Chapter 5
Submanifolds

Many familiar manifolds appear naturally as subsets of other manifolds. We have already seen that open subsets of smooth manifolds can be viewed as smooth manifolds in their own right; but there are many interesting examples beyond the open ones. In this chapter we explore *smooth submanifolds*, which are smooth manifolds that are subsets of other smooth manifolds. As you will soon discover, the situation is quite a bit more subtle than the analogous theory of topological subspaces.

We begin by defining the most important type of smooth submanifolds, called *embedded submanifolds*. These have the subspace topology inherited from their containing manifold, and turn out to be exactly the images of smooth embeddings. As we will see in this chapter, they are modeled locally on linear subspaces of Euclidean spaces. Because embedded submanifolds are most often presented as level sets of smooth maps, we devote some time to analyzing the conditions under which level sets are smooth submanifolds. We will see, for example, that level sets of constant-rank maps (in particular, smooth submersions) are always embedded submanifolds.

Next, we introduce a more general kind of submanifolds, called *immersed submanifolds*, which turn out to be the images of injective immersions. An immersed submanifold looks locally like an embedded one, but globally it may have a topology that is different from the subspace topology.

After introducing these basic concepts, we address two crucial technical questions about submanifolds: When is it possible to restrict the domain or codomain of a smooth map to a smooth submanifold and still retain smoothness? How can we identify the tangent space to a smooth submanifold as a subspace of the tangent space of its ambient manifold? Then we show how the theory of submanifolds can be generalized to the case of submanifolds with boundary.

Embedded Submanifolds

Suppose M is a smooth manifold with or without boundary. An ***embedded submanifold of*** M is a subset $S \subseteq M$ that is a manifold (without boundary) in the subspace topology, endowed with a smooth structure with respect to which the inclusion map

$S \hookrightarrow M$ is a smooth embedding. Embedded submanifolds are also called ***regular submanifolds*** by some authors.

If S is an embedded submanifold of M, the difference $\dim M - \dim S$ is called the **codimension of S in M**, and the containing manifold M is called the **ambient manifold** for S. An **embedded hypersurface** is an embedded submanifold of codimension 1. The empty set is an embedded submanifold of any dimension.

The easiest embedded submanifolds to understand are those of codimension 0. Recall that in Example 1.26, for any smooth manifold M we defined an *open submanifold of M* to be any open subset with the subspace topology and with the smooth charts obtained by restricting those of M.

Proposition 5.1 (Open Submanifolds). *Suppose M is a smooth manifold. The embedded submanifolds of codimension* 0 *in M are exactly the open submanifolds.*

Proof. Suppose $U \subseteq M$ is an open submanifold, and let $\iota: U \hookrightarrow M$ be the inclusion map. Example 1.26 showed that U is a smooth manifold of the same dimension as M, so it has codimension 0. In terms of the smooth charts for U constructed in Example 1.26, ι is represented in coordinates by an identity map, so it is a smooth immersion; and because U has the subspace topology, ι is a smooth embedding. Thus U is an embedded submanifold. Conversely, suppose U is any codimension-0 embedded submanifold of M. Then inclusion $\iota: U \hookrightarrow M$ is a smooth embedding by definition, and therefore it is a local diffeomorphism by Proposition 4.8, and an open map by Proposition 4.6. Thus U is an open subset of M. $\qquad\square$

The next few propositions demonstrate several other ways to produce embedded submanifolds.

Proposition 5.2 (Images of Embeddings as Submanifolds). *Suppose M is a smooth manifold with or without boundary, N is a smooth manifold, and $F: N \to M$ is a smooth embedding. Let $S = F(N)$. With the subspace topology, S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of M with the property that F is a diffeomorphism onto its image.*

Proof. If we give S the subspace topology that it inherits from M, then the assumption that F is an embedding means that F can be considered as a homeomorphism from N onto S, and thus S is a topological manifold. We give S a smooth structure by taking the smooth charts to be those of the form $\left(F(U), \varphi \circ F^{-1}\right)$, where (U, φ) is any smooth chart for N; smooth compatibility of these charts follows immediately from the smooth compatibility of the corresponding charts for N. With this smooth structure on S, the map F is a diffeomorphism onto its image (essentially by definition), and this is obviously the only smooth structure with this property. The inclusion map $S \hookrightarrow M$ is equal to the composition of a diffeomorphism followed by a smooth embedding:

$$ S \xrightarrow{\ F^{-1}\ } N \xrightarrow{\ F\ } M, \tag{5.1} $$

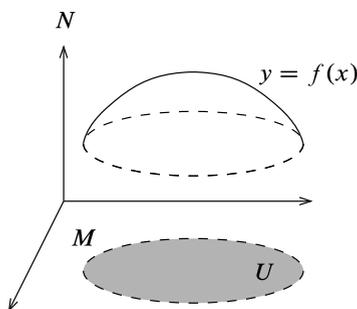and therefore it is a smooth embedding. $\qquad\square$

Fig. 5.1 A graph is an embedded submanifold

Since every embedded submanifold is the image of a smooth embedding (namely its own inclusion map), the previous proposition shows that embedded submanifolds are exactly the images of smooth embeddings.

Proposition 5.3 (Slices of Product Manifolds). *Suppose M and N are smooth manifolds. For each $p \in N$, the subset $M \times \{p\}$ (called a **slice** of the product manifold) is an embedded submanifold of $M \times N$ diffeomorphic to M.*

Proof. The set $M \times \{p\}$ is the image of the smooth embedding $x \mapsto (x, p)$.   □

Proposition 5.4 (Graphs as Submanifolds). *Suppose M is a smooth m-manifold (without boundary), N is a smooth n-manifold with or without boundary, $U \subseteq M$ is open, and $f : U \to N$ is a smooth map. Let $\Gamma(f) \subseteq M \times N$ denote the graph of f:*

$$\Gamma(f) = \big\{(x, y) \in M \times N : x \in U, \ y = f(x)\big\}.$$

Then $\Gamma(f)$ is an embedded m-dimensional submanifold of $M \times N$ (see Fig. 5.1).

Proof. Define a map $\gamma_f : U \to M \times N$ by

$$\gamma_f(x) = \big(x, f(x)\big). \tag{5.2}$$

It is a smooth map whose image is $\Gamma(f)$. Because the projection $\pi_M : M \times N \to M$ satisfies $\pi_M \circ \gamma_f(x) = x$ for $x \in U$, the composition $d(\pi_M)_{(x, f(x))} \circ d(\gamma_f)_x$ is the identity on $T_x M$ for each $x \in U$. Thus, $d(\gamma_f)_x$ is injective, so γ_f is a smooth immersion. It a homeomorphism onto its image because $\pi_M|_{\Gamma(f)}$ is a continuous inverse for it. Thus, $\Gamma(f)$ is an embedded submanifold diffeomorphic to U.   □

For some purposes, merely being an embedded submanifold is not quite a strong enough condition. (See, e.g., Lemma 5.34 below.) An embedded submanifold $S \subseteq M$ is said to be ***properly embedded*** if the inclusion $S \hookrightarrow M$ is a proper map.

Proposition 5.5. *Suppose M is a smooth manifold with or without boundary and $S \subseteq M$ is an embedded submanifold. Then S is properly embedded if and only if it is a closed subset of M.*

Proof. If S is properly embedded, then it is closed by Theorem A.57. Conversely, if S is closed in M, then Proposition A.53(c) shows that the inclusion map $S \hookrightarrow M$ is proper. ☐

Corollary 5.6. *Every compact embedded submanifold is properly embedded.*

Proof. Compact subsets of Hausdorff spaces are closed. ☐

Graphs of globally defined functions are common examples of properly embedded submanifolds.

Proposition 5.7 (Global Graphs Are Properly Embedded). *Suppose M is a smooth manifold, N is a smooth manifold with or without boundary, and $f : M \to N$ is a smooth map. With the smooth manifold structure of Proposition 5.4, $\Gamma(f)$ is properly embedded in $M \times N$.*

Proof. In this case, the projection $\pi_M : M \times N \to M$ is a smooth left inverse for the embedding $\gamma_f : M \to M \times N$ defined by (5.2). Thus γ_f is proper by Proposition A.53. ☐

Slice Charts for Embedded Submanifolds

As our next theorem will show, embedded submanifolds are modeled locally on the standard embedding of $\mathbb{R}^k$ into $\mathbb{R}^n$, identifying $\mathbb{R}^k$ with the subspace

$$\left\{ \left(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n \right) : x^{k+1} = \cdots = x^n = 0 \right\} \subseteq \mathbb{R}^n.$$

Somewhat more generally, if U is an open subset of $\mathbb{R}^n$ and $k \in \{0, \ldots, n\}$, a *k-dimensional slice of U* (or simply a *k-slice*) is any subset of the form

$$S = \left\{ \left(x^1, \ldots, x^k, x^{k+1}, \ldots, x^n \right) \in U : x^{k+1} = c^{k+1}, \ldots, x^n = c^n \right\}$$

for some constants $c^{k+1}, \ldots, c^n$. (When $k = n$, this just means $S = U$.) Clearly, every k-slice is homeomorphic to an open subset of $\mathbb{R}^k$. (Sometimes it is convenient to consider slices defined by setting some subset of the coordinates other than the last ones equal to constants. The meaning should be clear from the context.)

