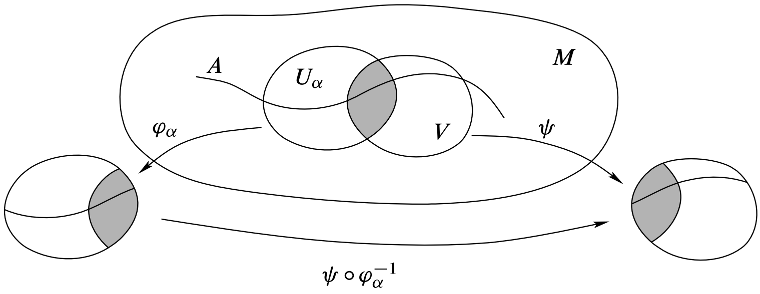

Fig. 6.4 Proof of Lemma 6.6

Proof. Let $\{(U_i, \varphi_i)\}$ be a countable cover of M by smooth charts. We need to show that for each smooth chart (V, ψ) for N, the set $\psi\big(F(A) \cap V\big)$ has measure zero in $\mathbb{R}^n$. Note that this set is the union of countably many subsets of the form $\psi \circ F \circ \varphi_i^{-1}\big(\varphi_i\big(A \cap U_i \cap F^{-1}(V)\big)\big)$, each of which has measure zero by the result of Proposition 6.5. □

Sard's Theorem

Here is the theorem that underlies all of our results about embedding, approximation, and transversality.

Theorem 6.10 (Sard's Theorem). *Suppose M and N are smooth manifolds with or without boundary and $F: M \to N$ is a smooth map. Then the set of critical values of F has measure zero in N.*

Proof. Let $m = \dim M$ and $n = \dim N$. We prove the theorem by induction on m. For $m = 0$, the result is immediate, because if $n = 0$, F has no critical points, while if $n > 0$, the entire image of F has measure zero because it is countable.

Now suppose $m \geq 1$, and assume the theorem holds for maps whose domains have dimensions less than m. By covering M and N with countably many smooth charts, we can reduce to the case in which F is a smooth map from an open subset $U \subseteq \mathbb{R}^m$ or $\mathbb{H}^m$ to $\mathbb{R}^n$. Write the coordinates in the domain U as $\big(x^1, \dots, x^m\big)$, and those in the codomain as $\big(y^1, \dots, y^n\big)$.

Let $C \subseteq U$ denote the set of critical points of F. We define a decreasing sequence of subsets $C \supseteq C_1 \supseteq C_2 \supseteq \cdots$ as follows:

$$C_k = \{x \in C : \text{for } 1 \leq i \leq k, \text{ all } i\text{th partial derivatives of } F \text{ vanish at } x\}.$$

By continuity, C and all of the C_k's are closed in U. We will prove in three steps that $F(C)$ has measure zero.

STEP 1: *$F(C \smallsetminus C_1)$ has measure zero.* Because C_1 is closed in U, we might as well replace U by $U \smallsetminus C_1$, and assume that $C_1 = \varnothing$. Let a be a point of C. Our assumption means that some first partial derivative of F is not zero at a.

By rearranging the coordinates in the domain and codomain, we may assume that $\partial F^1/\partial x^1(a) \neq 0$. This means that we can define new smooth coordinates $(u,v) = \left(u, v^2, \dots, v^m\right)$ in some neighborhood V_a of a in U by

$$u = F^1, \quad v^2 = x^2, \quad \dots, \quad v^m = x^m,$$

because the Jacobian of the coordinate transformation is nonsingular at a. Shrinking V_a if necessary, we can assume that $\overline{V}_a$ is a compact subset of U and the coordinates extend smoothly to $\overline{V}_a$. In these coordinates, F has the coordinate representation

$$F\left(u, v^2, \dots, v^m\right) = \left(u, F^2(u,v), \dots, F^n(u,v)\right), \tag{6.2}$$

and its Jacobian is

$$DF(u,v) = \begin{pmatrix} 1 & 0 \\ * & \dfrac{\partial F^i}{\partial v^j} \end{pmatrix}.$$

Therefore, $C \cap \overline{V}_a$ consists of exactly those points where the $(n-1) \times (m-1)$ matrix $\left(\partial F^i/\partial v^j\right)$ has rank less than $n-1$.

We wish to show that the set $F\left(C \cap \overline{V}_a\right)$ has measure zero in $\mathbb{R}^n$. Because this set is compact, by Lemma 6.2 it suffices to show that its intersection with each hyperplane $y^1 = c$ has $(n-1)$-dimensional measure zero.

For $c \in \mathbb{R}$, let $B_c = \left\{v : (c,v) \in \overline{V}_a\right\} \subseteq \mathbb{R}^{m-1}$, and define $F_c\colon B_c \to \mathbb{R}^{n-1}$ by

$$F_c(v) = \left(F^2(c,v), \dots, F^n(c,v)\right).$$

Because $F(c,v) = (c, F_c(v))$, the critical values of $F|_{\overline{V}_a}$ that lie in the hyperplane $y^1 = c$ are exactly the points of the form (c,w) with w a critical value of F_c. By the induction hypothesis, the set of critical values of each F_c has $(n-1)$-dimensional measure zero. Thus by Lemma 6.2, $F\left(C \cap \overline{V}_a\right)$ has measure zero.

Because U is covered by countably many sets of the form $\overline{V}_a$, it follows that $F(C \cap U)$ is a countable union of sets of measure zero and thus has measure zero. This completes the proof of Step 1.

STEP 2: *For each k, $F(C_k \smallsetminus C_{k+1})$ has measure zero.* Again, since C_{k+1} is closed in U, we can discard it and assume that at every point of C_k there is some $(k+1)$st partial derivative of F that does not vanish.

Let $a \in C_k$ be arbitrary, and let $y\colon U \to \mathbb{R}$ denote some kth partial derivative of F that has at least one nonvanishing first partial derivative at a. Then a is a regular point of the smooth map y, so there is a neighborhood V_a of a consisting entirely of regular points of y. Let Y be the zero set of y in V_a, which is a smooth hypersurface by the regular level set theorem. By definition of C_k, all kth derivatives of F (including y) vanish on C_k, so $C_k \cap V_a$ is contained in Y. At any $p \in C_k \cap V_a$, dF_p is not surjective, so certainly $d(F|_Y)_p = (dF_p)|_{T_pY}$ is not surjective. Thus, $F(C_k \cap V_a)$ is contained in the set of critical values of $F|_Y\colon Y \to \mathbb{R}^n$, which has measure zero by the induction hypothesis. Since U can be covered by countably many neighborhoods like V_a, it follows that $F(C_k \smallsetminus C_{k+1})$ is contained in a countable union of sets of the form $F(C_k \cap V_a)$, and thus has measure zero.

We are not yet finished, because there may be points of C at which all partial derivatives of F vanish, which means that they are neither in $C \smallsetminus C_1$ nor in $C_k \smallsetminus C_{k+1}$ for any k. This possibility is taken care of by the final step.

STEP 3: *For $k > m/n - 1$, $F(C_k)$ has measure zero.* For each $a \in U$, there is a closed cube E such that $a \in E \subseteq U$. Since U can be covered by countably many such cubes, it suffices to show that $F(C_k \cap E)$ has measure zero whenever E is a closed cube contained in U. Let E be such a cube, and let A be a constant that bounds the absolute values of all of the $(k+1)$st derivatives of F in E. Let R denote the side length of E, and let K be a large integer to be chosen later. We can subdivide E into K^m cubes of side length R/K, denoted by $\left(E_1, \dots, E_{K^m}\right)$. If E_i is one of these cubes and there is a point $a_i \in C_k \cap E_i$, then Corollary C.16 to Taylor's theorem implies that for all $x \in E_i$ we have

$$\left|F(x) - F(a_i)\right| \le A'|x - a_i|^{k+1},$$

for some constant A' that depends only on A, k, and m. Thus, $F(E_i)$ is contained in a ball of radius $A'(R/K)^{k+1}$. This implies that $F(C_k \cap E)$ is contained in a union of K^m balls, the sum of whose n-dimensional volumes is no more than

$$K^m (A')^n (R/K)^{n(k+1)} = A'' K^{m-nk-n},$$

where $A'' = (A')^n R^{n(k+1)}$. Since $k > m/n - 1$, this can be made as small as desired by choosing K large, so $F(C_k \cap E)$ has measure zero. □

Corollary 6.11. *Suppose M and N are smooth manifolds with or without boundary, and $F\colon M \to N$ is a smooth map. If* $\dim M < \dim N$, *then $F(M)$ has measure zero in N.*

Proof. In this case, each point of M is a critical point for F. □

Problem 6-1 outlines a simple proof of the preceding corollary that does not depend on the full strength of Sard's theorem.

