Chapter 2
Smooth Maps

The main reason for introducing smooth structures was to enable us to define smooth functions on manifolds and smooth maps between manifolds. In this chapter we carry out that project.

We begin by defining smooth real-valued and vector-valued functions, and then generalize this to smooth maps between manifolds. We then focus our attention for a while on the special case of *diffeomorphisms*, which are bijective smooth maps with smooth inverses. If there is a diffeomorphism between two smooth manifolds, we say that they are *diffeomorphic*. The main objects of study in smooth manifold theory are properties that are invariant under diffeomorphisms.

At the end of the chapter, we introduce a powerful tool for blending together locally defined smooth objects, called *partitions of unity*. They are used throughout smooth manifold theory for building global smooth objects out of local ones.

Smooth Functions and Smooth Maps

Although the terms *function* and *map* are technically synonymous, in studying smooth manifolds it is often convenient to make a slight distinction between them. Throughout this book we generally reserve the term ***function*** for a map whose codomain is $\mathbb{R}$ (a ***real-valued function***) or $\mathbb{R}^k$ for some $k > 1$ (a ***vector-valued function***). Either of the words ***map*** or ***mapping*** can mean any type of map, such as a map between arbitrary manifolds.

Smooth Functions on Manifolds

Suppose M is a smooth n-manifold, k is a nonnegative integer, and $f : M \to \mathbb{R}^k$ is any function. We say that f is a ***smooth function*** if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$ (Fig. 2.1). If M is a smooth manifold with boundary, the definition is exactly the same, except that

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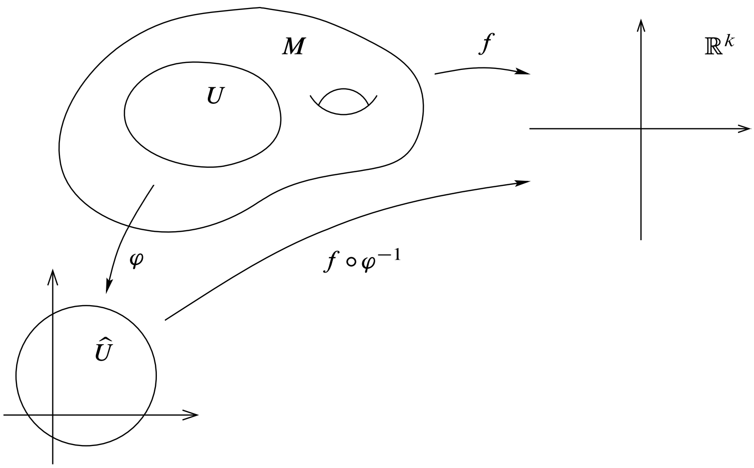

Fig. 2.1 Definition of smooth functions

$\varphi(U)$ is now an open subset of either $\mathbb{R}^n$ or $\mathbb{H}^n$, and in the latter case we interpret smoothness of $f \circ \varphi^{-1}$ to mean that each point of $\varphi(U)$ has a neighborhood (in $\mathbb{R}^n$) on which $f \circ \varphi^{-1}$ extends to a smooth function in the ordinary sense.

The most important special case is that of smooth real-valued functions $f: M \to \mathbb{R}$; the set of all such functions is denoted by $C^\infty(M)$. Because sums and constant multiples of smooth functions are smooth, $C^\infty(M)$ is a vector space over $\mathbb{R}$.

► **Exercise 2.1.** Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$. (See Appendix B, p. 624, for the definition of an algebra.)

► **Exercise 2.2.** Let U be an open submanifold of $\mathbb{R}^n$ with its standard smooth manifold structure. Show that a function $f: U \to \mathbb{R}^k$ is smooth in the sense just defined if and only if it is smooth in the sense of ordinary calculus. Do the same for an open submanifold with boundary in $\mathbb{H}^n$ (see Exercise 1.44).

► **Exercise 2.3.** Let M be a smooth manifold with or without boundary, and suppose $f: M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \varphi^{-1}: \varphi(U) \to \mathbb{R}^k$ is smooth for *every* smooth chart (U, φ) for M.

Given a function $f: M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function $\widehat{f}: \varphi(U) \to \mathbb{R}^k$ defined by $\widehat{f}(x) = f \circ \varphi^{-1}(x)$ is called the ***coordinate representation of f***. By definition, f is smooth if and only if its coordinate representation is smooth in some smooth chart around each point. By the preceding exercise, smooth functions have smooth coordinate representations in every smooth chart.

For example, consider the real-valued function $f(x, y) = x^2 + y^2$ defined on the plane. In polar coordinates on, say, the set $U = \{(x, y) : x > 0\}$, it has the coordinate representation $\widehat{f}(r, \theta) = r^2$. In keeping with our practice of using local coordinates

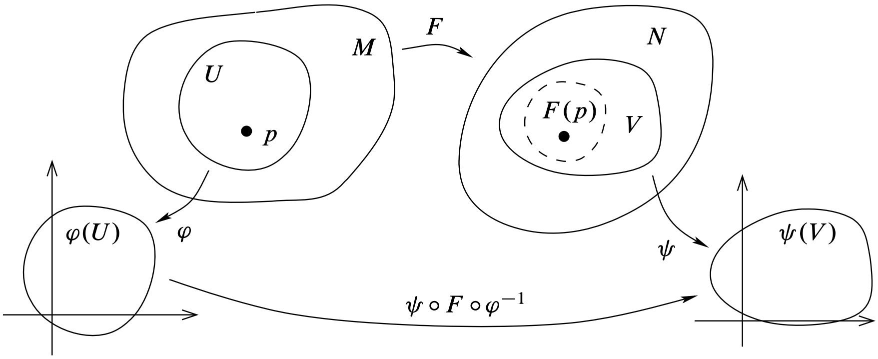

Fig. 2.2 Definition of smooth maps

to identify an open subset of a manifold with an open subset of Euclidean space, in cases where it causes no confusion we often do not even observe the distinction between $\widehat{f}$ and f itself, and instead say something like "f is smooth on U because its coordinate representation $f(r,\theta) = r^2$ is smooth."