Let M be a smooth n-manifold, and let (U, φ) be a smooth chart on M. If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U)$, then we say that *S is a k-slice of U*. (Although in general we allow our slices to be defined by arbitrary constants $c^{k+1}, \ldots, c^n$, it is sometimes useful to have slice coordinates for which the constants are all zero, which can easily be achieved by subtracting a constant from each coordinate function.) Given a subset $S \subseteq M$ and a nonnegative integer k, we say that S satisfies the *local k-slice condition* if each point of S is contained in the domain of a smooth chart (U, φ) for M such that $S \cap U$ is a single k-slice in U. Any such chart is called a *slice chart for S in M*, and the corresponding coordinates $\left(x^1, \ldots, x^n \right)$ are called *slice coordinates*.

Theorem 5.8 (Local Slice Criterion for Embedded Submanifolds). *Let M be a smooth n-manifold. If $S \subseteq M$ is an embedded k-dimensional submanifold, then S*
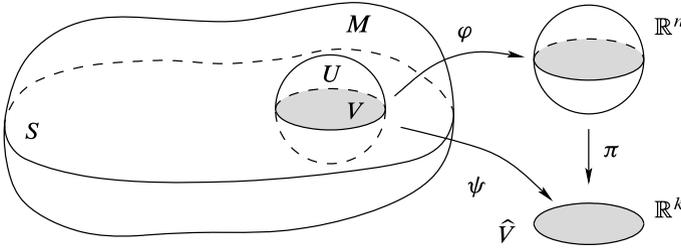
Fig. 5.2 A chart for a subset satisfying the k-slice condition

satisfies the local k-slice condition. Conversely, if $S \subseteq M$ is a subset that satisfies the local k-slice condition, then with the subspace topology, S is a topological manifold of dimension k, and it has a smooth structure making it into a k-dimensional embedded submanifold of M.

Proof. First suppose that $S \subseteq M$ is an embedded k-dimensional submanifold. Since the inclusion map $S \hookrightarrow M$ is an immersion, the rank theorem shows that for any $p \in S$ there are smooth charts (U, φ) for S (in its given smooth manifold structure) and (V, ψ) for M, both centered at p, in which the inclusion map $\iota|_U : U \to V$ has the coordinate representation

$$\left(x^1, \ldots, x^k\right) \mapsto \left(x^1, \ldots, x^k, 0, \ldots, 0\right).$$

Choose $\varepsilon > 0$ small enough that both U and V contain coordinate balls of radius ε centered at p, and denote these coordinate balls by $U_0 \subseteq U$ and $V_0 \subseteq V$. It follows that $U_0 = \iota(U_0)$ is exactly a single slice in V_0. Because S has the subspace topology, the fact that U_0 is open in S means that there is an open subset $W \subseteq M$ such that $U_0 = W \cap S$. Setting $V_1 = V_0 \cap W$, we obtain a smooth chart $\left(V_1, \psi|_{V_1}\right)$ for M containing p such that $V_1 \cap S = U_0$, which is a single slice of V_1.

Conversely, suppose S satisfies the local k-slice condition. With the subspace topology, S is Hausdorff and second-countable, because both properties are inherited by subspaces. To see that S is locally Euclidean, we construct an atlas. The basic idea of the construction is that if $\left(x^1, \ldots, x^n\right)$ are slice coordinates for S in M, we can use $\left(x^1, \ldots, x^k\right)$ as local coordinates for S.

For this proof, let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ denote the projection onto the first k coordinates. Let (U, φ) be any slice chart for S in M (Fig. 5.2), and define

$$V = U \cap S, \qquad \widehat{V} = \pi \circ \varphi(V), \qquad \psi = \pi \circ \varphi|_V : V \to \widehat{V}.$$

By definition of slice charts, $\varphi(V)$ is the intersection of $\varphi(U)$ with a certain k-slice $A \subseteq \mathbb{R}^n$ defined by setting $x^{k+1} = c^{k+1}, \ldots, x^n = c^n$, and therefore $\varphi(V)$ is open in A. Since $\pi|_A$ is a diffeomorphism from A to $\mathbb{R}^k$, it follows that $\widehat{V}$ is open in $\mathbb{R}^k$. Moreover, ψ is a homeomorphism because it has a continuous inverse given by $\varphi^{-1} \circ j|_{\widehat{V}}$, where $j : \mathbb{R}^k \to \mathbb{R}^n$ is the map

$$j\left(x^1, \ldots, x^k\right) = \left(x^1, \ldots, x^k, c^{k+1}, \ldots, c^n\right).$$
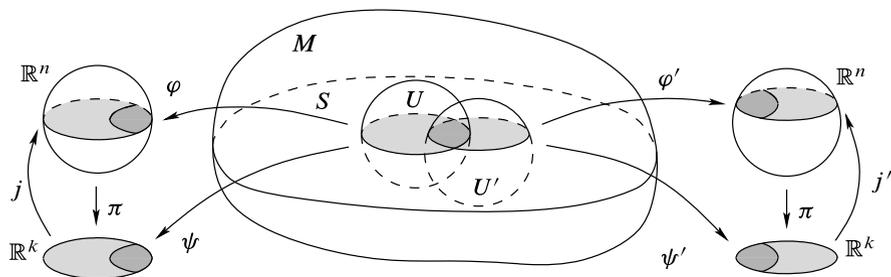
Fig. 5.3 Smooth compatibility of slice charts

Thus S is a topological k-manifold, and the inclusion map $\iota\colon S \hookrightarrow M$ is a topological embedding.

To put a smooth structure on S, we need to verify that the charts constructed above are smoothly compatible. Suppose (U, φ) and (U', φ') are two slice charts for S in M, and let (V, ψ), (V', ψ') be the corresponding charts for S. The transition map is given by $\psi' \circ \psi^{-1} = \pi \circ \varphi' \circ \varphi^{-1} \circ j$, which is a composition of four smooth maps (Fig. 5.3). Thus the atlas we have constructed is in fact a smooth atlas, and it defines a smooth structure on S. In terms of a slice chart (U, φ) for M and the corresponding chart (V, ψ) for S, the inclusion map $S \hookrightarrow M$ has a coordinate representation of the form

$$\left(x^1, \dots, x^k\right) \mapsto \left(x^1, \dots, x^k, c^{k+1}, \dots, c^n\right),$$

which is a smooth immersion. Since the inclusion is a smooth immersion and a topological embedding, S is an embedded submanifold. □

Notice that the local slice condition for $S \subseteq M$ is a condition on the *subset S* only; it does not presuppose any particular topology or smooth structure on S. As we will see later (Theorem 5.31), the smooth manifold structure constructed in the preceding theorem is the *unique* one in which S can be considered as a submanifold, so a subset satisfying the local slice condition is an embedded submanifold in only one way.

Example 5.9 (Spheres as Submanifolds). For any $n \geq 0$, $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$, because it is locally the graph of a smooth function: as we showed in Example 1.4, the intersection of $\mathbb{S}^n$ with the open subset $\{x : x^i > 0\}$ is the graph of the smooth function

$$x^i = f\left(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{n+1}\right),$$

where $f\colon \mathbb{B}^n \to \mathbb{R}$ is given by $f(u) = \sqrt{1 - |u|^2}$. Similarly, the intersection of $\mathbb{S}^n$ with $\{x : x^i < 0\}$ is the graph of $-f$. Since every point in $\mathbb{S}^n$ is in one of these sets, $\mathbb{S}^n$ satisfies the local n-slice condition and is thus an embedded submanifold of $\mathbb{R}^{n+1}$. The smooth structure thus induced on $\mathbb{S}^n$ is the same as the one we defined in Chapter 1: in fact, the coordinates for $\mathbb{S}^n$ determined by these slice charts are exactly the graph coordinates defined in Example 1.31. //
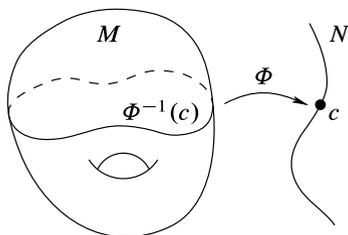
Fig. 5.4 A level set

▶ **Exercise 5.10.** Show that spherical coordinates (Example C.38) form a slice chart for $\mathbb{S}^2$ in $\mathbb{R}^3$ on any open subset where they are defined.

If M is a smooth manifold with nonempty boundary and $S \subseteq M$ is an embedded submanifold, then S might intersect ∂M in very complicated ways, so we will not attempt to prove any general results about the existence of slice charts for S in M in that case. However, in the special case in which the submanifold is the boundary of M itself, the boundary charts for M play the role of slice charts for ∂M in M, and we do have the following result.

Theorem 5.11. *If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-dimensional manifold (without boundary), and has a smooth structure such that it is a properly embedded submanifold of M.*

Proof. See Problem 5-2.   □

We will see later that the smooth structure on ∂M is unique. (See Theorem 5.31.)

In order to analyze more general submanifolds of M when M has a boundary, the most effective technique is often to find an embedding of M into a larger smooth manifold $\widetilde{M}$ without boundary, and apply the preceding results in that context. Example 9.32 will show that every smooth manifold with boundary can be embedded in such a way.

Level Sets

In practice, embedded submanifolds are most often presented as solution sets of equations or systems of equations. Extending the terminology we introduced in Example 1.32, if $\Phi \colon M \to N$ is any map and c is any point of N, we call the set $\Phi^{-1}(c)$ a ***level set of Φ*** (Fig. 5.4). (In the special case $N = \mathbb{R}^k$ and $c = 0$, the level set $\Phi^{-1}(0)$ is usually called the ***zero set of Φ***.)