It is important to be aware that Corollary 6.11 is false if F is merely assumed to be continuous. For example, there is a continuous map $F\colon [0,1] \to \mathbb{R}^2$ whose image is the entire unit square $[0,1] \times [0,1]$. (Such a map is called a ***space-filling curve***. See [Rud76, p. 168] for an example.)

Corollary 6.12. *Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold with or without boundary. If* $\dim S < \dim M$, *then S has measure zero in M.*

Proof. Apply Corollary 6.11 to the inclusion map $S \hookrightarrow M$. □

The Whitney Embedding Theorem

Our first application of Sard's theorem is to show that every smooth manifold can be embedded into a Euclidean space. In fact, we will show that every smooth n-manifold with or without boundary is diffeomorphic to a properly embedded submanifold (with or without boundary) of $\mathbb{R}^{2n+1}$.

The first step is to show that an injective immersion of an n-manifold into $\mathbb{R}^N$ can be turned into an injective immersion into a lower-dimensional Euclidean space if $N > 2n + 1$.

Lemma 6.13. *Suppose $M \subseteq \mathbb{R}^N$ is a smooth n-dimensional submanifold with or without boundary. For any $v \in \mathbb{R}^N \smallsetminus \mathbb{R}^{N-1}$, let $\pi_v\colon \mathbb{R}^N \to \mathbb{R}^{N-1}$ be the projection with kernel $\mathbb{R}v$ (where we identify $\mathbb{R}^{N-1}$ with the subspace of $\mathbb{R}^N$ consisting of points with last coordinate zero). If $N > 2n + 1$, then there is a dense set of vectors $v \in \mathbb{R}^N \smallsetminus \mathbb{R}^{N-1}$ for which $\pi_v|_M$ is an injective immersion of M into $\mathbb{R}^{N-1}$.*

Proof. In order for $\pi_v|_M$ to be injective, it is necessary and sufficient that $p - q$ never be parallel to v when p and q are distinct points in M. Similarly, in order for $\pi_v|_M$ to be a smooth immersion, it is necessary and sufficient that T_pM not contain any nonzero vectors in $\operatorname{Ker} d(\pi_v)_p$ for any $p \in M$. Because π_v is linear, its differential is the same linear map (under the usual identification $T_p\mathbb{R}^N \cong \mathbb{R}^N$), so this condition is equivalent to the requirement that T_pM not contain any nonzero vectors parallel to v.

Let $\Delta_M \subseteq M \times M$ denote the closed set $\Delta_M = \{(p,p) : p \in M\}$ (called the ***diagonal of $M \times M$***), and let $M_0 \subseteq TM$ denote the closed set $M_0 = \{(p,0) \in TM : p \in M\}$ (the set of zero vectors at all points of M). Consider the following two maps into the real projective space $\mathbb{RP}^{N-1}$:

$$\kappa : (M \times M) \smallsetminus \Delta_M \to \mathbb{RP}^{N-1}, \quad \kappa(p,q) = [p - q],$$
$$\tau : TM \smallsetminus M_0 \to \mathbb{RP}^{N-1}, \quad \tau(p,w) = [w],$$

where the brackets mean the equivalence class of a vector in $\mathbb{R}^N \smallsetminus \{0\}$ considered as a point in $\mathbb{RP}^{N-1}$. These are both smooth because they are compositions of smooth maps with the projection $\mathbb{R}^N \smallsetminus \{0\} \to \mathbb{RP}^{N-1}$, and the condition that $\pi_v|_M$ be an injective smooth immersion is precisely the condition that $[v]$ not be in the image of either κ or τ. Because the domains of both κ and τ have dimension $2n < N - 1 = \dim \mathbb{RP}^{N-1}$, Corollary 6.11 to Sard's theorem implies that the image of each map has measure zero, and so their union has measure zero as well. Thus, the set of vectors whose equivalence classes are not in either image is dense. □

By applying the preceding lemma repeatedly, we can conclude that if an n-manifold admits an injective immersion into *some* Euclidean space, then it admits one into $\mathbb{R}^{2n+1}$. When M is compact, this map is actually an embedding by Proposition 4.22(c); but if M is not compact, we need to work a little harder to ensure that our injective immersions are also embeddings.

Lemma 6.14. *Let M be a smooth n-manifold with or without boundary. If M admits a smooth embedding into $\mathbb{R}^N$ for some N, then it admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.*

Proof. For this proof, given a one-dimensional linear subspace $S \subseteq \mathbb{R}^N$ and a positive number R, let us define the ***tube with axis S and radius R*** to be the open subset $T_R(S) \subseteq \mathbb{R}^N$ consisting of points whose distance from S is less than R:

$$T_R(S) = \big\{x \in \mathbb{R}^N : |x - y| < R \text{ for some } y \in S\big\}.$$

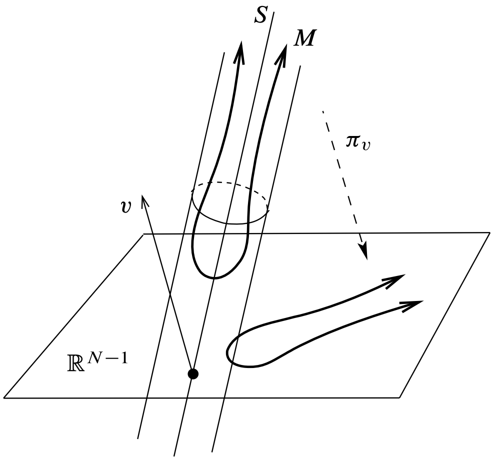

Fig. 6.5 Reducing the codimension of a proper embedding

Suppose $F: M \to \mathbb{R}^N$ is an arbitrary smooth embedding. Let $G: \mathbb{R}^N \to \mathbb{B}^N$ be a diffeomorphism, and let $f: M \to \mathbb{R}$ be a smooth exhaustion function (see Proposition 2.28). Define $\Psi: M \to \mathbb{R}^N \times \mathbb{R}$ by $\Psi(p) = \big(G \circ F(p), f(p)\big)$. Because $G \circ F$ is an embedding, it follows that Ψ is injective and $d\Psi_p$ is injective for each p, so Ψ is an injective immersion. It is proper because the preimage of any compact set is a closed subset of the compact set $f^{-1}\big((-\infty, c]\big)$ for some c, so Ψ is a smooth embedding by Proposition 4.22(b). By construction, the image of Ψ is contained in the tube $\mathbb{B}^N \times \mathbb{R}$.

Henceforth (after replacing $N+1$ by N), we assume that M admits a proper smooth embedding into $\mathbb{R}^N$ that takes its values in some tube $T_R(S)$ (Fig. 6.5). Identifying M with its image, we may consider M as a properly embedded submanifold of $\mathbb{R}^N$ contained in the tube.

If $N > 2n+1$, Lemma 6.13 shows that there exists $v \in \mathbb{R}^N \smallsetminus \mathbb{R}^{N-1}$ so that $\pi_v|_M$ is an injective immersion of M into $\mathbb{R}^{N-1}$. Moreover, we may choose v so that it does not lie in the subspace S; it follows that $\pi_v(S)$ is a one-dimensional subspace of $\mathbb{R}^{N-1}$, and $\pi_v(M)$ lies in a tube around $\pi_v(S)$ because π_v is a bounded linear map. We will show that $\pi_v|_M$ is proper, so it is an embedding by Proposition 4.22(b).

Suppose $K \subseteq \mathbb{R}^{N-1}$ is a compact set. Then K is contained in the open ball around 0 of some radius R_1. For any $x \in \pi_v^{-1}(K)$, there is some $c \in \mathbb{R}$ such that $\pi_v(x) = x - cv$; since $|\pi_v(x)| < R_1$, this means that x is in the tube of radius R_1 around the line $\mathbb{R}v$ spanned by v. It follows that $M \cap \pi_v^{-1}(K)$ is contained in two tubes, one around S and the other around $\mathbb{R}v$. A simple geometric argument shows that the intersection of two tubes is bounded when their axes are not parallel, so $M \cap \pi_v^{-1}(K)$ is compact. Thus $\pi_v|_M$ is proper, which implies that $\pi_v(M)$ is a properly embedded submanifold of $\mathbb{R}^{N-1}$ contained in a tube. We can now iterate this argument until we achieve a proper smooth embedding of M into $\mathbb{R}^{2n+1}$. $\square$

Theorem 6.15 (Whitney Embedding Theorem). *Every smooth n-manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.*

Proof. Let M be a smooth n-manifold with or without boundary. By Lemma 6.14, it suffices to show that M admits a smooth embedding into some Euclidean space.