Smooth Maps Between Manifolds

The definition of smooth functions generalizes easily to maps between manifolds. Let M, N be smooth manifolds, and let $F\colon M \to N$ be any map. We say that F is a ***smooth map*** if for every $p \in M$, there exist smooth charts (U,φ) containing p and (V,ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$ (Fig. 2.2). If M and N are smooth manifolds with boundary, smoothness of F is defined in exactly the same way, with the usual understanding that a map whose domain is a subset of $\mathbb{H}^n$ is smooth if it admits an extension to a smooth map in a neighborhood of each point, and a map whose codomain is a subset of $\mathbb{H}^n$ is smooth if it is smooth as a map into $\mathbb{R}^n$. Note that our previous definition of smoothness of real-valued or vector-valued functions can be viewed as a special case of this one, by taking $N = V = \mathbb{R}^k$ and $\psi = \mathrm{Id}\colon \mathbb{R}^k \to \mathbb{R}^k$.

The first important observation about our definition of smooth maps is that, as one might expect, smoothness implies continuity.

Proposition 2.4. *Every smooth map is continuous.*

Proof. Suppose M and N are smooth manifolds with or without boundary, and $F\colon M \to N$ is smooth. Given $p \in M$, smoothness of F means there are smooth charts (U,φ) containing p and (V,ψ) containing $F(p)$, such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to \psi(V)$ is smooth, hence continuous. Since $\varphi\colon U \to \varphi(U)$ and $\psi\colon V \to \psi(V)$ are homeomorphisms, this implies in turn that

$$F|_U = \psi^{-1} \circ \left(\psi \circ F \circ \varphi^{-1}\right) \circ \varphi\colon U \to V,$$

which is a composition of continuous maps. Since F is continuous in a neighborhood of each point, it is continuous on M. $\square$

To prove that a map $F: M \to N$ is smooth directly from the definition requires, in part, that for each $p \in M$ we prove the existence of coordinate domains U containing p and V containing $F(p)$ such that $F(U) \subseteq V$. This requirement is included in the definition precisely so that smoothness automatically implies continuity. (Problem 2-1 illustrates what can go wrong if this requirement is omitted.) There are other ways of characterizing smoothness of maps between manifolds that accomplish the same thing. Here are two of them.

Proposition 2.5 (Equivalent Characterizations of Smoothness). *Suppose M and N are smooth manifolds with or without boundary, and $F: M \to N$ is a map. Then F is smooth if and only if either of the following conditions is satisfied*:

(a) *For every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $U \cap F^{-1}(V)$ is open in M and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi\big(U \cap F^{-1}(V)\big)$ to $\psi(V)$.*
(b) *F is continuous and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha\big(U_\alpha \cap F^{-1}(V_\beta)\big)$ to $\psi_\beta(V_\beta)$.*

Proposition 2.6 (Smoothness Is Local). *Let M and N be smooth manifolds with or without boundary, and let $F: M \to N$ be a map.*

(a) *If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth.*
(b) *Conversely, if F is smooth, then its restriction to every open subset is smooth.*

► **Exercise 2.7.** Prove the preceding two propositions.

The next corollary is essentially just a restatement of the previous proposition, but it gives a highly useful way of constructing smooth maps.

Corollary 2.8 (Gluing Lemma for Smooth Maps). *Let M and N be smooth manifolds with or without boundary, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Suppose that for each $\alpha \in A$, we are given a smooth map $F_\alpha: U_\alpha \to N$ such that the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all α and β. Then there exists a unique smooth map $F: M \to N$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.* $\square$

If $F: M \to N$ is a smooth map, and (U, φ) and (V, ψ) are any smooth charts for M and N, respectively, we call $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ the ***coordinate representation of F*** with respect to the given coordinates. It maps the set $\varphi\big(U \cap F^{-1}(V)\big)$ to $\psi(V)$.

► **Exercise 2.9.** Suppose $F: M \to N$ is a smooth map between smooth manifolds with or without boundary. Show that the coordinate representation of F with respect to *every* pair of smooth charts for M and N is smooth.

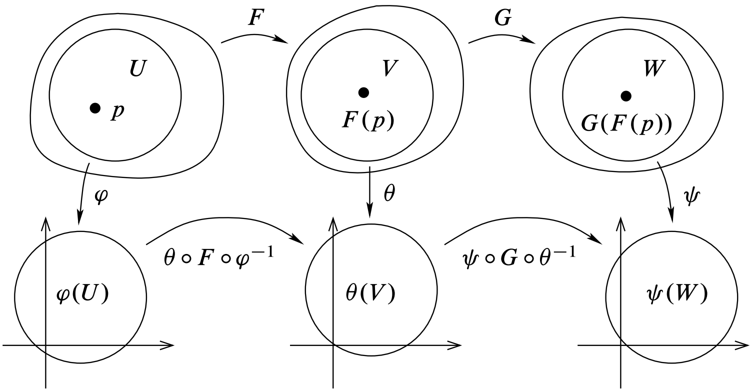

Fig. 2.3 A composition of smooth maps is smooth

As with real-valued or vector-valued functions, once we have chosen specific local coordinates in both the domain and codomain, we can often ignore the distinction between F and $\widehat{F}$.

Next we examine some simple classes of maps that are automatically smooth.

Proposition 2.10. *Let M, N, and P be smooth manifolds with or without boundary.*

(a) *Every constant map $c\colon M \to N$ is smooth.*
(b) *The identity map of M is smooth.*
(c) *If $U \subseteq M$ is an open submanifold with or without boundary, then the inclusion map $U \hookrightarrow M$ is smooth.*
(d) *If $F\colon M \to N$ and $G\colon N \to P$ are smooth, then so is $G \circ F\colon M \to P$.*

Proof. We prove (d) and leave the rest as exercises. Let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps, and let $p \in M$. By definition of smoothness of G, there exist smooth charts (V,θ) containing $F(p)$ and (W,ψ) containing $G(F(p))$ such that $G(V) \subseteq W$ and $\psi \circ G \circ \theta^{-1}\colon \theta(V) \to \psi(W)$ is smooth. Since F is continuous, $F^{-1}(V)$ is a neighborhood of p in M, so there is a smooth chart (U,φ) for M such that $p \in U \subseteq F^{-1}(V)$ (Fig. 2.3). By Exercise 2.9, $\theta \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\theta(V)$. Then we have $G \circ F(U) \subseteq G(V) \subseteq W$, and $\psi \circ (G \circ F) \circ \varphi^{-1} = \left(\psi \circ G \circ \theta^{-1}\right) \circ \left(\theta \circ F \circ \varphi^{-1}\right)\colon \varphi(U) \to \psi(W)$ is smooth because it is a composition of smooth maps between subsets of Euclidean spaces. □

► **Exercise 2.11.** Prove parts (a)–(c) of the preceding proposition.