It is easy to find level sets of smooth functions that are not smooth submanifolds. For instance, consider the three smooth functions $\Theta, \Phi, \Psi \colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\Theta(x, y) = x^2 - y, \qquad \Phi(x, y) = x^2 - y^2, \qquad \Psi(x, y) = x^2 - y^3.$$

(See Fig. 5.5.) Although the zero set of Θ (a parabola) is an embedded submanifold of $\mathbb{R}^2$ because it is the graph of the smooth function $f(x) = x^2$, Problem 5-11 asks

you to show that neither the zero set of Φ nor that of Ψ is an embedded submanifold. In fact, without further assumptions on the smooth function, the situation is about as bad as could be imagined: as Theorem 2.29 showed, *every* closed subset of M can be expressed as the zero set of some smooth real-valued function.

The argument we used in Example 1.32 (based on the implicit function theorem) to show that certain level sets in $\mathbb{R}^n$ are smooth manifolds can be adapted to show that those level sets are in fact embedded submanifolds of $\mathbb{R}^n$. But using the rank theorem, we can prove something much stronger.

Theorem 5.12 (Constant-Rank Level Set Theorem). *Let M and N be smooth manifolds, and let $\Phi\colon M \to N$ be a smooth map with constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M.*

Proof. Write $m = \dim M$, $n = \dim N$, and $k = m - r$. Let $c \in N$ be arbitrary, and let S denote the level set $\Phi^{-1}(c) \subseteq M$. From the rank theorem, for each $p \in S$ there are smooth charts (U, φ) centered at p and (V, ψ) centered at $c = \Phi(p)$ in which Φ has a coordinate representation of the form (4.1), and therefore $S \cap U$ is the slice

$$\left\{\left(x^1, \dots, x^r, x^{r+1}, \dots, x^m\right) \in U : x^1 = \cdots = x^r = 0\right\}.$$

Thus S satisfies the local k-slice condition, so it is an embedded submanifold of dimension k. It is closed in M by continuity, so it is properly embedded by Proposition 5.5. $\qquad\square$

Corollary 5.13 (Submersion Level Set Theorem). *If M and N are smooth manifolds and $\Phi\colon M \to N$ is a smooth submersion, then each level set of Φ is a properly embedded submanifold whose codimension is equal to the dimension of N.*

Proof. Every smooth submersion has constant rank equal to the dimension of its codomain. $\qquad\square$

This result should be compared to the corresponding result in linear algebra: if $L\colon \mathbb{R}^m \to \mathbb{R}^r$ is a surjective linear map, then the kernel of L is a linear subspace of codimension r by the rank-nullity law. The vector equation $Lx = 0$ is equivalent to r linearly independent scalar equations, each of which can be thought of as cutting down one of the degrees of freedom in $\mathbb{R}^m$, leaving a subspace of codimension r. In the context of smooth manifolds, the analogue of a surjective linear map is a smooth submersion, each of whose (local) component functions cuts down the dimension by one.

Corollary 5.13 can be strengthened considerably, because we need only check the submersion condition on the level set we are interested in. If $\Phi\colon M \to N$ is a smooth map, a point $p \in M$ is said to be a ***regular point of Φ*** if $d\Phi_p\colon T_pM \to T_{\Phi(p)}N$ is surjective; it is a ***critical point of Φ*** otherwise. This means, in particular, that every point of M is critical if $\dim M < \dim N$, and every point is regular if and only if F is a submersion. Note that the set of regular points of Φ is always an open subset of M by Proposition 4.1. A point $c \in N$ is said to be a ***regular value of Φ*** if every point of the level set $\Phi^{-1}(c)$ is a regular point, and a ***critical value*** otherwise. In particular, if $\Phi^{-1}(c) = \varnothing$, then c is a regular value. Finally, a level set $\Phi^{-1}(c)$ is

$$x^2 - y = 0 \qquad\qquad x^2 - y^2 = 0 \qquad\qquad x^2 - y^3 = 0$$
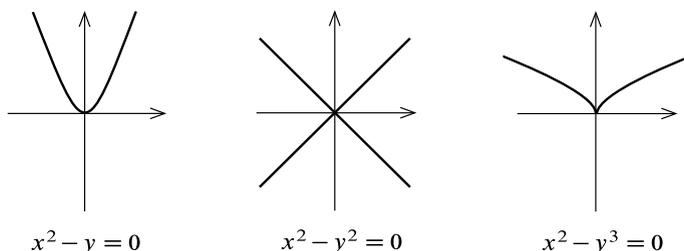
Fig. 5.5   Level sets may or may not be embedded submanifolds

called a **_regular level set_** if c is a regular value of Φ; in other words, a regular level set is a level set consisting entirely of regular points of Φ (points p such that $d\Phi_p$ is surjective).

Corollary 5.14 (Regular Level Set Theorem). *Every regular level set of a smooth map between smooth manifolds is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.*

Proof. Let $\Phi\colon M \to N$ be a smooth map and let $c \in N$ be a regular value. The set U of points $p \in M$ where rank $d\Phi_p = \dim N$ is open in M by Proposition 4.1, and contains $\Phi^{-1}(c)$ because of the assumption that c is a regular value. It follows that $\Phi|_U\colon U \to N$ is a smooth submersion, and the preceding corollary shows that $\Phi^{-1}(c)$ is an embedded submanifold of U. Since the composition of smooth embeddings $\Phi^{-1}(c) \hookrightarrow U \hookrightarrow M$ is again a smooth embedding, it follows that $\Phi^{-1}(c)$ is an embedded submanifold of M, and it is closed by continuity. $\square$

It is worth noting that the previous corollary also applies to empty level sets, which are both regular level sets and properly embedded submanifolds.

Example 5.15 (Spheres). Now we can give a much easier proof that $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$. The sphere is a regular level set of the smooth function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ given by $f(x) = |x|^2$, since $df_x(v) = 2\sum_i x^i v^i$, which is surjective except at the origin.   ⫽

Not all embedded submanifolds can be expressed as level sets of smooth submersions. However, the next proposition shows that every embedded submanifold is at least locally of this form.

Proposition 5.16. *Let S be a subset of a smooth m-manifold M. Then S is an embedded k-submanifold of M if and only if every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi\colon U \to \mathbb{R}^{m-k}$.*

Proof. First suppose S is an embedded k-submanifold. If $\left(x^1,\ldots,x^m\right)$ are slice coordinates for S on an open subset $U \subseteq M$, the map $\Phi\colon U \to \mathbb{R}^{m-k}$ given in coordinates by $\Phi(x) = \left(x^{k+1},\ldots,x^m\right)$ is easily seen to be a smooth submersion, one of whose level sets is $S \cap U$ (Fig. 5.6). Conversely, suppose that around every point $p \in S$ there is a neighborhood U and a smooth submersion $\Phi\colon U \to \mathbb{R}^{m-k}$
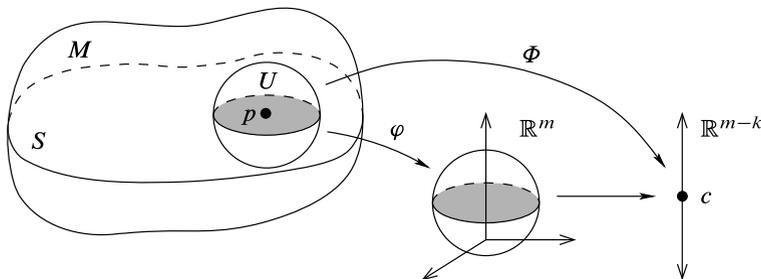
Fig. 5.6 An embedded submanifold is locally a level set

such that $S \cap U$ is a level set of Φ. By the submersion level set theorem, $S \cap U$ is an embedded submanifold of U, so it satisfies the local slice condition; it follows that S is itself an embedded submanifold of M. $\qquad\square$

If $S \subseteq M$ is an embedded submanifold, a smooth map $\Phi\colon M \to N$ such that S is a regular level set of Φ is called a ***defining map for*** S. In the special case $N = \mathbb{R}^{m-k}$ (so that Φ is a real-valued or vector-valued function), it is usually called a ***defining function***. Example 5.15 shows that $f(x) = |x|^2$ is a defining function for the sphere. More generally, if U is an open subset of M and $\Phi\colon U \to N$ is a smooth map such that $S \cap U$ is a regular level set of Φ, then Φ is called a ***local defining map*** (or ***local defining function***) ***for*** S. Proposition 5.16 says that every embedded submanifold admits a local defining function in a neighborhood of each of its points.

In specific examples, finding a (local or global) defining function for a submanifold is usually just a matter of using geometric information about how the submanifold is defined together with some computational ingenuity. Here is an example.