First suppose M is compact. In this case M admits a finite cover $\{B_1, \dots, B_m\}$ in which each B_i is a regular coordinate ball or half-ball. This means that for each i there exist a coordinate domain $B_i' \supseteq \overline{B}_i$ and a smooth coordinate map $\varphi_i \colon B_i' \to \mathbb{R}^n$ that restricts to a diffeomorphism from $\overline{B}_i$ to a compact subset of $\mathbb{R}^n$. For each i, let $\rho_i \colon M \to \mathbb{R}$ be a smooth bump function that is equal to 1 on $\overline{B}_i$ and supported in B_i'. Define a smooth map $F \colon M \to \mathbb{R}^{nm+m}$ by

$$F(p) = \big(\rho_1(p)\varphi_1(p), \dots, \rho_m(p)\varphi_m(p), \rho_1(p), \dots, \rho_m(p)\big),$$

where, as usual, $\rho_i\varphi_i$ is extended to be zero away from the support of ρ_i. We will show that F is an injective smooth immersion; because M is compact, this implies that F is a smooth embedding.

To see that F is injective, suppose $F(p) = F(q)$. Because the sets B_i cover M, there is some i such that $p \in B_i$. Then $\rho_i(p) = 1$, and the fact that $\rho_i(q) = \rho_i(p) = 1$ implies that $q \in \operatorname{supp} \rho_i \subseteq B_i'$, and

$$\varphi_i(q) = \rho_i(q)\varphi_i(q) = \rho_i(p)\varphi_i(p) = \varphi_i(p).$$

Since φ_i is injective on B_i', it follows that $p = q$.

Next, to see that F is a smooth immersion, let $p \in M$ be arbitrary and choose i such that $p \in B_i$. Because $\rho_i \equiv 1$ on a neighborhood of p, we have $d(\rho_i\varphi_i)_p = d(\varphi_i)_p$, which is injective. It follows easily that dF_p is injective. Thus, F is an injective smooth immersion and hence an embedding.

Now suppose M is noncompact. Let $f \colon M \to \mathbb{R}$ be a smooth exhaustion function. Sard's theorem shows that for each nonnegative integer i, there are regular values a_i, b_i of f such that $i < a_i < b_i < i + 1$. Define subsets $D_i, E_i \subseteq M$ by

$$\begin{aligned} D_0 &= f^{-1}\big((-\infty, 1]\big), \quad & E_0 &= f^{-1}\big((-\infty, a_1]\big); \\ D_i &= f^{-1}\big([i, i+1]\big), \quad & E_i &= f^{-1}\big([b_{i-i}, a_{i+1}]\big), \quad i \geq 1. \end{aligned}$$

By Proposition 5.47, each E_i is a compact regular domain. We have $D_i \subseteq \operatorname{Int} E_i$, $M = \bigcup_i D_i$, and $E_i \cap E_j = \varnothing$ unless $j = i - 1$, i, or $i + 1$. The first part of the proof shows that for each i there is a smooth embedding of E_i into some Euclidean space, and therefore by Lemma 6.14 there is an embedding $\varphi_i \colon E_i \to \mathbb{R}^{2n+1}$. For each i, let $\rho_i \colon M \to \mathbb{R}$ be a smooth bump function that is equal to 1 on a neighborhood of D_i and supported in $\operatorname{Int} E_i$, and define $F \colon M \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$ by

$$F(p) = \left(\sum_{i \text{ even}} \rho_i(p)\varphi_i(p), \sum_{i \text{ odd}} \rho_i(p)\varphi_i(p), f(p)\right).$$

Then F is smooth because only one term in each sum is nonzero in a neighborhood of each point, and F is proper because f is. We will show that F is also an injective smooth immersion, hence a smooth embedding.

Suppose $F(p) = F(q)$. Then $p \in D_j$ for some j, and $f(q) = f(p)$ implies that $q \in D_j$ as well. Arguing just as in the compact case above, we conclude that $p = q$.

Now let $p \in M$ be arbitrary, and choose j such that $p \in D_j$. Then $\rho_j \equiv 1$ on a neighborhood of p. Assuming j is odd, for all q in that neighborhood we have

$$F(q) = \big(\varphi_j(q), \dots, \dots, \dots\big),$$

which implies that dF_p is injective. A similar argument applies when j is even. □

Corollary 6.16. *Every smooth n-dimensional manifold with or without boundary is diffeomorphic to a properly embedded submanifold* (*with or without boundary*) *of* $\mathbb{R}^{2n+1}$. □

Corollary 6.17. *Suppose M is a compact smooth n-manifold with or without boundary. If $N \geq 2n+1$, then every smooth map from M to $\mathbb{R}^N$ can be uniformly approximated by embeddings.*

Proof. Assume $N \geq 2n+1$, and let $f: M \to \mathbb{R}^N$ be a smooth map. By the Whitney embedding theorem, there is a smooth embedding $F: M \to \mathbb{R}^{2n+1}$. The map $G = f \times F: M \to \mathbb{R}^N \times \mathbb{R}^{2n+1}$ is also a smooth embedding, and f is equal to the composition $\pi \circ G$, where $\pi: \mathbb{R}^N \times \mathbb{R}^{2n+1} \to \mathbb{R}^N$ is the projection. Let $\widetilde{M} = G(M) \subseteq \mathbb{R}^N \times \mathbb{R}^{2n+1}$. Lemma 6.13 shows that there is a vector $v_{N+2n+1} \in \mathbb{R}^N \times \mathbb{R}^{2n+1}$ arbitrarily close to $e_{N+2n+1} = (0, \dots, 0, 1)$ such that $\pi_{v_{N+2n+1}}|_{\widetilde{M}}$ is an embedding. This implies that $\pi_{v_{N+2n+1}}$ is arbitrarily close to $\pi_{e_{N+2n+1}}$. Iterating this, we obtain vectors $v_{N+2n+1}, v_{N+2n}, \dots, v_{N+1}$ such that $\pi_{v_{N+1}} \circ \dots \circ \pi_{v_{N+2n+1}}$ restricts to an embedding of $\widetilde{M}$ that is arbitrarily close to $\pi_{e_{N+1}} \circ \dots \circ \pi_{e_{N+2n+1}} = \pi$, and therefore $\pi_{v_{N+1}} \circ \dots \circ \pi_{v_{N+2n+1}} \circ G$ is an embedding of M into $\mathbb{R}^N$ that is arbitrarily close to f. □

If we require only immersions rather than embeddings, we can lower the dimension by one.

Theorem 6.18 (Whitney Immersion Theorem). *Every smooth n-manifold with or without boundary admits a smooth immersion into* $\mathbb{R}^{2n}$.

Proof. See Problem 6-2 for the case $\partial M = \varnothing$, and Problem 9-14 for the general case. □

Theorem 6.15, first proved by Hassler Whitney (in the case of empty boundary) in 1936 [Whi36], answered a question that had been nagging mathematicians since the notion of an abstract manifold was first introduced: Are there abstract smooth manifolds that are not diffeomorphic to embedded submanifolds of Euclidean space? Now we know that there are not.

Although the version of the embedding theorem that we have proved is quite sufficient for our purposes, it is interesting to note that eight years later, using much more sophisticated techniques of algebraic topology, Whitney was able to obtain the following improvements [Whi44a, Whi44b].

Theorem 6.19 (Strong Whitney Embedding Theorem). *If $n > 0$, every smooth n-manifold admits a smooth embedding into* $\mathbb{R}^{2n}$.

Theorem 6.20 (Strong Whitney Immersion Theorem). *If $n > 1$, every smooth n-manifold admits a smooth immersion into $\mathbb{R}^{2n-1}$.*

Because of these results, Theorems 6.15 and 6.18 are sometimes called the *easy* or *weak Whitney embedding and immersion theorems*.

In fact, not even the strong Whitney theorems are sharp in all dimensions. In 1985, Ralph Cohen proved that every compact smooth n-manifold can be immersed in $\mathbb{R}^{2n-a(n)}$, where $a(n)$ is the number of 1's in the binary expression for n. Thus, for example, every 3-manifold can be immersed in $\mathbb{R}^4$, while 4-manifolds require $\mathbb{R}^7$. This result is the best possible in every dimension. On the other hand, the best possible embedding dimension is known only for certain dimensions. For example, Whitney's dimension $2n$ is optimal for manifolds of dimensions $n = 1$ and $n = 2$, but C.T.C. Wall showed in 1965 [Wal65] that every 3-manifold can be embedded in $\mathbb{R}^5$. A good summary of the state of the art with references can be found in [Osb82].