Proposition 2.12. *Suppose $M_1, \dots, M_k$ and N are smooth manifolds with or without boundary, such that at most one of $M_1, \dots, M_k$ has nonempty boundary. For each i, let $\pi_i\colon M_1 \times \dots \times M_k \to M_i$ denote the projection onto the M_i factor. A map $F\colon N \to M_1 \times \dots \times M_k$ is smooth if and only if each of the component maps $F_i = \pi_i \circ F\colon N \to M_i$ is smooth.*

Proof. Problem 2-2. □

Although most of our efforts in this book are devoted to the study of smooth manifolds and smooth maps, we also need to work with topological manifolds and continuous maps on occasion. For the sake of consistency, we adopt the following conventions: without further qualification, the words "function" and "map" are to be understood purely in the set-theoretic sense, and carry no assumptions of continuity or smoothness. Most other objects we study, however, will be understood to carry some minimal topological structure by default. Unless otherwise specified, a "manifold" or "manifold with boundary" is always to be understood as a topological one, and a "coordinate chart" is to be understood in the topological sense, as a homeomorphism from an open subset of the manifold to an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. If we wish to restrict attention to smooth manifolds or smooth coordinate charts, we will say so. Similarly, our default assumptions for many other specific types of geometric objects and the maps between them will be continuity at most; smoothness will not be assumed unless explicitly specified. The only exceptions will be a few concepts that require smoothness for their very definitions.

This convention requires a certain discipline, in that we have to remember to state the smoothness hypothesis whenever it is needed; but its advantage is that it frees us (for the most part) from having to remember which types of maps are assumed to be smooth and which are not.

On the other hand, because the definition of a smooth map requires smooth structures in the domain and codomain, if we say "$F: M \to N$ is a smooth map" without specifying what M and N are, it should always be understood that they are smooth manifolds with or without boundaries.

We now have enough information to produce a number of interesting examples of smooth maps. In spite of the apparent complexity of the definition, it is usually not hard to prove that a particular map is smooth. There are basically only three common ways to do so:

- Write the map in smooth local coordinates and recognize its component functions as compositions of smooth elementary functions.
- Exhibit the map as a composition of maps that are known to be smooth.
- Use some special-purpose theorem that applies to the particular case under consideration.

Example 2.13 (Smooth Maps).

(a) Any map from a zero-dimensional manifold into a smooth manifold with or without boundary is automatically smooth, because each coordinate representation is constant.

(b) If the circle $\mathbb{S}^1$ is given its standard smooth structure, the map $\varepsilon: \mathbb{R} \to \mathbb{S}^1$ defined by $\varepsilon(t) = e^{2\pi i t}$ is smooth, because with respect to any angle coordinate θ for $\mathbb{S}^1$ (see Problem 1-8) it has a coordinate representation of the form $\widehat{\varepsilon}(t) = 2\pi t + c$ for some constant c, as you can check.

(c) The map $\varepsilon^n: \mathbb{R}^n \to \mathbb{T}^n$ defined by $\varepsilon^n\left(x^1, \dots, x^n\right) = \left(e^{2\pi i x^1}, \dots, e^{2\pi i x^n}\right)$ is smooth by Proposition 2.12.

(d) Now consider the n-sphere $\mathbb{S}^n$ with its standard smooth structure. The inclusion map $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is certainly continuous, because it is the inclusion map of a topological subspace. It is a smooth map because its coordinate representation with respect to any of the graph coordinates of Example 1.31 is

$$\begin{aligned}\hat{\iota}\left(u^1,\dots,u^n\right) &= \iota\circ\left(\varphi_i^{\pm}\right)^{-1}\left(u^1,\dots,u^n\right)\\ &= \left(u^1,\dots,u^{i-1},\pm\sqrt{1-|u|^2},u^i,\dots,u^n\right),\end{aligned}$$

which is smooth on its domain (the set where $|u|^2 < 1$).

(e) The quotient map $\pi\colon \mathbb{R}^{n+1}\smallsetminus\{0\}\to\mathbb{RP}^n$ used to define $\mathbb{RP}^n$ is smooth, because its coordinate representation in terms of any of the coordinates for $\mathbb{RP}^n$ constructed in Example 1.33 and standard coordinates on $\mathbb{R}^{n+1}\smallsetminus\{0\}$ is

$$\begin{aligned}\hat{\pi}\left(x^1,\dots,x^{n+1}\right) &= \varphi_i\circ\pi\left(x^1,\dots,x^{n+1}\right) = \varphi_i\left[x^1,\dots,x^{n+1}\right]\\ &= \left(\frac{x^1}{x^i},\dots,\frac{x^{i-1}}{x^i},\frac{x^{i+1}}{x^i},\dots,\frac{x^{n+1}}{x^i}\right).\end{aligned}$$

(f) Define $q\colon \mathbb{S}^n\to\mathbb{RP}^n$ as the restriction of $\pi\colon \mathbb{R}^{n+1}\smallsetminus\{0\}\to\mathbb{RP}^n$ to $\mathbb{S}^n\subseteq\mathbb{R}^{n+1}\smallsetminus\{0\}$. It is a smooth map, because it is the composition $q=\pi\circ\iota$ of the maps in the preceding two examples.

(g) If $M_1,\dots,M_k$ are smooth manifolds, then each projection map $\pi_i\colon M_1\times\dots\times M_k\to M_i$ is smooth, because its coordinate representation with respect to any of the product charts of Example 1.8 is just a coordinate projection. //

Diffeomorphisms

If M and N are smooth manifolds with or without boundary, a ***diffeomorphism from M to N*** is a smooth bijective map $F\colon M\to N$ that has a smooth inverse. We say that ***M and N are diffeomorphic*** if there exists a diffeomorphism between them. Sometimes this is symbolized by $M\approx N$.

Example 2.14 (Diffeomorphisms).

(a) Consider the maps $F\colon \mathbb{B}^n\to\mathbb{R}^n$ and $G\colon\mathbb{R}^n\to\mathbb{B}^n$ given by

$$F(x)=\frac{x}{\sqrt{1-|x|^2}},\qquad G(y)=\frac{y}{\sqrt{1+|y|^2}}. \tag{2.1}$$

These maps are smooth, and it is straightforward to compute that they are inverses of each other. Thus they are both diffeomorphisms, and therefore $\mathbb{B}^n$ is diffeomorphic to $\mathbb{R}^n$.