Example 5.17 (Surfaces of Revolution). Let H be the half-plane $\{(r, z) : r > 0\}$, and suppose $C \subseteq H$ is an embedded 1-dimensional submanifold. The ***surface of revolution*** determined by C is the subset $S_C \subseteq \mathbb{R}^3$ given by

$$S_C = \left\{ (x, y, z) : \left(\sqrt{x^2 + y^2}, z \right) \in C \right\}.$$

The set C is called its ***generating curve*** (see Fig. 5.7). If $\varphi\colon U \to \mathbb{R}$ is any local defining function for C in H, we get a local defining function Φ for S_C by

$$\Phi(x, y, z) = \varphi\left(\sqrt{x^2 + y^2}, z \right),$$

defined on the open subset

$$\tilde{U} = \left\{ (x, y, z) : \left(\sqrt{x^2 + y^2}, z \right) \in U \right\} \subseteq \mathbb{R}^3.$$

A computation shows that the Jacobian matrix of Φ is

$$D\Phi(x, y, z) = \left(\frac{x}{r} \frac{\partial \varphi}{\partial r}(r, z) \quad \frac{y}{r} \frac{\partial \varphi}{\partial r}(r, z) \quad \frac{\partial \varphi}{\partial z}(r, z) \right),$$
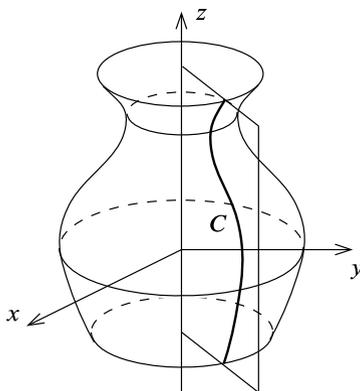
Fig. 5.7   A surface of revolution

where we have written $r = \sqrt{x^2 + y^2}$. At any point $(x, y, z) \in S_C$, at least one of the components of $D\Phi(x, y, z)$ is nonzero, so S_C is a regular level set of Φ and is thus an embedded 2-dimensional submanifold of $\mathbb{R}^3$.

For a specific example, the doughnut-shaped torus of revolution D described in Example 4.2(d) is the surface of revolution obtained from the circle $(r - 2)^2 + z^2 = 1$. It is a regular level set of the function $\Phi(x, y, z) = \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2$, which is smooth on $\mathbb{R}^3$ minus the z-axis.   //

Immersed Submanifolds

Although embedded submanifolds are the most natural and common submanifolds and suffice for most purposes, it is sometimes important to consider a more general notion of submanifold. In particular, when we study Lie subgroups in Chapter 7 and foliations in Chapter 19, we will encounter subsets of smooth manifolds that are images of injective immersions, but not necessarily of embeddings. To see some of the kinds of phenomena that occur, look back at the two examples we introduced in Chapter 4 of sets that are images of injective immersions that are not embeddings: the figure-eight curve of Example 4.19 and the dense curve on the torus of Example 4.20. Neither of these sets is an embedded submanifold (see Problems 5-4 and 5-5).

So as to have a convenient language for talking about examples like these, we introduce the following definition. Let M be a smooth manifold with or without boundary. An ***immersed submanifold of*** M is a subset $S \subseteq M$ endowed with a topology (not necessarily the subspace topology) with respect to which it is a topological manifold (without boundary), and a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth immersion. As for embedded submanifolds, we define the ***codimension of*** S ***in*** M to be $\dim M - \dim S$.

Every embedded submanifold is also an immersed submanifold. Because immersed submanifolds are the more general of the two types of submanifolds, we

adopt the convention that the term **smooth submanifold** without further qualifica-
tion means an immersed one, which includes an embedded submanifold as a special
case. Similarly, the term **smooth hypersurface** without qualification means an im-
mersed submanifold of codimension 1.

You should be aware that there are variations in how smooth submanifolds are
defined in the literature. Some authors reserve the unqualified term "submanifold"
to mean what we call an embedded submanifold. If there is room for confusion, it is
safest to specify explicitly which type of submanifold—embedded or immersed—is
meant. Even though both terms "smooth submanifold" and "immersed submani-
fold" encompass embedded ones as well, when we are considering general sub-
manifolds we sometimes use the phrase **immersed or embedded submanifold** as a
reminder that the discussion applies equally to the embedded case.

(Some authors define immersed submanifolds even more generally than we have,
as images of smooth immersions with no injectivity requirement. Such a subman-
ifold can have "self-crossings" at points where the immersion fails to be injective.
We do not consider such sets as submanifolds, but it is good to be aware that some
authors do.)

There are also various notions of submanifolds in the topological category. For
example, if M is a topological manifold, one could define an **immersed topological
submanifold of M** to be a subset $S \subseteq M$ endowed with a topology such that it
is a topological manifold and such that the inclusion map is a topological immer-
sion. It is an **embedded topological submanifold** if the inclusion is a topological
embedding. To be entirely consistent with our convention of assuming by default
only continuity rather than smoothness, we would have to distinguish the types of
submanifolds we have defined in this chapter by calling them *smooth embedded
submanifolds* and *smooth immersed submanifolds*, respectively; but since we have
no reason to treat topological submanifolds in this book, for the sake of simplicity
let us agree that the terms *embedded submanifold* and *immersed submanifold* always
refer to the smooth kind.

Immersed submanifolds often arise in the following way.

Proposition 5.18 (Images of Immersions as Submanifolds). *Suppose M is a
smooth manifold with or without boundary, N is a smooth manifold, and $F: N \to
M$ is an injective smooth immersion. Let $S = F(N)$. Then S has a unique topol-
ogy and smooth structure such that it is a smooth submanifold of M and such that
$F: N \to S$ is a diffeomorphism onto its image.*

Proof. The proof is very similar to that of Proposition 5.2, except that now we also
have to define the topology on S. We give S a topology by declaring a set $U \subseteq S$ to
be open if and only if $F^{-1}(U) \subseteq N$ is open, and then give it a smooth structure by
taking the smooth charts to be those of the form $\left(F(U), \varphi \circ F^{-1}\right)$, where (U, φ) is
any smooth chart for N. As in the proof of Proposition 5.2, the smooth compatibility
condition follows from that for N. With this topology and smooth structure on S,
the map F is a diffeomorphism onto its image, and these are the only topology and
smooth structure on S with this property. As in the embedding case, the inclusion

$S \hookrightarrow M$ can be written as the composition

$$ S \xrightarrow{\ F^{-1}\ } N \xrightarrow{\ F\ } M \, ; $$

in this case, the first map is a diffeomorphism and the second is a smooth immersion, so the composition is a smooth immersion.   □

Example 5.19 (The Figure-Eight and the Dense Curve on the Torus). Look back at the two examples we introduced in Chapter 4 of injective smooth immersions that are not embeddings: because the figure-eight of Example 4.19 and the dense curve of Example 4.20 are images of injective smooth immersions, they are immersed submanifolds when given appropriate topologies and smooth structures. As smooth manifolds, they are diffeomorphic to $\mathbb{R}$. They are *not* embedded submanifolds, because neither one has the subspace topology. In fact, their image sets cannot be made into embedded submanifolds even if we are allowed to change their topologies and smooth structures (see Problems 5-4 and 5-5).   ⫽

The following observation is sometimes useful when thinking about the topology of an immersed submanifold.

▶ **Exercise 5.20.** Suppose M is a smooth manifold and $S \subseteq M$ is an immersed submanifold. Show that every subset of S that is open in the subspace topology is also open in its given submanifold topology; and the converse is true if and only if S is embedded.

Given a smooth submanifold that is known only to be immersed, it is often useful to have simple criteria that guarantee that it is embedded. The next proposition gives several such criteria.

Proposition 5.21. *Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold. If any of the following holds, then S is embedded.*

(a) *S has codimension 0 in M.*
(b) *The inclusion map $S \subseteq M$ is proper.*
(c) *S is compact.*

Proof. Problem 5-3.   □

Although many immersed submanifolds are not embedded, the next proposition shows that the *local* structure of an immersed submanifold is the same as that of an embedded one.

Proposition 5.22 (Immersed Submanifolds Are Locally Embedded). *If M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold, then for each $p \in S$ there exists a neighborhood U of p in S that is an embedded submanifold of M.*

Proof. Theorem 4.25 shows that each $p \in S$ has a neighborhood U in S such that the inclusion $\iota|_U : U \hookrightarrow M$ is an embedding.   □
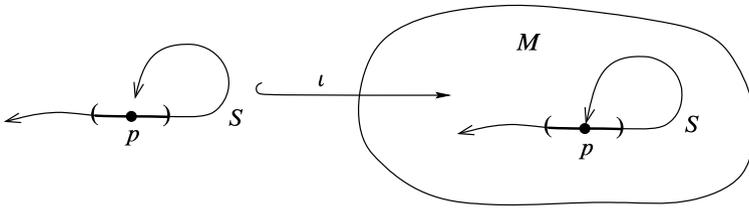
Fig. 5.8 An immersed submanifold is locally embedded

It is important to be clear about what this proposition does and does not say: given an immersed submanifold $S \subseteq M$ and a point $p \in S$, it is possible to find a neighborhood U of p (in S) such that U is embedded; but it may not be possible to find a neighborhood V of p *in M* such that $V \cap S$ is embedded (see Fig. 5.8).

Suppose $S \subseteq M$ is an immersed k-dimensional submanifold. A ***local parametrization of S*** is a continuous map $X \colon U \to M$ whose domain is an open subset $U \subseteq \mathbb{R}^k$, whose image is an open subset of S, and which, considered as a map into S, is a homeomorphism onto its image. It is called a ***smooth local parametrization*** if it is a diffeomorphism onto its image (with respect to S's smooth manifold structure). If the image of X is all of S, it is called a ***global parametrization***.