The Whitney Approximation Theorems

In this section we prove the two theorems mentioned at the beginning of the chapter on approximation of continuous maps by smooth ones. Both of these theorems, like the embedding theorem we just proved, are due to Hassler Whitney [Whi36].

We begin with a theorem about smoothly approximating functions into Euclidean spaces. Our first theorem shows, in particular, that any continuous function from a smooth manifold M into $\mathbb{R}^k$ can be uniformly approximated by a smooth function. In fact, we will prove something stronger. If $\delta\colon M \to \mathbb{R}$ is a positive continuous function, we say that two functions $F, \widetilde{F}\colon M \to \mathbb{R}^k$ are ***δ-close*** if $\left|F(x) - \widetilde{F}(x)\right| < \delta(x)$ for all $x \in M$.

Theorem 6.21 (Whitney Approximation Theorem for Functions). *Suppose M is a smooth manifold with or without boundary, and $F\colon M \to \mathbb{R}^k$ is a continuous function. Given any positive continuous function $\delta\colon M \to \mathbb{R}$, there exists a smooth function $\widetilde{F}\colon M \to \mathbb{R}^k$ that is δ-close to F. If F is smooth on a closed subset $A \subseteq M$, then $\widetilde{F}$ can be chosen to be equal to F on A.*

Proof. If F is smooth on the closed subset A, then by the extension lemma for smooth functions (Lemma 2.26), there is a smooth function $F_0\colon M \to \mathbb{R}^k$ that agrees with F on A. Let

$$U_0 = \left\{y \in M : \left|F_0(y) - F(y)\right| < \delta(y)\right\}.$$

Then U_0 is an open subset containing A. (If there is no such set A, we just take $U_0 = A = \varnothing$ and $F_0 \equiv 0$.)

We will show that there are countably many points $\{x_i\}_{i=1}^{\infty}$ in $M \smallsetminus A$ and neighborhoods U_i of x_i in $M \smallsetminus A$ such that $\{U_i\}_{i=1}^{\infty}$ is an open cover of $M \smallsetminus A$ and

$$\left|F(y) - F(x_i)\right| < \delta(y) \quad \text{for all } y \in U_i. \tag{6.3}$$

To see this, for any $x \in M \smallsetminus A$, let U_x be a neighborhood of x contained in $M \smallsetminus A$ and small enough that

$$\delta(y) > \tfrac{1}{2}\delta(x) \quad \text{and} \quad \left|F(y) - F(x)\right| < \tfrac{1}{2}\delta(x)$$

for all $y \in U_x$. (Such a neighborhood exists by continuity of δ and F.) Then if $y \in U_x$, we have

$$\left|F(y) - F(x)\right| < \tfrac{1}{2}\delta(x) < \delta(y).$$

The collection $\{U_x : x \in M \smallsetminus A\}$ is an open cover of $M \smallsetminus A$. Choosing a countable subcover $\{U_{x_i}\}_{i=1}^{\infty}$ and setting $U_i = U_{x_i}$, we have (6.3).

Let $\{\varphi_0, \varphi_i\}$ be a smooth partition of unity subordinate to the cover $\{U_0, U_i\}$ of M, and define $\widetilde{F}\colon M \to \mathbb{R}^k$ by

$$\widetilde{F}(y) = \varphi_0(y)F_0(y) + \sum_{i \geq 1} \varphi_i(y)F(x_i).$$

Then clearly $\widetilde{F}$ is smooth, and is equal to F on A. For any $y \in M$, the fact that $\sum_{i \geq 0} \varphi_i \equiv 1$ implies that

$$\begin{aligned}
\left|\widetilde{F}(y) - F(y)\right| &= \left|\varphi_0(y)F_0(y) + \sum_{i \geq 1} \varphi_i(y)F(x_i) - \left(\varphi_0(y) + \sum_{i \geq 1} \varphi_i(y)\right)F(y)\right| \\
&\leq \varphi_0(y)\left|F_0(y) - F(y)\right| + \sum_{i \geq 1} \varphi_i(y)\left|F(x_i) - F(y)\right| \\
&< \varphi_0(y)\delta(y) + \sum_{i \geq 1} \varphi_i(y)\delta(y) = \delta(y).
\end{aligned}$$

□

Corollary 6.22. *If M is a smooth manifold with or without boundary and $\delta\colon M \to \mathbb{R}$ is a positive continuous function, there is a smooth function $e\colon M \to \mathbb{R}$ such that $0 < e(x) < \delta(x)$ for all $x \in M$.*

Proof. Use the Whitney approximation theorem to construct a smooth function $e\colon M \to \mathbb{R}$ that satisfies $\left|e(x) - \tfrac{1}{2}\delta(x)\right| < \tfrac{1}{2}\delta(x)$ for all $x \in M$. □

Tubular Neighborhoods

We would like to find a way to apply the Whitney approximation theorem to produce smooth approximations to continuous maps between smooth manifolds. If $F\colon N \to M$ is such a map, then by the Whitney embedding theorem we can consider M as an embedded submanifold of some Euclidean space $\mathbb{R}^n$, and approximate F by a smooth map into R^n. However, in general, the image of this smooth map will not lie in M. To correct for this, we need to know that there is a smooth retraction from some neighborhood of M onto M. For this purpose, we introduce tubular neighborhoods.

For each $x \in \mathbb{R}^n$, the tangent space $T_x\mathbb{R}^n$ is canonically identified with $\mathbb{R}^n$ itself, and the tangent bundle $T\mathbb{R}^n$ is canonically diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$. By virtue of this identification, each tangent space $T_x\mathbb{R}^n$ inherits a Euclidean dot product. Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold. For each $x \in M$, we define the ***normal space to M at x*** to be the $(n-m)$-dimensional subspace $N_xM \subseteq T_x\mathbb{R}^n$ consisting of all vectors that are orthogonal to T_xM with respect to the Euclidean dot product. The ***normal bundle of M***, denoted by NM, is the subset of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$ consisting of vectors that are normal to M:

$$NM = \big\{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n : x \in M,\ v \in N_xM\big\}.$$

There is a natural projection $\pi_{NM}\colon NM \to M$ defined as the restriction to NM of $\pi\colon T\mathbb{R}^n \to \mathbb{R}^n$.

Theorem 6.23. *If $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, then NM is an embedded n-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.*

Proof. Let x_0 be any point of M, and let (U,φ) be a slice chart for M in $\mathbb{R}^n$ centered at x_0. Write $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$, and write the coordinate functions of φ as $\big(u^1,\dots,u^n\big)$, so that $M \cap U$ is the set where $u^{m+1} = \dots = u^n = 0$. At each point $x \in U$, the vectors $E_j|_x = (d\varphi_x)^{-1}\big(\partial/\partial u^j|_{\varphi(x)}\big)$ form a basis for $T_x\mathbb{R}^n$. We can expand each $E_j|_x$ in terms of the standard coordinate frame as

$$E_j\big|_x = E_j^i(x)\,\frac{\partial}{\partial x^i}\bigg|_x,$$

where each $E_j^i(x)$ is a partial derivative of φ^{-1} evaluated at $\varphi(x)$, and thus is a smooth function of x.

Define a smooth function $\Phi\colon U \times \mathbb{R}^n \to \widehat{U} \times \mathbb{R}^n$ by

$$\Phi(x,v) = \big(u^1(x),\dots,u^n(x), v \cdot E_1\big|_x,\dots, v\cdot E_n\big|_x\big).$$

The total derivative of Φ at a point (x,v) is

$$D\Phi_{(x,v)} = \begin{pmatrix} \dfrac{\partial u^i}{\partial x^j}(x) & 0 \\ * & E_j^i(x) \end{pmatrix},$$

which is invertible, so Φ is a local diffeomorphism. If $\Phi(x,v) = \Phi(x',v')$, then $x = x'$ because φ is injective, and then the fact that $v \cdot E_i|_x = v' \cdot E_i|_x$ for each i implies that $v - v'$ is orthogonal to the span of $(E_1|_x,\dots,E_n|_x)$ and is therefore zero. Thus Φ is injective, so it defines a smooth coordinate chart on $U \times \mathbb{R}^n$. The definitions imply that $(x,v) \in NM$ if and only if $\Phi(x,v)$ is in the slice

$$\big\{(y,z) \in \mathbb{R}^n \times \mathbb{R}^n : y^{m+1} = \dots = y^n = 0,\ z^1 = \dots = z^m = 0\big\}.$$

Thus Φ is a slice chart for NM in $\mathbb{R}^n \times \mathbb{R}^n$. □

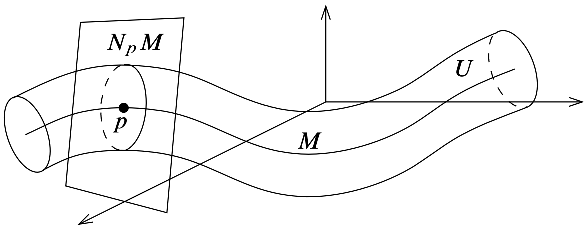

Fig. 6.6 A tubular neighborhood

(We will be able to give a shorter proof of this theorem in Chapter 10; see Corollary 10.36.)