(b) If M is any smooth manifold and (U,φ) is a smooth coordinate chart on M, then $\varphi\colon U\to\varphi(U)\subseteq\mathbb{R}^n$ is a diffeomorphism. (In fact, it has an identity map as a coordinate representation.) //

Proposition 2.15 (Properties of Diffeomorphisms).

(a) *Every composition of diffeomorphisms is a diffeomorphism.*
(b) *Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.*
(c) *Every diffeomorphism is a homeomorphism and an open map.*
(d) *The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.*
(e) *"Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.*

► **Exercise 2.16.** Prove the preceding proposition.

The following theorem is a weak version of invariance of dimension, which suffices for many purposes.

Theorem 2.17 (Diffeomorphism Invariance of Dimension). *A nonempty smooth manifold of dimension m cannot be diffeomorphic to an n-dimensional smooth manifold unless $m = n$.*

Proof. Suppose M is a nonempty smooth m-manifold, N is a nonempty smooth n-manifold, and $F: M \to N$ is a diffeomorphism. Choose any point $p \in M$, and let (U, φ) and (V, ψ) be smooth coordinate charts containing p and $F(p)$, respectively. Then (the restriction of) $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is a diffeomorphism from an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$, so it follows from Proposition C.4 that $m = n$. □

There is a similar invariance statement for boundaries.

Theorem 2.18 (Diffeomorphism Invariance of the Boundary). *Suppose M and N are smooth manifolds with boundary and $F: M \to N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from* $\operatorname{Int} M$ *to* $\operatorname{Int} N$.

► **Exercise 2.19.** Use Theorem 1.46 to prove the preceding theorem.

Just as two topological spaces are considered to be "the same" if they are homeomorphic, two smooth manifolds with or without boundary are essentially indistinguishable if they are diffeomorphic. The central concern of smooth manifold theory is the study of properties of smooth manifolds that are preserved by diffeomorphisms. Theorem 2.17 shows that dimension is one such property.

It is natural to wonder whether the smooth structure on a given topological manifold is unique. This straightforward version of the question is easy to answer: we observed in Example 1.21 that every zero-dimensional manifold has a unique smooth structure, but as Problem 1-6 showed, each positive-dimensional manifold admits many distinct smooth structures as soon as it admits one.

A more subtle and interesting question is whether a given topological manifold admits smooth structures that are not diffeomorphic to each other. For example, let $\widetilde{\mathbb{R}}$ denote the topological manifold $\mathbb{R}$, but endowed with the smooth structure described in Example 1.23 (defined by the global chart $\psi(x) = x^3$). It turns out that $\widetilde{\mathbb{R}}$ is diffeomorphic to $\mathbb{R}$ with its standard smooth structure. Define a map $F: \mathbb{R} \to \widetilde{\mathbb{R}}$ by $F(x) = x^{1/3}$. The coordinate representation of this map is $\widehat{F}(t) = \psi \circ F \circ \mathrm{Id}_{\mathbb{R}}^{-1}(t) = t$, which is clearly smooth. Moreover, the coordinate representation of its inverse is

$$\widehat{F^{-1}}(y) = \mathrm{Id}_{\mathbb{R}} \circ F^{-1} \circ \psi^{-1}(y) = y,$$

which is also smooth, so F is a diffeomorphism. (This is a case in which it *is* important to maintain the distinction between a map and its coordinate representation!)

In fact, as you will see later, there is only one smooth structure on $\mathbb{R}$ up to diffeomorphism (see Problem 15-13). More precisely, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are any two smooth structures on $\mathbb{R}$, there exists a diffeomorphism $F: (\mathbb{R}, \mathcal{A}_1) \to (\mathbb{R}, \mathcal{A}_2)$. In fact, it follows from work of James Munkres [Mun60] and Edwin Moise [Moi77] that every topological manifold of dimension less than or equal to 3 has a smooth structure that is unique up to diffeomorphism. The analogous question in higher dimensions turns out to be quite deep, and is still largely unanswered. Even for Euclidean spaces, the question of uniqueness of smooth structures was not completely settled until late in the twentieth century. The answer is surprising: as long as $n \neq 4$, $\mathbb{R}^n$ has a unique smooth structure (up to diffeomorphism); but $\mathbb{R}^4$ has uncountably many distinct smooth structures, no two of which are diffeomorphic to each other! The existence of nonstandard smooth structures on $\mathbb{R}^4$ (called ***fake*** $\boldsymbol{\mathbb{R}^4}$***'s***) was first proved by Simon Donaldson and Michael Freedman in 1984 as a consequence of their work on the geometry and topology of compact 4-manifolds; the results are described in [DK90] and [FQ90].

For compact manifolds, the situation is even more fascinating. In 1956, John Milnor [Mil56] showed that there are smooth structures on $\mathbb{S}^7$ that are not diffeomorphic to the standard one. Later, he and Michel Kervaire [KM63] showed (using a deep theorem of Steve Smale [Sma62]) that there are exactly 15 diffeomorphism classes of such structures (or 28 classes if you restrict to diffeomorphisms that preserve a property called *orientation*, which will be discussed in Chapter 15).

On the other hand, in all dimensions greater than 3 there are compact topological manifolds that have no smooth structures at all. The problem of identifying the number of smooth structures (if any) on topological 4-manifolds is an active subject of current research.

Partitions of Unity

A frequently used tool in topology is the gluing lemma (Lemma A.20), which shows how to construct continuous maps by "gluing together" maps defined on open or closed subsets. We have a version of the gluing lemma for smooth maps defined on *open* subsets (Corollary 2.8), but we cannot expect to glue together smooth maps defined on *closed* subsets and obtain a smooth result. For example, the two functions $f_+: [0,\infty) \to \mathbb{R}$ and $f_-: (-\infty, 0] \to \mathbb{R}$ defined by

$$f_+(x) = +x, \quad x \in [0,\infty),$$
$$f_-(x) = -x, \quad x \in (-\infty, 0],$$

are both smooth and agree at the point 0 where they overlap, but the continuous function $f : \mathbb{R} \to \mathbb{R}$ that they define, namely $f(x) = |x|$, is not smooth at the origin.

A disadvantage of Corollary 2.8 is that in order to use it, we must construct maps that agree exactly on relatively large subsets of the manifold, which is too restrictive for some purposes. In this section we introduce *partitions of unity*, which are tools for "blending together" local smooth objects into global ones without necessarily assuming that they agree on overlaps. They are indispensable in smooth manifold theory and will reappear throughout the book.