Proposition 5.23. *Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed k-submanifold, $\iota \colon S \hookrightarrow M$ is the inclusion map, and U is an open subset of $\mathbb{R}^k$. A map $X \colon U \to M$ is a smooth local parametrization of S if and only if there is a smooth coordinate chart (V, φ) for S such that $X = \iota \circ \varphi^{-1}$. Therefore, every point of S is in the image of some local parametrization.*

▶ **Exercise 5.24.** Prove the preceding proposition.

Example 5.25 (Graph Parametrizations). Suppose $U \subseteq \mathbb{R}^n$ is an open subset and $f \colon U \to \mathbb{R}^k$ is a smooth function. The map $\gamma_f \colon U \to \mathbb{R}^n \times \mathbb{R}^k$ given by $\gamma_f(u) = (u, f(u))$ is a smooth global parametrization of $\Gamma(f)$, called a ***graph parametrization***. Its inverse is the graph coordinate map constructed in Example 1.3. For example, the map $F \colon \mathbb{B}^2 \to \mathbb{R}^3$ given by

$$F(u, v) = \left(u, v, \sqrt{1 - u^2 - v^2} \right)$$

is a smooth local parametrization of $\mathbb{S}^2$ whose image is the open upper hemisphere, and whose inverse is one of the graph coordinate maps described in Example 1.4. //

Example 5.26 (Parametrization of the Figure-Eight Curve). Let $S \subseteq \mathbb{R}^2$ be the figure-eight curve of Example 5.19, considered as an immersed submanifold of $\mathbb{R}^2$. The map $\beta \colon (-\pi, \pi) \to \mathbb{R}^2$ of Example 4.19 is a smooth global parametrization of S.   //
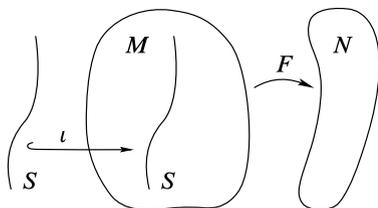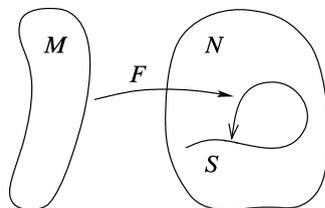
Fig. 5.9 Restricting the domain

Fig. 5.10 Restricting the codomain

Restricting Maps to Submanifolds

Given a smooth map $F\colon M \to N$, it is important to know whether F is still smooth when its domain or codomain is restricted to a submanifold. In the case of restricting the domain, the answer is easy.

Theorem 5.27 (Restricting the Domain of a Smooth Map). *If M and N are smooth manifolds with or without boundary, $F\colon M \to N$ is a smooth map, and $S \subseteq M$ is an immersed or embedded submanifold (Fig. 5.9), then $F|_S\colon S \to N$ is smooth.*

Proof. The inclusion map $\iota\colon S \hookrightarrow M$ is smooth by definition of an immersed submanifold. Since $F|_S = F \circ \iota$, the result follows.   □

When the codomain is restricted, however, the resulting map may not be smooth, as the following example shows.

Example 5.28. Let $S \subseteq \mathbb{R}^2$ be the figure-eight submanifold, with the topology and smooth structure induced by the immersion β of Example 4.19. Define a smooth map $G\colon \mathbb{R} \to \mathbb{R}^2$ by

$$G(t) = (\sin 2t, \sin t).$$

(This is the same formula that we used to define β, but now the domain is extended to the whole real line instead of being just a subinterval.) It is easy to check that the image of G lies in S. However, as a map from $\mathbb{R}$ to S, G is not even continuous, because $\beta^{-1} \circ G$ is not continuous at $t = \pi$.   //

The next theorem gives sufficient conditions for a map to be smooth when its codomain is restricted to an immersed submanifold. It shows that the failure of continuity is the only thing that can go wrong.

Theorem 5.29 (Restricting the Codomain of a Smooth Map). *Suppose M is a smooth manifold (without boundary), $S \subseteq M$ is an immersed submanifold, and $F\colon N \to M$ is a smooth map whose image is contained in S (Fig. 5.10). If F is continuous as a map from N to S, then $F\colon N \to S$ is smooth.*

Remark. This theorem is stated only for the case in which the ambient manifold M is a manifold without boundary, because it is only in that case that we have constructed slice charts for embedded submanifolds of M. But the conclusion of the theorem is still true when M has nonempty boundary; see Problem 9-13.
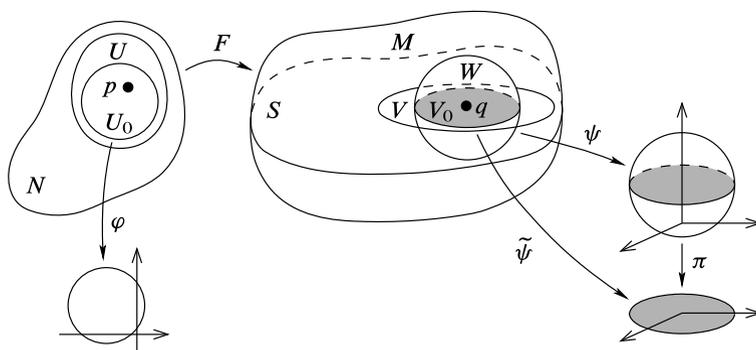
Fig. 5.11 Proof of Theorem 5.29

Proof. Let $p \in N$ be arbitrary and let $q = F(p) \in S$. Proposition 5.22 guarantees that there is a neighborhood V of q in S such that $\iota|_V : V \hookrightarrow M$ is a smooth embedding. Thus there exists a smooth chart (W, ψ) for M that is a slice chart for V in M centered at q (Fig. 5.11). (It might not be a slice chart for S in M.) The fact that (W, ψ) is a slice chart means that $\left(V_0, \widetilde{\psi}\right)$ is a smooth chart for V, where $V_0 = W \cap V$ and $\widetilde{\psi} = \pi \circ \psi$, with $\pi \colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the first $k = \dim S$ coordinates. Since $V_0 = (\iota|_V)^{-1}(W)$ is open in V, it is open in S in its given topology, and so $\left(V_0, \widetilde{\psi}\right)$ is also a smooth chart for S.

Let $U = F^{-1}(V_0)$, which is an open subset of N containing p. (Here is where we use the hypothesis that F is continuous into S.) Choose a smooth chart (U_0, φ) for N such that $p \in U_0 \subseteq U$. Then the coordinate representation of $F \colon N \to S$ with respect to the charts (U_0, φ) and $\left(V_0, \widetilde{\psi}\right)$ is

$$\widetilde{\psi} \circ F \circ \varphi^{-1} = \pi \circ \left(\psi \circ F \circ \varphi^{-1}\right),$$

which is smooth because $F \colon N \to M$ is smooth. $\qquad\square$

In the special case in which the submanifold S is embedded, the continuity hypothesis is always satisfied.

Corollary 5.30 (Embedded Case). *Let M be a smooth manifold and $S \subseteq M$ be an embedded submanifold. Then every smooth map $F \colon N \to M$ whose image is contained in S is also smooth as a map from N to S.*

Proof. Since $S \subseteq M$ has the subspace topology, a continuous map $F \colon N \to M$ whose image is contained in S is automatically continuous into S, by the characteristic property of the subspace topology (Proposition A.17(a)). $\qquad\square$

Although the conclusion of the preceding corollary fails for some immersed submanifolds such as the figure-eight curve (see Example 5.28), it turns out that there are certain immersed but nonembedded submanifolds for which it holds. To distinguish them, we introduce the following definition. If M is a smooth manifold and $S \subseteq M$ is an immersed submanifold, then S is said to be ***weakly embedded in***

M if every smooth map $F: N \to M$ whose image lies in S is smooth as a map from N to S. (Weakly embedded submanifolds are called ***initial submanifolds*** by some authors.) Corollary 5.30 shows that every embedded submanifold is weakly embedded. It follows from Example 5.28 that the figure-eight curve is not weakly embedded. However, the dense curve on the torus *is* weakly embedded; see Problem 5-13. In Chapter 19, we will encounter some classes of submanifolds that are automatically weakly embedded (see Theorems 19.17 and 19.25).

Uniqueness of Smooth Structures on Submanifolds

Using the preceding results about restricting maps to submanifolds, we can prove the promised uniqueness theorem for the smooth manifold structure on an embedded submanifold.

Theorem 5.31. *Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. The subspace topology on S and the smooth structure described in Theorem 5.8 are the only topology and smooth structure with respect to which S is an embedded or immersed submanifold.*

Proof. Suppose $S \subseteq M$ is an embedded k-dimensional submanifold. Theorem 5.8 shows that it satisfies the local k-slice condition, so it is an embedded submanifold with the subspace topology and the smooth structure of Theorem 5.8. Suppose there were some other topology and smooth structure on S making it into an immersed submanifold of some dimension. Let $\widetilde{S}$ denote the same set S, considered as a smooth manifold with the non-standard topology and smooth structure, and let $\tilde{\iota}: \widetilde{S} \hookrightarrow M$ denote the inclusion map, which by assumption is an injective immersion (but not necessarily an embedding). Because $\tilde{\iota}(\widetilde{S}) = S$, Corollary 5.30 implies that $\tilde{\iota}$ is also smooth when considered as a map from $\widetilde{S}$ to S. For each $p \in \widetilde{S}$, the differential $d\tilde{\iota}_p: T_p\widetilde{S} \to T_pM$ is equal to the composition

$$T_p\widetilde{S} \xrightarrow{\ d\tilde{\iota}_p\ } T_pS \xrightarrow{\ d\iota_p\ } T_pM,$$

where $\iota: S \hookrightarrow M$ is also inclusion. Because this composition is injective (since $\widetilde{S}$ is assumed to be a smooth submanifold of M), $d\tilde{\iota}_p$ must be injective. In particular, this means that $\tilde{\iota}: \widetilde{S} \to S$ is an immersion. Because it is bijective, it follows from the global rank theorem that it is a diffeomorphism. In other words, the topology and smooth manifold structure of $\widetilde{S}$ are the same as those of S. $\square$

Thanks to this uniqueness result, we now know that a subset $S \subseteq M$ is an embedded submanifold if and only if it satisfies the local slice condition, and if so, its topology and smooth structure are uniquely determined. Because the local slice condition is a local condition, if every point $p \in S$ has a neighborhood $U \subseteq M$ such that $U \cap S$ is an embedded k-submanifold of U, then S is an embedded k-submanifold of M.