Thinking of NM as a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$, we define $E\colon NM \to \mathbb{R}^n$ by

$$E(x,v) = x + v.$$

This maps each normal space N_xM affinely onto the affine subspace through x and orthogonal to T_xM. Clearly, E is smooth, because it is the restriction to NM of the addition map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. A ***tubular neighborhood of M*** is a neighborhood U of M in $\mathbb{R}^n$ that is the diffeomorphic image under E of an open subset $V \subseteq NM$ of the form

$$V = \big\{(x,v) \in NM : |v| < \delta(x)\big\}, \tag{6.4}$$

for some positive continuous function $\delta\colon M \to \mathbb{R}$ (Fig. 6.6).

Theorem 6.24 (Tubular Neighborhood Theorem). *Every embedded submanifold of $\mathbb{R}^n$ has a tubular neighborhood.*

Proof. Let $M_0 \subseteq NM$ be the subset $\{(x,0) : x \in M\}$. We begin by showing that E is a local diffeomorphism on a neighborhood of M_0. By the inverse function theorem, it suffices to show that the differential $dE_{(x,0)}$ is bijective at each point $(x,0) \in M_0$. This follows easily from the following two facts: First, the restriction of E to M_0 is the obvious diffeomorphism $M_0 \to M$, so $dE_{(x,0)}$ maps the subspace $T_{(x,0)}M_0 \subseteq T_{(x,0)}NM$ isomorphically onto T_xM. Second, the restriction of E to the fiber N_xM is the affine map $w \mapsto x + w$, so $dE_{(x,0)}$ maps $T_{(x,0)}(N_xM) \subseteq T_{(x,0)}NM$ isomorphically onto N_xM. Since $T_x\mathbb{R}^n = T_xM \oplus N_xM$, this shows that $dE_{(x,0)}$ is surjective, and hence is bijective for dimensional reasons. Thus, E is a diffeomorphism on a neighborhood of $(x,0)$ in NM, which we can take to be of the form $V_\delta(x) = \{(x',v') \in NM : |x - x'| < \delta,\ |v'| < \delta\}$ for some $\delta > 0$. (This uses the fact that NM is embedded in $\mathbb{R}^n \times \mathbb{R}^n$, and therefore its topology is induced by the Euclidean metric.)

To complete the proof, we need to show that there is an open subset V of the form (6.4) on which E is a global diffeomorphism. For each point $x \in M$, let $\rho(x)$ be the supremum of all $\delta \le 1$ such that E is a diffeomorphism from $V_\delta(x)$ to its image. The argument in the preceding paragraph implies that $\rho\colon M \to \mathbb{R}$ is positive. To show it is continuous, let $x, x' \in M$ be arbitrary, and suppose first that $|x - x'| <$

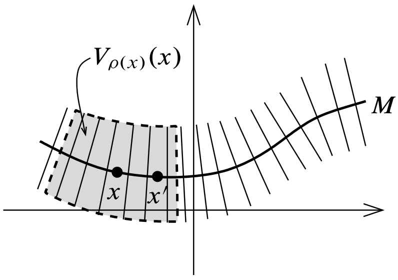

Fig. 6.7 Continuity of ρ

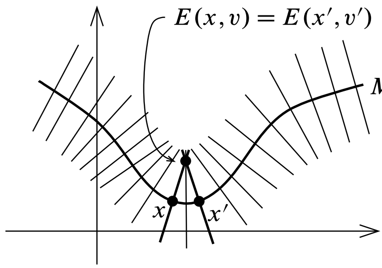

Fig. 6.8 Injectivity of E

$\rho(x)$. By the triangle inequality, $V_\delta(x')$ is contained in $V_{\rho(x)}(x)$ for $\delta = \rho(x) - |x - x'|$ (Fig. 6.7), which implies that $\rho(x') \geq \rho(x) - |x - x'|$, or

$$\rho(x) - \rho(x') \leq |x - x'|. \tag{6.5}$$

On the other hand, if $|x - x'| \geq \rho(x)$, then (6.5) holds for trivial reasons. Reversing the roles of x and x' yields an analogous inequality, which shows that $|\rho(x) - \rho(x')| \leq |x - x'|$, so ρ is continuous. Note that E is injective on the entire set $V_{\rho(x)}(x)$, because any two points $(x_1, v_1), (x_2, v_2)$ in this set are in $V_\delta(x)$ for some $\delta < \rho(x)$.

Now let $V = \left\{(x,v) \in NM : |v| < \frac{1}{2}\rho(x)\right\}$. We will show that E is injective on V. Suppose that (x,v) and (x',v') are points in V such that $E(x,v) = E(x',v')$ (Fig. 6.8). Assume without loss of generality that $\rho(x') \leq \rho(x)$. It follows from $x + v = x' + v'$ that

$$|x - x'| = |v - v'| \leq |v| + |v'| < \tfrac{1}{2}\rho(x) + \tfrac{1}{2}\rho(x') \leq \rho(x).$$

Therefore, both (x,v) and (x',v') are in $V_{\rho(x)}(x)$. Since E is injective on this set, this implies $(x,v) = (x',v')$.

The set $U = E(V)$ is open in $\mathbb{R}^n$ because $E|_V$ is a local diffeomorphism and thus an open map. It follows that $E\colon V \to U$ is a smooth bijection and a local diffeomorphism, hence a diffeomorphism by Proposition 4.6. Therefore, U is a tubular neighborhood of M. □

One of the most useful features of tubular neighborhoods is expressed in the next proposition. A ***retraction*** of a topological space X onto a subspace $M \subseteq X$ is a continuous map $r\colon X \to M$ such that $r|_M$ is the identity map of M.

Proposition 6.25. *Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. If U is any tubular neighborhood of M, there exists a smooth map $r\colon U \to M$ that is both a retraction and a smooth submersion.*

Proof. Let $NM \subseteq T\mathbb{R}^n$ be the normal bundle of M, and let $M_0 \subseteq NM$ be the set $M_0 = \{(x,0) : x \in M\}$. By definition of a tubular neighborhood, there exists an open subset $V \subseteq NM$ containing M_0 such that $E\colon V \to U$ is a diffeomorphism.

Define $r\colon U \to M$ by $r = \pi_{NM} \circ E^{-1}$, where $\pi_{NM}\colon NM \to M$ is the natural projection. Then r is smooth by composition. For $x \in M$, note that $E(x,0) = x$, so $r(x) = \pi \circ E^{-1}(x) = \pi(x,0) = x$, which shows that r is a retraction. Since π is a smooth submersion and E^{-1} is a diffeomorphism, it follows that r is a smooth submersion. □

Smooth Approximation of Maps Between Manifolds

Now we can extend the Whitney approximation theorem to maps between manifolds. This extension will have important applications to line integrals in Chapter 16 and to de Rham cohomology in Chapters 17–18.

Theorem 6.26 (Whitney Approximation Theorem). *Suppose N is a smooth manifold with or without boundary, M is a smooth manifold (without boundary), and $F\colon N \to M$ is a continuous map. Then F is homotopic to a smooth map. If F is already smooth on a closed subset $A \subseteq N$, then the homotopy can be taken to be relative to A.*

Proof. By the Whitney embedding theorem, we may as well assume that M is a properly embedded submanifold of $\mathbb{R}^n$. Let U be a tubular neighborhood of M in $\mathbb{R}^n$, and let $r\colon U \to M$ be the smooth retraction given by Proposition 6.25. For any $x \in M$, let

$$\delta(x) = \sup\{\varepsilon \le 1 : B_\varepsilon(x) \subseteq U\}. \tag{6.6}$$

By a triangle-inequality argument just like the one in the proof of the tubular neighborhood theorem, $\delta\colon M \to \mathbb{R}^+$ is continuous. Let $\widetilde{\delta} = \delta \circ F\colon N \to \mathbb{R}^+$. By Theorem 6.21, there exists a smooth function $\widetilde{F}\colon N \to \mathbb{R}^n$ that is $\widetilde{\delta}$-close to F, and is equal to F on A (which might be the empty set). Let $H\colon N \times I \to M$ be the composition of r with the straight-line homotopy between F and $\widetilde{F}$:

$$H(p,t) = r\big((1-t)F(p) + t\widetilde{F}(p)\big).$$

This is well defined, because our condition on $\widetilde{F}$ guarantees that for each $p \in N$, $\big|\widetilde{F}(p) - F(p)\big| < \widetilde{\delta}(p) = \delta(F(p))$, which means that $\widetilde{F}(p)$ is contained in the ball of radius $\delta\big(F(p)\big)$ around $F(p)$; since this ball is contained in U, so is the entire line segment from $F(p)$ to $\widetilde{F}(p)$.