All of our constructions in this section are based on the existence of smooth functions that are positive in a specified part of a manifold and identically zero in some other part. We begin by defining a smooth function on the real line that is zero for $t \le 0$ and positive for $t > 0$.

Lemma 2.20. *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

is smooth.

Proof. The function in question is pictured in Fig. 2.4. It is smooth on $\mathbb{R} \smallsetminus \{0\}$ by composition, so we need only show f has continuous derivatives of all orders at the origin. Because existence of the $(k+1)$st derivative implies continuity of the kth, it suffices to show that each such derivative exists. We begin by noting that f is continuous at 0 because $\lim_{t \searrow 0} e^{-1/t} = 0$. In fact, a standard application of l'Hôpital's rule and induction shows that for any integer $k \ge 0$,

$$\lim_{t \searrow 0} \frac{e^{-1/t}}{t^k} = \lim_{t \searrow 0} \frac{t^{-k}}{e^{1/t}} = 0. \tag{2.2}$$

We show by induction that for $t > 0$, the kth derivative of f is of the form

$$f^{(k)}(t) = p_k(t) \frac{e^{-1/t}}{t^{2k}} \tag{2.3}$$

for some polynomial p_k of degree at most k. This is clearly true (with $p_0(t) = 1$) for $k = 0$, so suppose it is true for some $k \ge 0$. By the product rule,

$$\begin{aligned} f^{(k+1)}(t) &= p_k'(t) \frac{e^{-1/t}}{t^{2k}} + p_k(t) \frac{t^{-2} e^{-1/t}}{t^{2k}} - 2k p_k(t) \frac{e^{-1/t}}{t^{2k+1}} \\ &= \left(t^2 p_k'(t) + p_k(t) - 2kt p_k(t) \right) \frac{e^{-1/t}}{t^{2(k+1)}}, \end{aligned}$$

which is of the required form.

Finally, we prove by induction that $f^{(k)}(0) = 0$ for each integer $k \ge 0$. For $k = 0$ this is true by definition, so assume that it is true for some $k \ge 0$. To prove that $f^{(k+1)}(0)$ exists, it suffices to show that $f^{(k)}$ has one-sided derivatives from both sides at $t = 0$ and that they are equal. Clearly, the derivative from the left is zero.

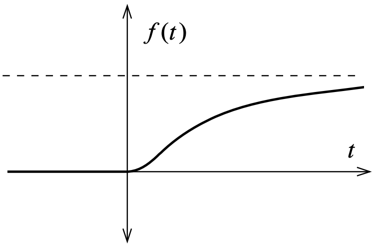

Fig. 2.4 $f(t) = e^{-1/t}$

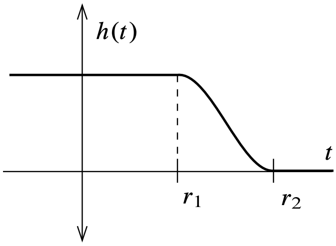

Fig. 2.5 A cutoff function

Using (2.3) and (2.2) again, we find that the derivative of $f^{(k)}$ from the right at $t = 0$ is equal to

$$\lim_{t \searrow 0} \frac{p_k(t)\dfrac{e^{-1/t}}{t^{2k}} - 0}{t} = \lim_{t \searrow 0} p_k(t)\frac{e^{-1/t}}{t^{2k+1}} = p_k(0) \lim_{t \searrow 0} \frac{e^{-1/t}}{t^{2k+1}} = 0.$$

Thus $f^{(k+1)}(0) = 0$. □

Lemma 2.21. *Given any real numbers r_1 and r_2 such that $r_1 < r_2$, there exists a smooth function $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(t) \equiv 1$ for $t \le r_1$, $0 < h(t) < 1$ for $r_1 < t < r_2$, and $h(t) \equiv 0$ for $t \ge r_2$.*

Proof. Let f be the function of the previous lemma, and set

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

(See Fig. 2.5.) Note that the denominator is positive for all t, because at least one of the expressions $r_2 - t$ and $t - r_1$ is always positive. The desired properties of h follow easily from those of f. □

A function with the properties of h in the preceding lemma is usually called a ***cutoff function***.

Lemma 2.22. *Given any positive real numbers $r_1 < r_2$, there is a smooth function $H\colon \mathbb{R}^n \to \mathbb{R}$ such that $H \equiv 1$ on $\overline{B}_{r_1}(0)$, $0 < H(x) < 1$ for all $x \in B_{r_2}(0) \smallsetminus \overline{B}_{r_1}(0)$, and $H \equiv 0$ on $\mathbb{R}^n \smallsetminus B_{r_2}(0)$.*

Proof. Just set $H(x) = h(|x|)$, where h is the function of the preceding lemma. Clearly, H is smooth on $\mathbb{R}^n \smallsetminus \{0\}$, because it is a composition of smooth functions there. Since it is identically equal to 1 on $B_{r_1}(0)$, it is smooth there too. □

The function H constructed in this lemma is an example of a ***smooth bump function***, a smooth real-valued function that is equal to 1 on a specified set and is zero outside a specified neighborhood of that set. Later in this chapter, we will generalize this notion to manifolds.

If f is any real-valued or vector-valued function on a topological space M, the ***support of f***, denoted by $\operatorname{supp} f$, is the closure of the set of points where f is nonzero:

$$\operatorname{supp} f = \overline{\{p \in M : f(p) \neq 0\}}.$$

(For example, if H is the function constructed in the preceding lemma, then $\operatorname{supp} H = \overline{B}_{r_2}(0)$.) If $\operatorname{supp} f$ is contained in some set $U \subseteq M$, we say that f is ***supported in U***. A function f is said to be ***compactly supported*** if $\operatorname{supp} f$ is a compact set. Clearly, every function on a compact space is compactly supported.

The next construction is the most important application of paracompactness. Suppose M is a topological space, and let $\mathcal{X} = (X_\alpha)_{\alpha\in A}$ be an arbitrary open cover of M, indexed by a set A. A ***partition of unity subordinate to $\mathcal{X}$*** is an indexed family $(\psi_\alpha)_{\alpha\in A}$ of continuous functions $\psi_\alpha\colon M \to \mathbb{R}$ with the following properties:

(i) $0 \le \psi_\alpha(x) \le 1$ for all $\alpha \in A$ and all $x \in M$.
(ii) $\operatorname{supp} \psi_\alpha \subseteq X_\alpha$ for each $\alpha \in A$.
(iii) The family of supports $(\operatorname{supp} \psi_\alpha)_{\alpha\in A}$ is locally finite, meaning that every point has a neighborhood that intersects $\operatorname{supp} \psi_\alpha$ for only finitely many values of α.
(iv) $\sum_{\alpha\in A} \psi_\alpha(x) = 1$ for all $x \in M$.