The preceding theorem is false in general if S is merely immersed; but we do have the following uniqueness theorem for the smooth structure of an immersed submanifold once the topology is known.

Theorem 5.32. *Suppose M is a smooth manifold and $S \subseteq M$ is an immersed sub-manifold. For the given topology on S, there is only one smooth structure making S into an immersed submanifold.*

Proof. See Problem 5-14.   □

It is certainly possible for a given subset of M to have more than one topology making it into an immersed submanifold (see Problem 5-15). However, for weakly embedded submanifolds we have a stronger uniqueness result.

Theorem 5.33. *If M is a smooth manifold and $S \subseteq M$ is a weakly embedded sub-manifold, then S has only one topology and smooth structure with respect to which it is an immersed submanifold.*

Proof. See Problem 5-16.   □

Extending Functions from Submanifolds

Complementary to the restriction problem is the problem of extending smooth functions from a submanifold to the ambient manifold. Let M be a smooth manifold with or without boundary, and let $S \subseteq M$ be a smooth submanifold. If $f : S \to \mathbb{R}$ is a function, there are two ways we might interpret the statement "f is smooth": it might mean that f is smooth as a function on the smooth manifold S (i.e., each coordinate representation is smooth), or it might mean that it is smooth as a function on the subset $S \subseteq M$ (i.e., it admits a smooth extension to a neighborhood of each point). We adopt the convention that the notation $f \in C^{\infty}(S)$ always means that f is smooth in the former sense (as a function on the manifold S).

Lemma 5.34 (Extension Lemma for Functions on Submanifolds). *Suppose M is a smooth manifold, $S \subseteq M$ is a smooth submanifold, and $f \in C^{\infty}(S)$.*

(a) *If S is embedded, then there exist a neighborhood U of S in M and a smooth function $\tilde{f} \in C^{\infty}(U)$ such that $\tilde{f}|_S = f$.*
(b) *If S is properly embedded, then the neighborhood U in part (a) can be taken to be all of M.*

Proof. Problem 5-17.   □

Problem 5-18 shows that the hypotheses in both (a) and (b) are necessary.

The Tangent Space to a Submanifold

If S is a smooth submanifold of $\mathbb{R}^n$, we intuitively think of the tangent space $T_p S$ at a point of S as a subspace of the tangent space $T_p \mathbb{R}^n$. Similarly, the tangent space to a smooth submanifold of an abstract smooth manifold can be viewed as a subspace of the tangent space to the ambient manifold, once we make appropriate identifications.
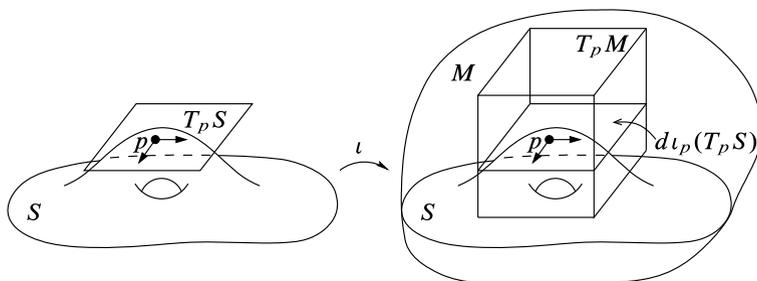
Fig. 5.12 The tangent space to an embedded submanifold

Let M be a smooth manifold with or without boundary, and let $S \subseteq M$ be an immersed or embedded submanifold. Since the inclusion map $\iota\colon S \hookrightarrow M$ is a smooth immersion, at each point $p \in S$ we have an injective linear map $d\iota_p\colon T_p S \to T_p M$. In terms of derivations, this injection works in the following way: for any vector $v \in T_p S$, the image vector $\tilde{v} = d\iota_p(v) \in T_p M$ acts on smooth functions on M by

$$\tilde{v} f = d\iota_p(v) f = v(f \circ \iota) = v\left(f|_S\right).$$

We adopt the convention of *identifying $T_p S$* with its image under this map, thereby thinking of $T_p S$ as a certain linear subspace of $T_p M$ (Fig. 5.12). This identification makes sense regardless of whether S is embedded or immersed.

There are several alternative ways of characterizing $T_p S$ as a subspace of $T_p M$. The first one is the most general; it is just a straightforward generalization of Proposition 3.23.

Proposition 5.35. *Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed or embedded submanifold, and $p \in S$. A vector $v \in T_p M$ is in $T_p S$ if and only if there is a smooth curve $\gamma\colon J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$.*

▶ **Exercise 5.36.** Prove the preceding proposition.

The next proposition gives a useful way to characterize $T_p S$ in the embedded case. (Problem 5-20 shows that this does not work in the nonembedded case.)

Proposition 5.37. *Suppose M is a smooth manifold, $S \subseteq M$ is an embedded submanifold, and $p \in S$. As a subspace of $T_p M$, the tangent space $T_p S$ is characterized by*

$$T_p S = \left\{ v \in T_p M : vf = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f|_S = 0 \right\}.$$

Proof. First suppose $v \in T_p S \subseteq T_p M$. This means, more precisely, that $v = d\iota_p(w)$ for some $w \in T_p S$, where $\iota\colon S \to M$ is inclusion. If f is any smooth real-valued function on M that vanishes on S, then $f \circ \iota \equiv 0$, so

$$vf = d\iota_p(w) f = w(f \circ \iota) = 0.$$

Conversely, if $v \in T_p M$ satisfies $vf = 0$ whenever f vanishes on S, we need to show that there is a vector $w \in T_p S$ such that $v = d\iota_p(w)$. Let $(x^1, \dots, x^n)$ be slice coordinates for S in some neighborhood U of p, so that $U \cap S$ is the subset of U where $x^{k+1} = \cdots = x^n = 0$, and $(x^1, \dots, x^k)$ are coordinates for $U \cap S$. Because the inclusion map $\iota\colon S \cap U \hookrightarrow M$ has the coordinate representation

$$\iota(x^1, \dots, x^k) = (x^1, \dots, x^k, 0, \dots, 0)$$

in these coordinates, it follows that $T_p S$ (that is, $d\iota_p(T_p S)$) is exactly the subspace of $T_p M$ spanned by $\partial/\partial x^1|_p, \dots, \partial/\partial x^k|_p$. If we write the coordinate representation of v as

$$v = \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_p,$$

we see that $v \in T_p S$ if and only if $v^i = 0$ for $i > k$.

Let φ be a smooth bump function supported in U that is equal to 1 in a neighborhood of p. Choose an index $j > k$, and consider the function $f(x) = \varphi(x)x^j$, extended to be zero on $M \smallsetminus \operatorname{supp}\varphi$. Then f vanishes identically on S, so

$$0 = vf = \sum_{i=1}^{n} v^i \frac{\partial\left(\varphi(x)x^j\right)}{\partial x^i}(p) = v^j.$$

Thus $v \in T_p S$ as desired. $\qquad\square$

If an embedded submanifold is characterized by a defining map, the defining map gives a concise characterization of its tangent space at each point, as the next proposition shows.

Proposition 5.38. *Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold. If $\Phi\colon U \to N$ is any local defining map for S, then $T_p S = \operatorname{Ker} d\Phi_p\colon T_p M \to T_{\Phi(p)} N$ for each $p \in S \cap U$.*

Proof. Recall that we identify $T_p S$ with the subspace $d\iota_p(T_p S) \subseteq T_p M$, where $\iota\colon S \hookrightarrow M$ is the inclusion map. Because $\Phi \circ \iota$ is constant on $S \cap U$, it follows that $d\Phi_p \circ d\iota_p$ is the zero map from $T_p S$ to $T_{\Phi(p)} N$, and therefore $\operatorname{Im} d\iota_p \subseteq \operatorname{Ker} d\Phi_p$. On the other hand, $d\Phi_p$ is surjective by the definition of a defining map, so the rank–nullity law implies that

$$\dim \operatorname{Ker} d\Phi_p = \dim T_p M - \dim T_{\Phi(p)} N = \dim T_p S = \dim \operatorname{Im} d\iota_p,$$

which implies that $\operatorname{Im} d\iota_p = \operatorname{Ker} d\Phi_p$. $\qquad\square$

When the defining function Φ takes its values in $\mathbb{R}^k$, it is useful to restate the proposition in terms of component functions of Φ. The proof of the next corollary is immediate.