Thus H is a homotopy between $H(p,0) = F(p)$ and $H(p,1) = r\big(\widetilde{F}(p)\big)$, which is a smooth map by composition. It satisfies $H(p,t) = F(p)$ for all $p \in A$, since $F = \widetilde{F}$ there. □

Corollary 6.27 (Extension Lemma for Smooth Maps). *Suppose N is a smooth manifold with or without boundary, M is a smooth manifold, $A \subseteq N$ is a closed subset, and $f\colon A \to M$ is a smooth map. Then f has a smooth extension to N if and only if it has a continuous extension to N.*

Proof. If $F\colon N \to M$ is a continuous extension of f to all of N, the Whitney approximation theorem guarantees the existence of a smooth map $\widetilde{F}$ (homotopic to F,

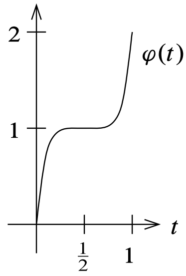

Fig. 6.9 The function φ in the proof of Lemma 6.28

in fact, though we do not need that here) that agrees with f on A; in other words, $\widetilde{F}$ is a smooth extension of f. The converse is obvious. □

If N and M are two smooth manifolds with or without boundary, a homotopy $H: N \times I \to M$ is called a ***smooth homotopy*** if it is also a smooth map, in the sense that it extends to a smooth map on some neighborhood of $N \times I$ in $N \times \mathbb{R}$. Two maps are said to be ***smoothly homotopic*** if there is a smooth homotopy between them.

Lemma 6.28. *If N and M are smooth manifolds with or without boundary, smooth homotopy is an equivalence relation on the set of all smooth maps from N to M.*

Proof. Reflexivity and symmetry are proved just as for ordinary homotopy. To prove transitivity, suppose $F, G, K: N \to M$ are smooth maps, and $H_1, H_2: N \times I \to M$ are smooth homotopies from F to G and G to K, respectively. Let $\varphi: [0,1] \to [0,2]$ be a smooth map such that $0 \le \varphi(t) \le 1$ for $t \in \left[0, \frac{1}{2}\right]$, $1 \le \varphi(t) \le 2$ for $t \in \left[\frac{1}{2}, 1\right]$, $\varphi(0) = 0$, $\varphi(1) = 2$, and $\varphi(t) \equiv 1$ for t in a neighborhood of $\frac{1}{2}$ (see Fig. 6.9).Define $H: N \times I \to M$ by

$$H(x,t) = \begin{cases} H_1\big(x, \varphi(t)\big), & t \in \left[0, \frac{1}{2}\right], \\ H_2\big(x, \varphi(t) - 1\big), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then it is easy to check that H is a smooth homotopy from F to K. □

Theorem 6.29. *Suppose N is a smooth manifold with or without boundary, M is a smooth manifold, and $F, G: N \to M$ are smooth maps. If F and G are homotopic, then they are smoothly homotopic. If F and G are homotopic relative to some closed subset $A \subseteq N$, then they are smoothly homotopic relative to A.*

Proof. Suppose $F, G: N \to M$ are smooth, and let $H: N \times I \to M$ be a homotopy from F to G (relative to A, which may be empty). We wish to show that H can be replaced by a smooth homotopy.

Define $\bar{H}: N \times \mathbb{R} \to M$ by

$$\bar{H}(x,t) = \begin{cases} H(x,t), & t \in [0,1], \\ H(x,0), & t \le 0, \\ H(x,1), & t \ge 1. \end{cases}$$

This is continuous by the gluing lemma. The restriction of $\bar{H}$ to $N \times \{0\} \cup N \times \{1\}$ is smooth, because it is equal to $F \circ \pi_1$ on $N \times \{0\}$ and $G \circ \pi_1$ on $N \times \{1\}$ (where $\pi_1: N \times \mathbb{R} \to N$ is the projection). If H is a homotopy relative to A, then $\bar{H}$ is also smooth on $A \times I$. Because $N \times \mathbb{R}$ is a smooth manifold with (possibly empty) boundary, the Whitney approximation theorem guarantees that there is a smooth map $\widetilde{H}: N \times \mathbb{R} \to M$ (homotopic to $\bar{H}$, but we do not need that here) whose restriction to $N \times \{0\} \cup N \times \{1\} \cup A \times I$ agrees with $\bar{H}$ (and therefore H). Restricting back to $N \times I$ again, we see that $\widetilde{H}|_{N \times I}$ is a smooth homotopy (relative to A) between F and G. □

When the target manifold has nonempty boundary, the analogues of Theorems 6.26 and 6.29 do not hold, because it might not be possible to find a smooth map that agrees with F on A (see Problem 6-7). However, if we do not insist on homotopy relative to a subset, the rest of the results can be extended to maps into manifolds with boundary. The proofs will have to wait until Chapter 9 (see Theorems 9.27 and 9.28).

Transversality

As our final application of Sard's theorem, we show how submanifolds can be perturbed so that they intersect "nicely." To explain what this means, we introduce the concept of *transversality*.

The intersection of two linear subspaces of a vector space is always another linear subspace. The analogous statement for submanifolds is certainly not true: it is easy to come up with examples of smooth submanifolds whose intersection is not a submanifold. (See Problem 6-14.) But with an additional assumption about the submanifolds, it is possible to show that their intersection is again a submanifold.

Suppose M is a smooth manifold. Two embedded submanifolds $S, S' \subseteq M$ are said to ***intersect transversely*** if for each $p \in S \cap S'$, the tangent spaces T_pS and T_pS' together span T_pM (where we consider T_pS and T_pS' as subspaces of T_pM).

For many purposes, it is more convenient to work with the following more general definition. If $F: N \to M$ is a smooth map and $S \subseteq M$ is an embedded submanifold, we say that F is ***transverse to S*** if for every $x \in F^{-1}(S)$, the spaces $T_{F(x)}S$ and $dF_x(T_xN)$ together span $T_{F(x)}M$. One special case is worth noting: if F is a smooth submersion, then it is automatically transverse to every embedded submanifold of M. Two embedded submanifolds intersect transversely if and only if the inclusion of either one is transverse to the other.

The next result, a generalization of the regular level set theorem, shows why transversality is desirable.

Theorem 6.30. *Suppose N and M are smooth manifolds and $S \subseteq M$ is an embedded submanifold.*

(a) *If $F: N \to M$ is a smooth map that is transverse to S, then $F^{-1}(S)$ is an embedded submanifold of N whose codimension is equal to the codimension of S in M.*
(b) *If $S' \subseteq M$ is an embedded submanifold that intersects S transversely, then $S \cap S'$ is an embedded submanifold of M whose codimension is equal to the sum of the codimensions of S and S'.*

Proof. The second statement follows easily from the first, simply by taking F to be the inclusion map $S' \hookrightarrow M$, and noting that a composition of smooth embeddings $S \cap S' \hookrightarrow S \hookrightarrow M$ is again a smooth embedding.

To prove (a), let m denote the dimension of M and k the codimension of S in M. Given $x \in F^{-1}(S)$, we can find a neighborhood U of $F(x)$ in M and a local defining function $\varphi: U \to \mathbb{R}^k$ for S, with $S \cap U = \varphi^{-1}(0)$. The theorem will be proved if we can show that 0 is a regular value of $\varphi \circ F$, because $F^{-1}(S) \cap F^{-1}(U)$ is the zero set of $\varphi \circ F|_{F^{-1}(U)}$.