Because of the local finiteness condition (iii), the sum in (iv) actually has only finitely many nonzero terms in a neighborhood of each point, so there is no issue of convergence. If M is a smooth manifold with or without boundary, a ***smooth partition of unity*** is one for which each of the functions ψ_α is smooth.

Theorem 2.23 (Existence of Partitions of Unity). *Suppose M is a smooth manifold with or without boundary, and $\mathcal{X} = (X_\alpha)_{\alpha\in A}$ is any indexed open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.*

Proof. For simplicity, suppose for this proof that M is a smooth manifold without boundary; the general case is left as an exercise. Each set X_α is a smooth manifold in its own right, and thus has a basis $\mathcal{B}_\alpha$ of regular coordinate balls by Proposition 1.19, and it is easy to check that $\mathcal{B} = \bigcup_\alpha \mathcal{B}_\alpha$ is a basis for the topology of M. It follows from Theorem 1.15 that $\mathcal{X}$ has a countable, locally finite refinement $\{B_i\}$ consisting of elements of $\mathcal{B}$. By Lemma 1.13(a), the cover $\{\overline{B}_i\}$ is also locally finite.

For each i, the fact that B_i is a regular coordinate ball in some X_α guarantees that there is a coordinate ball $B_i' \subseteq X_\alpha$ such that $B_i' \supseteq \overline{B}_i$, and a smooth coordinate map $\varphi_i\colon B_i' \to \mathbb{R}^n$ such that $\varphi_i\big(\overline{B}_i\big) = \overline{B}_{r_i}(0)$ and $\varphi_i(B_i') = B_{r_i'}(0)$ for some $r_i < r_i'$. For each i, define a function $f_i\colon M \to \mathbb{R}$ by

$$f_i = \begin{cases} H_i \circ \varphi_i & \text{on } B_i', \\ 0 & \text{on } M \smallsetminus \overline{B}_i, \end{cases}$$

where $H_i\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function that is positive in $B_{r_i}(0)$ and zero elsewhere, as in Lemma 2.22. On the set $B_i' \smallsetminus \overline{B}_i$ where the two definitions overlap, both definitions yield the zero function, so f_i is well defined and smooth, and $\operatorname{supp} f_i = \overline{B}_i$.

Define $f\colon M \to \mathbb{R}$ by $f(x) = \sum_i f_i(x)$. Because of the local finiteness of the cover $\{\bar{B}_i\}$, this sum has only finitely many nonzero terms in a neighborhood of each point and thus defines a smooth function. Because each f_i is nonnegative everywhere and positive on B_i, and every point of M is in some B_i, it follows that $f(x) > 0$ everywhere on M. Thus, the functions $g_i\colon M \to \mathbb{R}$ defined by $g_i(x) = f_i(x)/f(x)$ are also smooth. It is immediate from the definition that $0 \le g_i \le 1$ and $\sum_i g_i \equiv 1$.

Finally, we need to reindex our functions so that they are indexed by the same set A as our open cover. Because the cover $\{B_i'\}$ is a refinement of $\mathcal{X}$, for each i we can choose some index $a(i) \in A$ such that $B_i' \subseteq X_{a(i)}$. For each $\alpha \in A$, define $\psi_\alpha\colon M \to \mathbb{R}$ by

$$\psi_\alpha = \sum_{i:a(i)=\alpha} g_i.$$

If there are no indices i for which $a(i) = \alpha$, then this sum should be interpreted as the zero function. It follows from Lemma 1.13(b) that

$$\operatorname{supp}\psi_\alpha = \overline{\bigcup_{i:a(i)=\alpha} B_i} = \bigcup_{i:a(i)=\alpha} \bar{B}_i \subseteq X_\alpha.$$

Each ψ_α is a smooth function that satisfies $0 \le \psi_\alpha \le 1$. Moreover, the family of supports $(\operatorname{supp}\psi_\alpha)_{\alpha\in A}$ is still locally finite, and $\sum_\alpha \psi_\alpha \equiv \sum_i g_i \equiv 1$, so this is the desired partition of unity. □

► **Exercise 2.24.** Show how the preceding proof needs to be modified for the case in which M has nonempty boundary.

There are basically two different strategies for patching together locally defined smooth maps to obtain a global one. If you can define a map in a neighborhood of each point in such a way that the locally defined maps all agree where they overlap, then the local definitions piece together to yield a global smooth map by Corollary 2.8. (This usually requires some sort of uniqueness result.) But if the local definitions are not guaranteed to agree, then you usually have to resort to a partition of unity. The trick then is showing that the patched-together objects still have the required properties. We use both strategies repeatedly throughout the book.

Applications of Partitions of Unity

As our first application of partitions of unity, we extend the notion of bump functions to arbitrary closed subsets of manifolds. If M is a topological space, $A \subseteq M$ is a closed subset, and $U \subseteq M$ is an open subset containing A, a continuous function $\psi\colon M \to \mathbb{R}$ is called a ***bump function for A supported in U*** if $0 \le \psi \le 1$ on M, $\psi \equiv 1$ on A, and $\operatorname{supp}\psi \subseteq U$.

Proposition 2.25 (Existence of Smooth Bump Functions). *Let M be a smooth manifold with or without boundary. For any closed subset $A \subseteq M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.*

Proof. Let $U_0 = U$ and $U_1 = M \smallsetminus A$, and let $\{\psi_0, \psi_1\}$ be a smooth partition of unity subordinate to the open cover $\{U_0, U_1\}$. Because $\psi_1 \equiv 0$ on A and thus $\psi_0 = \sum_i \psi_i = 1$ there, the function ψ_0 has the required properties. □

Our second application is an important result concerning the possibility of extending smooth functions from closed subsets. Suppose M and N are smooth manifolds with or without boundary, and $A \subseteq M$ is an arbitrary subset. We say that a map $F: A \to N$ is ***smooth on A*** if it has a smooth extension in a neighborhood of each point: that is, if for every $p \in A$ there is an open subset $W \subseteq M$ containing p and a smooth map $\widetilde{F}: W \to N$ whose restriction to $W \cap A$ agrees with F.