Corollary 5.39. *Suppose $S \subseteq M$ is a level set of a smooth submersion $\Phi = (\Phi^1, \dots, \Phi^k)\colon M \to \mathbb{R}^k$. A vector $v \in T_p M$ is tangent to S if and only if $v\Phi^1 = \cdots = v\Phi^k = 0$.* $\qquad\square$
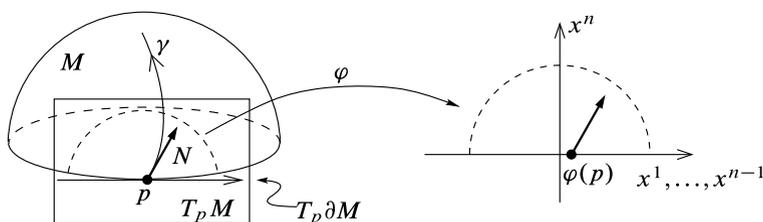
Fig. 5.13 An inward-pointing vector

▶ **Exercise 5.40.** Suppose $S \subseteq M$ is a level set of a smooth map $\Phi \colon M \to N$ with constant rank. Show that $T_p S = \operatorname{Ker} d\Phi_p$ for each $p \in S$.

If M is a smooth manifold with boundary and $p \in \partial M$, it is intuitively evident that the vectors in $T_p M$ can be separated into three classes: those tangent to the boundary, those pointing inward, and those pointing outward. Formally, we make the following definition. If $p \in \partial M$, a vector $v \in T_p M \smallsetminus T_p \partial M$ is said to be ***inward-pointing*** if for some $\varepsilon > 0$ there exists a smooth curve $\gamma \colon [0, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, and it is ***outward-pointing*** if there exists such a curve whose domain is $(-\varepsilon, 0]$. The following proposition gives another characterization of inward-pointing and outward-pointing vectors, which is usually much easier to check. (See Fig. 5.13.)

Proposition 5.41. *Suppose M is a smooth n-dimensional manifold with boundary, $p \in \partial M$, and (x^i) are any smooth boundary coordinates defined on a neighborhood of p. The inward-pointing vectors in $T_p M$ are precisely those with positive x^n-component, the outward-pointing ones are those with negative x^n-component, and the ones tangent to ∂M are those with zero x^n-component. Thus, $T_p M$ is the disjoint union of $T_p \partial M$, the set of inward-pointing vectors, and the set of outward-pointing vectors, and $v \in T_p M$ is inward-pointing if and only if $-v$ is outward-pointing.*

▶ **Exercise 5.42.** Prove Proposition 5.41.

If M is a smooth manifold with boundary, a ***boundary defining function*** for M is a smooth function $f \colon M \to [0, \infty)$ such that $f^{-1}(0) = \partial M$ and $df_p \neq 0$ for all $p \in \partial M$. For example, $f(x) = 1 - |x|^2$ is a boundary defining function for the closed unit ball $\bar{\mathbb{B}}^n$.

Proposition 5.43. *Every smooth manifold with boundary admits a boundary defining function.*

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a collection of smooth charts whose domains cover M. For each α, define a smooth function $f_\alpha \colon U_\alpha \to [0, \infty)$ as follows: if U_α is an interior chart, let $f_\alpha \equiv 1$; while if U_α is a boundary chart, let $f_\alpha(x^1, \ldots, x^n) = x^n$ (the nth coordinate function in that chart). Thus, $f_\alpha(p)$ is positive if $p \in \operatorname{Int} M$ and zero if $p \in \partial M$. Let $\{\psi_\alpha\}$ be a partition of unity subordinate to this cover, and let

$f = \sum_\alpha \psi_\alpha f_\alpha$. Then f is smooth, identically zero on ∂M, and strictly positive in Int M. To see that df does not vanish on ∂M, suppose $p \in \partial M$ and v is an inward-pointing vector at p. For each α such that $p \in U_\alpha$, we have $f_\alpha(p) = 0$ and $df_\alpha|_p(v) = dx^n|_p(v) > 0$ by Proposition 5.41. Thus

$$df_p(v) = \sum_\alpha \left(f_\alpha(p) d\psi_\alpha|_p(v) + \psi_\alpha(p) df_\alpha|_p(v) \right).$$

For each α, the first term in parentheses is zero and the second is nonnegative, and there is at least one α for which the second term is positive. Thus $df_p(v) > 0$, which implies that $df_p \neq 0$. $\qquad\square$

▶ **Exercise 5.44.** Suppose M is a smooth manifold with boundary, f is a boundary defining function, and $p \in \partial M$. Show that a vector $v \in T_p M$ is inward-pointing if and only if $vf > 0$, outward-pointing if and only if $vf < 0$, and tangent to ∂M if and only if $vf = 0$.

The results of this section have important applications to the problem of deciding whether a given subset of a smooth manifold is a submanifold. Given a smooth manifold M and a subset $S \subseteq M$, it is important to bear in mind that there are two very different questions one can ask. The simplest question is whether S is an embedded submanifold. Because embedded submanifolds are exactly those subsets satisfying the local slice condition, this is simply a question about the subset S itself: either it is an embedded submanifold or it is not, and if so, the topology and smooth structure making it into an embedded submanifold are uniquely determined (Theorem 5.31).

A more subtle question is whether S can be an immersed submanifold. In this case, neither the topology nor the smooth structure is known in advance, so one needs to ask whether there exist *any* topology and smooth structure on S making it into an immersed submanifold. This question is not always straightforward to answer, and it can be especially tricky to prove that S is *not* a smooth submanifold. A typical approach is to assume that it is, and then use one or more of the following phenomena to derive a contradiction:

- At each $p \in S$, the tangent space $T_p S$ is a linear subspace of $T_p M$, with the same dimension at each point.
- Each point of S is in the image of a local parametrization of S.
- Each vector tangent to S is the velocity vector of some smooth curve in S.
- Each vector tangent to S annihilates every smooth function that is constant on S.

Here is one example of how this can be done; others can be found in Problems 5-4 through 5-11.

Example 5.45. Consider the subset $S = \{(x, y) : y = |x|\} \subseteq \mathbb{R}^2$. It is easy to check that $S \smallsetminus \{(0, 0)\}$ is an embedded 1-dimensional submanifold of $\mathbb{R}^2$, so if S itself is a smooth submanifold at all, it must be 1-dimensional. Suppose there were some smooth manifold structure on S making it into an immersed submanifold. Then $T_{(0,0)} S$ would be a 1-dimensional subspace of $T_{(0,0)} \mathbb{R}^2$, so by Proposition 5.35,
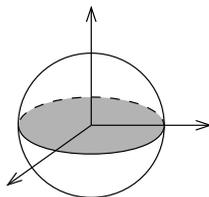
Fig. 5.14 A submanifold with boundary in a manifold with boundary

there would be a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ whose image lies in S, and that satisfies $\gamma(0) = (0,0)$ and $\gamma'(0) \neq 0$. Writing $\gamma(t) = \big(x(t), y(t)\big)$, we see that $y(t)$ takes a global minimum at $t = 0$, so $y'(0) = 0$. On the other hand, because every point $(x, y) \in S$ satisfies $x^2 = y^2$, we have $x(t)^2 = y(t)^2$ for all t. Differentiating twice and setting $t = 0$, we conclude that $2x'(0)^2 = 2y'(0)^2 = 0$, which is a contradiction. Thus, there is no such smooth manifold structure. //

Submanifolds with Boundary

So far in this chapter, all of our submanifolds have been manifolds without boundary. For some purposes (notably in the theory of integration), it is important also to consider submanifolds that have boundaries. The definitions are straightforward generalizations of the ones for ordinary submanifolds. If M is a smooth manifold with or without boundary, a ***smooth submanifold with boundary in M*** is a subset $S \subseteq M$ endowed with a topology and smooth structure making it into a smooth manifold with boundary such that the inclusion map is a smooth immersion. If the inclusion map is an embedding, then it is called an ***embedded submanifold with boundary***; in the general case, it is an ***immersed submanifold with boundary***. The terms ***codimension*** and ***properly embedded*** are defined just as in the submanifold case.

   For example, for any positive integers $k \leq n$, the closed unit k-dimensional ball $\overline{\mathbb{B}}^k$ is a properly embedded submanifold with boundary in $\overline{\mathbb{B}}^n$, because the inclusion map $\overline{\mathbb{B}}^k \hookrightarrow \overline{\mathbb{B}}^n$ is easily seen to be a proper smooth embedding (Fig. 5.14).

   One particular type of submanifold with boundary is especially important. If M is a smooth manifold with or without boundary, a ***regular domain in M*** is a properly embedded codimension-0 submanifold with boundary. Familiar examples are the closed upper half space $\mathbb{H}^n \subseteq \mathbb{R}^n$, the closed unit ball $\overline{\mathbb{B}}^n \subseteq \mathbb{R}^n$, and the closed upper hemisphere in $\mathbb{S}^n$.

Proposition 5.46. *Suppose M is a smooth manifold without boundary and $D \subseteq M$ is a regular domain. The topological interior and boundary of D are equal to its manifold interior and boundary, respectively.*

Proof. Suppose $p \in D$ is arbitrary. If p is in the manifold boundary of D, Theorem 4.15 shows that there exist a smooth boundary chart (U, φ) for D centered at p

and a smooth chart (V, ψ) for M centered at p in which F has the coordinate representation $F(x^1, \ldots, x^n) = (x^1, \ldots, x^n)$, where $n = \dim M = \dim D$. Since D has the subspace topology, $U = D \cap W$ for some open subset $W \subseteq M$, so $V_0 = V \cap W$ is a neighborhood of p in M such that $V_0 \cap D$ consists of all the points in V_0 whose x^m coordinate is nonnegative. Thus every neighborhood of p intersects both D and $M \smallsetminus D$, so p is in the topological boundary of D.