Given $z \in T_0\mathbb{R}^k$ and $p \in (\varphi \circ F)^{-1}(0)$, the fact that 0 is a regular value of φ means there is a vector $y \in T_{F(p)}M$ such that $d\varphi_{F(p)}(y) = z$. The fact that F is transverse to S means we can write $y = y_0 + dF_p(v)$ for some $y_0 \in T_{F(p)}S$ and some $v \in T_pN$. Because φ is constant on $S \cap U$, it follows that $d\varphi_{F(p)}(y_0) = 0$, so

$$d(\varphi \circ F)_p(v) = d\varphi_{F(p)}\big(dF_p(v)\big) = d\varphi_{F(p)}\big(y_0 + dF_p(v)\big) = d\varphi_{F(p)}(y) = z.$$

Thus $F^{-1}(S)$ is an embedded submanifold of codimension k. □

For example, in $\mathbb{R}^3$, this theorem shows that a smooth curve and a smooth surface intersecting transversely have only isolated points in their intersection, while two smooth surfaces intersect transversely in a smooth curve. Two smooth curves in $\mathbb{R}^3$ intersect transversely if and only if their intersection is empty, because at any intersection point, the two one-dimensional tangent spaces to the curves would have to span the tangent space to $\mathbb{R}^3$.

Because a submersion is transverse to every embedded submanifold, the next corollary is immediate.

Corollary 6.31. *Suppose N and M are smooth manifolds, $S \subseteq M$ is an embedded submanifold of codimension k, and $F: N \to M$ is a submersion. Then $F^{-1}(S)$ is an embedded codimension-k submanifold of N.* □

Transversality also provides a convenient criterion for recognizing a submanifold as a graph. The next theorem is a global version of the implicit function theorem.

Theorem 6.32 (Global Characterization of Graphs). *Suppose M and N are smooth manifolds and $S \subseteq M \times N$ is an immersed submanifold. Let π_M and π_N denote the projections from $M \times N$ onto M and N, respectively. The following are equivalent.*

(a) *S is the graph of a smooth map $f: M \to N$.*

(b) $\pi_M|_S$ *is a diffeomorphism from* S *onto* M.
(c) *For each* $p \in M$, *the submanifolds* S *and* $\{p\} \times N$ *intersect transversely in exactly one point.*

If these conditions hold, then S *is the graph of the map* $f: M \to N$ *defined by* $f = \pi_N \circ (\pi_M|_S)^{-1}$.

Proof. Problem 6-15. □

Corollary 6.33 (Local Characterization of Graphs). *Suppose* M *and* N *are smooth manifolds,* $S \subseteq M \times N$ *is an immersed submanifold, and* $(p,q) \in S$. *If* S *intersects the submanifold* $\{p\} \times N$ *transversely at* (p,q), *then there exist a neighborhood* U *of* p *in* M *and a neighborhood* V *of* (p,q) *in* S *such that* V *is the graph of a smooth map* $f: U \to N$.

Proof. The hypothesis guarantees that $d(\pi_M)_{(p,q)}: T_{(p,q)}S \to T_pM$ is an isomorphism, so $\pi_M|_S$ restricts to a diffeomorphism from a neighborhood V of (p,q) in S to a neighborhood U of p. The result then follows from Theorem 6.32(b). □

The surprising thing about transversely intersecting submanifolds and transverse maps is that they are "generic," as we will soon see. To set the stage, we need to consider families of maps that are somewhat more general than smooth homotopies.

Suppose N, M, and S are smooth manifolds, and for each $s \in S$ we are given a map $F_s: N \to M$. The collection $\{F_s : s \in S\}$ is called a ***smooth family of maps*** if the map $F: M \times S \to N$ defined by $F(x,s) = F_s(x)$ is smooth. You should think of such a family as a higher-dimensional analogue of a homotopy. The next proposition shows how such families are related to ordinary homotopies.

Proposition 6.34. *If* $\{F_s : s \in S\}$ *is a smooth family of maps from* N *to* M *and* S *is connected, then for any* $s_1, s_2 \in S$, *the maps* $F_{s_1}, F_{s_2}: N \to M$ *are homotopic.*

Proof. Because S is connected, it is path-connected. If $\gamma: [0,1] \to S$ is any path from s_1 to s_2, then $H(x,s) = F\big(x, \gamma(s)\big)$ is a homotopy from F_{s_1} to F_{s_2}. □

The key to finding transverse maps is the following application of Sard's theorem, which gives a simple sufficient condition for a family of smooth maps to contain at least one map that is transverse to a given submanifold. If S is a smooth manifold and $B \subseteq S$ is a subset whose complement has measure zero in S, we say that B contains ***almost every element of*** S.

Theorem 6.35 (Parametric Transversality Theorem). *Suppose* N *and* M *are smooth manifolds,* $X \subseteq M$ *is an embedded submanifold, and* $\{F_s : s \in S\}$ *is a smooth family of maps from* N *to* M. *If the map* $F: N \times S \to M$ *is transverse to* X, *then for almost every* $s \in S$, *the map* $F_s: N \to M$ *is transverse to* X.

Proof. The hypothesis implies that $W = F^{-1}(X)$ is an embedded submanifold of $N \times S$ by Theorem 6.30. Let $\pi: N \times S \to S$ be the projection onto the second factor. What we will actually show is that if $s \in S$ is a regular value of the restriction $\pi|_W$, then F_s is transverse to X. Since almost every s is a regular value by Sard's theorem, this proves the theorem.

Suppose $s \in S$ is a regular value of $\pi|_W$. Let $p \in F_s^{-1}(X)$ be arbitrary, and set $q = F_s(p) \in X$. We need to show that $T_q M = T_q X + d(F_s)(T_p N)$. Here is what we know. First, because of our hypothesis on F,

$$T_q M = T_q X + dF\big(T_{(p,s)}(N \times S)\big). \tag{6.7}$$

Second, because s is a regular value and $(p,s) \in W$,

$$T_s S = d\pi\big(T_{(p,s)} W\big). \tag{6.8}$$

Third, by the result of Problem 6-10, we have $T_{(p,s)} W = \big(dF_{(p,s)}\big)^{-1}(T_q X)$, which implies

$$dF\left(T_{(p,s)} W\right) = T_q X. \tag{6.9}$$

Now let $w \in T_q M$ be arbitrary. We need to find $v \in T_q X$ and $y \in T_p N$ such that

$$w = v + d(F_s)(y). \tag{6.10}$$

Because of (6.7), there exist $v_1 \in T_q X$ and $(y_1, z_1) \in T_p N \times T_s S \cong T_{(p,s)}(N \times S)$ such that

$$w = v_1 + dF(y_1, z_1). \tag{6.11}$$

By (6.8), there exists $(y_2, z_2) \in T_{(p,s)} W$ such that $d\pi(y_2, z_2) = z_1$. Since π is a projection, this means $z_2 = z_1$. By linearity, we can write

$$dF(y_1, z_1) = dF(y_2, z_1) + dF(y_1 - y_2, 0).$$

On the one hand, (6.9) implies $dF(y_2, z_1) = dF(y_2, z_2) \in dF\big(T_{(p,s)} W\big) = T_q X$. On the other hand, if $\iota_s\colon N \to N \times S$ is the map $\iota_s(p') = (p', s)$, then we have $F_s = F \circ \iota_s$ and $d(\iota_s)(y_1 - y_2) = (y_1 - y_2, 0)$, and therefore $dF(y_1 - y_2, 0) = dF \circ d(\iota_s)(y_1 - y_2) = d(F_s)(y_1 - y_2)$. By virtue of (6.11), therefore, (6.10) is satisfied with $v = v_1 + dF(y_2, z_1)$ and $y = y_1 - y_2$, and the proof is complete. □

In order to make use of the parametric transversality theorem, we need to construct a smooth family of maps satisfying the hypothesis. The proof of the next theorem shows that it is always possible to do so.

Theorem 6.36 (Transversality Homotopy Theorem). *Suppose M and N are smooth manifolds and $X \subseteq M$ is an embedded submanifold. Every smooth map $f\colon N \to M$ is homotopic to a smooth map $g\colon N \to M$ that is transverse to X.*

Proof. The crux of the proof is constructing a smooth map $F\colon N \times S \to M$ that is transverse to X, where $S = \mathbb{B}^k$ for some k and $F_0 = f$. It then follows from the parametric transversality theorem that there is some $s \in S$ such that $F_s\colon N \to M$ is transverse to X, and from Proposition 6.34 that F_s is homotopic to f.