Lemma 2.26 (Extension Lemma for Smooth Functions). *Suppose M is a smooth manifold with or without boundary, $A \subseteq M$ is a closed subset, and $f: A \to \mathbb{R}^k$ is a smooth function. For any open subset U containing A, there exists a smooth function $\widetilde{f}: M \to \mathbb{R}^k$ such that $\widetilde{f}|_A = f$ and* $\operatorname{supp} \widetilde{f} \subseteq U$.

Proof. For each $p \in A$, choose a neighborhood W_p of p and a smooth function $\widetilde{f}_p: W_p \to \mathbb{R}^k$ that agrees with f on $W_p \cap A$. Replacing W_p by $W_p \cap U$, we may assume that $W_p \subseteq U$. The family of sets $\{W_p : p \in A\} \cup \{M \smallsetminus A\}$ is an open cover of M. Let $\{\psi_p : p \in A\} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subseteq W_p$ and $\operatorname{supp} \psi_0 \subseteq M \smallsetminus A$.

For each $p \in A$, the product $\psi_p \widetilde{f}_p$ is smooth on W_p, and has a smooth extension to all of M if we interpret it to be zero on $M \smallsetminus \operatorname{supp} \psi_p$. (The extended function is smooth because the two definitions agree on the open subset $W_p \smallsetminus \operatorname{supp} \psi_p$ where they overlap.) Thus we can define $\widetilde{f}: M \to \mathbb{R}^k$ by

$$\widetilde{f}(x) = \sum_{p \in A} \psi_p(x) \widetilde{f}_p(x).$$

Because the collection of supports $\{\operatorname{supp} \psi_p\}$ is locally finite, this sum actually has only a finite number of nonzero terms in a neighborhood of any point of M, and therefore defines a smooth function. If $x \in A$, then $\psi_0(x) = 0$ and $\widetilde{f}_p(x) = f(x)$ for each p such that $\psi_p(x) \neq 0$, so

$$\widetilde{f}(x) = \sum_{p \in A} \psi_p(x) f(x) = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x)\right) f(x) = f(x),$$

so $\widetilde{f}$ is indeed an extension of f. It follows from Lemma 1.13(b) that

$$\operatorname{supp} \widetilde{f} = \overline{\bigcup_{p \in A} \operatorname{supp} \psi_p} = \bigcup_{p \in A} \operatorname{supp} \psi_p \subseteq U.$$

□

► **Exercise 2.27.** Give a counterexample to show that the conclusion of the extension lemma can be false if A is not closed.

The assumption in the extension lemma that the codomain of f is $\mathbb{R}^k$, and not some other smooth manifold, is needed: for other codomains, extensions can fail to exist for topological reasons. (For example, the identity map $\mathbb{S}^1 \to \mathbb{S}^1$ is smooth, but does not have even a *continuous* extension to a map from $\mathbb{R}^2$ to $\mathbb{S}^1$.) Later we will show that a smooth map from a closed subset of a smooth manifold into a smooth manifold has a smooth extension if and only if it has a continuous one (see Corollary 6.27).

This extension lemma, by the way, illustrates an essential difference between smooth manifolds and real-analytic manifolds. The analogue of the extension lemma for real-analytic functions on real-analytic manifolds is decidedly false, because a real-analytic function that is defined on a connected domain and vanishes on an open subset must be identically zero.

Next, we use partitions of unity to construct a special kind of smooth function. If M is a topological space, an ***exhaustion function for M*** is a continuous function $f: M \to \mathbb{R}$ with the property that the set $f^{-1}\big((-\infty, c]\big)$ (called a ***sublevel set of f***) is compact for each $c \in \mathbb{R}$. The name comes from the fact that as n ranges over the positive integers, the sublevel sets $f^{-1}\big((-\infty, n]\big)$ form an exhaustion of M by compact sets; thus an exhaustion function provides a sort of continuous version of an exhaustion by compact sets. For example, the functions $f: \mathbb{R}^n \to \mathbb{R}$ and $g: \mathbb{B}^n \to \mathbb{R}$ given by

$$f(x) = |x|^2, \qquad g(x) = \frac{1}{1 - |x|^2}$$

are smooth exhaustion functions. Of course, if M is compact, any continuous real-valued function on M is an exhaustion function, so such functions are interesting only for noncompact manifolds.

Proposition 2.28 (Existence of Smooth Exhaustion Functions). *Every smooth manifold with or without boundary admits a smooth positive exhaustion function.*

Proof. Let M be a smooth manifold with or without boundary, let $\{V_j\}_{j=1}^\infty$ be any countable open cover of M by precompact open subsets, and let $\{\psi_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) = \sum_{j=1}^{\infty} j\,\psi_j(p).$$

Then f is smooth because only finitely many terms are nonzero in a neighborhood of any point, and positive because $f(p) \ge \sum_j \psi_j(p) = 1$.

To see that f is an exhaustion function, let $c \in \mathbb{R}$ be arbitrary, and choose a positive integer $N > c$. If $p \notin \bigcup_{j=1}^N \overline{V}_j$, then $\psi_j(p) = 0$ for $1 \le j \le N$, so

$$f(p) = \sum_{j=N+1}^{\infty} j\,\psi_j(p) \ge \sum_{j=N+1}^{\infty} N\psi_j(p) = N\sum_{j=1}^{\infty} \psi_j(p) = N > c.$$

Equivalently, if $f(p) \le c$, then $p \in \bigcup_{j=1}^N \overline{V}_j$. Thus $f^{-1}\big((-\infty, c]\big)$ is a closed subset of the compact set $\bigcup_{j=1}^N \overline{V}_j$ and is therefore compact. □

As our final application of partitions of unity, we will prove the remarkable fact that every closed subset of a manifold can be expressed as a level set of some smooth real-valued function. We will not use this result in this book (except in a few of the problems), but it provides an interesting contrast with the result of Example 1.32.

Theorem 2.29 (Level Sets of Smooth Functions). *Let M be a smooth manifold. If K is any closed subset of M, there is a smooth nonnegative function $f\colon M \to \mathbb{R}$ such that $f^{-1}(0) = K$.*

Proof. We begin with the special case in which $M = \mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ is a closed subset. For each $x \in M \smallsetminus K$, there is a positive number $r \le 1$ such that $B_r(x) \subseteq M \smallsetminus K$. By Proposition A.16, $M \smallsetminus K$ is the union of countably many such balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$.

Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function that is equal to 1 on $\bar{B}_{1/2}(0)$ and supported in $B_1(0)$. For each positive integer i, let $C_i \ge 1$ be a constant that bounds the absolute values of h and all of its partial derivatives up through order i. Define $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{\infty} \frac{(r_i)^i}{2^i C_i}\, h\left(\frac{x - x_i}{r_i}\right).$$

The terms of the series are bounded in absolute value by those of the convergent series $\sum_i 1/2^i$, so the entire series converges uniformly to a continuous function by the Weierstrass M-test. Because the ith term is positive exactly when $x \in B_{r_i}(x_i)$, it follows that f is zero in K and positive elsewhere.

It remains only to show that f is smooth. We have already shown that it is continuous, so suppose $k \ge 1$ and assume by induction that all partial derivatives of f of order less than k exist and are continuous. By the chain rule and induction, every kth partial derivative of the ith term in the series can be written in the form

$$\frac{(r_i)^{i-k}}{2^i C_i}\, D_k h\left(\frac{x - x_i}{r_i}\right),$$

where $D_k h$ is some kth partial derivative of h. By our choices of r_i and C_i, as soon as $i \ge k$, each of these terms is bounded in absolute value by $1/2^i$, so the differentiated series also converges uniformly to a continuous function. It then follows from Theorem C.31 that the kth partial derivatives of f exist and are continuous. This completes the induction, and shows that f is smooth.

Now let M be an arbitrary smooth manifold, and $K \subseteq M$ be any closed subset. Let $\{B_\alpha\}$ be an open cover of M by smooth coordinate balls, and let $\{\psi_\alpha\}$ be a subordinate partition of unity. Since each B_α is diffeomorphic to $\mathbb{R}^n$, the preceding argument shows that for each α there is a smooth nonnegative function $f_\alpha\colon B_\alpha \to \mathbb{R}$ such that $f_\alpha^{-1}(0) = B_\alpha \cap K$. The function $f = \sum_\alpha \psi_\alpha f_\alpha$ does the trick. □

Problems

2-1. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi\big(U \cap f^{-1}(V)\big)$ to $\psi(V)$, but f is not smooth in the sense we have defined in this chapter.

2-2. Prove Proposition 2.12 (smoothness of maps into product manifolds).

2-3. For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.
(a) $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ is the ***nth power map*** for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.
(b) $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ is the ***antipodal map*** $\alpha(x) = -x$.
(c) $F\colon \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(w,z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$, where we think of $\mathbb{S}^3$ as the subset $\big\{(w,z) : |w|^2 + |z|^2 = 1\big\}$ of $\mathbb{C}^2$.

2-4. Show that the inclusion map $\overline{\mathbb{B}}^n \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{\mathbb{B}}^n$ is regarded as a smooth manifold with boundary.

2-5. Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\widetilde{\mathbb{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.
(a) Show that f is also smooth as a map from $\mathbb{R}$ to $\widetilde{\mathbb{R}}$.
(b) Show that f is smooth as a map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

2-6. Let $P\colon \mathbb{R}^{n+1} \smallsetminus \{0\} \to \mathbb{R}^{k+1} \smallsetminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \smallsetminus \{0\}$ and $x \in \mathbb{R}^{n+1} \smallsetminus \{0\}$. (Such a function is said to be ***homogeneous of degree d***.) Show that the map $\widetilde{P}\colon \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\widetilde{P}([x]) = [P(x)]$ is well defined and smooth.

2-7. Let M be a nonempty smooth n-manifold with or without boundary, and suppose $n \geq 1$. Show that the vector space $C^\infty(M)$ is infinite-dimensional. [Hint: show that if $f_1, \dots, f_k$ are elements of $C^\infty(M)$ with nonempty disjoint supports, then they are linearly independent.]

2-8. Define $F\colon \mathbb{R}^n \to \mathbb{RP}^n$ by $F\big(x^1, \dots, x^n\big) = \big[x^1, \dots, x^n, 1\big]$. Show that F is a diffeomorphism onto a dense open subset of $\mathbb{RP}^n$. Do the same for $G\colon \mathbb{C}^n \to \mathbb{CP}^n$ defined by $G\big(z^1, \dots, z^n\big) = \big[z^1, \dots, z^n, 1\big]$ (see Problem 1-9).

2-9. Given a polynomial p in one variable with complex coefficients, not identically zero, show that there is a unique smooth map $\widetilde{p}\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ that makes the following diagram commute, where $\mathbb{CP}^1$ is 1-dimensional complex projective space and $G\colon \mathbb{C} \to \mathbb{CP}^1$ is the map of Problem 2-8:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{CP}^1 \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{p}} \\ \mathbb{C} & \xrightarrow[G]{} & \mathbb{CP}^1. \end{array}$$

(*Used on p. 465.*)

2-10. For any topological space M, let $C(M)$ denote the algebra of continuous functions $f\colon M \to \mathbb{R}$. Given a continuous map $F\colon M \to N$, define $F^*\colon C(N) \to C(M)$ by $F^*(f) = f \circ F$.

(a) Show that F^* is a linear map.

(b) Suppose M and N are smooth manifolds. Show that $F\colon M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

(c) Suppose $F\colon M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

[Remark: this result shows that in a certain sense, the entire smooth structure of M is encoded in the subset $C^\infty(M) \subseteq C(M)$. In fact, some authors *define* a smooth structure on a topological manifold M to be a subalgebra of $C(M)$ with certain properties; see, e.g., [Nes03].] (*Used on p. 75.*)

2-11. Suppose V is a real vector space of dimension $n \geq 1$. Define the ***projectivization of V***, denoted by $\mathbb{P}(V)$, to be the set of 1-dimensional linear subspaces of V, with the quotient topology induced by the map $\pi\colon V \smallsetminus \{0\} \to \mathbb{P}(V)$ that sends x to its span. (Thus $\mathbb{P}(\mathbb{R}^n) = \mathbb{RP}^{n-1}$.) Show that $\mathbb{P}(V)$ is a topological $(n-1)$-manifold, and has a unique smooth structure with the property that for each basis $(E_1, \dots, E_n)$ for V, the map $E\colon \mathbb{RP}^{n-1} \to \mathbb{P}(V)$ defined by $E\left[v^1, \dots, v^n\right] = \left[v^i E_i\right]$ (where brackets denote equivalence classes) is a diffeomorphism. (*Used on p. 561.*)

2-12. State and prove an analogue of Problem 2-11 for complex vector spaces.

2-13. Suppose M is a topological space with the property that for every indexed open cover $\mathcal{X}$ of M, there exists a partition of unity subordinate to $\mathcal{X}$. Show that M is paracompact.

2-14. Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \leq f(x) \leq 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.