On the other hand, suppose p is in the manifold interior of D. The manifold interior is a smooth embedded codimension-0 submanifold without boundary in M, so it is an open subset by Proposition 5.1. Thus p is in the topological interior of D.

Conversely, if p is in the topological interior of D, then it is not in the topological boundary, so the preceding argument shows that it is not in the manifold boundary and hence must be in the manifold interior. Similarly, if p is in the topological boundary, it is also in the manifold boundary. □

Here are some ways in which regular domains often arise.

Proposition 5.47. *Suppose M is a smooth manifold and $f \in C^\infty(M)$.*

(a) *For each regular value b of f, the sublevel set $f^{-1}((-\infty, b])$ is a regular domain in M.*

(b) *If a and b are two regular values of f with $a < b$, then $f^{-1}([a, b])$ is a regular domain in M.*

Proof. Problem 5-21. □

A set of the form $f^{-1}((-\infty, b])$ for b a regular value of f is called a ***regular sublevel set of f***. Part (a) of the preceding theorem shows that every regular sublevel set of a smooth real-valued function is a regular domain. If $D \subseteq M$ is a regular domain and $f \in C^\infty(M)$ is a smooth function such that D is a regular sublevel set of f, then f is called a ***defining function for D***.

Theorem 5.48. *If M is a smooth manifold and $D \subseteq M$ is a regular domain, then there exists a defining function for D. If D is compact, then f can be taken to be a smooth exhaustion function for M.*

Proof. Problem 5-22. □

Many (though not all) of the earlier results in this chapter have analogues for submanifolds with boundary. Since we will have little reason to consider nonembedded submanifolds with boundary, we focus primarily on the embedded case. The statements in the following proposition can be proved in the same way as their submanifold counterparts.

Proposition 5.49 (Properties of Submanifolds with Boundary). *Suppose M is a smooth manifold with or without boundary.*

(a) *Every open subset of M is an embedded codimension-0 submanifold with (possibly empty) boundary.*

(b) *If N is a smooth manifold with boundary and $F \colon N \to M$ is a smooth embedding, then with the subspace topology $F(N)$ is a topological manifold with*

boundary, and it has a smooth structure making it into an embedded submanifold with boundary in M.

(c) *An embedded submanifold with boundary in M is properly embedded if and only if it is closed.*

(d) *If $S \subseteq M$ is an immersed submanifold with boundary, then for each $p \in S$ there exists a neighborhood U of p in S that is embedded in M.*

▶ **Exercise 5.50.** Prove the preceding proposition.

In order to adapt the results that depended on the existence of local slice charts, we have to generalize the local k-slice condition as follows. Suppose M is a smooth manifold (without boundary). If $(U, (x^i))$ is a chart for M, a ***k*-dimensional half-slice of U** is a subset of the following form for some constants $c^{k+1}, \ldots, c^n$:

$$\left\{ (x^1, \ldots, x^n) \in U : x^{k+1} = c^{k+1}, \ldots, x^n = c^n, \text{ and } x^k \geq 0 \right\}.$$

We say that a subset $S \subseteq M$ satisfies the ***local k-slice condition for submanifolds with boundary*** if each point of S is contained in the domain of a smooth chart $(U, (x^i))$ such that $S \cap U$ is either an ordinary k-dimensional slice or a k-dimensional half-slice. In the former case, the chart is called an ***interior slice chart for S in M***, and in the latter, it is a ***boundary slice chart for S in M***.

Theorem 5.51. *Let M be a smooth n-manifold without boundary. If $S \subseteq M$ is an embedded k-dimensional submanifold with boundary, then S satisfies the local k-slice condition for submanifolds with boundary. Conversely, if $S \subseteq M$ is a subset that satisfies the local k-slice condition for submanifolds with boundary, then with the subspace topology, S is a topological k-manifold with boundary, and it has a smooth structure making it into an embedded submanifold with boundary in M.*

▶ **Exercise 5.52.** Prove the preceding theorem.

Using the preceding theorem in place of Theorem 5.8, one can readily prove the following theorem.

Theorem 5.53 (Restricting Maps to Submanifolds with Boundary). *Suppose M and N are smooth manifolds with boundary and $S \subseteq M$ is an embedded submanifold with boundary.*

(a) RESTRICTING THE DOMAIN: *If $F : M \to N$ is a smooth map, then $F|_S : S \to N$ is smooth.*

(b) RESTRICTING THE CODOMAIN: *If $\partial M = \varnothing$ and $F : N \to M$ is a smooth map whose image is contained in S, then F is smooth as a map from N to S.*

Remark. The requirement that $\partial M = \varnothing$ can be removed in part (b) just as for Theorem 5.29; see Problem 9-13.

▶ **Exercise 5.54.** Prove Theorem 5.53.

Problems

5-1. Consider the map $\Phi\colon \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$\Phi(x, y, s, t) = \left(x^2 + y, x^2 + y^2 + s^2 + t^2 + y\right).$$

Show that $(0, 1)$ is a regular value of Φ, and that the level set $\Phi^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.

5-2. Prove Theorem 5.11 (the boundary of a manifold with boundary is an embedded submanifold).

5-3. Prove Proposition 5.21 (sufficient conditions for immersed submanifolds to be embedded).

5-4. Show that the image of the curve $\beta\colon (-\pi, \pi) \to \mathbb{R}^2$ of Example 4.19 is not an embedded submanifold of $\mathbb{R}^2$. [Be careful: this is not the same as showing that β is not an embedding.]

5-5. Let $\gamma\colon \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 4.20. Show that $\gamma(\mathbb{R})$ is not an embedded submanifold of the torus. [Remark: the warning in Problem 5-4 applies in this case as well.]

5-6. Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$UM = \left\{(x, v) \in T\mathbb{R}^n : x \in M,\ v \in T_x M,\ |v| = 1\right\}.$$

It is called the ***unit tangent bundle of*** M. Prove that UM is an embedded $(2m - 1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$. (*Used on p. 147.*)

5-7. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

5-8. Suppose M is a smooth n-manifold and $B \subseteq M$ is a regular coordinate ball. Show that $M \smallsetminus B$ is a smooth manifold with boundary, whose boundary is diffeomorphic to $\mathbb{S}^{n-1}$. (*Used on p. 225.*)

5-9. Let $S \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin (see Problem 3-5). Show that S does not have a topology and smooth structure in which it is an immersed submanifold of $\mathbb{R}^2$.

5-10. For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \left\{(x, y) : y^2 = x(x - 1)(x - a)\right\}.$$

For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values can M_a be given a topology and smooth structure making it into an immersed submanifold?

5-11. Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $\Phi(x, y) = x^2 - y^2$.
(a) Show that $\Phi^{-1}(0)$ is not an embedded submanifold of $\mathbb{R}^2$.
(b) Can $\Phi^{-1}(0)$ be given a topology and smooth structure making it into an immersed submanifold of $\mathbb{R}^2$?

(c) Answer the same two questions for $\Psi \colon \mathbb{R}^2 \to \mathbb{R}$ defined by $\Psi(x, y) = x^2 - y^3$.

5-12. Suppose E and M are smooth manifolds with boundary, and $\pi \colon E \to M$ is a smooth covering map. Show that the restriction of π to each connected component of ∂E is a smooth covering map onto a component of ∂M. (*Used on p. 433.*)

5-13. Prove that the image of the dense curve on the torus described in Example 4.20 is a weakly embedded submanifold of $\mathbb{T}^2$.

5-14. Prove Theorem 5.32 (uniqueness of the smooth structure on an immersed submanifold once the topology is given).

5-15. Show by example that an immersed submanifold $S \subseteq M$ might have more than one topology and smooth structure with respect to which it is an immersed submanifold.

5-16. Prove Theorem 5.33 (uniqueness of the topology and smooth structure of a weakly embedded submanifold).

5-17. Prove Lemma 5.34 (the extension lemma for functions on submanifolds).

5-18. Suppose M is a smooth manifold and $S \subseteq M$ is a smooth submanifold.
   (a) Show that S is embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to a neighborhood of S in M. [Hint: if S is not embedded, let $p \in S$ be a point that is not in the domain of any slice chart. Let U be a neighborhood of p in S that is embedded, and consider a function $f \in C^\infty(S)$ that is supported in U and equal to 1 at p.]
   (b) Show that S is properly embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to all of M.

5-19. Suppose $S \subseteq M$ is an embedded submanifold and $\gamma \colon J \to M$ is a smooth curve whose image happens to lie in S. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$ for all $t \in J$. Give a counterexample if S is not embedded.

5-20. Show by giving a counterexample that the conclusion of Proposition 5.37 may be false if S is merely immersed.

5-21. Prove Proposition 5.47 (regular domains defined by smooth functions).

5-22. Prove Theorem 5.48 (existence of defining functions for regular domains).

5-23. Suppose M is a smooth manifold with boundary, N is a smooth manifold, and $F \colon M \to N$ is a smooth map. Let $S = F^{-1}(c)$, where $c \in N$ is a regular value for both F and $F|_{\partial M}$. Prove that S is a smooth submanifold with boundary in M, with $\partial S = S \cap \partial M$.