By the Whitney embedding theorem, we can assume that M is a properly embedded submanifold of $\mathbb{R}^k$ for some k. Let U be a tubular neighborhood of M in $\mathbb{R}^k$, and let $r\colon U \to M$ be a smooth retraction that is also a smooth submersion. If we define $\delta\colon M \to \mathbb{R}^+$ by (6.6), Corollary 6.22 shows that there exists a smooth function $e\colon N \to \mathbb{R}^+$ that satisfies $0 < e(p) < \delta\big(f(p)\big)$ everywhere. Let S be the unit ball in $\mathbb{R}^k$, and define $F\colon N \times S \to M$ by

$$F(p,s) = r\big(f(p) + e(p)s\big).$$

Note that $\big|e(p)s\big| < e(p) < \delta\big(f(p)\big)$, which implies that $f(p) + e(p)s \in U$, so F is well defined. Clearly, F is smooth, and $F_0 = f$ because r is a retraction.

For each $p \in N$, the restriction of F to $\{p\} \times S$ is the composition of the local diffeomorphism $s \mapsto f(p) + e(p)s$ followed by the smooth submersion r, so F is a smooth submersion and hence transverse to X. □

Problems

6-1. Use Proposition 6.5 to give a simpler proof of Corollary 6.11 that does not use Sard's theorem. [Hint: given a smooth map $F\colon M \to N$, define a suitable map from $M \times \mathbb{R}^k$ to N, where $k = \dim N - \dim M$.]

6-2. Prove Theorem 6.18 (the Whitney immersion theorem) in the special case $\partial M = \varnothing$. [Hint: without loss of generality, assume that M is an embedded n-dimensional submanifold of $\mathbb{R}^{2n+1}$. Let $UM \subseteq T\mathbb{R}^{2n+1}$ be the unit tangent bundle of M (Problem 5-6), and let $G\colon UM \to \mathbb{RP}^{2n}$ be the map $G(x,v) = [v]$. Use Sard's theorem to conclude that there is some $v \in \mathbb{R}^{2n+1} \smallsetminus \mathbb{R}^{2n}$ such that $[v]$ is not in the image of G, and show that the projection from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n}$ with kernel $\mathbb{R}v$ restricts to an immersion of M into $\mathbb{R}^{2n}$.]

6-3. Let M be a smooth manifold, let $B \subseteq M$ be a closed subset, and let $\delta\colon M \to \mathbb{R}$ be a positive continuous function. Show that there is a smooth function $\widetilde{\delta}\colon M \to \mathbb{R}$ that is zero on B, positive on $M \smallsetminus B$, and satisfies $\widetilde{\delta}(x) < \delta(x)$ everywhere. [Hint: consider $f/(f+1)$, where f is a smooth nonnegative function that vanishes exactly on B, and use Corollary 6.22.]

6-4. Let M be a smooth manifold, let B be a closed subset of M, and let $\delta\colon M \to \mathbb{R}$ be a positive continuous function.

(a) Given any continuous function $f\colon M \to \mathbb{R}^k$, show that there is a continuous function $\widetilde{f}\colon M \to \mathbb{R}^k$ that is smooth on $M \smallsetminus B$, agrees with f on B, and is δ-close to f. [Hint: use Problem 6-3.]

(b) Given a smooth manifold N and a continuous map $F\colon M \to N$, show that F is homotopic relative to B to a map that is smooth on $M \smallsetminus B$.

6-5. Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. Show that M has a tubular neighborhood U with the following property: for each $y \in U$, $r(y)$ is the unique point in M closest to y, where $r\colon U \to M$ is the retraction defined in Proposition 6.25. [Hint: first show that if $y \in \mathbb{R}^n$ has a closest point $x \in M$, then $(y-x) \perp T_xM$. Then, using the notation of the proof of Theorem 6.24, show that for each $x \in M$, it is possible to choose $\delta > 0$ such that every $y \in E\big(V_\delta(x)\big)$ has a closest point in M, and that point is equal to $r(y)$.]

6-6. Suppose $M \subseteq \mathbb{R}^n$ is a compact embedded submanifold. For any $\varepsilon > 0$, let M_ε be the set of points in $\mathbb{R}^n$ whose distance from M is less than ε. Show that for sufficiently small ε, ∂M_ε is a compact embedded hypersurface in $\mathbb{R}^n$, and $\overline{M}_\varepsilon$ is a compact regular domain in $\mathbb{R}^n$ whose interior contains M.

6-7. By considering the map $F\colon \mathbb{R} \to \mathbb{H}^2$ given by $F(t) = (t, |t|)$ and the subset $A = [0, \infty) \subseteq \mathbb{R}$, show that the conclusions of Theorem 6.26 and Corollary 6.27 can be false when M has nonempty boundary.

6-8. Prove that every proper continuous map between smooth manifolds is homotopic to a proper smooth map. [Hint: show that the map $\widetilde{F}$ constructed in the proof of Theorem 6.26 is proper if F is.]

6-9. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the map $F(x, y) = \left(e^y \cos x, e^y \sin x, e^{-y}\right)$. For which positive numbers r is F transverse to the sphere $S_r(0) \subseteq \mathbb{R}^3$? For which positive numbers r is $F^{-1}\left(S_r(0)\right)$ an embedded submanifold of $\mathbb{R}^2$?

6-10. Suppose $F\colon N \to M$ is a smooth map that is transverse to an embedded submanifold $X \subseteq M$, and let $W = F^{-1}(X)$. For each $p \in W$, show that $T_p W = (dF_p)^{-1}\left(T_{F(p)} X\right)$. Conclude that if two embedded submanifolds $X, X' \subseteq M$ intersect transversely, then $T_p(X \cap X') = T_p X \cap T_p X'$ for every $p \in X \cap X'$. (*Used on p. 146.*)

6-11. Suppose $F\colon M \to N$ and $G\colon N \to P$ are smooth maps, and G is transverse to an embedded submanifold $X \subseteq P$. Show that F is transverse to the submanifold $G^{-1}(X)$ if and only if $G \circ F$ is transverse to X.

6-12. Let M be a compact smooth n-manifold. Prove that if $N \geq 2n$, every smooth map from M to $\mathbb{R}^N$ can be uniformly approximated by smooth immersions.

6-13. Let M be a smooth manifold. In this chapter, we defined what it means for two embedded submanifolds of M to intersect transversely, and for a smooth map into M to be transverse to an embedded submanifold. More generally, if $F\colon N \to M$ and $F'\colon N' \to M$ are smooth maps into M, we say that F and F' are ***transverse to each other*** if for every $x \in N$ and $x' \in N'$ such that $F(x) = F'(x')$, the spaces $dF_x(T_x N)$ and $dF'_{x'}(T_{x'} N')$ together span $T_{F(x)} M$. Prove the following statements.

(a) With N, N', F, F' as above, F and F' are transverse to each other if and only if the map $F \times F'\colon N \times N' \to M \times M$ is transverse to the diagonal $\Delta_M = \{(x, x) : x \in M\}$.

(b) If S is an embedded submanifold of M, a smooth map $F\colon N \to M$ is transverse to S if and only if it is transverse to the inclusion $\iota\colon S \hookrightarrow M$.

(c) If $F\colon N \to M$ and $F'\colon N' \to M$ are smooth maps that are transverse to each other, then $F^{-1}\left(F'(N')\right)$ is an embedded submanifold of N of dimension equal to $\dim N + \dim N' - \dim M$.

6-14. This problem illustrates how badly Theorem 6.30 can fail if the transversality hypothesis is removed. Let $S = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$, and suppose A is an arbitrary closed subset of S. Prove that there is a properly embedded hypersurface $S' \subseteq \mathbb{R}^{n+1}$ such that $S \cap S' = A$. [Hint: use Theorem 2.29.]

6-15. Prove Theorem 6.32 (global characterization of graphs).

6-16. Suppose M and N are smooth manifolds. A class $\mathcal{F}$ of smooth maps from N to M is said to be ***stable*** if it has the following property: whenever $\{F_s : s \in S\}$ is a smooth family of maps from N to M, and $F_{s_0} \in \mathcal{F}$ for some $s_0 \in S$, then there is a neighborhood U of s_0 in S such that $F_s \in \mathcal{F}$ for all $s \in U$. (Roughly speaking, a property of smooth maps is stable if it persists under small deformations.) Prove that if N is compact, then the following classes of smooth maps from N to M are stable:
(a) immersions
(b) submersions
(c) embeddings
(d) diffeomorphisms
(e) local diffeomorphisms
(f) maps that are transverse to a given properly embedded submanifold $X \subseteq M$

6-17. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a compactly supported smooth function such that $\varphi(0) = 1$. Use the family $\{F_s : s \in \mathbb{R}\}$ of maps from $\mathbb{R}$ to $\mathbb{R}$ given by $F_s(x) = x\varphi(sx)$ to show that the classes of maps described in Problem 6-16 need not be stable when N is not compact.